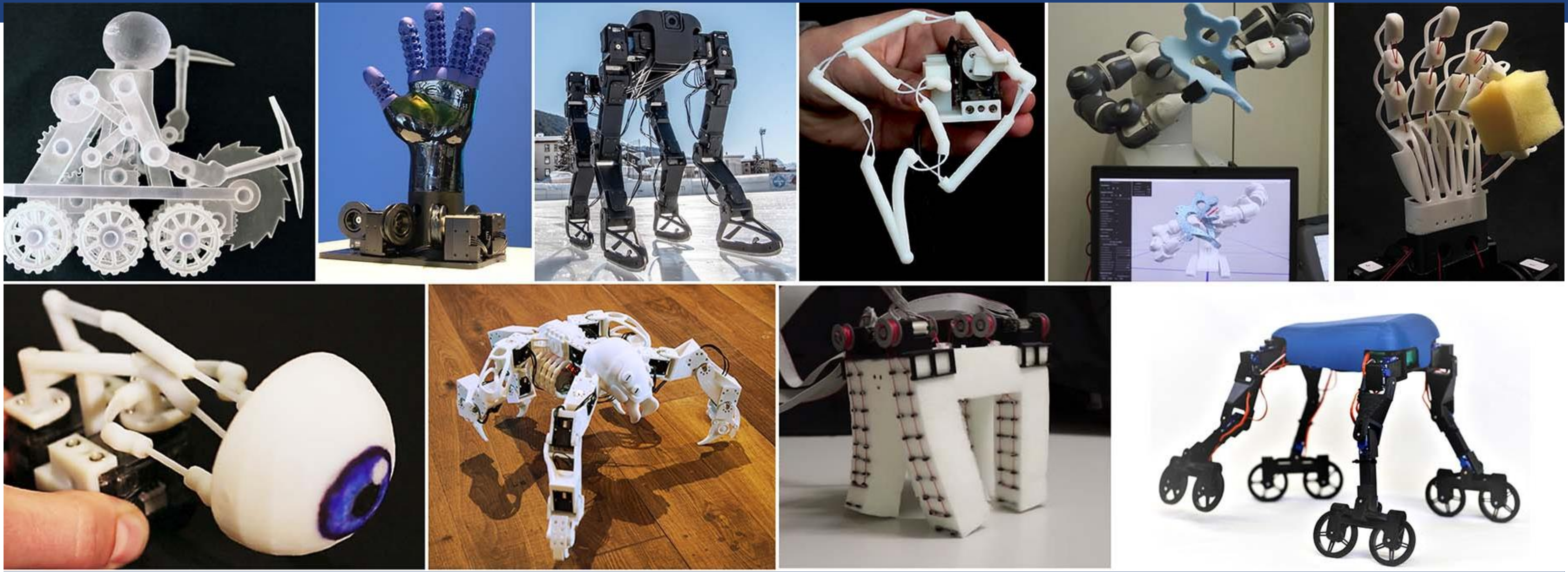
ETH zürich



Computational Models of Motion

Continuum Mechanics and FEM

CRL

ETH zürich

Studierendenbefragung 2021
Ihre Meinung zählt!
13. – 25.04.2021
Wie zufrieden sind Sie mit Ihrer Studiensituation? Gestalten Sie mit!
www.ethz.ch/studierendenbefragung

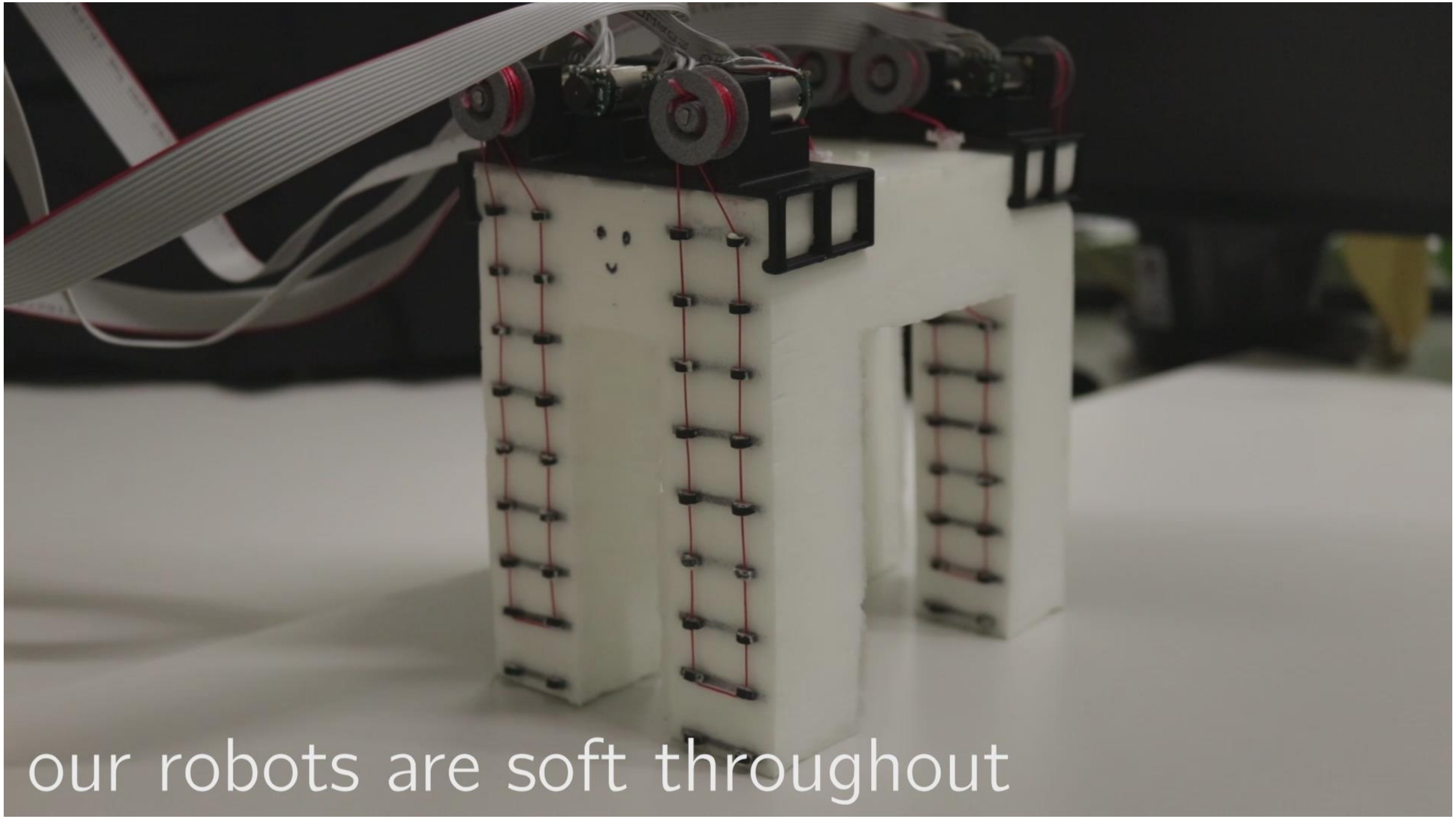
Do robots have to be rigid?

© Disney, Big Hero 6

Soft Robots

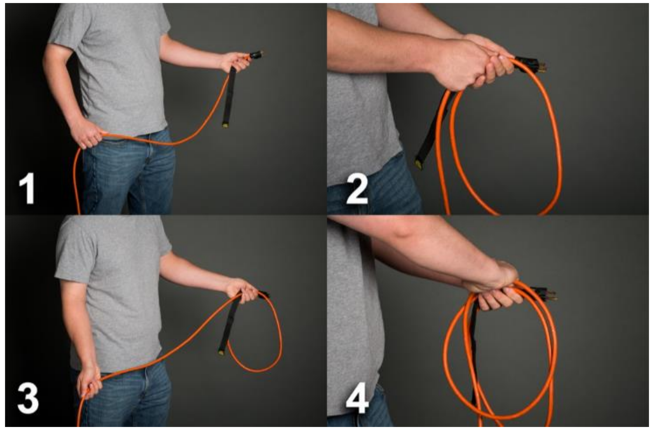
https://www.youtube.com/watch?v=2DsbS9cMOAE

Soft Robots



[CRL, RSS 2019]

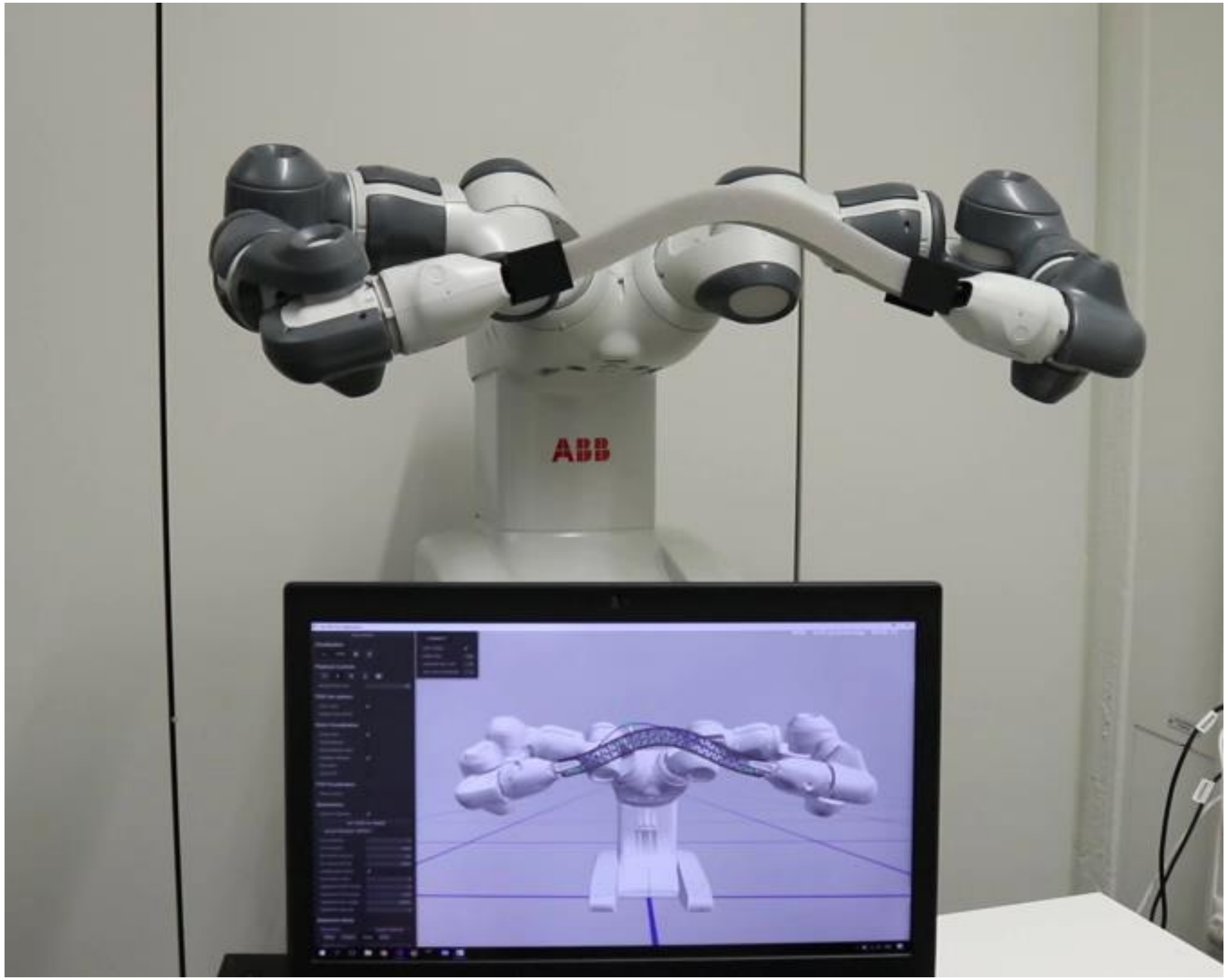
Real-world human manipulation







Robotic manipulation: soft materials



[CRL, IROS ‘18]

