VE281

Data Structures and Algorithms

Comparison Sort

Learning Objectives:

- Know the difference between comparison sort and non-comparison sort
- Know the procedures of merge sort and quick sort
- Know the master theorem
- Know different characteristics of sorting algorithms, such as time complexity, stability, etc.

Outline

- Sorting Basics
- Merge Sort
- Quick Sort
- Comparison Sort Summary

Sorting

- Given array A of size N, reorder A so that its elements are in order.
  - "In order" with respect to a consistent comparison function, such as "$\leq$" or "$\geq$".

- Sorting order
  - Ascending order
  - Descending order
- Unless otherwise specified, we consider sorting in ascending order.

Characteristics of Sorting Algorithms

- Average-case time complexity
- Worst-case time complexity
- Space usage: **in place** or not?
  - **in place**: requires $O(1)$ additional memory
  - Don't forget the stack space used in recursive calls
  - **In place is better**
    - Why? The data can fit into cache, not main memory
  - Real example: quick sort versus merge sort. Both have average-case time complexity of $O(n \log n)$. Quick sort is faster, due to in place

Characteristics of Sorting Algorithms

- **Stability**: whether the algorithm maintains the relative order of records with equal keys

(4, b), (3, e), (3, b), (5, b) → (3, e), (3, b), (4, b), (5, b)

Sort on the first number

Stable!

- Usually there is a secondary key whose ordering you want to keep. Stable sort is thus useful for sorting over multiple keys
- Example: sort complex numbers a+bi
  - Ordering rule: first compare a; when there is a tie, compare b
  - One sorting method: first sort b, then sort a

3+5i, 2+6i, 3+4i, 5+2i → Sort on b → 5+2i, 3+4i, 3+5i, 2+6i

… sort on a → 2+6i, 3+4i, 3+5i, 5+2i

Stability is important!

Types of Sorting Algorithms

- Sorting algorithms can be classified as **comparison sort** and **non-comparison sort**.

- **Comparison sort**: each item is compared against others to determine its order.

- **Non-comparison sort**: each item is put into predefined "bins" independent of the other items presented.
  - No comparison with other items needed.
  - It is also known as **distribution-based sort**.

Types of Sorting Algorithms

- General types of comparison sort
  - Insertion-based: insertion sort
  - Selection-based: selection sort, heap sort
  - Exchange-based: bubble sort, quick sort
  - Merging-based: merge sort

- Non-comparison sort:
  counting sort, bucket sort, radix sort

Insertion Sort

- **`A[0]`** alone is a sorted array.
- For **`i=1`** to **`N-1`**
  - **Insert** **`A[i]`** into the appropriate location in the sorted array **`A[0], …, A[i-1]`**, so that **`A[0], …, A[i]`** is sorted.
  - To do so, save **`A[i]`** in a temporary variable **`t`**, shift sorted elements greater than **`t`** right, and then insert **`t`** in the gap.

Example

	i=1	2	3	4	5	6	7
42	20	17	13	13	13	13	13
20	42	20	17	17	14	14	14
17	17	42	20	20	17	17	15
13	13	13	42	28	20	20	17
28	28	28	28	42	28	23	20
14	14	14	14	14	42	28	23
23	23	23	23	23	23	42	28
15	15	15	15	15	15	15	42

Insertion Sort

- **`A[0]`** alone is a sorted array.
-
```
void* insertsort(int* a, int n){
  for i=1 to n-1{
    int tmp = a[i], j = i-1;
    while (j>=0 && tmp<a[j]){a[j+1] = a[j]; --j;}
    a[j+1] = tmp;
  }
}
```

- Time complexity?
- In place?
- Stable?

