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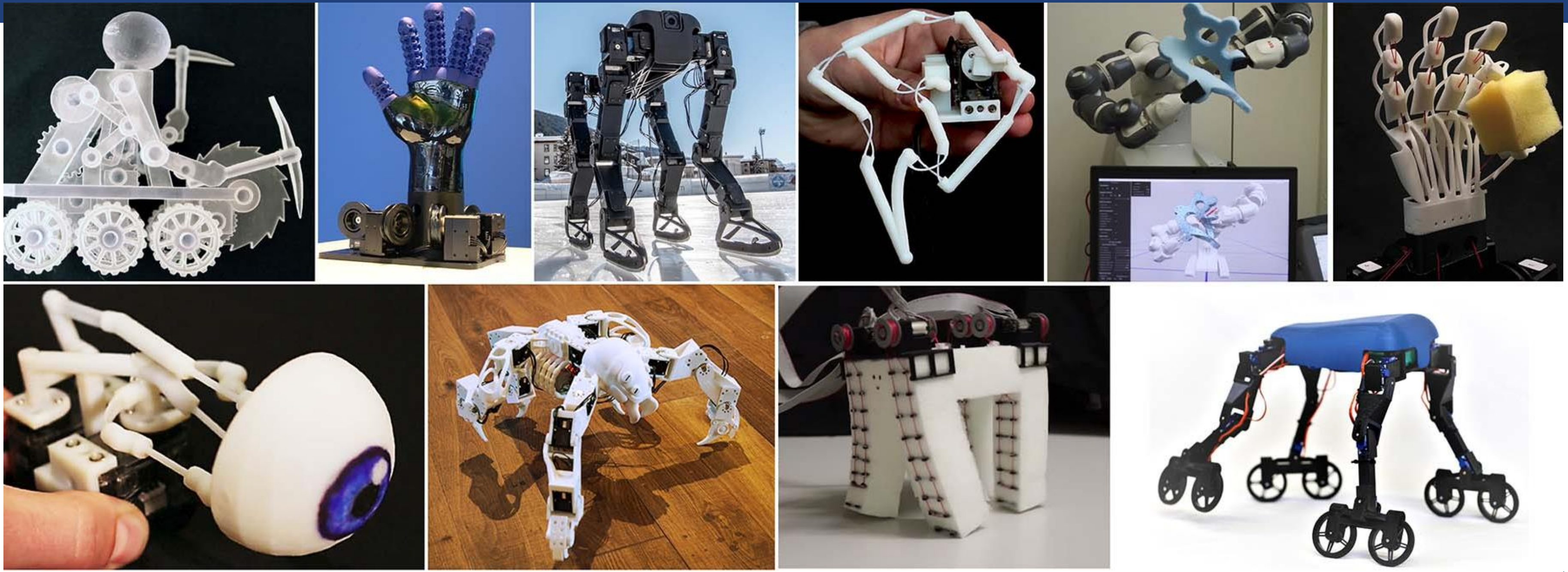

Computational Models of Motion

Contact Mechanics

Slides adapted from David Hahn

CRL

Motivation

ETH zürich

Motivation

Motivation

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Overview

- Contact constraints
- Coulomb friction
- Hard vs. soft constraints
- Soft body simulation

Contact Constraints

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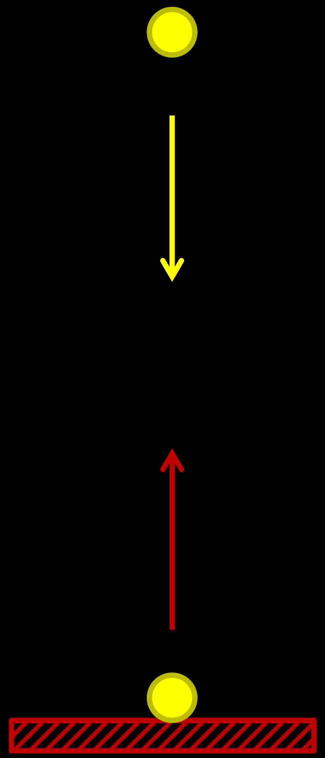
- 1D point mass (x, m) $m\ddot{x} = f$
- Gravity $f_{\text{ext}} = -9.81m$
- Obstacle (floor) $x \geq 0$
- Contact force $f_{\text{floor}} = \lambda$



Contact Constraints

- 1D point mass (x, m)

$$m\ddot{x} = -9.81m + \lambda$$

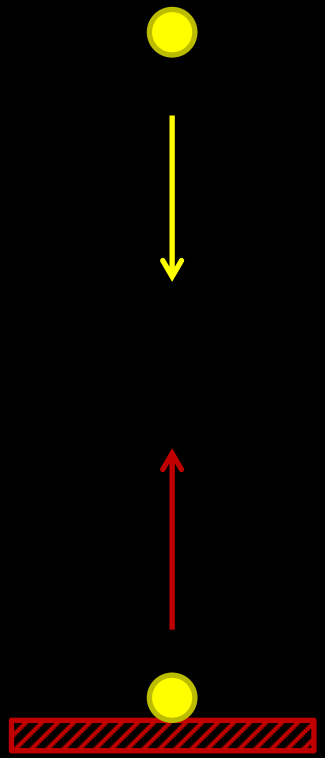
$$x \geq 0$$

- Contact force must never pull

$$\lambda \geq 0$$

- No contact force without contact

$$x\,\lambda = 0$$



Complementarity Formulation

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- Complementarity problem: find $\boldsymbol{z} \in \mathbf{R}^n$ such that
$$\mathbf{0} \leq \boldsymbol{z} \perp \boldsymbol{f}(\boldsymbol{z}) \geq \mathbf{0} \qquad (1)$$
where $\boldsymbol{f}(\boldsymbol{z}): \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a continuous function

- The complementarity condition (1) is short for
$$\boldsymbol{z} \geq \mathbf{0}, \quad \boldsymbol{f}(\boldsymbol{z}) \geq \mathbf{0} \qquad \text{and} \qquad z_i f_i(\boldsymbol{z}) = 0 \quad \forall i$$

- Complementarity conditions avoid discontinuous functions
  – Modeling $\lambda = \lambda(x)$ would require $\lambda \rightarrow \infty$
  – Using λ and x as independent variables with complementarity conditions avoids these problems

Linear Complementarity Problems

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- Linear Complementarity problem (LCP): find $\boldsymbol{z} \in \mathbf{R}^n$ such that

$$\mathbf{0} \leq \boldsymbol{z} \perp \boldsymbol{f}(\boldsymbol{z}) \geq \mathbf{0}$$

where $\boldsymbol{f}(\boldsymbol{z}) = \boldsymbol{M}\boldsymbol{z} + \boldsymbol{q} : \mathbf{R}^n \to \mathbf{R}^n$ is an affine function

Remarks

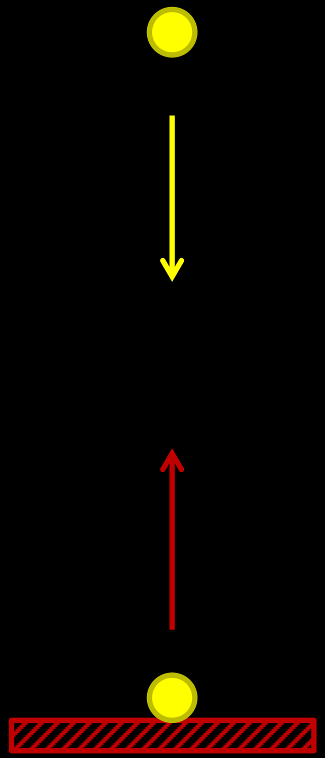
- LCP are truly nonlinear problems
- Useful for numerical modeling of contact mechanics problems

Contact constraints

- 1D point mass (x, m)

$$m\ddot{x} = -9.81m + \lambda$$
$$0 \le \lambda \perp x \ge 0$$

- What happens on impact?
  - – Stay on the floor or
  - – Bounce back?
  - – If so, how fast?



Contact constraints

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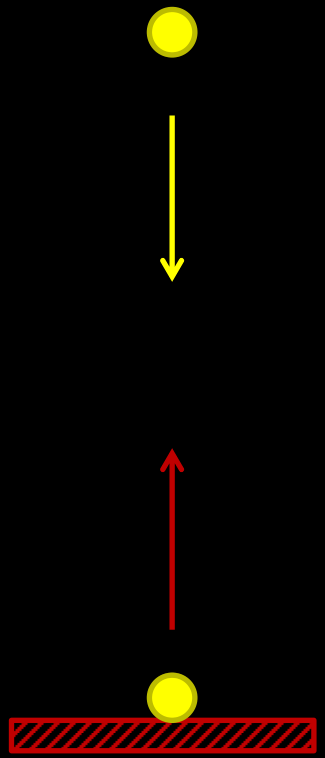
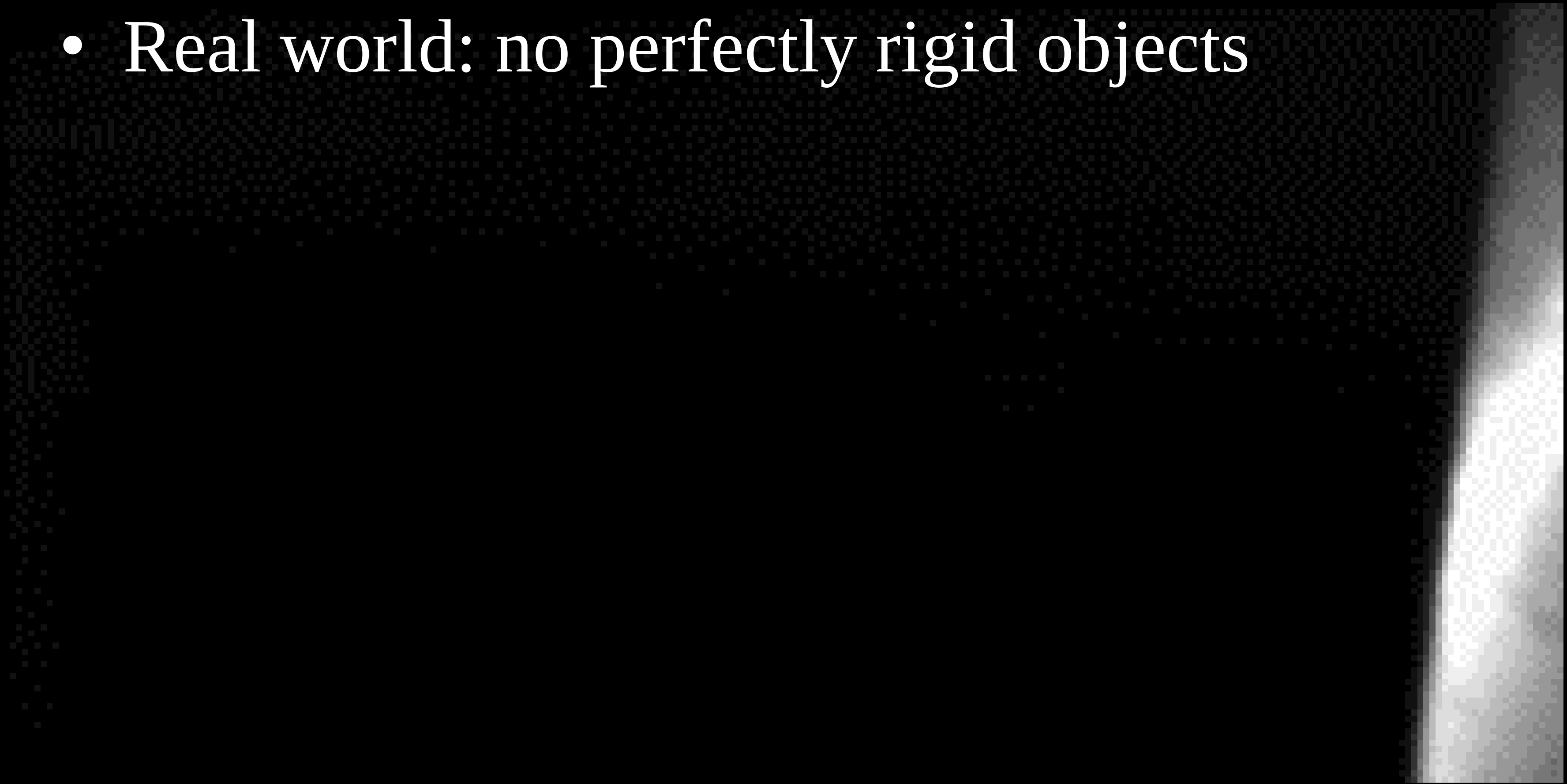
- Real world: no perfectly rigid objects
- What happens on impact?
- Rigid model: no deformation allowed