Continuum Mechanics

Elasticity

Elasticity
The ability of a material to resist a deforming force and to return to its original size and shape when that force is removed.

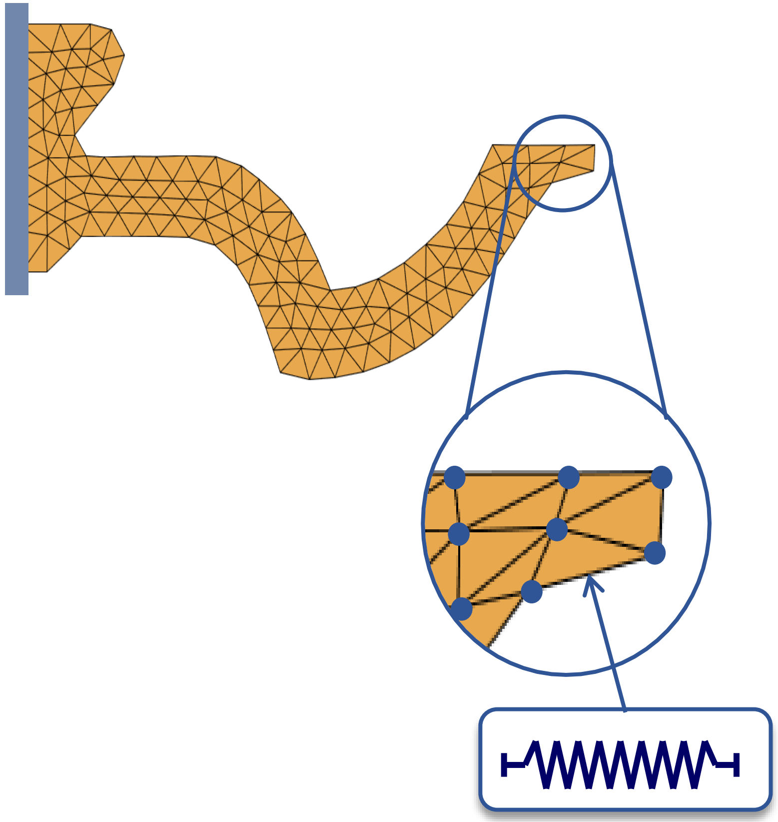
Modeling Elasticity

How to model elastic materials?

- Atomic or molecular mechanics
- Mass spring systems
- Continuum mechanics

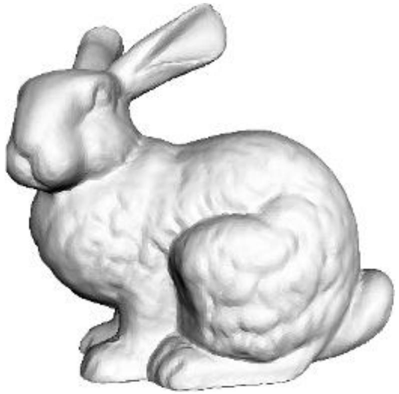
Mass Spring Systems

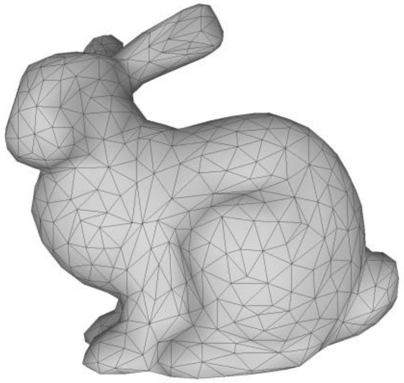
- Mass spring model
  - A simple model for deformable materials
  - Mass points & spring forces
  - Easy to understand and implement
- Limited accuracy
  - Behavior depends on mesh
  - Finding spring stiffness coefficients to best approximate a given real material is difficult
  - No volume and area preservation



Continuum Mechanics and FEM

- Start from continuous model
  - Deformation, stress, energy
  - Equilibrium conditions
- Discretize with Finite Elements
  - Decompose model into elements (e.g., tetrahedra)
  - Formulate energy and derivatives per element
  - Minimize sum of per-element energies
- Advantages
  - Accurate material behavior
  - Convergence under refinement





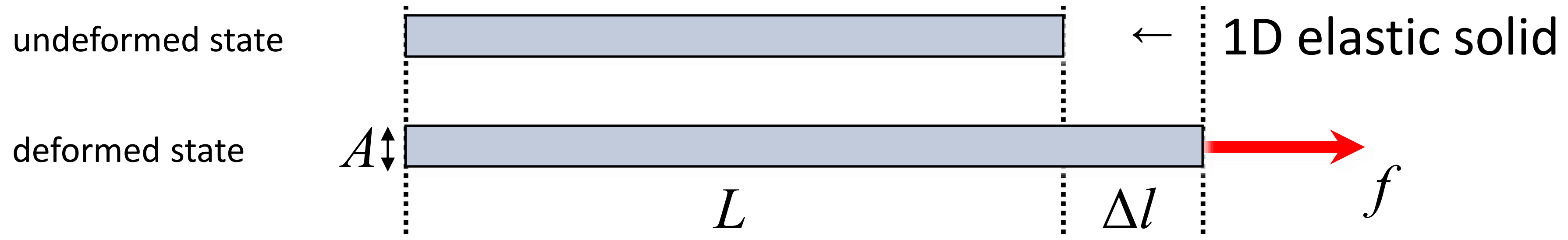
Overview

- Continuum Mechanics in 1D
  - Principles (strain, stress, energy, equilibrium)
  - Governing equations (strong and weak form)
  - Discretization (discrete energy approach)
- Continuum Mechanics in 3D
  - Strain
  - Stress
  - Linear elasticity
- Discretization with FEM
- Nonlinear Elasticity

1D Continuous Elasticity

← 1D elastic solid

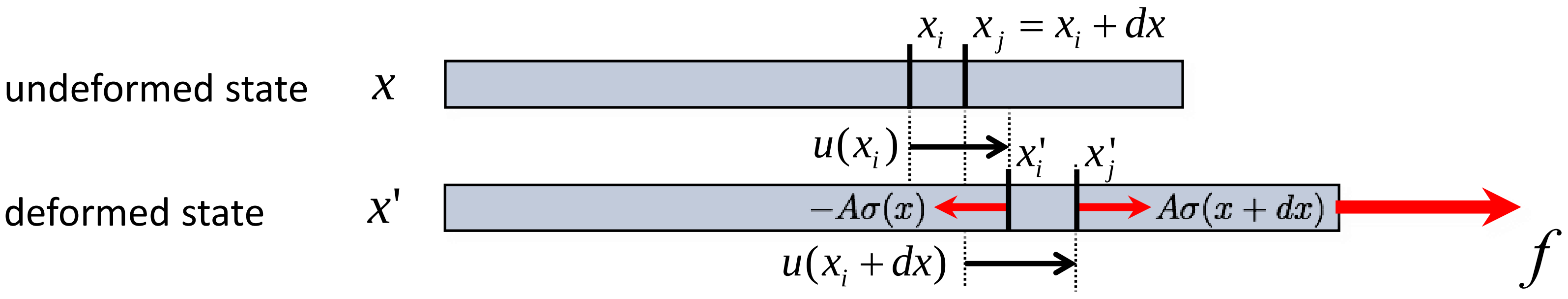
1D Continuous Elasticity



- Apply external force f
- Strain (relative stretch): $\varepsilon = \Delta l \,/\, L$
- Stress (internal force per area): $\sigma = f_{\text{int}} \,/\, A$
- Hooke's law: $\sigma = E\varepsilon$ (E: Young's elasticity modulus)

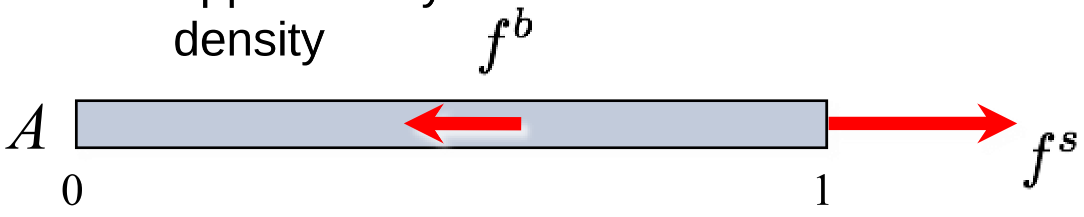
  Spring analogy: $F = k\Delta l$

1D Continuous Elasticity



- Consider segment $[x_i, x_j]$ with $L_{ij} = x_j - x_i$ and $l_{ij} = x'_j - x'_i$
- Introduce displacement field $u(x) = x' - x$
- Strain on segment $\varepsilon_{ij} = \frac{l_{ij} - L_{ij}}{L_{ij}} = \frac{u(x_i + dx) - u(x_i)}{dx}$
- Strain at arbitrary point $\xrightarrow{dx \to 0} \varepsilon = \frac{\partial u}{\partial x} =: \partial_x u$
- Force density on segment $f_{ij}^{\text{int}} = \frac{A\sigma(x_i + dx) - A\sigma(x_i)}{dx}$
- Force density at arbitrary point $\xrightarrow{dx \to 0} f^{\text{int}} = A\partial_x \sigma = AE\partial_{xx} u$