Insertion Sort

- Time complexity? $O(N^2)$
- In place? Yes. O(1) additional memory.
- Stable? Yes, because elements are visited in order and equal elements are inserted after its equals.
- The **best case** time complexity is $O(N)$.
  - It happens when the array is already sorted.
  - For other sorting algorithms we will talk, their best case time complexity is $\Omega(N \log N)$.
- The **worst case** time complexity is $O(N^2)$.

$$\sum_{i=1}^{n-1} i = \frac{n(n-1)}{2}$$

Average Case Analysis

- Two assumptions:
  - A[0..n-1] contains the numbers 0 through n-1.
  - All n! permutations are equally likely.
- Suppose A[i] should be inserted at position j ($0 \leq j \leq i$).
  - When $j = 0$, we need i comparisons to insert A[i].
  - Otherwise, we need $i - j + 1$ comparisons. (Note when $j = 1$, we still need i comparisons to determine its proper position.)
- Since any integer in [0, i] is equally likely to be taken by j, i.e.,

$$P(j=0) = P(j=1) = \cdots = P(j=i) = \frac{1}{i+1}$$

Average Case Analysis

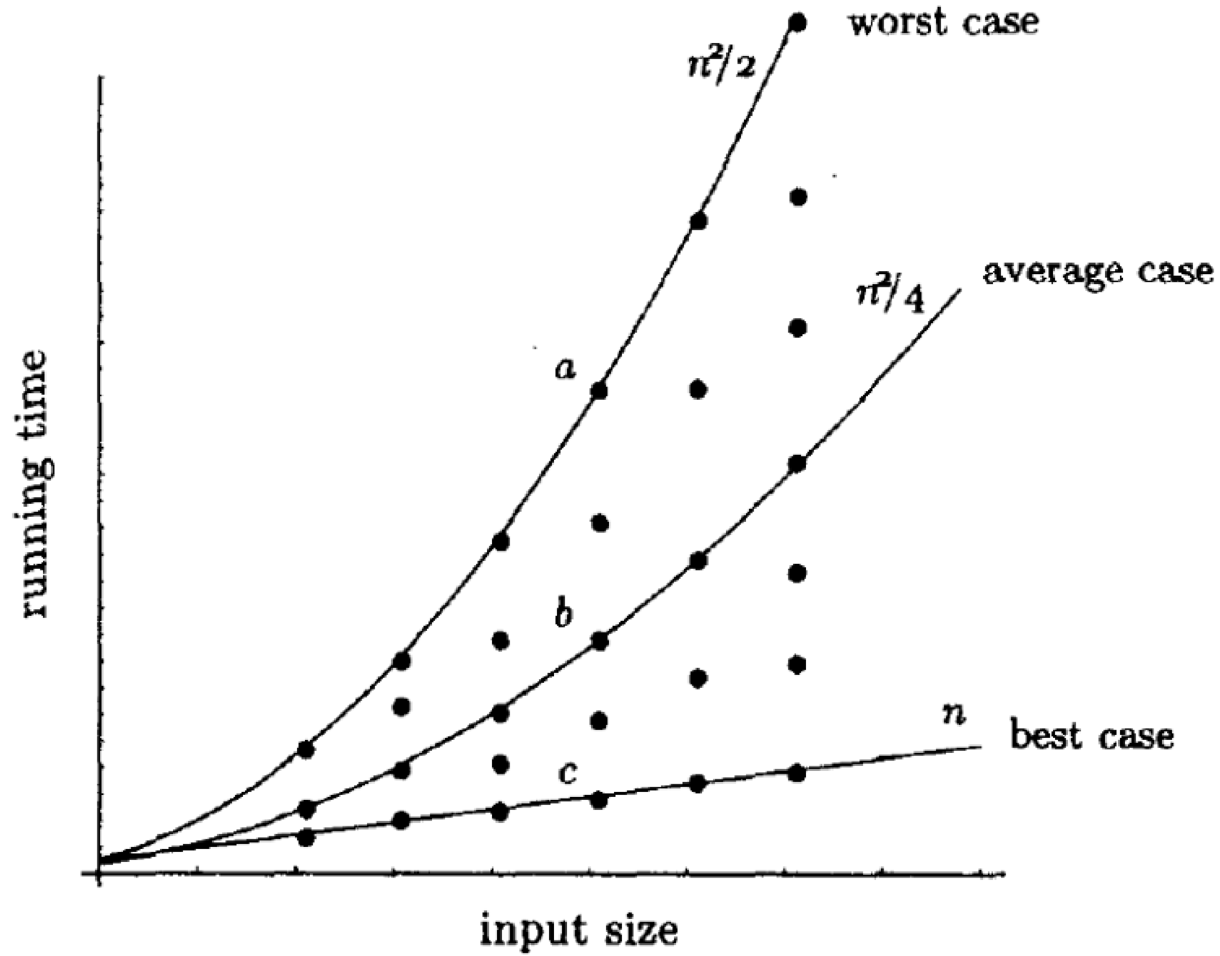
- The expectation number of comparisons for inserting element A[i] in its proper position, is

$$\frac{i}{i+1}+\sum_{j=1}^{i}\frac{i-j+1}{i+1}=\frac{i}{i+1}+\sum_{j=1}^{i}\frac{j}{i+1}=\frac{i}{2}-\frac{1}{i+1}+1$$