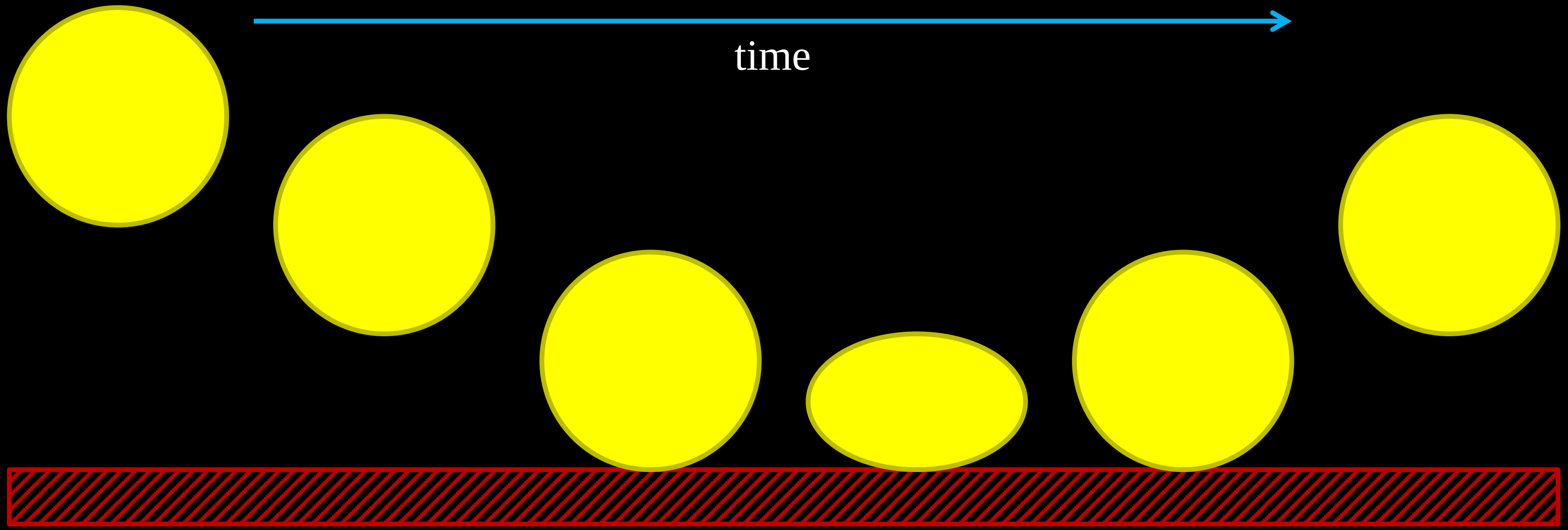
Elastic Contact

- Real world: no perfectly rigid objects



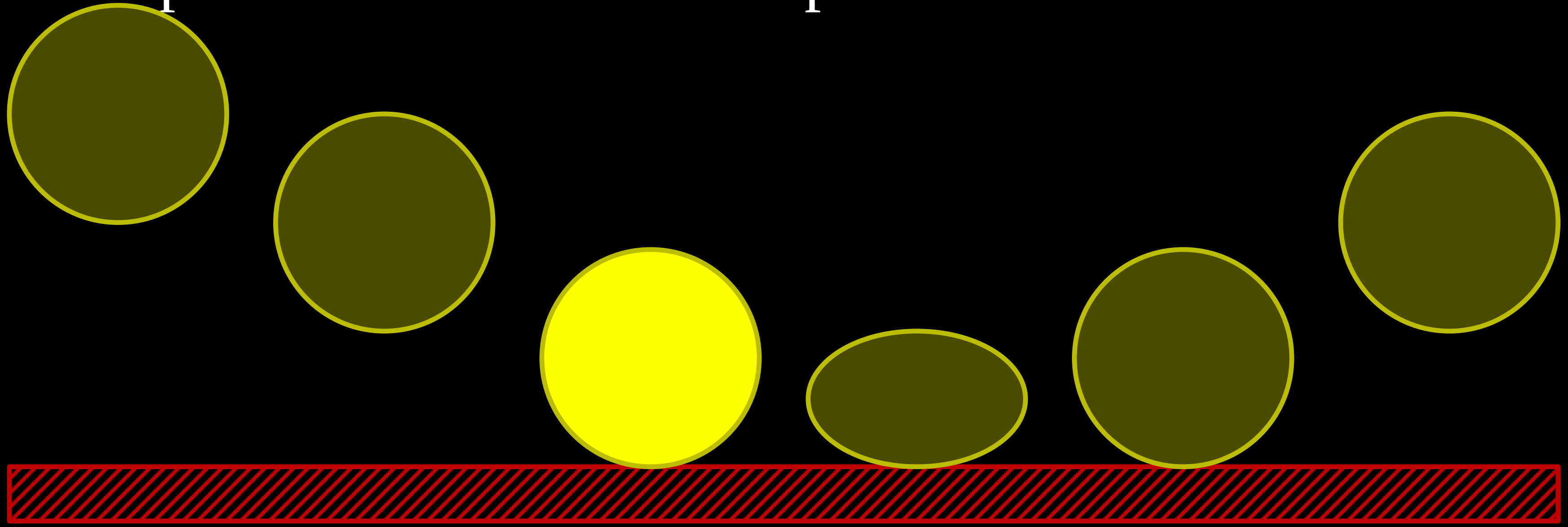
Elastic Contact

- Real world: no perfectly rigid objects



Elastic Contact

- Hertz theory of elastic contact
  - Moving down, surface in contact
  - Upward force causes compressive wave



Elastic Contact

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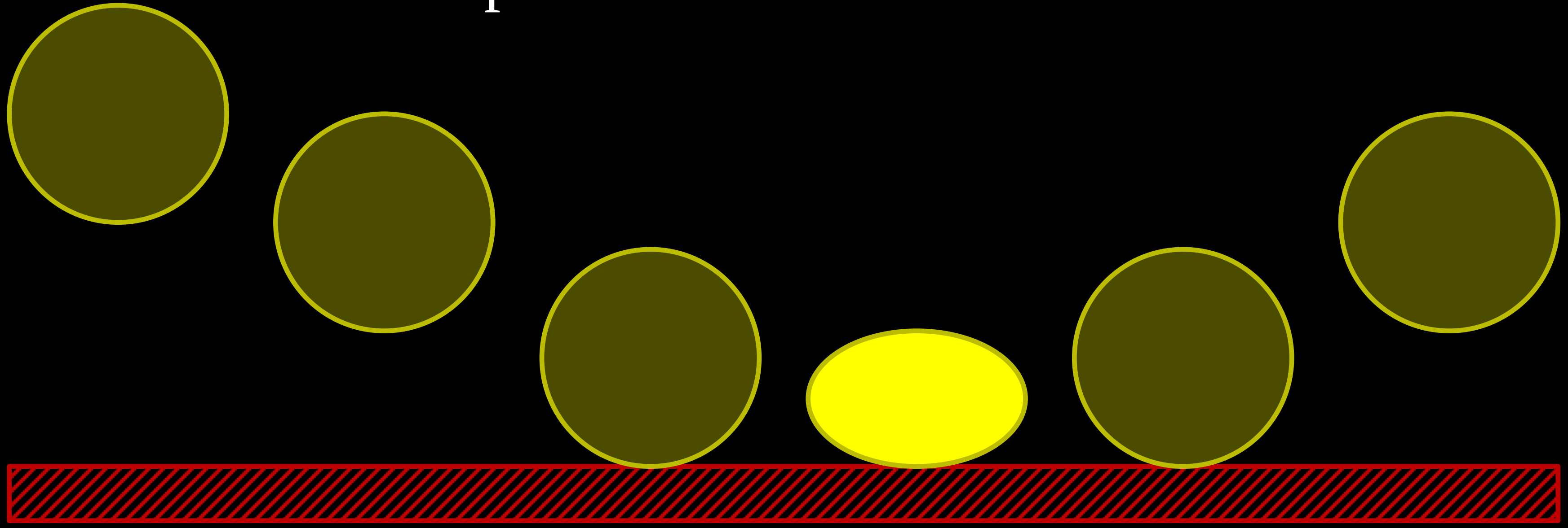
- Hertz theory of elastic contact
  - Shock wave reflects on the free surface
  - Maximal compression



Contact Constraints

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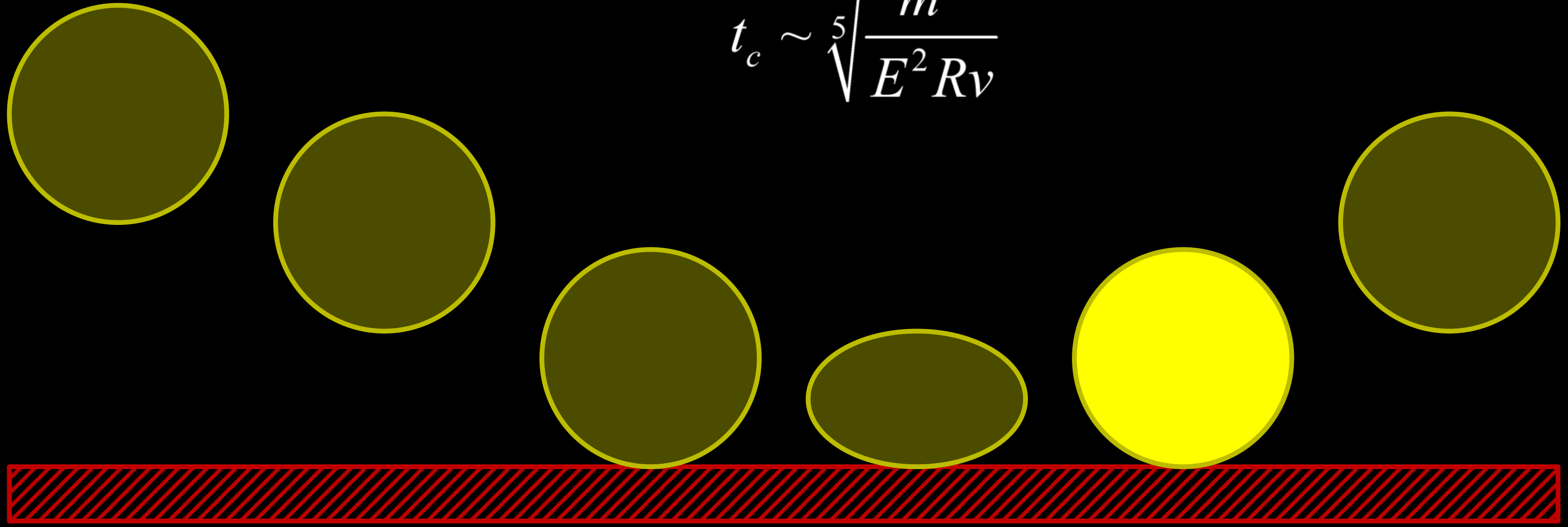
- Hertz theory of elastic contact
  - Moving back up, contact breaks
  - Duration of contact:

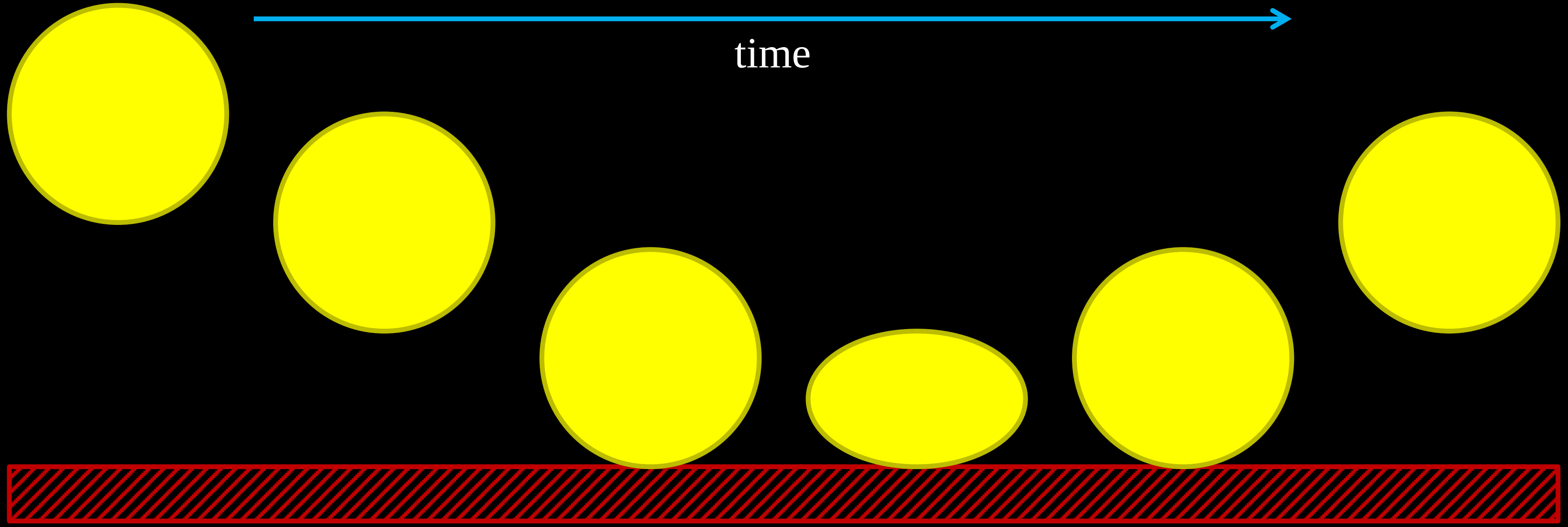
$$t_c \sim \sqrt[5]{\frac{m^2}{E^2 R v}}$$



Elastic Contact

ETH zürich

- Real world: no perfectly rigid objects

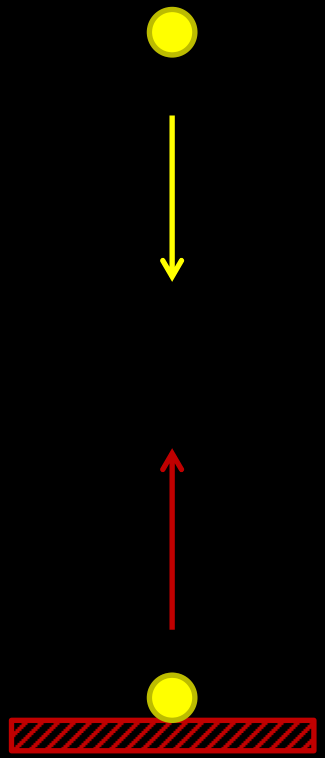


Bridging Rigidity and Elasticity

- Real world: no perfectly rigid objects

$$m\ddot{x} = -9.81m + \lambda$$
$$0 \leq \lambda \perp x \geq 0$$

- Rigid model: no deformation allowed



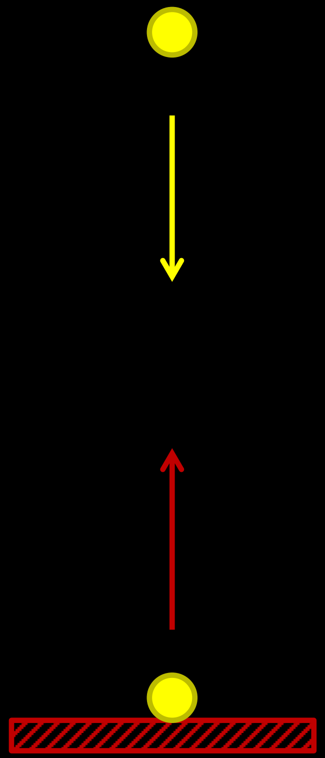
Impact Law

- Perfectly elastic contact

  → velocity reflection

$$x = 0 \wedge \dot{x}^{-} < 0:$$

$$\dot{x}^{+} = -\dot{x}^{-}$$



Contact with Impact Law

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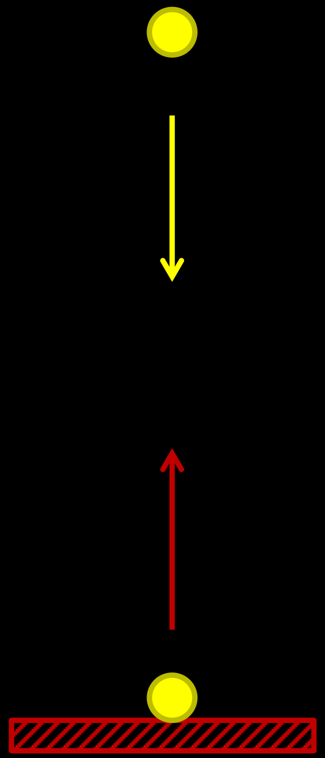
- 1D point mass (x, m)

$$x > 0:$$
$$m\ddot{x} = -9.81m$$

Free flight

$$x = 0 \wedge \dot{x}^- < 0:$$
$$m\ddot{x} = -9.81m + \lambda$$
$$0 \le \lambda \perp (\dot{x}^+ + \dot{x}^-) \ge 0$$

In contact



Time Discretization

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- BDF1 (aka. implicit Euler)

$$m\ddot{x} = f$$

$$m\dot{v} = f$$
$$\dot{x} = v$$

$$m(v_{i+1} - v_i) = \Delta_t f(x_{i+1})$$
$$x_{i+1} - x_i = \Delta_t v_{i+1}$$

$$\underbrace{mv_{i+1} - mv_i - \Delta_t f(x_i + \Delta_t v_{i+1})}_{r(v_{i+1})} = 0$$

$$\boxed{\begin{array}{l} \text{init: } \hat{v} = 0 \\ \mathbf{S} = dr / d\hat{v} \\ \Delta_v = -\mathbf{S}^{-1} r(\hat{v}) \\ \hat{v} \leftarrow \hat{v} + \Delta_v \\ \text{loop while } \|r(\hat{v})\| > \varepsilon \end{array}}$$

- How to include constraints?

Time Discretization

ETHzürich

- BDF1 (aka. implicit Euler)
- How to include constraints?

$$\underbrace{mv_{i+1} - mv_i - \Delta_t f(x_i + \Delta_t v_{i+1})}_{r(v_{i+1})} = 0$$

$$r(v_{i+1}) = \lambda$$

$$0 \le \lambda \perp (x_i + \Delta_t v_{i+1}) \ge 0$$

Time Discretization

ETH zürich

- BDF1 with contact constraints

$$r(v_{i+1}) = \lambda$$

$$0 \leq \lambda \perp (x_i + \Delta_t v_{i+1}) \geq 0$$

$$r(\hat{v} + \Delta_v) \approx r(\hat{v}) + \frac{dr}{d\hat{v}} \Delta_v$$

- Leads to LCP for the unknown Δ_v

$$\frac{dr}{d\hat{v}} \Delta_v = \lambda - r(\hat{v})$$

$$0 \leq \lambda \perp (x_i + \Delta_t(\hat{v} + \Delta_v)) \geq 0$$

Time Discretization

ETH zürich

- Leads to LCP for the unknown Δ_v

$$\frac{dr}{d\hat{v}}\Delta_v = \lambda - r(\hat{v})$$

$$0 \leq \lambda \perp (x_i + \Delta_t(\hat{v} + \Delta_v)) \geq 0$$

- Formulate as Quadratic Program (QP)

$$\min \frac{1}{2}\Delta_v^t \frac{dr}{d\hat{v}}\Delta_v + \Delta_v^t r(\hat{v})$$

$$\text{s.t. } \Delta_v \geq -(\hat{v} + \frac{1}{\Delta_t} x_i)$$

- Note: λ and complementarity conditions are implicit

Reminder: Inequality Constrained Problems

- Lagrangian

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{i \in \mathcal{E} \cup \mathcal{I}} \lambda_i c_i(x)$$

- First-order optimality (KKT) conditions

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0,$$

$$c_i(x^*) = 0, \quad \text{for all } i \in \mathcal{E},$$

$$c_i(x^*) \geq 0, \quad \text{for all } i \in \mathcal{I},$$

Feasibility: Inequality constraints have to be satisfied

$$\lambda_i^* \leq 0, \quad \text{for all } i \in \mathcal{I},$$

One-sidedness: Inequality constraints can only push, not pull

$$\lambda_i^* c_i(x^*) = 0, \quad \text{for all } i \in \mathcal{E} \cup \mathcal{I}.$$