Equilibrium Equations



- Distinguish between external body force densities and surface forces
- **Balance of forces**: the system is in mechanical equilibrium, if internal and external forces sum to zero in every point.

$$\boxed{\begin{aligned} EA\partial_{xx}u(x) &= -f^b(x) \quad x \in]0,1[\\ EA\partial_x u(1) &= -f^s \\ u(0) &= 0 \end{aligned}}$$

force constraint → $EA\partial_x u(1) = -f^s$

displacement constraint → $u(0) = 0$

boundary conditions

governing PDE in strong form

CRL

Weak Form

- Assume strong form is satisfied

$$EA\partial_{xx}u(x) + f^b(x) = 0 \text{ for all } x \in (0,1)$$

- Then, for *any* test function $\bar{u}(x), \quad \bar{u}(0) = 0$

$$\int_0^1 \left(EA\,\partial_{xx}u(x) + f^b(x)\right)\bar{u}(x)\,dx = 0$$

$$-\int_0^1 EA\,\partial_x u\,\partial_x \bar{u} + [EA\,\partial_x u\bar{u}]_0^1 + \int_0^1 f^b\bar{u} = 0$$

integration by parts

$$\int_0^1 f'g = [fg]_0^1 - \int_0^1 fg'$$

boundary conditions

$$\bar{u}(0) = 0, \quad EA\,\partial_x u(1) = f^s$$

$$\boxed{\begin{gathered}\int_0^1 EA\partial_x u\partial_x\bar{u}\,dx = \int_0^1 f^b\bar{u}\,dx - f^s\bar{u}(1)\\ u(0) = 0\\ \forall\bar{u} \text{ with } \bar{u}(0) = 0\end{gathered}}$$

governing PDE in weak form

Strong vs. Weak Form

- Strong form

$$
\begin{aligned}
EA\,\partial_{xx}u(x) &= -f^b(x) \quad x \in\,]0,1[\\
EA\partial_x u(1) &= -f^s \\
u(0) &= 0
\end{aligned}
$$

Requires $u \in C^1$ with displacement constraints and force constraints

$\Rightarrow$ Finite Difference Discretization

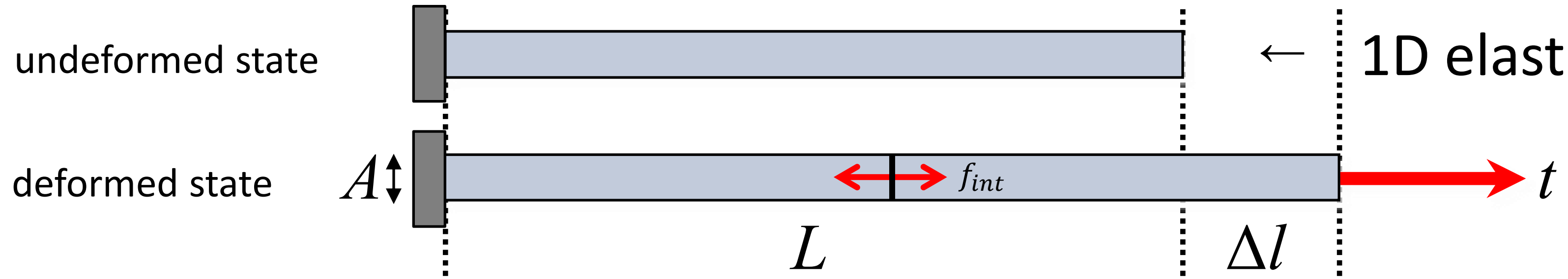
- Weak form

$$
\begin{aligned}
\int_0^1 EA\partial_x u \partial_x \bar{u}\, dx &= \int_0^1 f^b \bar{u}\, dx - f^s \bar{u}(1) \\
u(0) &= 0 \\
\forall \bar{u} \quad \text{with} \quad \bar{u}(0) &= 0
\end{aligned}
$$

Requires $u \in C^0$ with displacement constraints

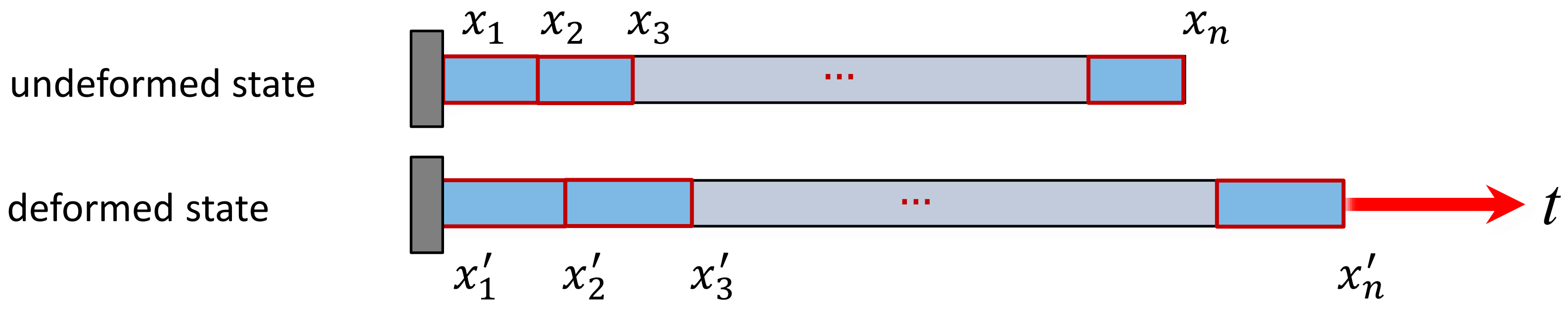
$\Rightarrow$ Finite Element Discretization

1D Continuous Elasticity – Discrete Energy Approach



- Strain: $\varepsilon = \frac{\Delta l}{L}$ (*relative stretch*)
- Stress: $\sigma = \frac{f_{int}}{A}$ (*internal force density*)
- Hooke's law: $\sigma = k\varepsilon$ (*k material constant*)
- Strain energy density: $\Psi = \frac{1}{2}k\varepsilon^2$ (*postulate via $\sigma = \frac{\partial \Psi}{\partial \varepsilon}$*)

1D Continuous Elasticity – Discrete Energy Approach



- Discretize domain into *finite* elements
- Element strain: $\varepsilon_i = \frac{x'_{i+1} - x'_i - L_i}{L_i}$ with $L_i = x_{i+1} - x_i$
- Element strain energy: $W_i = \int \Psi_i(x)dx = \Psi_i \cdot L_i = \frac{1}{2} k\varepsilon_i^2 \cdot L_i$
- Total strain energy: $W = \sum W_i$

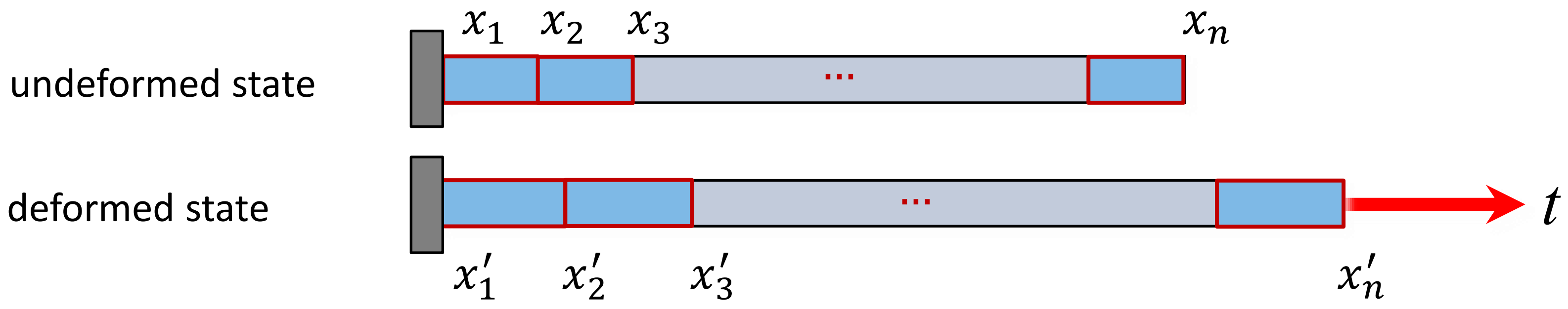
1D Continuous Elasticity – Discrete Energy Approach

Principle of minimum potential energy: at static equilibrium

- The system is in a state of minimum potential energy
- The total forces vanish for all nodes

- $W_i = \frac{1}{2} k \varepsilon_i^2 \cdot L_i$ and $\varepsilon_i = \frac{x'_{i+1} - x'_i - L_i}{L_i} \rightarrow \frac{\partial W_i}{\partial x'_i} = \frac{\partial W_i}{\partial \varepsilon_i} \frac{\partial \varepsilon_i}{\partial x'_i} = -k\varepsilon_i$
- $f_i = -\frac{\partial W}{\partial x'_i} = -\frac{\partial W_{i-1}}{\partial x'_i} - \frac{\partial W_i}{\partial x'_i} = -k(\varepsilon_{i-1} - \varepsilon_i)$ for $i = 2 \ldots n-1$
- $f_1 = k\varepsilon_1$ and $f_n = -k\varepsilon_{n-1}$

1D Continuous Elasticity – Discrete Energy Approach



Equilibrium conditions
$$f_i = \begin{cases} 0 & \forall i \in 2 \ldots n-1 \\ t & i = 1 \\ -t & i = n \end{cases}$$

→ *n-1* linear equations for *n-1* unknowns x_i'

→ solve linear system of equations to obtain deformed configuration.

In this case (constant material, no body forces), deformation is constant.