- The *average* number of comparisons performed by Algorithm InsertionSort is

$$\sum_{i=1}^{n-1}\left(\frac{i}{2}-\frac{1}{i+1}+1\right)=\frac{n^2}{4}+\frac{3n}{4}-\sum_{i=0}^{n-1}\frac{1}{i+1}$$

Performance of Insertion Sort



Selection Sort

- For `i=0` to `N-2`
  - Find the smallest item in the array `A[i], …, A[N-1]`. Then, swap that item with `A[i]`.

- Finding the smallest item requires **linear scan**.

Which Statements Are Correct for Selection Sort?

For **`i=0`** to **`N-2`**

Find the smallest item in the array **`A[i], …, A[N-1]`**. Then, swap that item with **`A[i]`**.

- **A.** Its worse-case time complexity is $O(N^2)$
- **B.** Its best-case time complexity is $\Omega(N^2)$
- **C.** It is not in-place
- **D.** It is stable

Bubble Sort

```
For i=N-2 downto 0
  For j=0 to i
    If A[j]>A[j+1] swap A[j] and A[j+1]
```

- Compares two adjacent items and swap them to keep them in ascending order.
  - From the beginning to the end. The last item will be the largest.

- Time complexity? $O(N^2)$
- In place? Yes.
- Stable?
  - Yes, because equal elements will not be swapped.

Two Problems with Simple Sorts

- They learn only one piece of information per comparison and hence might compare every pair of elements.
  - Contrast with binary search: learns N/2 pieces of information with first comparison.
- They often move elements one place at a time (bubble sort and insertion sort), even if the element is "far" from its **final place**.
  - Contrast with selection sort, which moves each element exactly to its final place.

- Fast sorts attack these two problems.
  - Two famous ones: **merge sort** and **quick sort**.

Outline

- Sorting Basics
- Merge Sort
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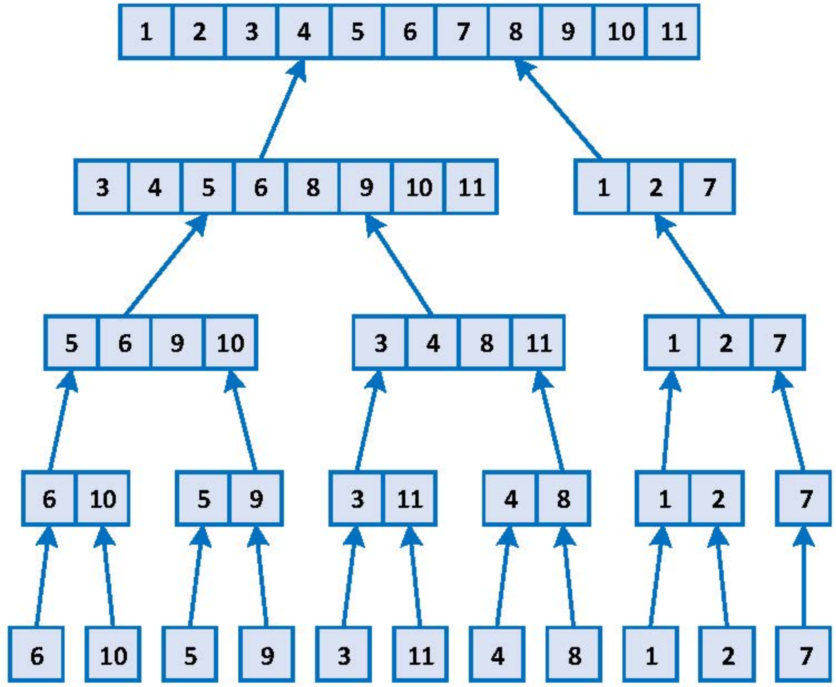
Merge Sort
Algorithm

- Spilt array into two (roughly) equal subarrays.
- Merge sort each subarray recursively.
  - The two subarrays will be sorted.
- Merge the two sorted subarrays into a sorted array.



