Ve492: Introduction to Artificial Intelligence
Constraint Satisfaction Problems II and Local Search

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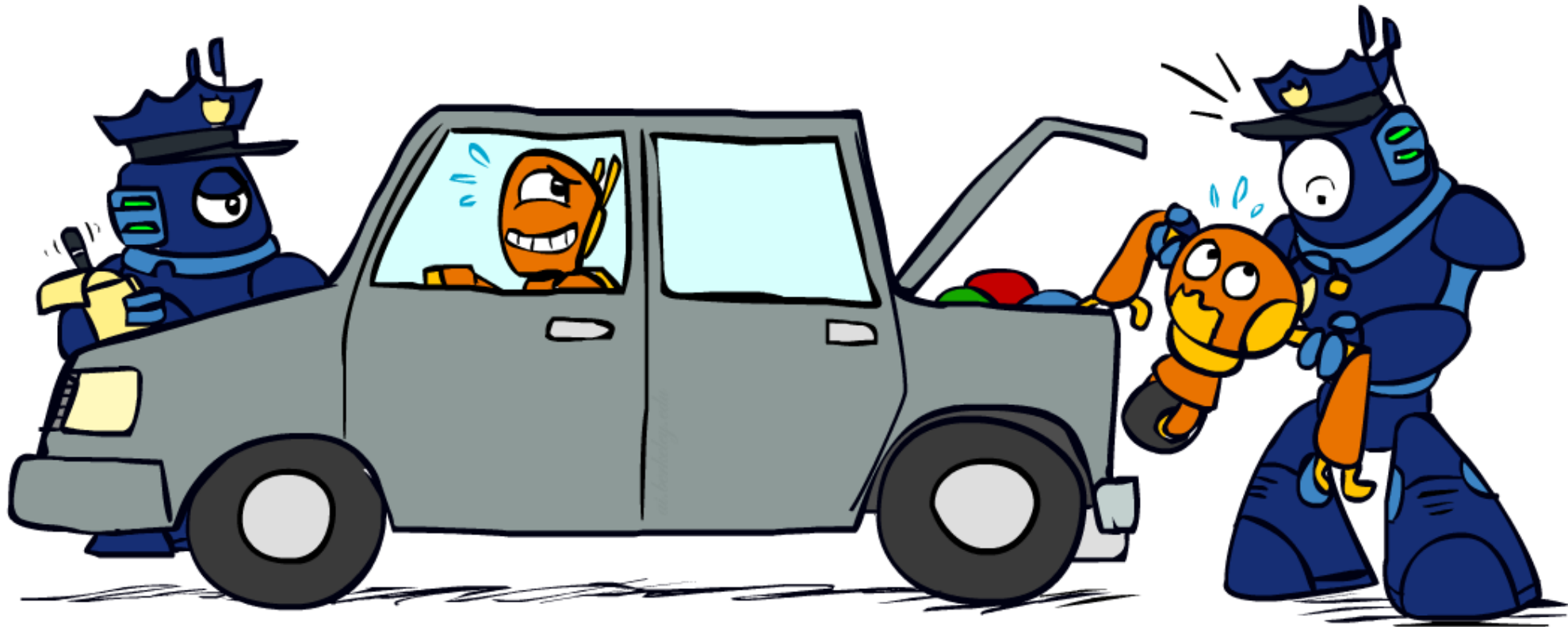Slides adapted from http://ai.berkeley.edu, AIMA, UM, CMU
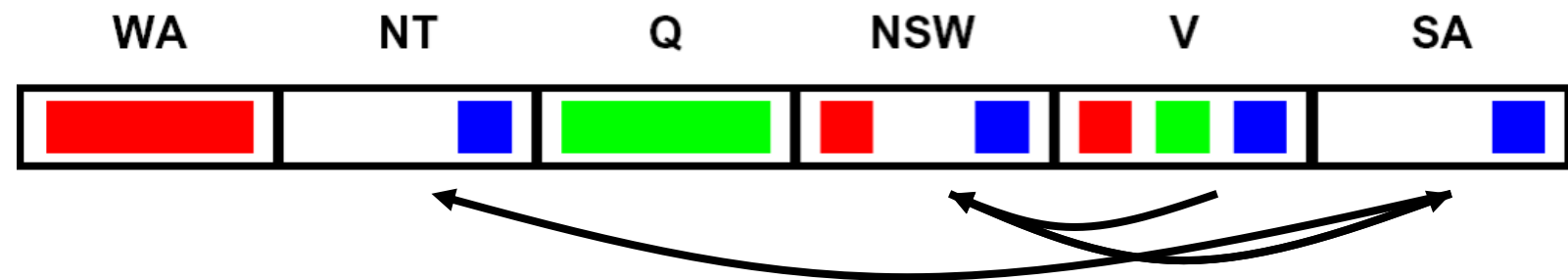
Today