General Concept

Continuous setting

Deformation measure
+
Material model
⇩
Strain energy

***Proper* FEM Discretization**

Governing equations (PDE)
⇩
Weak form
⇩
Discretization

Discrete Energy Approach

Discretization
⇩
Per-element energy
⇩
Minimum Energy Principle

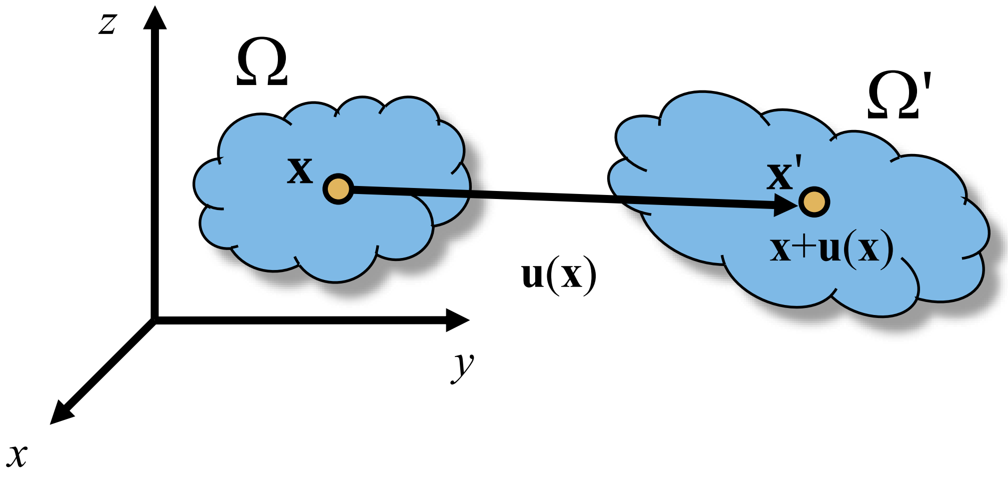
Algebraic equations

3D Continuum Mechanics

3D Deformations

- For a deformable body, identify the
  - undeformed state $\Omega \subset \mathbf{R}^3$ described by positions $\mathbf{x}$
  - deformed state $\Omega' \subset \mathbf{R}^3$ described by positions $\mathbf{x}'$
- Displacement field $\mathbf{u}$ describes Ω' in terms of Ω

$$\mathbf{u} : \Omega \to \Omega' \qquad \mathbf{x}' = \mathbf{x} + \mathbf{u}(\mathbf{x})$$



$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$$

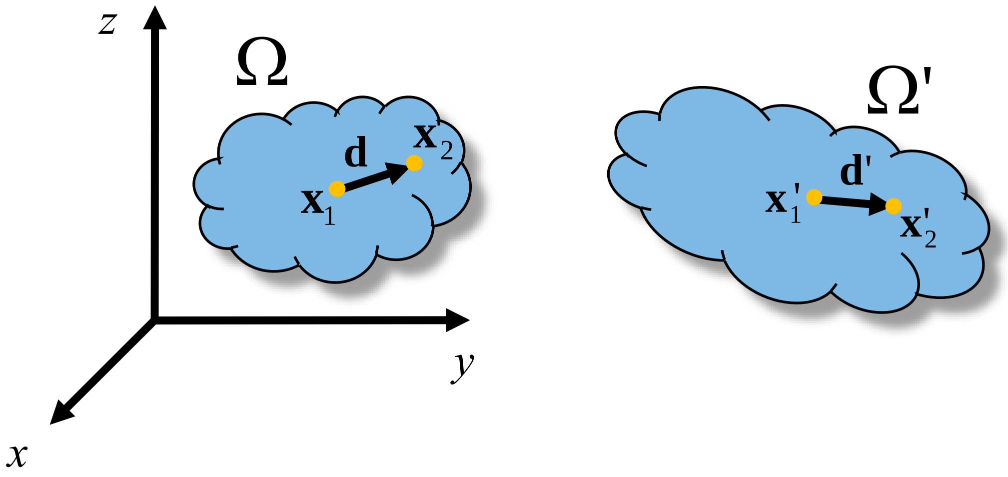
u is displacement in x direction
v is displacement in y direction
w is displacement in z direction

3D Deformations

- Consider material points $\mathbf{x}_1$ and $\mathbf{x}_2$ and $\mathbf{d} = \mathbf{x}_2 - \mathbf{x}_1$ such that $|\mathbf{d}|$ is infinitesimal

- Now consider deformed vector $\mathbf{d}'$

$$
\begin{aligned}
\mathbf{d}' = \mathbf{x}_2' - \mathbf{x}_1' &= \mathbf{x}_2 + \mathbf{u}(\mathbf{x}_2) - \mathbf{x}_1 - \mathbf{u}(\mathbf{x}_1) \\
&= \mathbf{d} + \mathbf{u}(\mathbf{x}_1 + \mathbf{d}) - \mathbf{u}(\mathbf{x}_1) \\
&\approx \mathbf{d} + \mathbf{u}(\mathbf{x}_1) + \nabla\mathbf{u}\mathbf{d} - \mathbf{u}(\mathbf{x}_1) = \overbrace{(\mathbf{I} + \nabla\mathbf{u})}^{\textit{Deformation gradient } \mathbf{F}}\mathbf{d}
\end{aligned}
$$



$$
\nabla\mathbf{u} = \begin{pmatrix} \partial_x u & \partial_y u & \partial_z u \\ \partial_x v & \partial_y v & \partial_z v \\ \partial_x w & \partial_y w & \partial_z w \end{pmatrix}
$$

3D Nonlinear Strain

- Deformation gradient $\mathbf{F} = (\mathbf{I} + \nabla\mathbf{u})$ maps undeformed vectors to deformed vectors, $\mathbf{d}' = \mathbf{F}\mathbf{d}$.

How can we quantify deformation at a given point?

- Measure change in length (squared) in all directions

$$|\mathbf{d}'|^2 - |\mathbf{d}|^2 = \mathbf{d}'^T\mathbf{d}' - \mathbf{d}^T\mathbf{d}$$

$$\Rightarrow \boxed{\textbf{Green strain} \quad \mathbf{E} = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I})}$$

3D Linear Strain

- Green strain is quadratic in displacements

$$\mathbf{E}=\frac{1}{2}(\mathbf{F}^T\mathbf{F}-\mathbf{I})=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^T+\nabla\mathbf{u}^T\nabla\mathbf{u})$$

- Discarding quadratic terms leads to the linear

$$\boxed{\textbf{Cauchy strain}\quad \boldsymbol{\varepsilon}=\frac{1}{2}(\nabla\mathbf{u}+\nabla\mathbf{u}^{t})=\frac{1}{2}(\mathbf{F}+\mathbf{F}^{t})-\mathbf{I}}$$

- Written out:

$$\varepsilon=\frac{1}{2}\begin{pmatrix} 2\partial_x u & \partial_y u+\partial_x v & \partial_z u+\partial_x w \\ \partial_x v+\partial_y u & 2\partial_y v & \partial_z v+\partial_y w \\ \partial_x w+\partial_z u & \partial_y w+\partial_z v & 2\partial_z w \end{pmatrix}$$

Notation

$$\mathbf{u}(\mathbf{x})=\begin{pmatrix} u(x,y,z) \\ v(x,y,z) \\ w(x,y,z) \end{pmatrix}$$

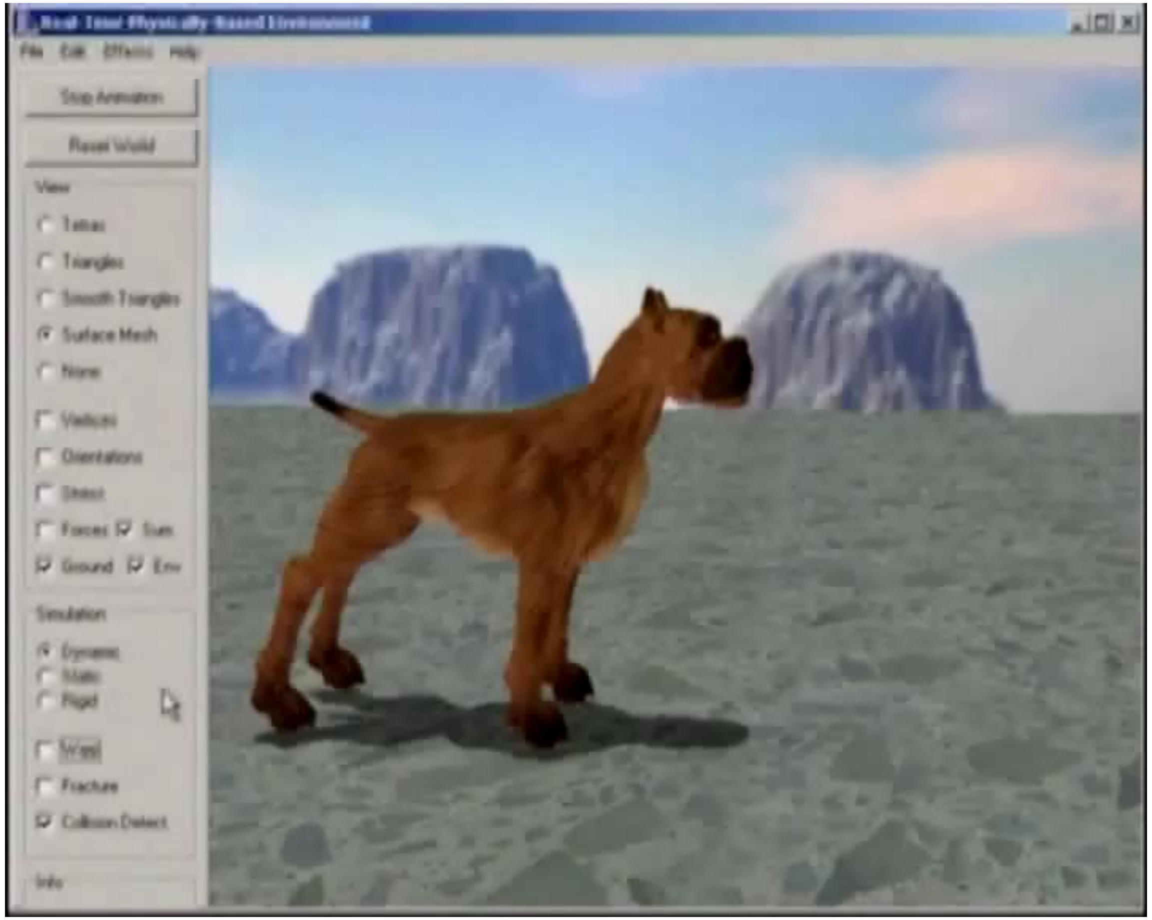
Cauchy vs. Green strain

- Nonlinear Green strain is rotation-invariant
  - Apply incremental rotation $\mathbf{R}$ to given deformation $\mathbf{F}$ to obtain $\mathbf{F}' = \mathbf{R}\mathbf{F}$
  - Then $\mathbf{E}' = \frac{1}{2}(\mathbf{F}'^T\mathbf{F}' - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T\mathbf{R}^T\mathbf{R}\mathbf{F} - \mathbf{I}) = \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{I}) = \mathbf{E}$
- Linear Cauchy strain is not rotation-invariant

$$\varepsilon' = \frac{1}{2}\left(\mathbf{F}' + \mathbf{F}'^{t}\right) - \mathbf{I} \neq \frac{1}{2}\left(\mathbf{F} + \mathbf{F}^{t}\right) - \mathbf{I} = \varepsilon$$

$\rightarrow$ artifacts for larger rotations

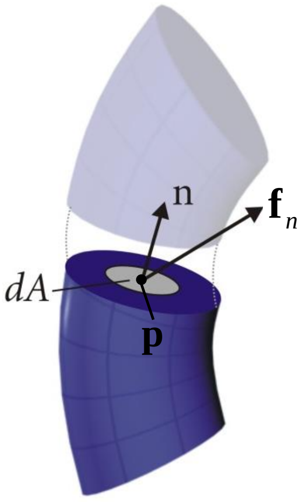
Stiffness Warping



Mueller and Gross, Interactive Virtual Materials, Graphics Interface ‘04
http://matthias-mueller-fischer.ch/publications/GI2004.pdf

3D Stress

- Virtual experiment on deformed solid
  - Insert cut plane with normal $\mathbf{n}$ through $\mathbf{p}$
  - Observe traction force $\mathbf{f}_n(\mathbf{n},\mathbf{p})$ on area dA
  - Traction force density $\mathbf{t}_n(\mathbf{n},\mathbf{p}) = \frac{d\mathbf{f}_n}{dA}$ as $dA \rightarrow 0$



How does $\mathbf{t}_n$ change with $\mathbf{n}$?