Merge Sort

Pseudo-code

```
void mergesort(int *a, int left, int
  right) {
    if (left >= right) return;
    int mid = (left+right)/2;
    mergesort(a, left, mid);
    mergesort(a, mid+1, right);
    merge(a, left, mid, right);
}
```

Merge Two Sorted Arrays

- For example, merge A = (2, 5, 6) and B = (1, 3, 8, 9, 10).

- Compare the smallest element in the two arrays A and B and move the smaller one to an additional array C.
- Repeat until one of the arrays becomes empty.
- Then append the other array at the end of array C.

Merge Two Sorted Arrays
Implementation

- We actually do not "remove" element from arrays A and B.
  - We just keep a pointer indicating the smallest element in each array.
  - We "remove" element by incrementing that pointer.

```
i = j = k = 0;
while(i < sizeA && j < sizeB) {
  if(A[i]<=B[j]) C[k++]=A[i++];
  else C[k++]=B[j++];
}
if(i == sizeA) append(C, B);
else append(C, A);
```

Time complexity?

Time complexity is $O(sizeA + sizeB)$

Merge Sort

Time Complexity

```
void mergesort(int *a, int left, int
  right) {
    if (left >= right) return;
    int mid = (left+right)/2;
    mergesort(a, left, mid);
    mergesort(a, mid+1, right);
    merge(a, left, mid, right);
}
```

$T(N/2)$

$T(N/2)$

$O(N)$

- Let $T(N)$ be the time required to merge sort N elements.
- Merge two sorted arrays with total size N takes $O(N)$.

Recursive relation: $T(N) = 2T(N/2) + O(N)$

How to solve the recurrence?

Solve Recurrence: Master Method

- A "black box" for solving recurrence.
- However, there is an important assumption: all sub-problems have roughly **equal** sizes.
  - E.g., merge sort
  - Not apply to unbalanced division.

Solve Recurrence: Master Method

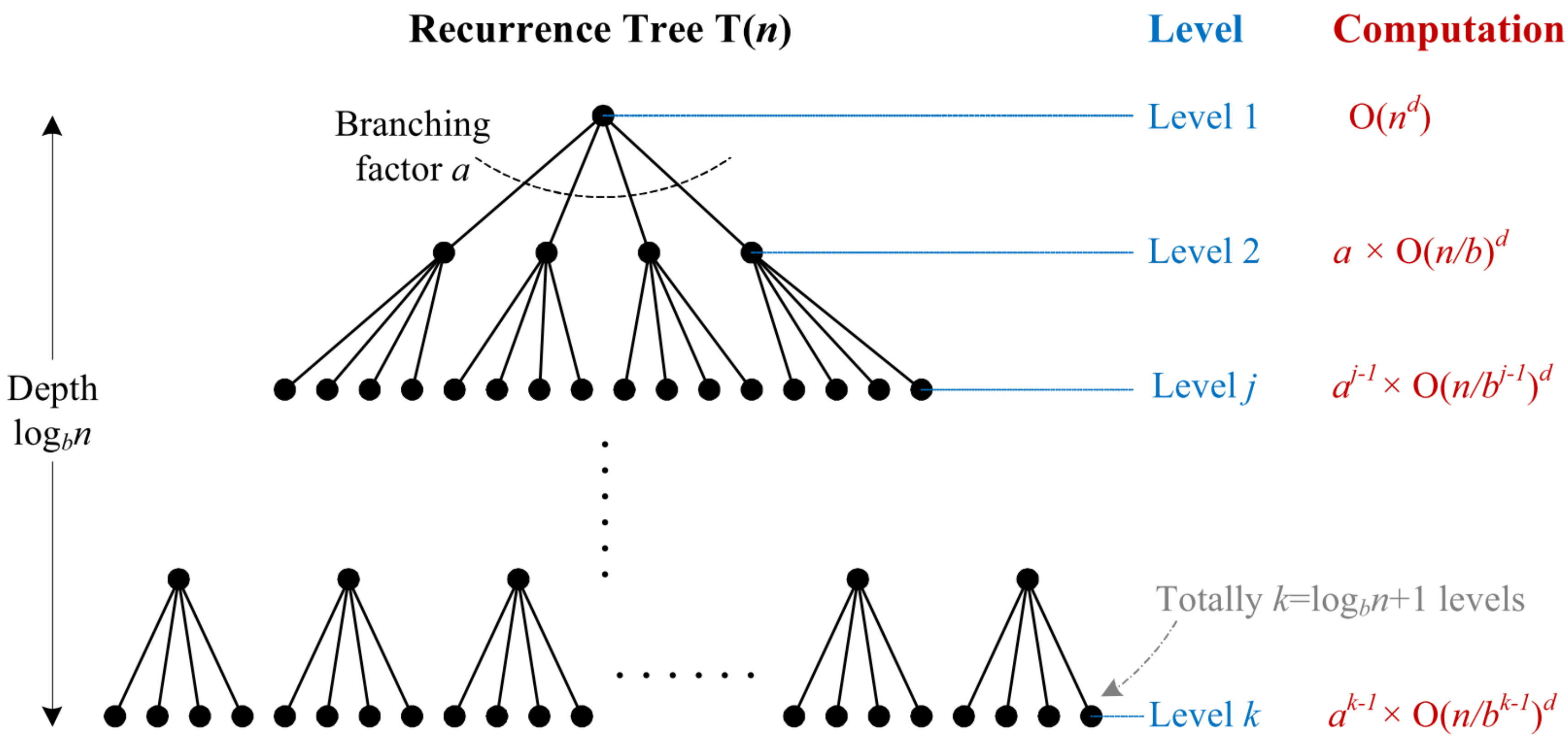
- Recurrence: $T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$
  - Base case: $T(n) \leq constant$ for all sufficiently small n.
  - a = number of recursive calls (integer ≥ 1)
  - b = input size shrinkage factor (integer > 1)
  - $O(n^d)$: the runtime of merging solutions. d is real value ≥ 0.
  - a, b, d are independent of n.
- Claim:

base doesn't matter

$$T(n) = \begin{cases} O(n^d \log n) & if\ a = b^d \\ O(n^d) & if\ a < b^d \\ O(n^{\log_b a}) & if\ a > b^d \end{cases}$$

base matters!

Master Theorem



Complexity of T(n) = Sum up all computations at each level.

Proof of Master Theorem:

- Assume that n is a power of b. This will not influence the final bound in any important way: n is at most a multiplicative factor of b away from some power of b.
- The size of the subproblems decreases by a factor of b with each level of recursion, and reaches the base case when

$$\frac{n}{b^{k-1}} = 1 \Rightarrow k = \log_b n + 1$$

(k is the level of the recursion tree, which equals to tree height + 1.)

- The branching factor of the recursion tree is a, so the j-th level of the tree is made up of a^{j-1} subproblems, each of size n/b^{j-1}.
- The total work done at the j-th level is

$$a^{j-1} \times O\left(\frac{n}{b^{j-1}}\right)^d = O(n^d) \times \left(\frac{a}{b^d}\right)^{j-1}.$$

Proof of Master Theorem

The total work done is

$$\sum_{j=1}^{\log_b n+1}\left(a^{j-1}\times O\left(\frac{n}{b^{j-1}}\right)^d\right)=\sum_{j=0}^{\log_b n}\left(O(n^d)\times\left(\frac{a}{b^d}\right)^j\right)=O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^j.$$

It's the sum of a *geometric series (GS)* with ratio a/b^d.

(1) $\frac{a}{b^d}<1\Rightarrow d>\log_b a$:

$$O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^j\le O(n^d)\frac{1}{1-\frac{a}{b^d}}=O(n^d).$$

(Sum of GS: $S_n=\sum_{j=1}^{n}a_1q^{j-1}=a_1\frac{1-q^n}{1-q}\le a_1\frac{1}{1-q}$ if $q<1$)

Proof of Master Theorem

$$(2)\ \frac{a}{b^d} = 1 \Rightarrow d = \log_b a:$$

$$O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^j = O(n^d)(\log_b n + 1) = O(n^d \log_b n) = O(n^d \log n).$$

$(\log_b n = \frac{\log n}{\log b} = \frac{1}{\log b}\log n = O(\log n)$ by changing the base)