Complementary slackness: Either constraint is active, or its LM is zero

Time Discretization

ETH zürich

- BDF1 w/ contact constraints

$$r(v_{i+1}) = \lambda$$

$$0 \le \lambda \perp (x_i + \Delta_t v_{i+1}) \ge 0$$

$$r(\hat{v} + \Delta_v) \approx r(\hat{v}) + \frac{dr}{d\hat{v}} \Delta_v$$

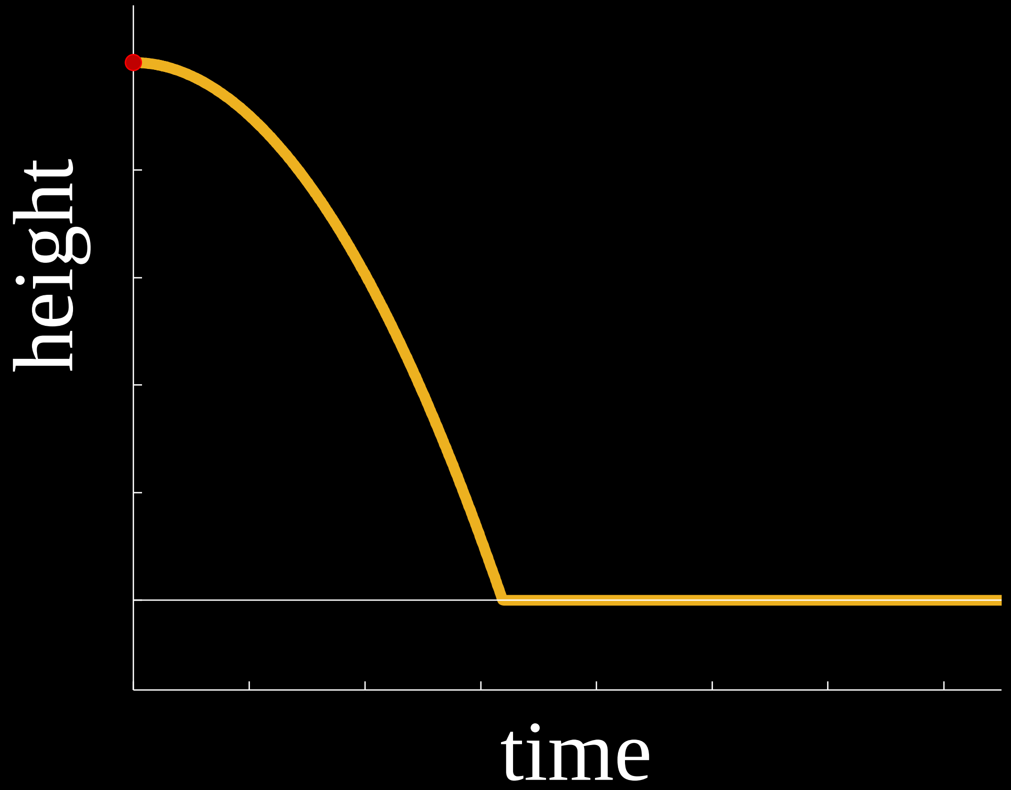
$$\min\left(\frac{1}{2}\Delta_v^\top \frac{dr}{d\hat{v}} \Delta_v + \Delta_v^\top r(\hat{v})\right)$$

$$\text{s.t. } \Delta_t \Delta_v \ge -(x_i + \Delta_t \hat{v})$$

Time Discretization

ETH zürich

- BDF1 with contact constraint



Time Discretization

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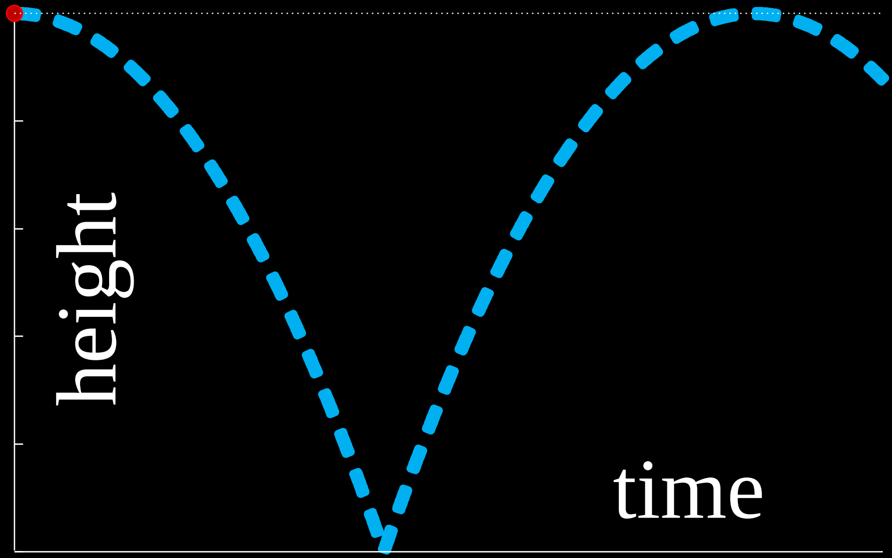
- BDF1 with impact law
  - Solve unconstrained equations of motion
  - Apply impact law when contact detected

$$x_{i+1} \leq 0 \wedge v_i < 0 :$$

$$0 \leq \lambda \perp (v_{i+1} + v_i) \geq 0$$



Hard vs. Soft Constraints

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Hard constraints

$$x \geq 0$$

$$f_{\text{floor}} = \lambda$$

Soft constraints

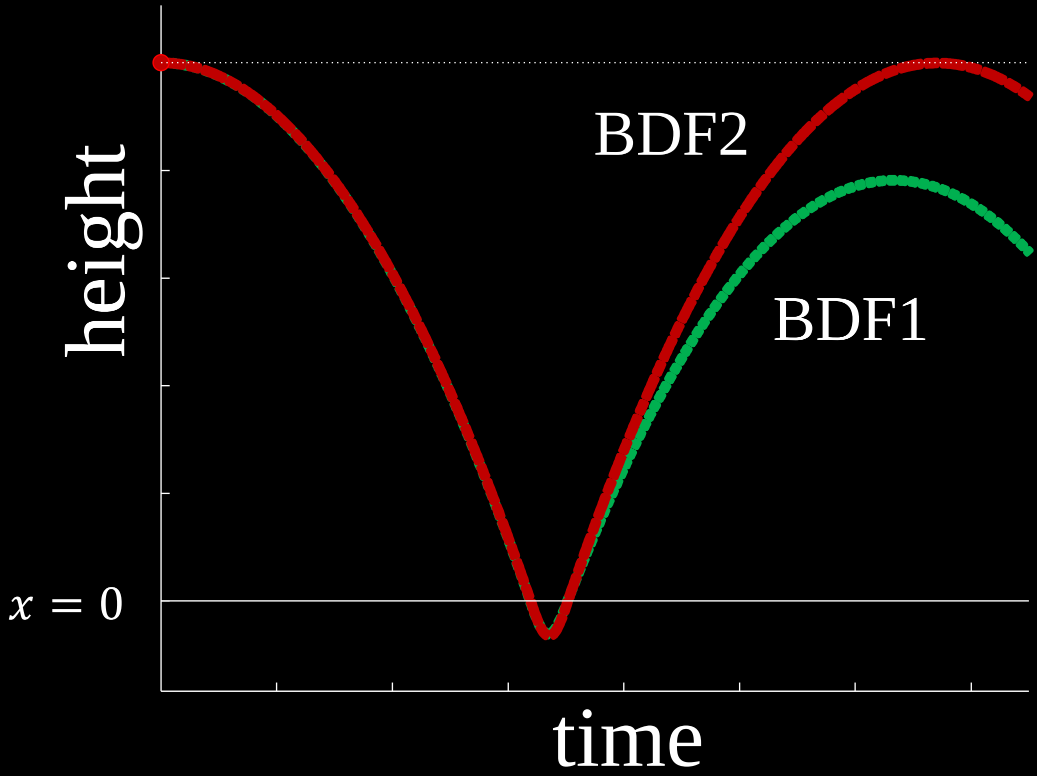
$$x < 0: \quad f_{\text{floor}} = kx$$

$$x \geq 0: \quad f_{\text{floor}} = 0$$

- Formally unconstrained
- Conservative force (spring analogy)
- Nonlinear force

Soft Constraints

- BDF1, BDF2 with soft constraints



Comparison

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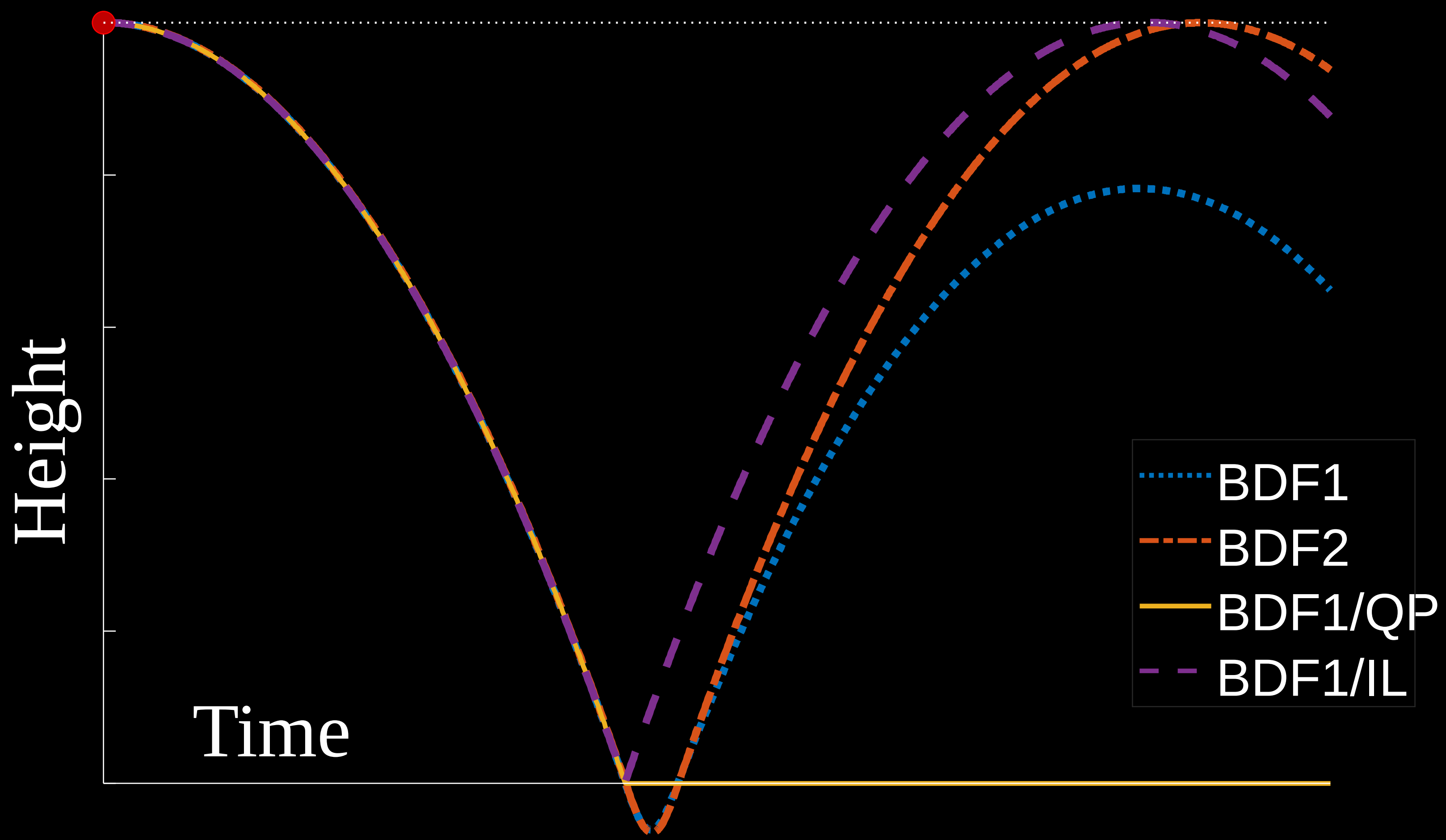
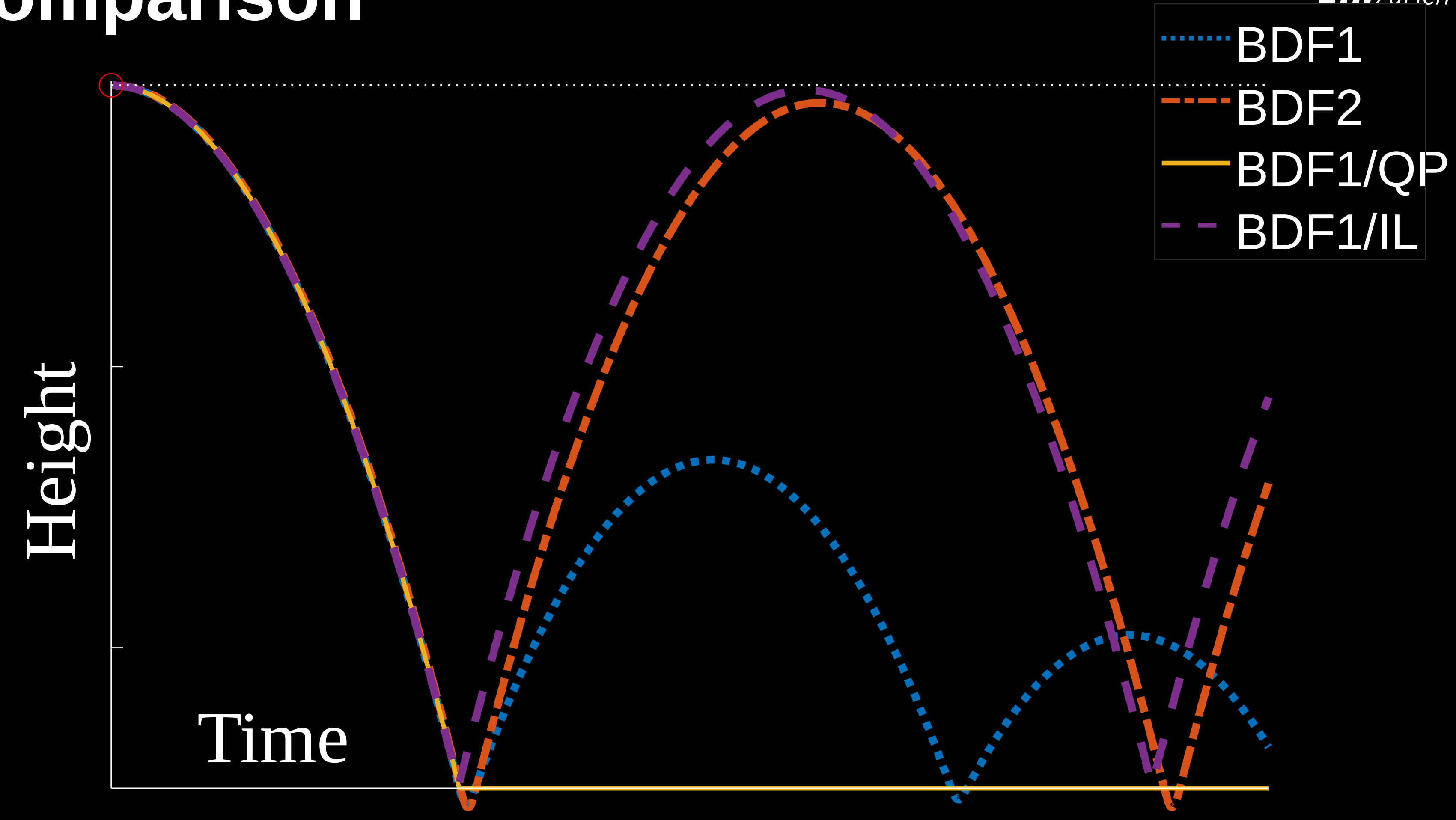