- Cauchy's stress theorem: $\mathbf{t}_n$ depends linearly on $\mathbf{n}$

$$\mathbf{t}(\mathbf{x},\mathbf{n}) = \underset{\uparrow}{\sigma}(\mathbf{x}) \cdot \mathbf{n}$$

Cauchy stress tensor

Linear Elasticity – Material Model

- Material model links strain to energy (and stress)
- Linear isotropic material (generalized Hooke's law)
  - Energy density $\Psi = \frac{1}{2}\lambda \mathrm{tr}(\boldsymbol{\varepsilon})^2 + \mu \mathrm{tr}(\boldsymbol{\varepsilon}^2)$
  - Cauchy stress $\sigma = \frac{\partial \Psi}{\partial \varepsilon} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}$
  - Lame parameters λ and μ are material constants

$\boxed{\mathrm{tr}(\boldsymbol{\varepsilon}) = \sum \varepsilon_{ii}}$

Linear Elasticity – Material Parameters

V · T · E **Elastic moduli for homogeneous isotropic materials** [hide]

Bulk modulus (K) · Young's modulus (E) · **Lamé's first parameter** (λ) · Shear modulus (G, μ) · Poisson's ratio (ν) · P-wave modulus (M)

Conversion formulas [hide]

Homogeneous isotropic linear elastic materials have their elastic properties uniquely determined by any two moduli among these; thus, given any two, any other of the elastic moduli can be calculated according to these formulas.

	$K=$	$E=$	$\lambda=$	$G=$	$\nu=$	$M=$	Notes
$(K,\,E)$	K	E	$\frac{3K(3K-E)}{9K-E}$	$\frac{3KE}{9K-E}$	$\frac{3K-E}{6K}$	$\frac{3K(3K+E)}{9K-E}$	
$(K,\,\lambda)$	K	$\frac{9K(K-\lambda)}{3K-\lambda}$	λ	$\frac{3(K-\lambda)}{2}$	$\frac{\lambda}{3K-\lambda}$	$3K-2\lambda$	
$(K,\,G)$	K	$\frac{9KG}{3K+G}$	$K-\frac{2G}{3}$	G	$\frac{3K-2G}{2(3K+G)}$	$K+\frac{4G}{3}$	
$(K,\,\nu)$	K	$3K(1-2\nu)$	$\frac{3K\nu}{1+\nu}$	$\frac{3K(1-2\nu)}{2(1+\nu)}$	ν	$\frac{3K(1-\nu)}{1+\nu}$	
$(K,\,M)$	K	$\frac{9K(M-K)}{3K+M}$	$\frac{3K-M}{2}$	$\frac{3(M-K)}{4}$	$\frac{3K-M}{3K+M}$	M	
$(E,\,\lambda)$	$\frac{E+3\lambda+R}{6}$	E	λ	$\frac{E-3\lambda+R}{4}$	$\frac{2\lambda}{E+\lambda+R}$	$\frac{E-\lambda+R}{2}$	$R=\sqrt{E^2+9\lambda^2+2E\lambda}$
$(E,\,G)$	$\frac{EG}{3(3G-E)}$	E	$\frac{G(E-2G)}{3G-E}$	G	$\frac{E}{2G}-1$	$\frac{G(4G-E)}{3G-E}$	
$(E,\,\nu)$	$\frac{E}{3(1-2\nu)}$	E	$\frac{E\nu}{(1+\nu)(1-2\nu)}$	$\frac{E}{2(1+\nu)}$	ν	$\frac{E(1-\nu)}{(1+\nu)(1-2\nu)}$	
$(E,\,M)$	$\frac{3M-E+S}{6}$	E	$\frac{M-E+S}{4}$	$\frac{3M+E-S}{8}$	$\frac{E-M+S}{4M}$	M	$S=\pm\sqrt{E^2+9M^2-10EM}$ There are two valid solutions. The plus sign leads to $\nu\geq 0$. The minus sign leads to $\nu\leq 0$.
$(\lambda,\,G)$	$\lambda+\frac{2G}{3}$	$\frac{G(3\lambda+2G)}{\lambda+G}$	λ	G	$\frac{\lambda}{2(\lambda+G)}$	$\lambda+2G$	
$(\lambda,\,\nu)$	$\frac{\lambda(1+\nu)}{3\nu}$	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	λ	$\frac{\lambda(1-2\nu)}{2\nu}$	ν	$\frac{\lambda(1-\nu)}{\nu}$	Cannot be used when $\nu=0\Leftrightarrow\lambda=0$
$(\lambda,\,M)$	$\frac{M+2\lambda}{3}$	$\frac{(M-\lambda)(M+2\lambda)}{M+\lambda}$	λ	$\frac{M-\lambda}{2}$	$\frac{\lambda}{M+\lambda}$	M	
$(G,\,\nu)$	$\frac{2G(1+\nu)}{3(1-2\nu)}$	$2G(1+\nu)$	$\frac{2G\nu}{1-2\nu}$	G	ν	$\frac{2G(1-\nu)}{1-2\nu}$	
$(G,\,M)$	$M-\frac{4G}{3}$	$\frac{G(3M-4G)}{M-G}$	$M-2G$	G	$\frac{M-2G}{2M-2G}$	M	
$(\nu,\,M)$	$\frac{M(1+\nu)}{3(1-\nu)}$	$\frac{M(1+\nu)(1-2\nu)}{1-\nu}$	$\frac{M\nu}{1-\nu}$	$\frac{M(1-2\nu)}{2(1-\nu)}$	ν	M	

https://en.wikipedia.org/wiki/Lam%C3%A9_parameters

Linear Elasticity – Material Model

- The material model links strain to stress (and energy)
- Linear isotropic material (*generalized Hooke's law*)
  - Energy density $\Psi = \frac{1}{2}\lambda \mathrm{tr}(\boldsymbol{\varepsilon})^2 + \mu \mathrm{tr}(\boldsymbol{\varepsilon}^2)$

    $\boxed{\mathrm{tr}(\boldsymbol{\varepsilon}) = \sum \varepsilon_{ii}}$
  - Cauchy stress $\sigma = \frac{\partial \Psi}{\partial \varepsilon} = \lambda \mathrm{tr}(\boldsymbol{\varepsilon})\mathbf{I} + 2\mu\boldsymbol{\varepsilon}$
  - Lame parameters λ and μ are material constants
- Interpretation
  - $\mathrm{tr}(\boldsymbol{\varepsilon}^2) = \mathrm{tr}(\boldsymbol{\varepsilon}^T\boldsymbol{\varepsilon}) = ||\boldsymbol{\varepsilon}||_F^2$ penalizes all strain components equally
  - $\lambda \mathrm{tr}(\boldsymbol{\varepsilon})^2$ penalizes dilatations, i.e., volume changes

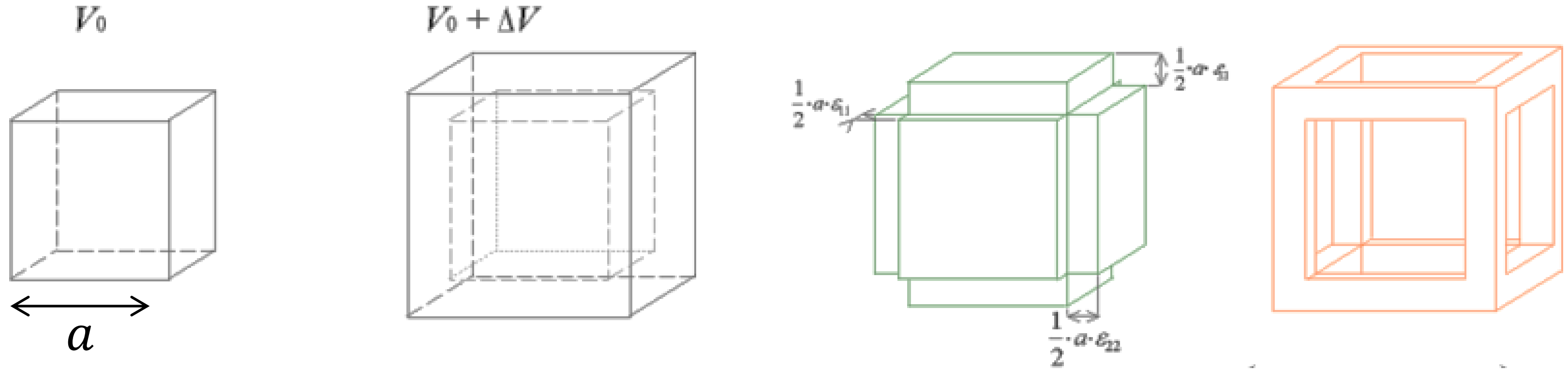
CRL

Linear Elasticity – Volumetric Strain

- Consider a cube with side length a
- For a given deformation $\boldsymbol{\varepsilon}$, the added volume is