Proof of Master Theorem

(3) $\frac{a}{b^d} > 1 \Rightarrow d < \log_b a$: (reverse the GS in decreasing order)

$$\begin{aligned} O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^j &= O(n^d)\sum_{j=0}^{\log_b n}\left(\frac{a}{b^d}\right)^{\log_b n}\cdot\left(\frac{b^d}{a}\right)^j \\ &= O(n^d)\sum_{j=0}^{\log_b n}\frac{a^{\log_b n}}{(b^{\log_b n})^d}\cdot\left(\frac{b^d}{a}\right)^j \\ &\leq O(n^d)\frac{n^{\log_b a}}{n^d}\cdot\frac{1}{1-\frac{b^d}{a}} \\ &= O(n^{\log_b a}) \end{aligned}$$

$\left(a^{\log_b n} = a^{(\log_a n)(\log_b a)} = n^{\log_b a}\right)$

Example of Merge Sort

Recurrence: $$T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$$

Claim: $$T(n) = \begin{cases} O\left(n^d \log n\right) & if\ a = b^d \\ O\left(n^d\right) & if\ a < b^d \\ O\left(n^{\log_b a}\right) & if\ a > b^d \end{cases}$$

- $a = 2, b = 2, d = 1 \quad \Rightarrow b^d = a$
- $T(n) = O(n \log n)$

What are a, b, d for Binary Search?

Recurrence: $$T(n) \leq aT\left(\frac{n}{b}\right) + O(n^d)$$

Claim: $$T(n) = \begin{cases} O\left(n^d \log n\right) & if\ a = b^d \\ O\left(n^d\right) & if\ a < b^d \\ O\left(n^{\log_b a}\right) & if\ a > b^d \end{cases}$$

A. a =2, b = 2, d = 0 **B.** a =1, b = 2, d =0

C. a =2, b = 2, d = 1 **D.** a =1, b = 2, d =1

Merge Sort

Characteristics

- Not in-place
  - For efficient merging two sorted arrays, we need an auxiliary $O(N)$ space.
  - Recursion needs up to $O(\log N)$ stack space.
- Stable if **`merge()`** **maintains** the relative order of equal keys.

Divide-and-Conquer Approach

- Merge sort uses the **divide-and-conquer** approach.

- Recursively **breaking** down a problem into two or more sub-problems of the same (or related) type, until these become simple enough to be solved directly.
  - For merge sort, split an array into two and sort them respectively.

- The solutions to the sub-problems are then **combined** to give a solution to the original problem.
  - For merge sort, merge two sorted arrays.

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Quick Sort

Algorithm

Another divide-and-conquer approach to sort

- Choose an array element as **pivot**.
- Put all elements < pivot to the left of pivot.
- Put all elements ≥ pivot to the right of pivot.
- Move pivot to its correct place in the array.
- Sort left and right subarrays recursively (not including pivot).

partition()

```
void quicksort(int *a, int left,
  int right) {
    int pivotat; // index of the pivot
    if(left >= right) return;
    pivotat = partition(a, left, right);
    quicksort(a, left, pivotat-1);
    quicksort(a, pivotat+1, right);
}
```

Choice of Pivot

- If your input is random, you can choose the **first** element.
  - But this is very bad for presorted input.

- A better strategy: **randomly** pick an element from the array as pivot.
  - **Claim**: **for any input**, the average running time is $O(n \log n)$.
    - **Note**: average is over random choice of pivots made by the algorithm, **not** on the input.

Partitioning the Array

- Once pivot is chosen, swap pivot to the beginning of the array.
- When another array B is available, scan original array A from left to right.
  - Put elements < pivot at the left end of B.
  - Put elements ≥ pivot at the right end of B.
  - The pivot is put at the remaining position of B.
  - Copy B back to A.

A	6	2	8	5	11	10	4	1	9	7	3
B	2	5	4	1	3	6	7	9	10	11	8

In-Place Partitioning the Array

1. Once pivot is chosen, swap pivot to the beginning of the array.
2. Start counters **`i=1`** and **`j=N-1`**.
3. Increment **`i`** until we find element **`A[i]>=pivot`**.
   - **`A[i]`** is the leftmost item $\geq$ pivot.
4. Decrement **`j`** until we find element **`A[j]<pivot`**.
   - **`A[j]`** is the rightmost item $<$ pivot.
5. If **`i<j`**, swap **`A[i]`** with **`A[j]`**. Go back to step 3.
6. Otherwise, swap the first element (pivot) with **`A[j]`**.