Comparison



Summary

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- So far: 1D motion normal to floor

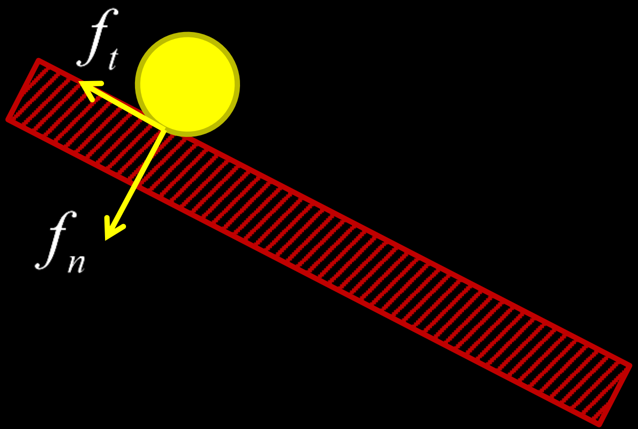
- Rigid contact: complementarity condition
  Restitution modelled via impact law

- Soft contact: allow (small) constraint violation
  Conservative forces – no need for impact law

Coulomb Friction

ETH zürich

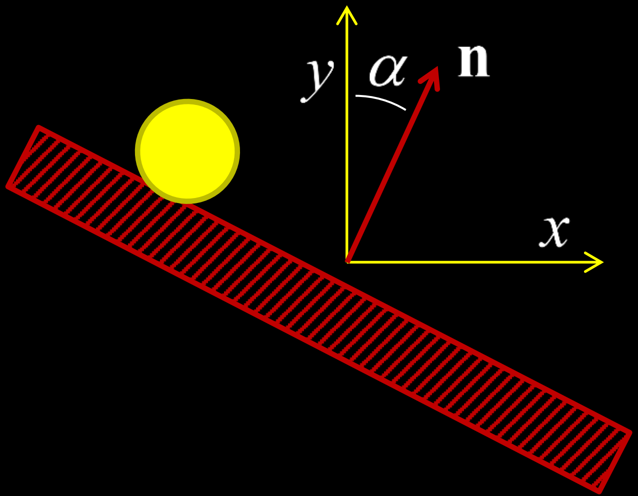
- Now: tangential motion in contact
- Coulomb model: $f_t \leq \mu f_n$
- Sticking: $f_t < \mu f_n \quad \wedge \quad v_t = 0$
- Sliding: $f_t = \mu f_n$



Coulomb Friction

- 2D point mass

$$m\ddot{\mathbf{x}} = \mathbf{f}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{f}_{\text{ext}} = \begin{pmatrix} 0 \\ -9.81m \end{pmatrix}$$



- Floor $\quad \mathbf{n}^{\mathsf{T}}\mathbf{x} \geq 0, \quad \mathbf{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix} = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}$

- Contact force $\quad \mathbf{f}_{\text{floor}} = \mathbf{f}_n + \mathbf{f}_t, \quad \mathbf{f}_n = \mathbf{n}\lambda, \quad \|\mathbf{f}_t\| \leq \mu \|\mathbf{f}_n\|$

- How to determine $\mathbf{f}_t$?

Maximum Dissipation Principle

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Out of all admissible tangential forces $\tilde{\mathbf{f}}_t \in \mathcal{F}$ with

$$\mathcal{F} = \left\{\tilde{\mathbf{f}}_t \,\middle|\, \left\|\tilde{\mathbf{f}}_t\right\| \le \mu \left\|\mathbf{f}_n\right\|\right\},$$

the friction force $\mathbf{f}_t$ is the one that maximizes the rate of energy dissipation, i.e.,

$$\mathbf{f}_t = \underset{\tilde{\mathbf{f}}_t \in \mathcal{F}}{\operatorname{argmin}} \; -\tilde{\mathbf{f}}_t^t \mathbf{v}_t$$

where $\mathbf{v}_t$ is the relative tangential velocity at the contact point.

J. Moreau: On unilateral constraints, friction, and plasticity. 1973

Coulomb Friction

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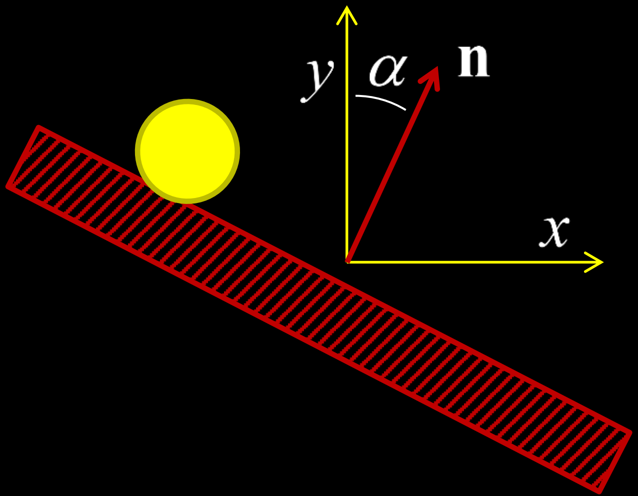
- 2D point mass

$$m\ddot{\mathbf{x}} = \mathbf{f}, \quad \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \mathbf{f}_{\text{ext}} = \begin{pmatrix} 0 \\ -9.81m \end{pmatrix}$$



- Floor $\quad \mathbf{n}^{\mathsf{T}}\mathbf{x} \geq 0, \quad \mathbf{n} = \begin{pmatrix} n_x \\ n_y \end{pmatrix} = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix}$

- Contact force $\quad \mathbf{f}_{\text{floor}} = \mathbf{f}_n + \mathbf{f}_t, \quad \mathbf{f}_n = \mathbf{n}\lambda, \quad \|\mathbf{f}_t\| \leq \mu \|\mathbf{f}_n\|$

- Max. dissipation sliding $\quad \max(-\mathbf{f}_t^{\mathsf{T}}\mathbf{T}\dot{\mathbf{x}}) \quad \text{s.t.} \|\mathbf{f}_t\| = \mu \|\mathbf{f}_n\|$

$$\mathbf{T} = (\mathbf{I} - \mathbf{n}\mathbf{n}^{\mathsf{T}})$$

Admissible Friction Forces

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- Contact force $\mathbf{f}_{\text{floor}} = \mathbf{f}_n + \mathbf{f}_t, \quad \mathbf{f}_n = \mathbf{n}\lambda, \quad \|\mathbf{f}_t\| \le \mu \|\mathbf{f}_n\|$

- 2D: friction force in interval $-\mu\lambda \le f_t \le \mu\lambda$

  Linear constraint

- 3D: friction force in disk $\|\mathbf{f}_t\| \le \mu\lambda$

  Nonlinear constraint

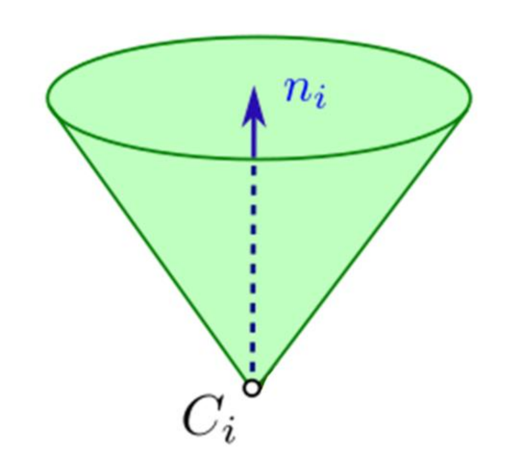
- Admissible friction force depends on normal force

  → friction cone

Friction cone

- Contact force $\mathbf{f}_{\text{floor}} = \mathbf{f}_n + \mathbf{f}_t, \quad \mathbf{f}_n = \mathbf{n}\lambda, \quad \|\mathbf{f}_t\| \leq \mu \|\mathbf{f}_n\|$

- Friction cone
  - Radius of friction force disk depends linearly on λ
  - Stacking up disks for all values of λ leads to friction cone



https://scaron.info/teaching/friction-cones.html

Friction cone

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- **Problem**: nonlinear friction constraint $\|\mathbf{f}_t\| \leq \mu\lambda$ is difficult to model
- **Idea**: linearize friction constraint → polygonal friction cone approximation