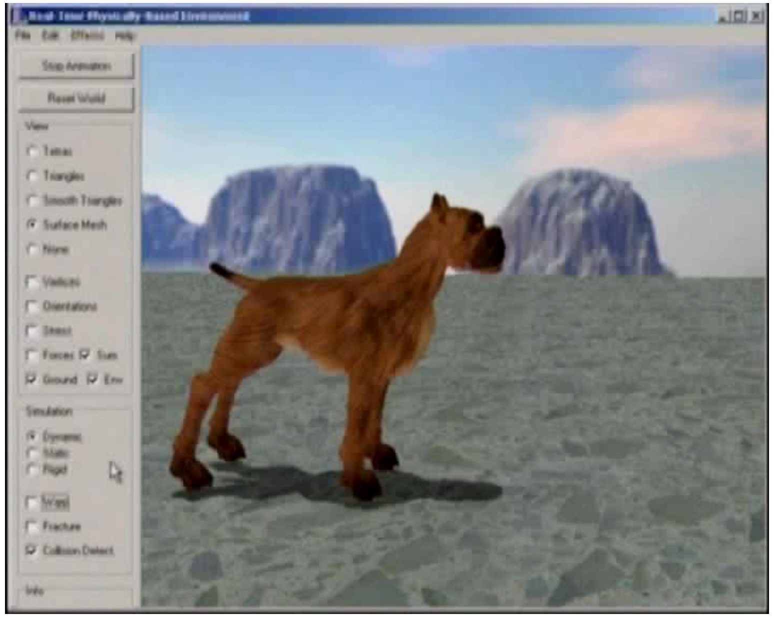
$$\Delta V = a(1+\varepsilon_{11}) \cdot a(1+\varepsilon_{22}) \cdot a(1+\varepsilon_{33}) - a^3$$
$$= a^3(\varepsilon_{11}+\varepsilon_{22}+\varepsilon_{33}) + O\left(\boldsymbol{\varepsilon}^2\right) \approx a^3 \mathrm{tr}(\boldsymbol{\varepsilon})$$

$$\varepsilon = \frac{\Delta l}{L} \rightarrow l = L + \Delta l = L(1+\varepsilon)$$



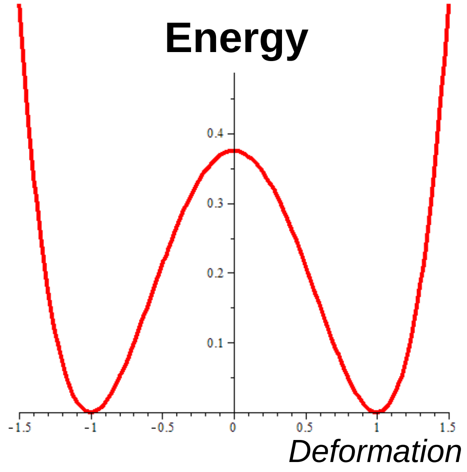
Linear Elasticity – Limitations

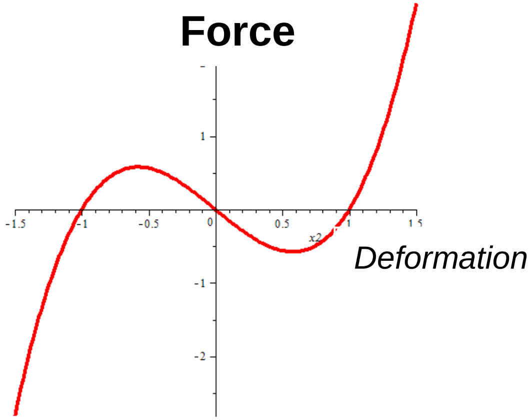
- Linear elasticity relies on linear Cauchy strain
- Problem: Cauchy strain is not invariant under rotations
  - → inaccuracies for rotations
- Solution: use nonlinear deformation measure
  - → nonlinear elasticity



Nonlinear Elasticity – St.Venant Kirchhoff Material

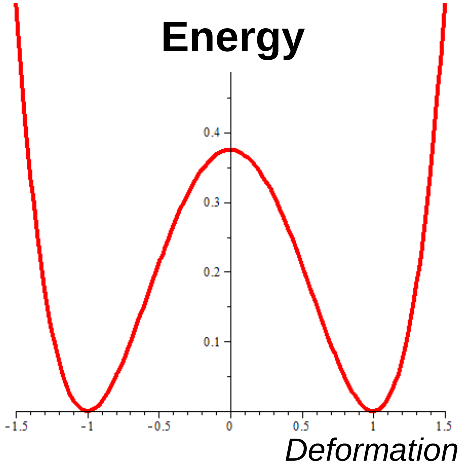
- First idea: simply replace Cauchy strain with Green strain
  → *St. Venant-Kirchhoff material* (StVK)
- Energy density $\Psi_{StVK} = \frac{1}{2}\lambda \mathrm{tr}(\mathbf{E})^2 + \mu \mathrm{tr}(\mathbf{E}^2)$

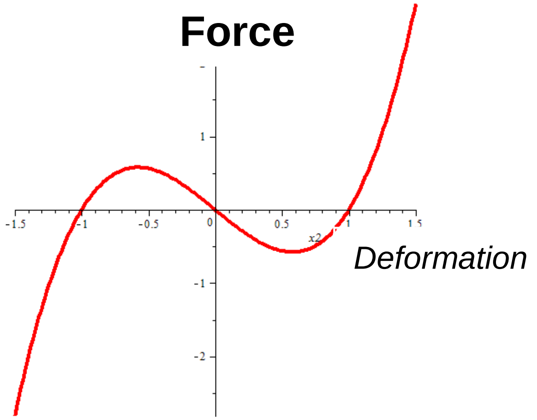




StVK Limitations

- *Problem*: StVK softens under compression





- *Reason*: Green strain $\mathbf{E} = \frac{1}{2}(\mathbf{F}^t\mathbf{F} - \mathbf{I}) \rightarrow -\frac{1}{2}\mathbf{I}$ *for* $\mathbf{F} \rightarrow \mathbf{0}$
- *Work around*: add volume term

$$\Psi_{StVK} = \frac{\lambda}{2}\mathrm{tr}(\mathbf{E})^2 + \mu\mathrm{tr}(\mathbf{E}^2) \quad \rightarrow \quad \Psi_{Mod} = \eta(\det(\mathbf{F}) - 1)^2 + \mu\mathrm{tr}(\mathbf{E}^2)$$

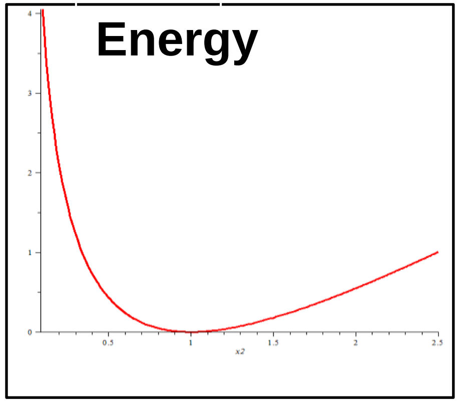
Neo-Hookean Material

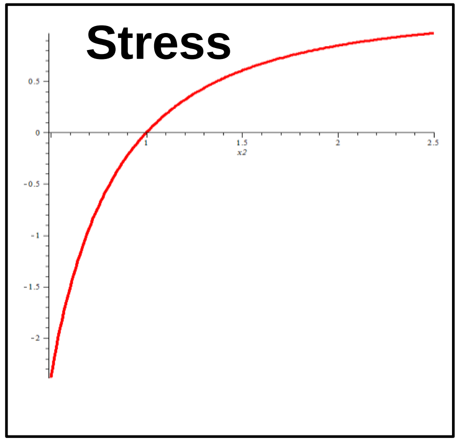
- The strain energy density for a *compressible* Neo-Hookean solid is defined as

$$\Psi_{NH} = \frac{\mu}{2}(\mathrm{tr}(\mathbf{C}) - 3) - \mu \ln J + \frac{\lambda}{2}\ln(J)^2$$

Observations:

- the first term penalizes all deformations equally
  (since $\mathrm{tr}(\boldsymbol{C}) = |\boldsymbol{F}|_F^2$)
- the second and third terms go to infinity for increasing compression ($J = \det \boldsymbol{F}$)
- the stress-strain behavior is initially linear, but goes into plateau for larger deformations
- Rule of thumb: NH is good for deformations of up to 20%

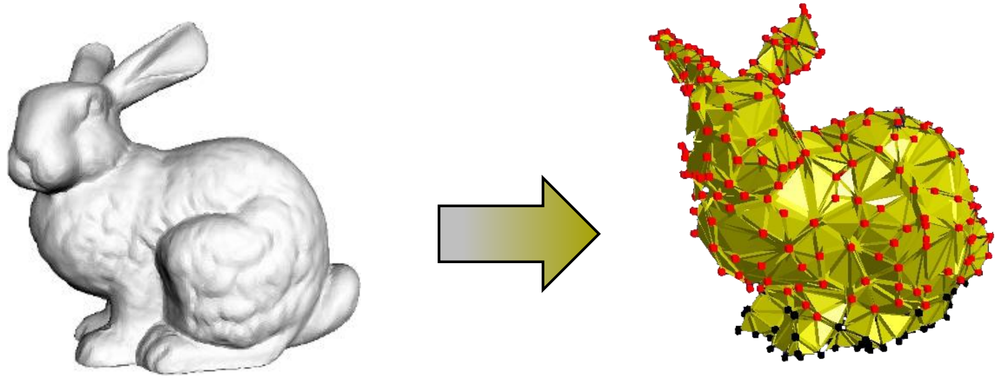




Finite Element Discretization

Finite Element Discretization – Overview

- Divide input model into elements (e.g., triangles in 2D, tetrahedra in 3D)



- For each element, evaluate its energy, the energy gradient, and the energy Hessian
- All quantities depend (only) on the deformation gradient $\mathbf{F}$

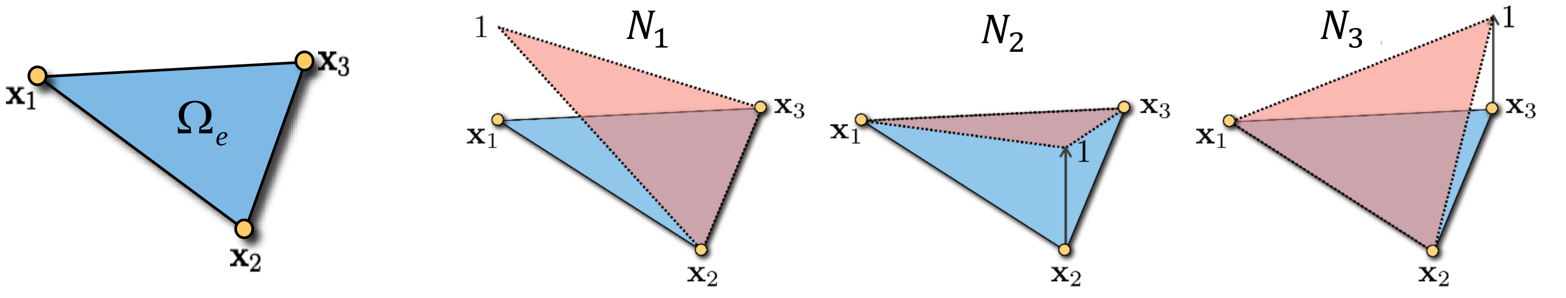
Finite Elements

What is a finite element?