In-Place Partitioning the Array

Example

`i` `j`

A	6	2	8	5	11	10	4	1	9	7	3
A	6	2	3	5	11	10	4	1	9	7	8
A	6	2	3	5	1	10	4	11	9	7	8
A	6	2	3	5	1	4	10	11	9	7	8

- Now, **`j < i`**, swap the first element (pivot) with **`A[j]`**.

A	4	2	3	5	1	6	10	11	9	7	8

In-Place Partitioning the Array

Time Complexity

1. Once pivot is chosen, swap pivot to the beginning of the array.
2. Start counters **`i=1`** and **`j=N-1`**.
3. Increment **`i`** until we find element **`A[i]>=pivot`**.
4. Decrement **`j`** until we find element **`A[j]<pivot`**.
5. If **`i<j`**, swap **`A[i]`** with **`A[j]`**. Go back to step 3.
6. Otherwise, swap the first element (pivot) with **`A[j]`**.

- Scan the entire array no more than twice.
- Time complexity is $O(N)$, where N is the size of the array.

Quick Sort

Time Complexity

```
void quicksort(int *a, int left,
  int right) {
    int pivotat; // index of the pivot
    if(left >= right) return;
    pivotat = partition(a, left, right);
    quicksort(a, left, pivotat-1);
    quicksort(a, pivotat+1, right);
}
```

$O(N)$

$T(LeftSz)$

$T(RightSz)$

- Let $T(N)$ be the time needed to sort N elements.
  - $T(0) = c$, where c is a constant.
- Recursive relation:

$$T(N) = T(LeftSz) + T(RightSz) + O(N)$$

  - $LeftSz + RightSz = N - 1$

Quick Sort

Worst Case Time Complexity

- Recursive relation:

$$T(N) = T(LeftSz) + T(RightSz) + O(N)$$

- Worst case happens when each time the pivot is the smallest item or the largest item
  - $T(N) = T(N-1) + T(0) + O(N)$

$$\begin{aligned} &\leq T(N-1) + T(0) + dN \\ &\leq T(N-2) + 2T(0) + d(N-1) + dN \\ &\quad \dots \\ &\leq T(0) + NT(0) + d + 2d + \cdots + d(N-1) + dN \\ &= O(N^2) \end{aligned}$$

Quick Sort

Best Case Time Complexity

- Recursive relation:

$$T(N) = T(LeftSz) + T(RightSz) + O(N)$$

- Best case happens when each time the pivot divides the array into two equal-sized ones.
  - $T(N) = T((N-1)/2) + T((N-1)/2) + O(N)$
  - The recursive relation is similar to that of merge sort.
  - $T(N) = O(N \log N)$

Quick Sort

Average Time Complexity

- Average time complexity of quick sort can be proved to be $O(N \log N)$.
  - Assume **randomly** pick an element from the array as pivot.
  - **Note**: average is over random choice of pivots made by the algorithm, **not** on the input.
  - The claim holds for any input.

Quick Sort

Other Characteristics

- In-place?
  - In-place partitioning.
  - Worst case needs $O(N)$ stack space.
  - Average case needs $O(\log N)$ stack space.
    - "Weakly" in-place.

- Not stable.

Quick Sort

Summary

- Like merge sort, quick sort is a divide-and-conquer algorithm.

- Merge sort: easy division, complex combination.
- Quick sort: complex division (partition with pivot step), easy combination.

- Insertion sort is faster than quick sort for small arrays.
  - Terminate quick sort when array size is below a threshold. Do insertion sort on subarrays.