Pyramidal friction cone

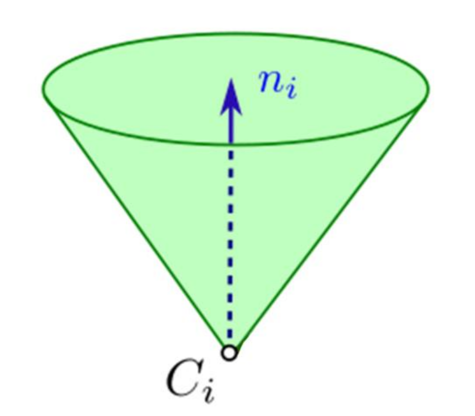
ETH zürich

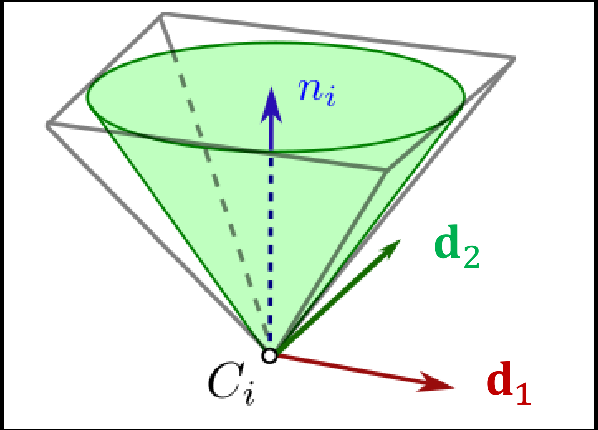
Nonlinear Cone

$\|\mathbf{f}_t\| \leq \mu \, \|\mathbf{f}_n\|$

Pyramid

$\mathbf{f}_t = \beta_1 \mathbf{d}_1 + \beta_2 \mathbf{d}_2$ with $-\tilde{\mu}\lambda \leq \beta_i \leq \tilde{\mu}\lambda$





Outer approximation:

$\tilde{\mu} = \mu$

https://scaron.info/teaching/friction-cones.html

General polygonal approximation

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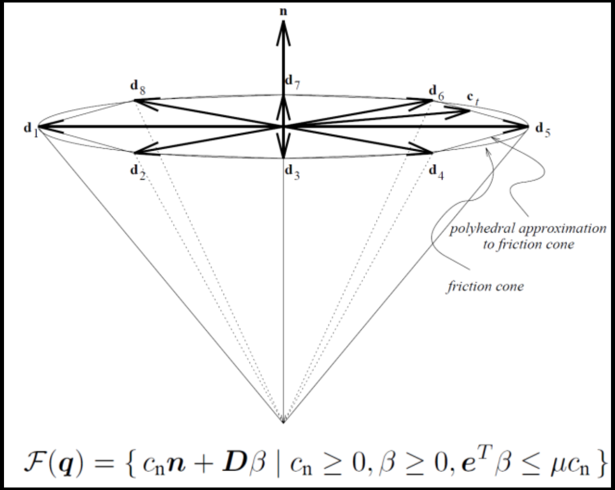
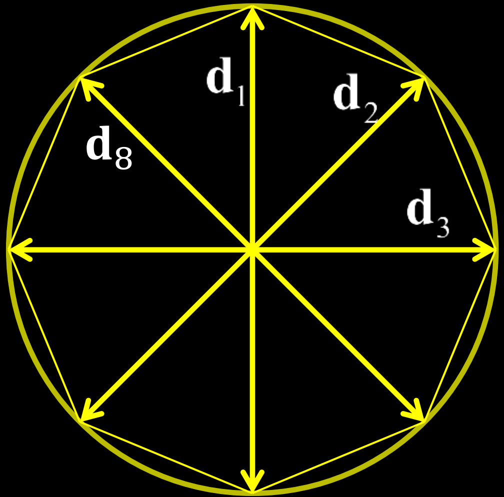
- *m*-sided linear approximation of friction cone

$$\mathbf{D} = \left(\mathbf{d}_1, \ldots, \mathbf{d}_m\right) \qquad \mathbf{f}_t = \mathbf{D}\boldsymbol{\beta}, \quad \mathbf{1}^\mathsf{T}\boldsymbol{\beta} \le \mu\lambda, \quad \boldsymbol{\beta} \ge 0$$





Images adapted from D. E. Stewart, SIAM Review 2000, 42 (1)

Coulomb friction

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see also Anitescu & Potra, Nonlinear Dynamics 14 (1997)

- Simulation time step
  - Collision detection: find contact point and normal
  - Apply impact law
  - Include approximate friction constraints
  - System of equations:

$$\mathbf{r}(\mathbf{v}_{i+1}) = \mathbf{n}\lambda + \mathbf{D}\boldsymbol{\beta}$$

Newtonian motion

$$0 \leq \lambda \perp \mathbf{n}^{\mathsf{T}}(\mathbf{v}_{i+1} + \mathbf{v}_i) \geq 0$$

Normal contact

$$\mathbf{1}^{\mathsf{T}}\boldsymbol{\beta} \leq \mu\lambda \perp \gamma \geq 0, \quad -\mathbf{D}^{\mathsf{T}}\mathbf{v}_{i+1} \leq \gamma\mathbf{1} \perp \boldsymbol{\beta} \geq 0$$

Linearized friction

Complementarity Conditions

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$$0 \le \lambda \perp \mathbf{n}^{\mathsf{T}}(\mathbf{v}_{i+1} + \mathbf{v}_i) \ge 0$$

$$\mathbf{1}^{\mathsf{T}}\boldsymbol{\beta} \le \mu\lambda \perp \gamma \ge 0, \quad -\mathbf{D}^{\mathsf{T}}\mathbf{v}_{i+1} \le \gamma\mathbf{1} \perp \boldsymbol{\beta} \ge 0$$

Case 1: if $\boldsymbol{n}^T(\boldsymbol{v}_{i+1} + \boldsymbol{v}_i) < \mathbf{0}$, no contact

$$\rightarrow \lambda = 0$$

$$\rightarrow \boldsymbol{\beta} = \mathbf{0}$$

$\rightarrow \gamma \ge 0$ such $D^T v_{i+1} + \gamma 1 \ge 0$ is satisfied.

Complementarity Conditions

ETH zürich

$$\mathbf{1}^{\mathsf{T}}\boldsymbol{\beta} \le \mu\lambda \perp \gamma \ge 0, \quad -\mathbf{D}^{\mathsf{T}}\mathbf{v}_{i+1} \le \gamma\mathbf{1} \perp \boldsymbol{\beta} \ge 0$$

Case 2: if $\mathbf{1}^T\boldsymbol{\beta} < \mu\lambda$

$$\rightarrow \gamma = 0$$

$$\rightarrow \mathbf{v}_{i+1} = 0$$

$\rightarrow$ static contact, no sliding.

Complementarity Conditions

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$$\mathbf{1}^{\top}\boldsymbol{\beta} \leq \mu\lambda \perp \gamma \geq 0, \quad -\mathbf{D}^{\top}\mathbf{v}_{i+1} \leq \gamma\mathbf{1} \perp \boldsymbol{\beta} \geq 0$$

Case 3: if $\mathbf{1}^T\boldsymbol{\beta} = \mu\lambda$

$\rightarrow \gamma > 0 \ \rightarrow \mathbf{v}_{i+1} \ \rightarrow$ sliding

- There is only one γ for all directions
- For $\beta_j > 0$, we must have $-\mathbf{d}_j^T\mathbf{v} = \gamma$
- For two active $\beta_j, \beta_k > 0$, we must have $\mathbf{d}_j^T\mathbf{v} = \mathbf{d}_k^T\mathbf{v}$

  $\rightarrow$ for direction pairs where $\boldsymbol{d}_j = -\boldsymbol{d}_k$, we must have $\beta_j\beta_k = 0$

  $\rightarrow$ typically, only one β_j will be nonzero

  $\rightarrow$ no guarantee that friction force is aligned with $\boldsymbol{v}_{i+1}$

Complementarity Conditions

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$$\mathbf{1}^{\mathsf{T}}\boldsymbol{\beta} \le \mu\lambda \perp \gamma \ge 0, \quad -\mathbf{D}^{\mathsf{T}}\mathbf{v}_{i+1} \le \gamma\mathbf{1} \perp \boldsymbol{\beta} \ge 0$$

Remark 3: from $-\mathbf{D}^T\mathrm{v}_{i+1} \le \gamma\mathbf{1} \perp \boldsymbol{\beta} = \mathbf{0}$

- We have $\beta_j\left(\boldsymbol{d}_j^t\boldsymbol{v}_{i+1} + \gamma\right) = 0 \rightarrow \ \beta_j\boldsymbol{d}_j^t\boldsymbol{v}_{i+1} \le 0\ \forall j$

  $\rightarrow$ friction force $\mathbf{D}\boldsymbol{\beta}$ acts against relative velocity (i.e., does negative work)

- It can be shown that $\boldsymbol{\beta}$ maximizes dissipation

LCP for Coulomb Friction

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- Simulation time step
  - Eliminate Δ_v
  - LCP form

$$\mathbf{A}\mathbf{z} + \mathbf{q} = \mathbf{w}$$
$$0 \le \mathbf{w} \perp \mathbf{z} \ge \mathbf{q}$$

$$\mathbf{r}(\mathbf{v}_{i+1}) = \mathbf{n}\lambda + \mathbf{D}\boldsymbol{\beta}$$
$$\mathbf{r}(\hat{\mathbf{v}} + \Delta_v) \approx \mathbf{r}(\hat{\mathbf{v}}) + \mathbf{S}\Delta_v$$
$$\mathbf{S} := d\mathbf{r} / d\hat{\mathbf{v}}$$
$$\Delta_v = \mathbf{S}^{-1}(\mathbf{n}\lambda + \mathbf{D}\beta - \mathbf{r}(\hat{\mathbf{v}}))$$

$$\mathbf{z} = \begin{pmatrix} \lambda \\ \boldsymbol{\beta} \\ \gamma \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{n}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{n} & \mathbf{n}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{D} & 0 \\ \mathbf{D}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{n} & \mathbf{D}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{D} & \mathbf{1} \\ \mu & -\mathbf{1}^{\mathsf{T}} & 0 \end{pmatrix}$$

see also Anitescu & Potra, Nonlinear Dynamics 14 (1997)

LCP for Coulomb Friction

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$$\mathbf{A}\mathbf{z} + \mathbf{q} = \mathbf{w}$$
$$0 \leq \mathbf{w} \perp \mathbf{z} \geq \mathbf{q}$$

$$\mathbf{z} = \begin{pmatrix} \lambda \\ \boldsymbol{\beta} \\ \gamma \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} \mathbf{n}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{n} & \mathbf{n}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{D} & 0 \\ \mathbf{D}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{n} & \mathbf{D}^{\mathsf{T}}\mathbf{S}^{-1}\mathbf{D} & \mathbf{1} \\ \mu & -\mathbf{1}^{\mathsf{T}} & 0 \end{pmatrix}$$

Existence of solution [Anitescu & Potra, 1997]

- The LCP is guaranteed to have a solution
- The solution can be computed using Lemke's algorithm

Coulomb Friction

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- Soft constraints

$$\mathbf{n}^\top \mathbf{x} \geq 0$$
$$\mathbf{f}_n = \mathbf{n}\lambda$$

$\Rightarrow$

$$\mathbf{n}^\top \mathbf{x} < 0: \quad \mathbf{f}_\mathrm{n} = -k_n\, \mathbf{n}\mathbf{n}^\top \mathbf{x}$$
$$\mathbf{n}^\top \mathbf{x} \geq 0: \quad \mathbf{f}_n = 0$$