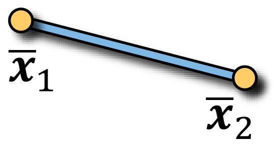
A finite element consists of

- a closed subset $\Omega_e \subset \boldsymbol{R}^d$ (*in d dimensions*)
- n nodal basis functions, $N_i : \Omega_e \to \boldsymbol{R}$
- n vectors of nodal variables $\bar{\boldsymbol{x}}_i \in \boldsymbol{R}^d$ describing the reference geometry
- n vectors of degrees of freedom (e.g., deformed positions $\boldsymbol{x}_i$)

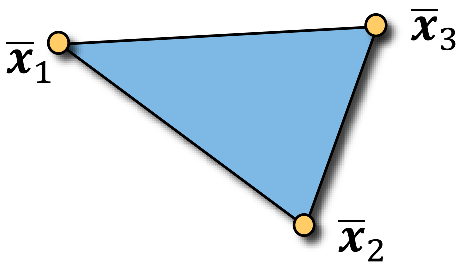
Example: 3-node triangle with linear basis functions



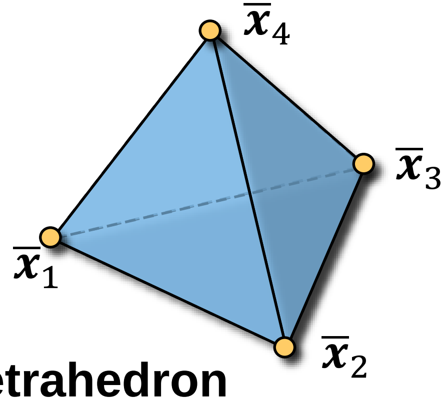
Linear Simplicial Elements



1D: **line segment**



2D: **triangle**



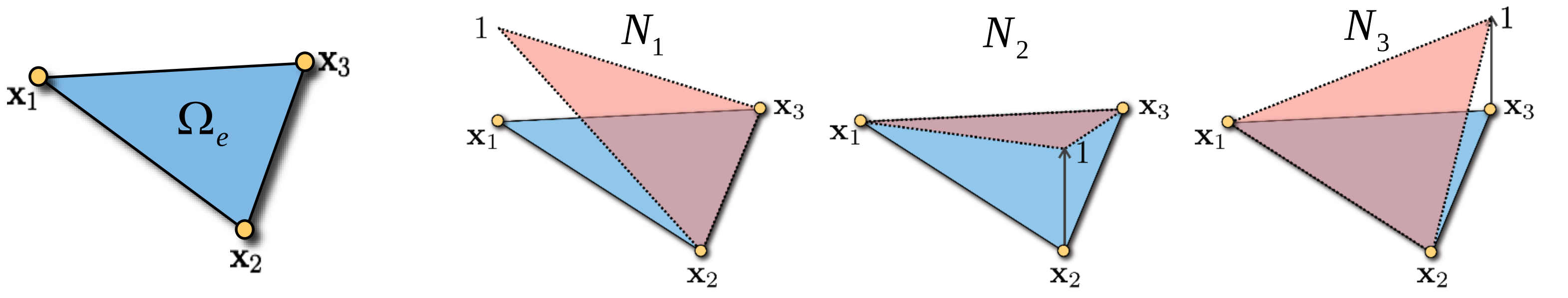
3D: **tetrahedron**

- Simplicial elements admit linear basis functions
- Basis functions are uniquely defined through
  - reference geometry $\bar{\boldsymbol{x}}_i$ and
  - interpolation requirement $N_i(\bar{\boldsymbol{x}}_j) = \delta_{ij}$

$\bar{\boldsymbol{x}}_i = \bar{x}_i$	in 1D
$\bar{\boldsymbol{x}}_i = (\bar{x}_i, \bar{y}_i)$	in 2D
$\bar{\boldsymbol{x}}_i = (\bar{x}_i, \bar{y}_i, \bar{z}_i)$	in 3D

Computing Basis Functions – 2D

Example: 3-node elements with linear basis functions



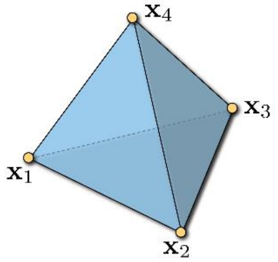
- Basis functions are linear: $N_i(x,y) = a_i x + b_i y + c$
- Due to $N_i(\mathbf{x}_j) = \delta_{ij}$, we have

$$\begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix} \cdot \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \end{bmatrix} \Rightarrow \begin{bmatrix} a_i \\ b_i \\ c_i \end{bmatrix} = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{bmatrix}^{-1} \cdot \begin{bmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \end{bmatrix}$$

CRL

Computing Basis Functions – 3D

4-node tetrahedron with 4 linear basis functions

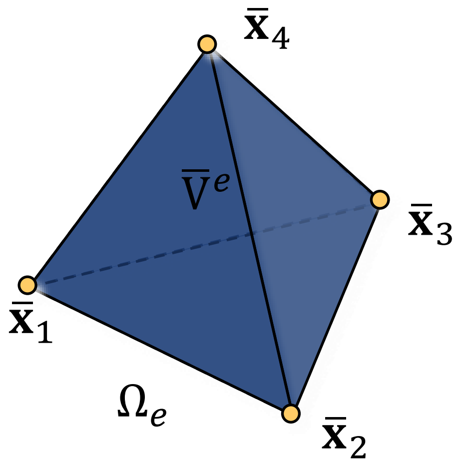


... ...

- Basis functions are linear, $N_i(\bar{x},\bar{y},\bar{z}) = a_i\bar{x} + b_i\bar{y} + c_i\bar{z} + d_i$
- From $N_i(\bar{\boldsymbol{x}}_j) = \delta_{ij}$ we obtain

$$N_i(\bar{x},\bar{y},\bar{z}) = a_i\bar{x} + b_i\bar{y} + c_i\bar{z} + d_i$$

$$\begin{pmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{pmatrix} \begin{pmatrix} a_i \\ b_i \\ c_i \\ d_i \end{pmatrix} = \begin{pmatrix} \delta_{1i} \\ \delta_{2i} \\ \delta_{3i} \\ \delta_{4i} \end{pmatrix}$$



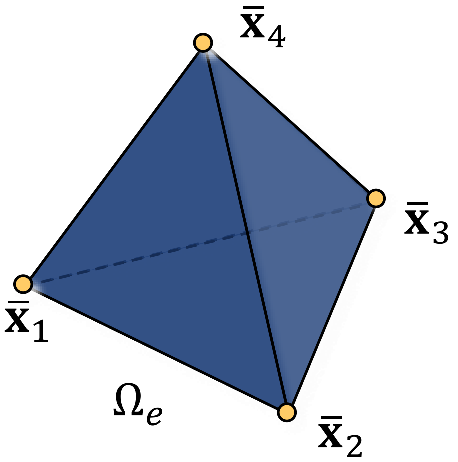
Deformation Gradient

- Use basis functions to define continuous geometry of element as

$$\bar{\mathbf{x}}(\bar{x},\bar{y},\bar{z}) = \sum N_i(\bar{x},\bar{y},\bar{z})\bar{\mathbf{x}}_i \quad \text{and} \quad \mathbf{x}(\bar{x},\bar{y},\bar{z}) = \sum N_i(\bar{x},\bar{y},\bar{z})\mathbf{x}_i$$

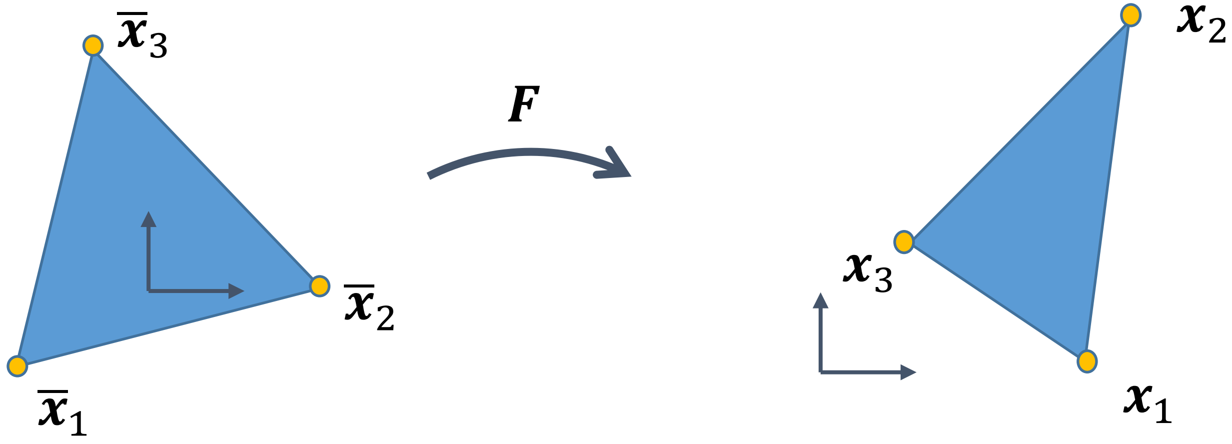
- Deformation gradient

$$\mathbf{F} = \boldsymbol{I} + \frac{\partial \boldsymbol{u}(\bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}} = \frac{\partial \mathbf{x}(\bar{\mathbf{x}})}{\partial \bar{\mathbf{x}}} = \sum_i \mathbf{x}_i \left(\frac{\partial N_i}{\partial \bar{\mathbf{x}}}\right)^t$$



- Note
  - $\mathbf{F} \in \mathbf{R}^{3\times 3}$ and $\mathbf{F}$ is linear in $\mathbf{x}_i$
  - N_i are linear on element → $\mathbf{F}$ is constant
  - Hence, $W^e = \int_{\Omega_e} \Psi = \Psi(\mathbf{F}) \cdot \bar{V}^e$

2D Example



$$e^1 = x_2 - x_1$$
$$e^2 = x_3 - x_1$$
$$\bar{e}^1 = \bar{x}_2 - \bar{x}_1$$
$$\bar{e}^2 = \bar{x}_3 - \bar{x}_1$$