❖ Efficient Solution of CSPs

❖ Local Search

Arc Consistency and Beyond

Arc Consistency of an Entire CSP

❖ A simple form of propagation makes sure all arcs are simultaneously consistent:

Remember: Delete from the tail!

❖ Arc consistency detects failure earlier than forward checking
❖ Important: If X loses a value, neighbors of X need to be rechecked!
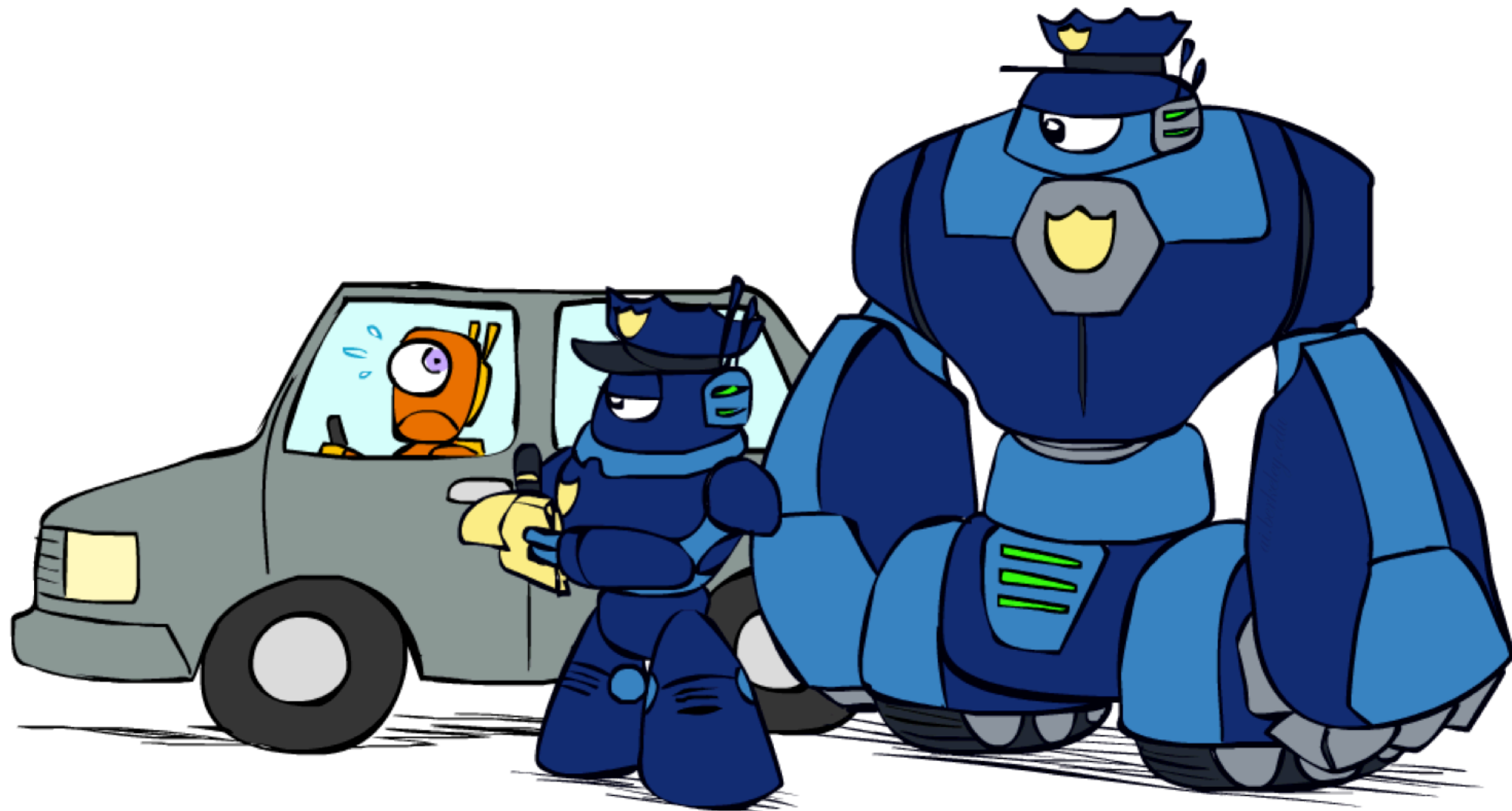❖ Must rerun after each assignment!