- Sticking

$$\mathbf{T}\dot{\mathbf{x}} = 0$$
$$\mathbf{T} := (\mathbf{I} - \mathbf{n}\mathbf{n}^\top)$$

$\Rightarrow$

$$\mathbf{f}_t = -k_t \mathbf{T}\dot{\mathbf{x}}$$
$$\text{if } \|\mathbf{f}_t\| < \mu \|\mathbf{f}_n\|$$

- Sliding

$$\max(-\mathbf{f}_t^\top \dot{\mathbf{x}})$$
$$\text{s.t. } \|\mathbf{f}_t\| = \mu \|\mathbf{f}_n\|$$

$\Rightarrow$

$$\mathbf{f}_t = -\mu\, \mathbf{n}^\top \mathbf{f}_n\, (\mathbf{T}\dot{\mathbf{x}} / \|\mathbf{T}\dot{\mathbf{x}}\|)$$

Coulomb Friction

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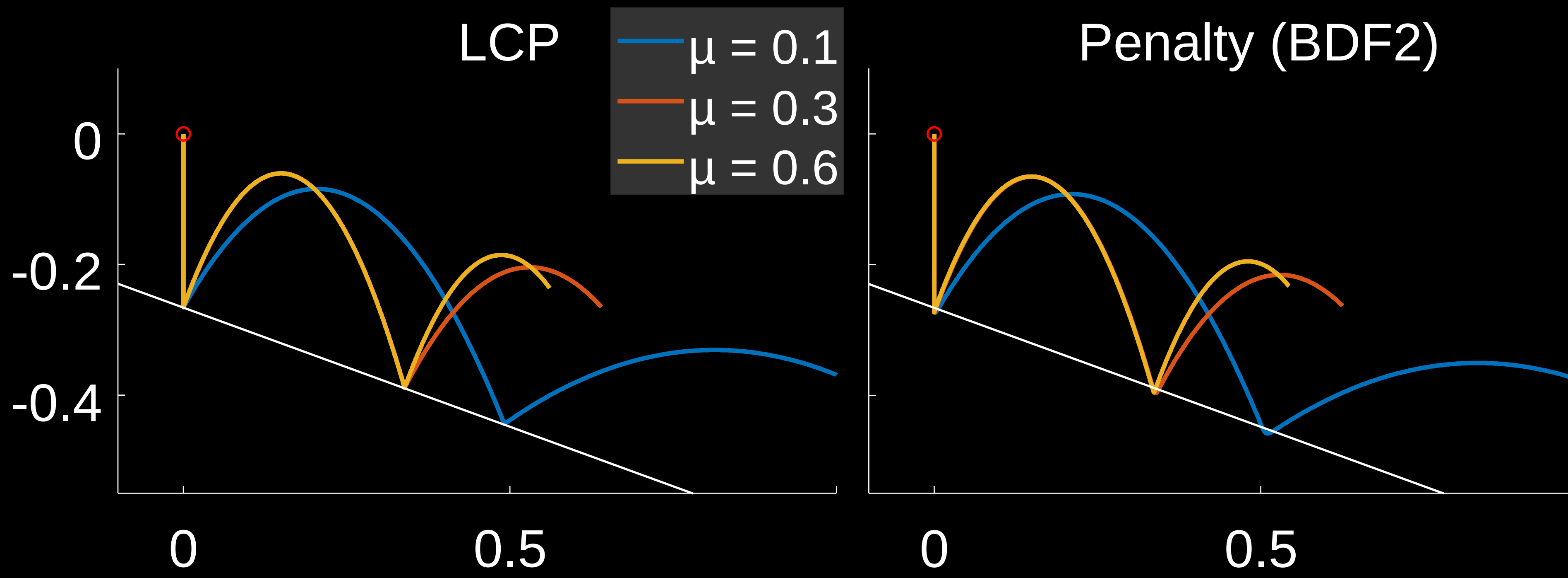

Coulomb Friction

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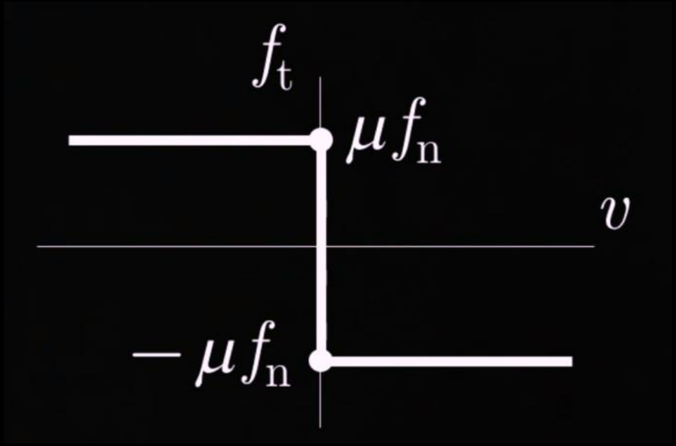
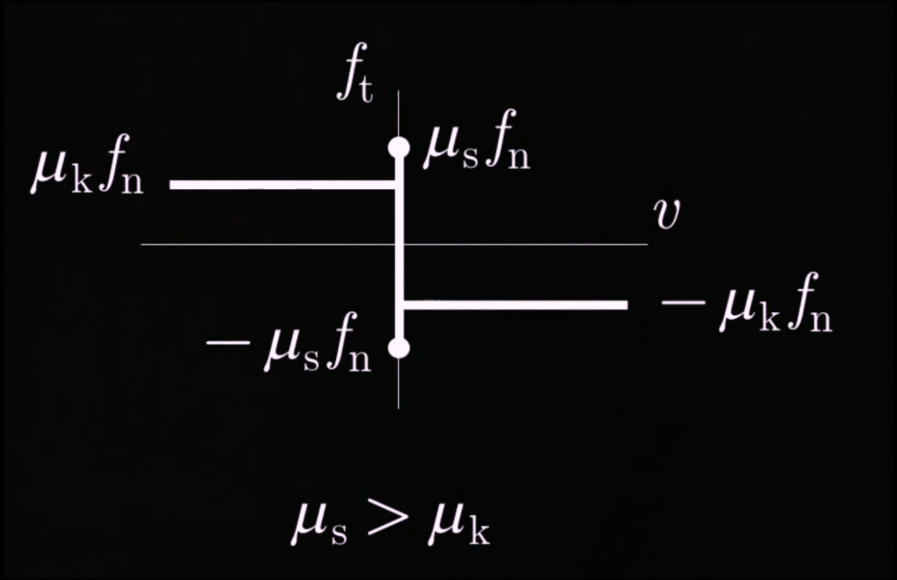
Extensions

- Separate μ_s for sticking and μ_k for sliding





Coulomb Friction

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Extensions

- Separate μ_s for sticking and μ_d for sliding
- Anisotropic models
  See also Erleben et al., "The Matchstick Model for Anisotropic Friction Cones" doi.org/10.1111/cgf.13885
- Adhesive contact

Summary

- Normal forces
  - Contact condition and normal force
  - Impact law for elastic contacts
- Tangential forces
  - Coulomb law for friction force magnitude
  - Max. dissipation principle for direction
- Complementarity as central modelling paradigm
- Linearization results in LCP

Generalization to Rigid Bodies

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- From point mass to rigid body simulation
  - Generalized coordinates: position and orientation
  - Constraints on contact points remain
  - Map contact points to generalized coordinates
  - Linearize (EoM, friction cone, rotations, ..) to obtain LCP

Soft Robots

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X-walker

Example application for soft robot control.

Soft Feet

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FEM with Contacts

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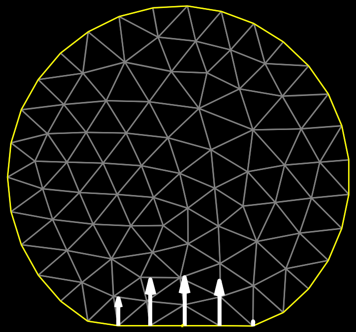
- Discretize soft body with finite elements
- Equations of motion

$$\mathbf{M}\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{v}) + \mathbf{N}\boldsymbol{\lambda} \qquad \mathbf{x} = \begin{pmatrix} x_1 \\ y_1 \\ x_2 \\ \vdots \end{pmatrix}$$

- Contact constraints

$$0 \le \boldsymbol{\lambda} \perp \mathbf{N}^{\mathsf{T}}\mathbf{x} \ge 0$$



- Note that $\mathbf{N} \in \mathbf{R}^{dn \times l}, \boldsymbol{\lambda} \in \mathbf{R}^{l}$

n ... nr. of nodes $\qquad$ d ... dimension (2 or 3)

l ... nr. of (possible) contacts

FEM with Contacts

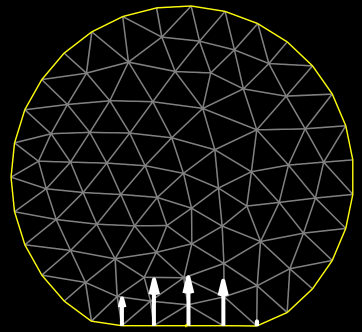
Soft body

- Motion $$\mathbf{M}\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{v}) + \mathbf{N}\boldsymbol{\lambda} + \mathbf{f}_t$$
- Contact $$0 \le \boldsymbol{\lambda} \perp \mathbf{N}^{\mathsf{T}}\mathbf{x} \ge 0$$
- Static friction (sticking)

$$\mathbf{T}\dot{\mathbf{x}} = 0 \quad \wedge \quad \|\mathbf{f}_t\| \le \mu \|\mathbf{N}\boldsymbol{\lambda}\|$$

- Dynamic friction (sliding)

$$\max(-\mathbf{f}_t^{\mathsf{T}}\mathbf{T}\dot{\mathbf{x}}) \quad \text{s.t.} \quad \|\mathbf{f}_t\| = \mu \|\mathbf{N}\boldsymbol{\lambda}\|$$



Options for Solution

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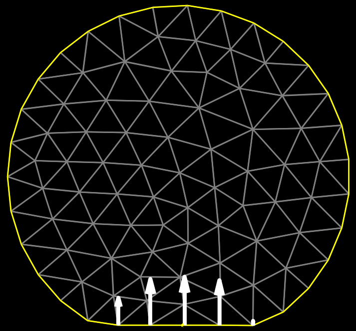
- Hard Constraints
  - Accurate solution
  - Easy for normal contact
  - Difficult for friction
- Soft constraints
  - Simple to implement and solve
  - (Small) penetrations
  - Drift for sticking contact



Soft Constraints

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- Equations of motion

$$\mathbf{M}\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mathbf{v}) + \mathbf{f}_n + \mathbf{f}_t$$

- Contact forces per node

$$\begin{aligned}
\mathbf{f}_n &= -k_n \mathbf{n}\mathbf{n}^\top \mathbf{x} && \text{if} \quad \mathbf{n}^\top \mathbf{x} \le 0 \\
\mathbf{f}_t &= -k_t \mathbf{T}\mathbf{v} && \text{if} \quad \left\| k_t \mathbf{T}\mathbf{v} \right\| < \mu \left\| \mathbf{f}_n \right\| \\
\mathbf{f}_t &= -\mu \left\| \mathbf{f}_n \right\| \mathbf{T}\mathbf{v} / \left\| \mathbf{T}\mathbf{v} \right\| && \text{otherwise}
\end{aligned}$$