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Comparison Sorts

Summary

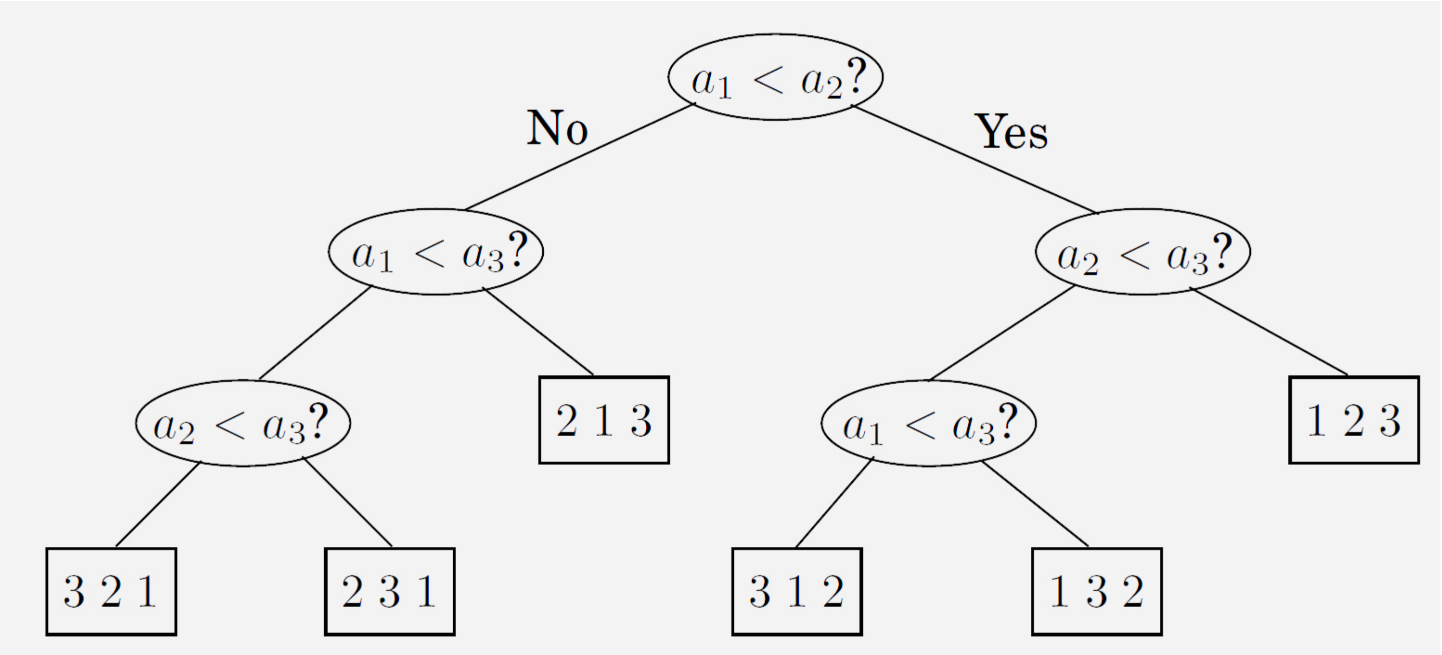
	Worst Case Time	Average Case Time	In Place	Stable
Insertion	$O(N^2)$	$O(N^2)$	Yes	Yes
Selection	$O(N^2)$	$O(N^2)$	Yes	No
Bubble	$O(N^2)$	$O(N^2)$	Yes	Yes
Merge Sort	$O(N \log N)$	$O(N \log N)$	No	Yes
Quick Sort	$O(N^2)$	$O(N \log N)$	Weakly	No

For comparison sort, is $O(N \log N)$ the best we can do in the **worst case**?

Comparison Sorts

Worst Case Time Complexity

- Theorem: A sorting algorithm that is based on pairwise comparisons must use $\Omega(N \log N)$ operations to sort in the worst case.
- An example sorting permutation tree for $\{a_1, a_2, a_3\}$:



An n log n Lower Bound for Sorting

- Sorting algorithms can be depicted as trees.
- The **depth** of the tree – the number of comparisons on the longest path from root to leaf, is exactly the worst-case time complexity of the algorithm.
- Consider any such tree that sorts an array of n elements. Each of its leaves is labeled by a *permutation* of {1, 2, . . . , n}.

  every permutation must appear as the label of a leaf.

- This is a binary tree with $n!$ leaves. Thus, the depth of our tree – and the complexity of our algorithm – must be at least

$$\log(n!) \approx \log\left(\sqrt{\pi\,(2n+1/3)}\cdot n^n \cdot e^{-n}\right) = \Omega(n \log n),$$

where we use Stirling's formula.