Limitations of Arc Consistency

❖ After enforcing arc consistency:

   ❖ Can have one solution left

   ❖ Can have multiple solutions left

   ❖ Can have no solutions left (and not know it)

❖ Arc consistency still runs inside a backtracking search!
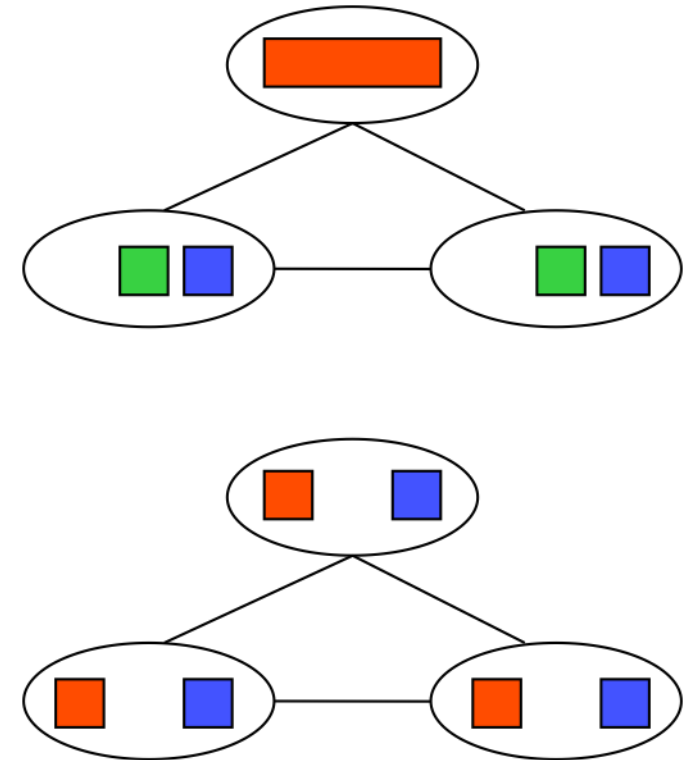
*What went wrong here?*

K-Consistency

K-Consistency

❖ **Increasing degrees of consistency**

   ❖ 1-Consistency (Node Consistency): Each single node's domain has a value which meets that node's unary constraints

   ❖ 2-Consistency (Arc Consistency): For each pair of nodes, any consistent assignment to one can be extended to the other

   ❖ K-Consistency: For each k nodes, any consistent assignment to k-1 can be extended to the kth node.
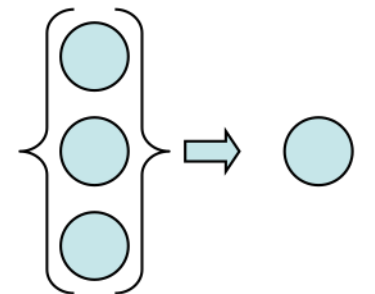
❖ **Higher k more expensive to compute**
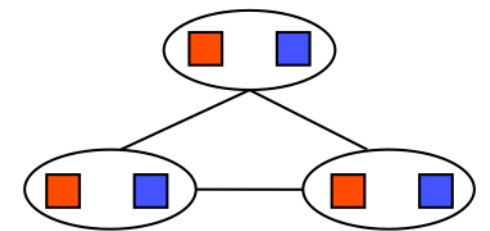
Strong K-Consistency

- Strong k-consistency: also k-1, k-2, … 1 consistent

- Claim: strong n-consistency means we can solve without backtracking!