FEM examples

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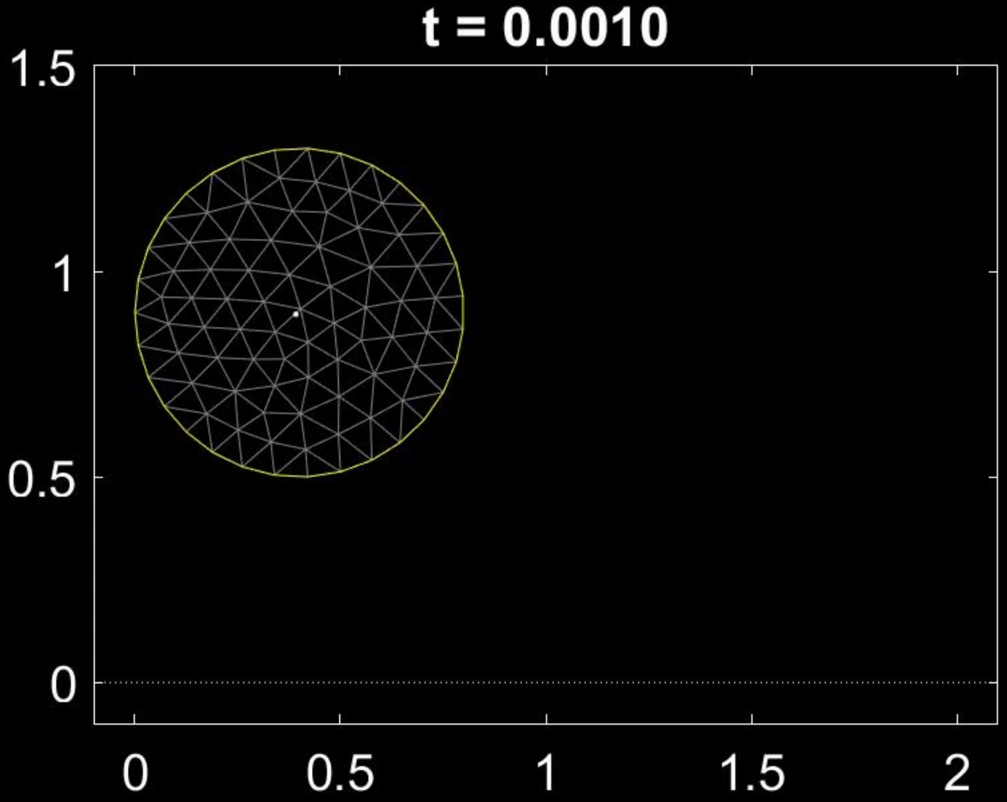
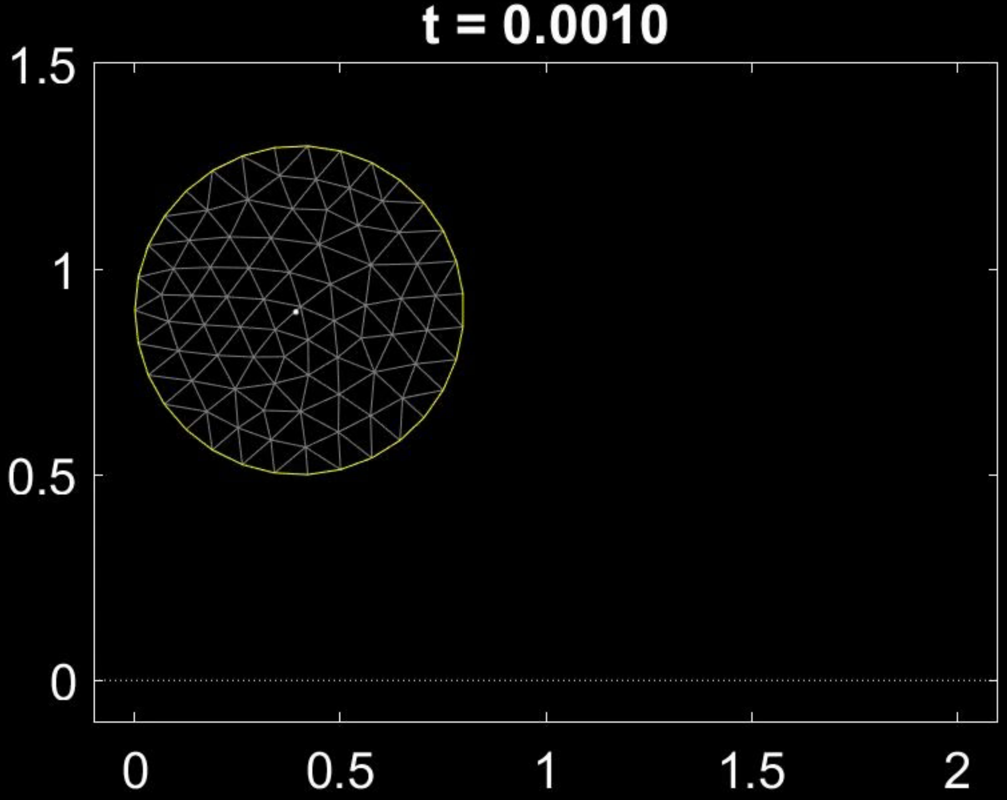
Frictionless case

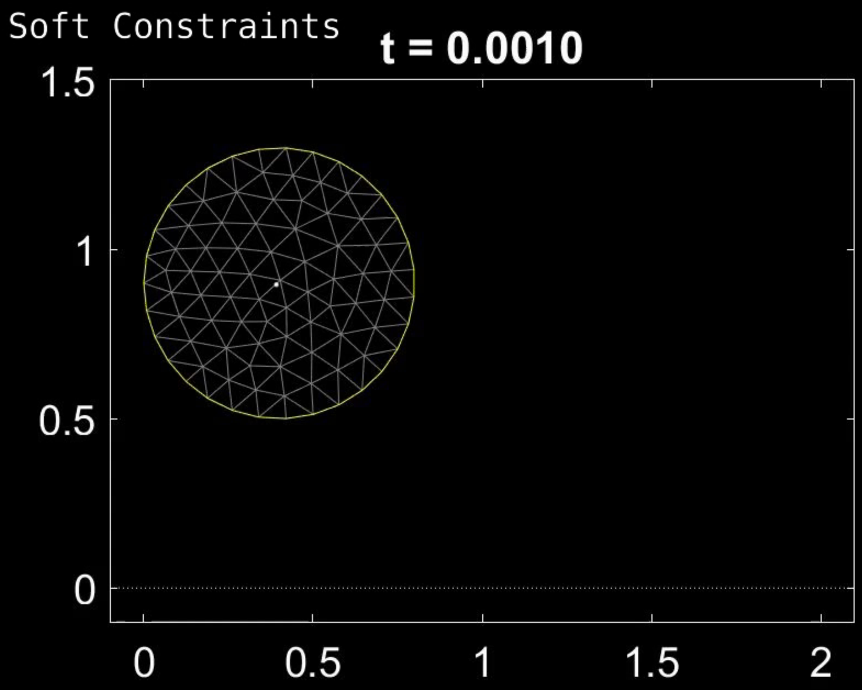


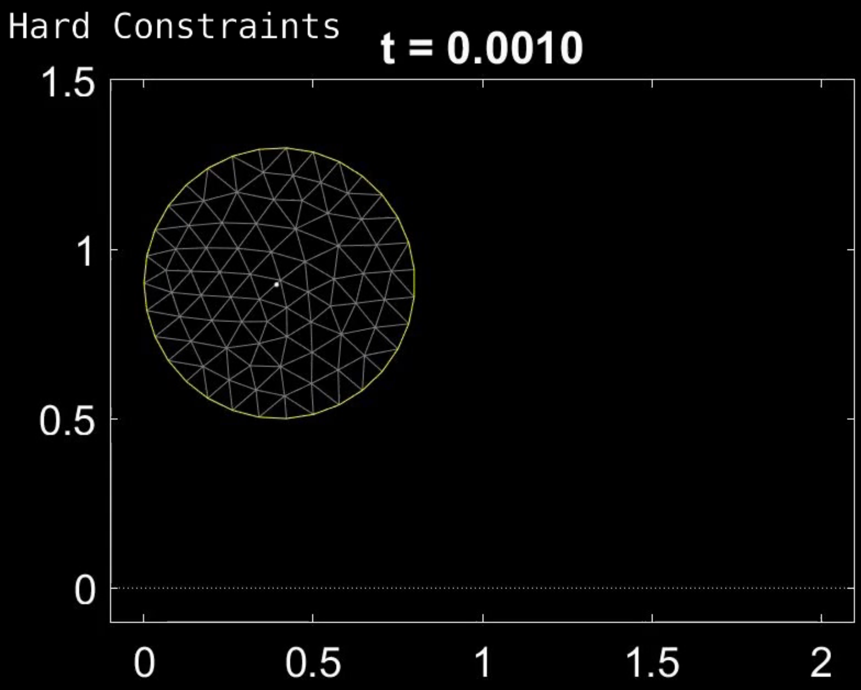
FEM Examples

Friction via soft constraints



FEM Examples





Examples

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ADD: Analytically Differentiable Dynamics for Multi-Body Systems with Frictional Contact

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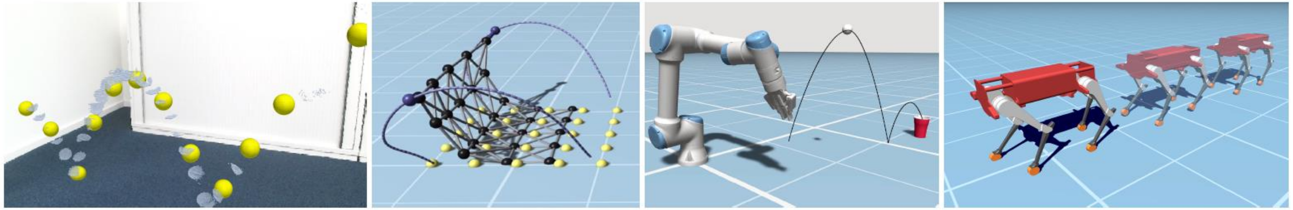

Fig. 1. Applications of our differentiable simulation framework (left-to-right): estimation of stiffness, damping and friction properties from real-world experiments, manipulation of multi-body systems, self-supervised learning of control policies for a throwing task, and physics-based motion planning for robotic creatures with compliant motors and soft feet.

M. Geilinger et al. “ADD: Analytically Differentiable Dynamics for Multi-Body Systems with Frictional Contact”. ACM Transactions on Graphics (Proc. SIGGRAPH ‘20)

ADD: Analytically Differentiable Dynamics for Multi-Body Systems with Frictional Contact

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Summary

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- Contact constraints $0 \leq \boldsymbol{\lambda} \perp \mathbf{N}^{\mathsf{T}}\mathbf{x} \geq 0$

- Coulomb friction $\left\|\mathbf{f}_t\right\| \leq \mu \left\|\mathbf{f}_n\right\|$

- Hard constraints → LCP or QP (linearized)

- Soft constraints → choose penalty factor

Further Reading

Hertz H., "Über die Berührung fester elastischer Körper", J. reine und angewandte Math. 92 (1881)

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V. L. Popov, "Contact Mechanics and Friction", Springer (2010)

Further Reading

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D.E. Stewart and J.C. Trinkle. “An implicit time-stepping scheme for rigid body dynamics with inelastic collisions and coulomb friction.” International Journal of Numerical Methods in Engineering, 39:2673–2691, 1996.

M. Geilinger et al. “ADD: Analytically Differentiable Dynamics for Multi-Body Systems with Frictional Contact”. ACM Transactions on Graphics (Proc. SIGGRAPH ‘20)