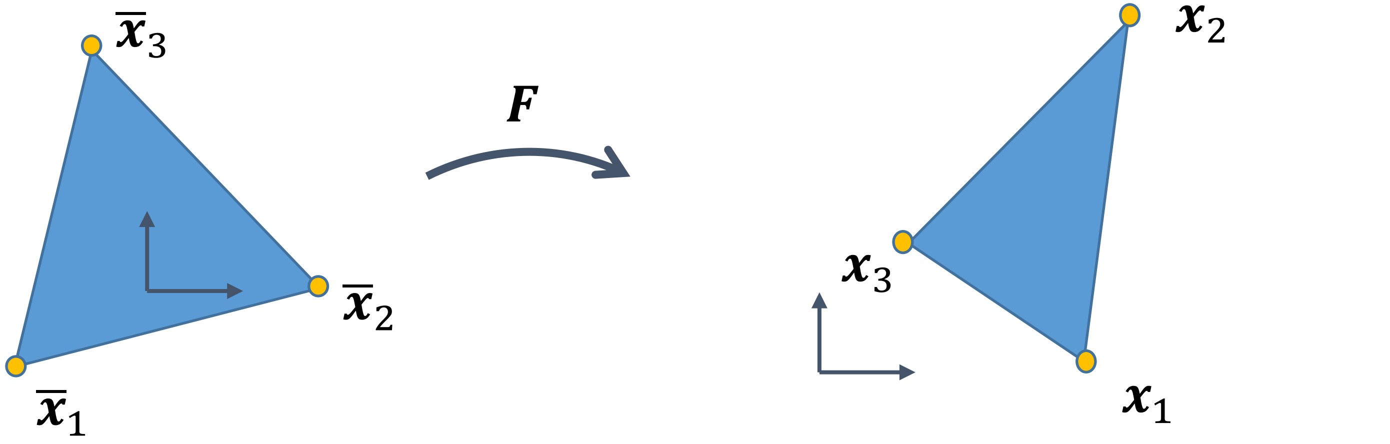
- Linear triangle element in 2D: N_i are linear on triangle → $\boldsymbol{F}$ is constant on triangle
- Deformation gradient $\boldsymbol{F}$ maps undeformed vectors to deformed vectors

$$\boldsymbol{v} = \boldsymbol{F}\bar{\boldsymbol{v}}$$

- In particular, $\boldsymbol{F}$ maps undeformed triangle edges to deformed triangle edges

$$\begin{bmatrix} e_x^1 & e_x^2 \\ e_y^1 & e_y^2 \end{bmatrix} = \boldsymbol{F} \begin{bmatrix} \bar{e}_x^1 & \bar{e}_x^2 \\ \bar{e}_y^1 & \bar{e}_y^2 \end{bmatrix}$$

2D Example



$$e^1 = x_2 - x_1$$
$$e^2 = x_3 - x_1$$
$$\bar{e}^1 = \bar{x}_2 - \bar{x}_1$$
$$\bar{e}^2 = \bar{x}_3 - \bar{x}_1$$

- Linear triangle element in 2D: N_i are linear on triangle → $\boldsymbol{F}$ is constant on triangle
- Deformation gradient $\boldsymbol{F}$ maps undeformed vectors to deformed vectors

$$\boldsymbol{v} = \boldsymbol{F}\bar{\boldsymbol{v}}$$

- In particular, $\boldsymbol{F}$ maps undeformed triangle edges to deformed triangle edges

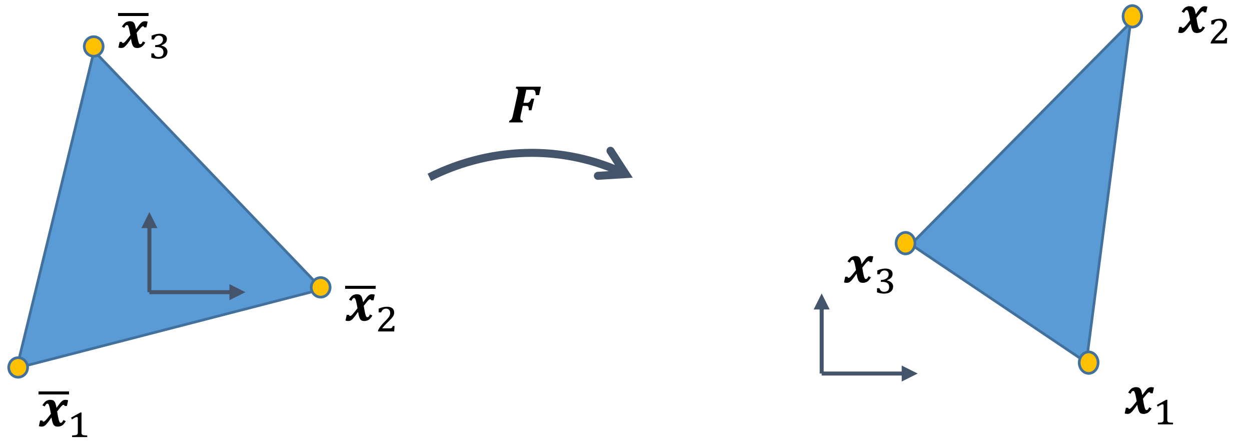
$$\boldsymbol{F}(\bar{\boldsymbol{x}}, \boldsymbol{x}) = \begin{bmatrix} \boldsymbol{e}_x^1 & \boldsymbol{e}_x^2 \\ \boldsymbol{e}_y^1 & \boldsymbol{e}_y^2 \end{bmatrix} \begin{bmatrix} \bar{\boldsymbol{e}}_x^1 & \bar{\boldsymbol{e}}_x^2 \\ \bar{\boldsymbol{e}}_y^1 & \bar{\boldsymbol{e}}_y^2 \end{bmatrix}^{-1} = \boldsymbol{B}(\boldsymbol{x})\boldsymbol{A}(\bar{\boldsymbol{x}})$$

- $\boldsymbol{A}$ does not depend on rest state $\bar{\boldsymbol{x}}$, can be precomputed

CRL

2D Example

- $\overline{\boldsymbol{x}}_1 = (-2,-1)$
- $\overline{\boldsymbol{x}}_2 = (2,0)$
- $\overline{\boldsymbol{x}}_3 = (-1,3)$

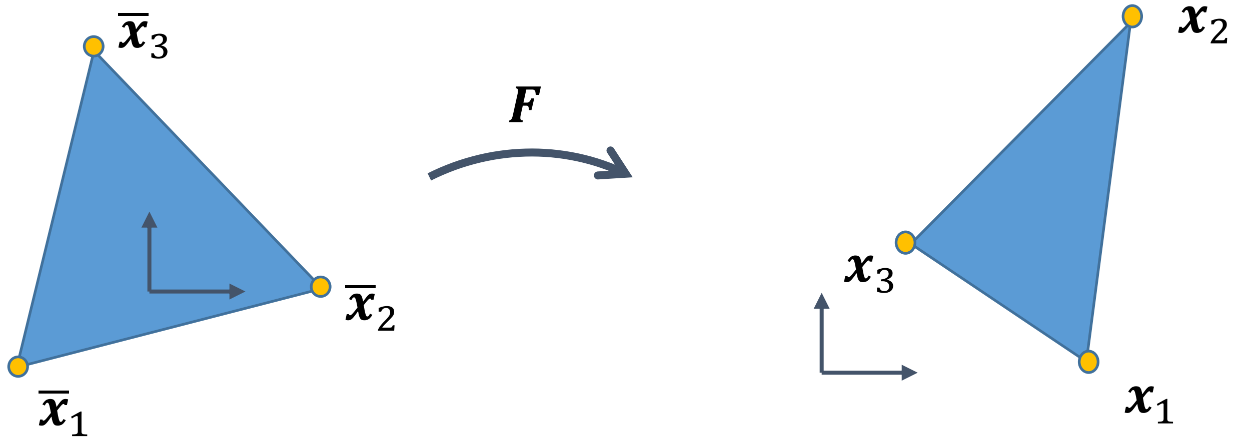


- $\boldsymbol{x}_1 = (3,0)$
- $\boldsymbol{x}_2 = (4,5)$
- $\boldsymbol{x}_3 = (1,2)$

- $\boldsymbol{F} = \boldsymbol{B}\boldsymbol{A}^{-1} = \begin{bmatrix} 1 & -2 \\ 5 & 2 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -2 \\ 5 & 2 \end{bmatrix} \frac{1}{15} \begin{bmatrix} 4 & -1 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 0.4 & -0.6 \\ 1.2 & 0.2 \end{bmatrix}$

2D Example

- $\bar{\boldsymbol{x}}_1 = (-2,-1)$
- $\bar{\boldsymbol{x}}_2 = (2,0)$
- $\bar{\boldsymbol{x}}_3 = (-1,3)$



- $\boldsymbol{x}_1 = (3,0)$
- $\boldsymbol{x}_2 = (4,5)$
- $\boldsymbol{x}_3 = (1,2)$

Alternative (general) way

- Compute basis functions N_i
- Compute basis function derivatives $\frac{\partial N_i}{\partial \bar{x}} = \nabla_{\bar{\boldsymbol{x}}} N_i$
- Compute $\mathbf{F}$ via $F_{kl} = \sum_i \mathrm{x}_{i,k} \, \nabla_{\bar{x}_l} N_i$

Nodal Forces and Force Jacobian

- To solve static and dynamic equilibrium problems, we must compute nodal forces and the force Jacobian

- Sketch
  - Forces on nodes $\boldsymbol{x}_i$ are negative gradients of energy, $\boldsymbol{f}_i = -\frac{\partial W}{\partial \boldsymbol{x}_i}$
  - Total energy is sum of elemental energies, $W = \sum_e W_e$
  - Per-element energy W_e depends nonlinearly on deformation gradient $\boldsymbol{F}_e$
  - $\boldsymbol{F}_e$ depends linearly on $\boldsymbol{x}_i$
  - Force Jacobian = - Energy Hessian

Further Reading

Textbooks

- Bonet and Wood, Nonlinear Continuum Mechanics
- Ogden, Nonlinear Elastic Deformations

Articles & Tutorials

- Sifakis & Barbic: *FEM Simulation of 3D Deformable Solids: A practitioner's guide to theory, discretization and model reduction*. SIGGRAPH ‘12 course (http://femdefo.org/)
- Kim and Eberle: *Dynamic Deformables: Implementation and Production Practicalities.* SIGGRAPH ‘20 course (http://www.tkim.graphics/DYNAMIC_DEFORMABLES/)