- Why?
  - Choose any assignment to any variable
  - Choose a new variable
  - By 2-consistency, there is a choice consistent with the first
  - Choose a new variable
  - By 3-consistency, there is a choice consistent with the first 2
  - …

- Lots of middle ground between arc consistency and n-consistency! (e.g. k=3, called path consistency)

Structure
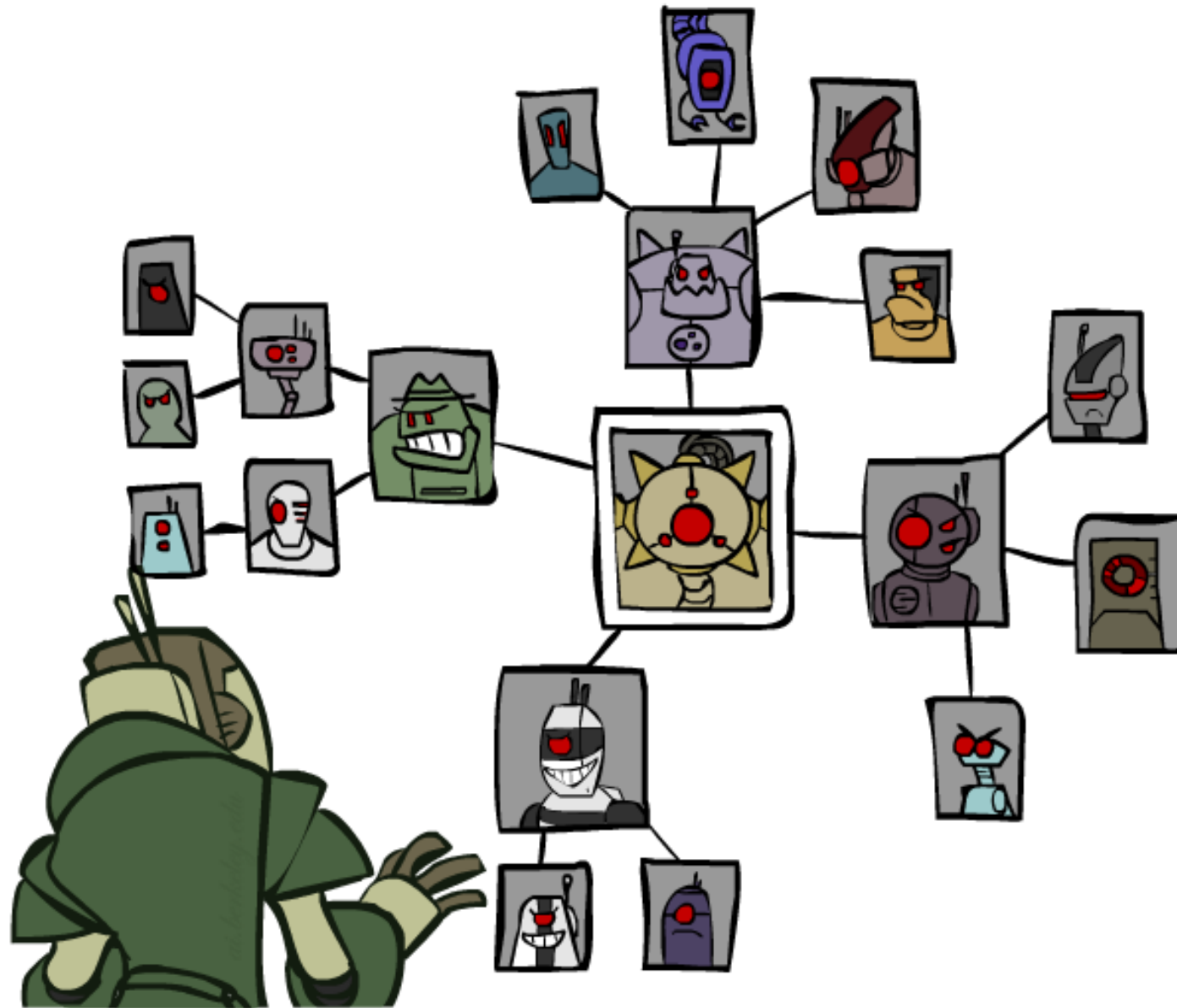
Problem Structure

- Extreme case: independent subproblems
  - Example: Tasmania and mainland do not interact

- Independent subproblems are identifiable as connected components of constraint graph

- Suppose a graph of n variables can be broken into subproblems of only c variables:
  - Worst-case solution cost is $O((n/c)(d^c))$, linear in n
  - E.g., n = 80, d = 2, c = 20
  - 2^{80} = 4 billion years at 10 million nodes/sec
 - $(4)(2^{20})$ = 0.4 seconds at 10 million nodes/sec
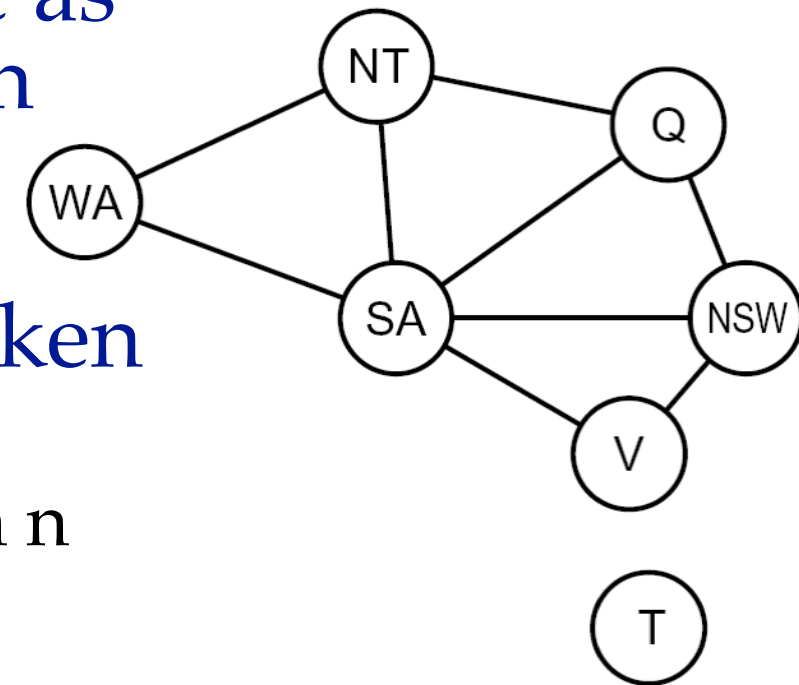
Tree-Structured CSPs
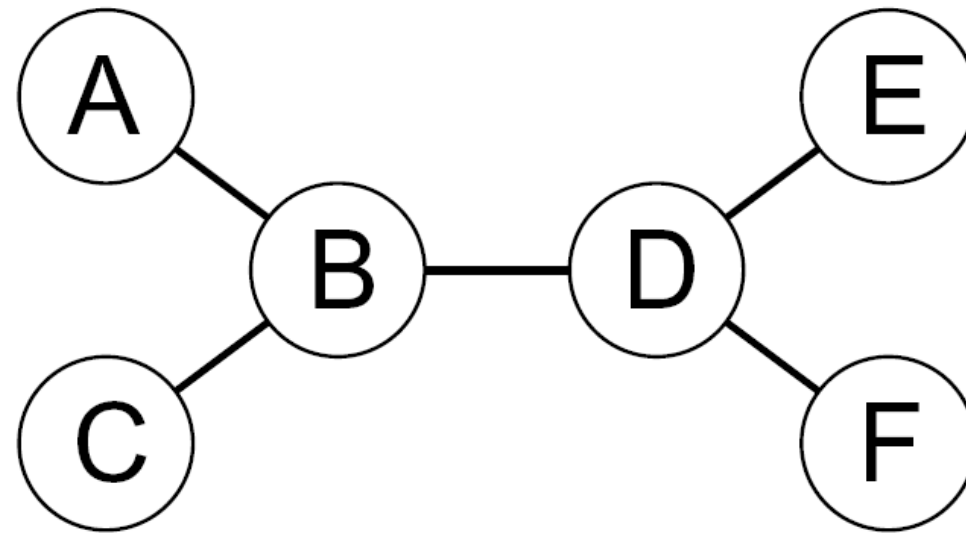
❖ Theorem: if the constraint graph has no loops, the CSP can be solved in $O(n\,d^2)$ time

    ❖ Compare to general CSPs, where worst-case time is $O(d^n)$

❖ This property also applies to probabilistic reasoning (later): an example of the relation between syntactic restrictions and the complexity of reasoning

Tree-Structured CSPs

- ❖ Algorithm for tree-structured CSPs:
  - ❖ Order: Choose a root variable, order variables so that parents precede children
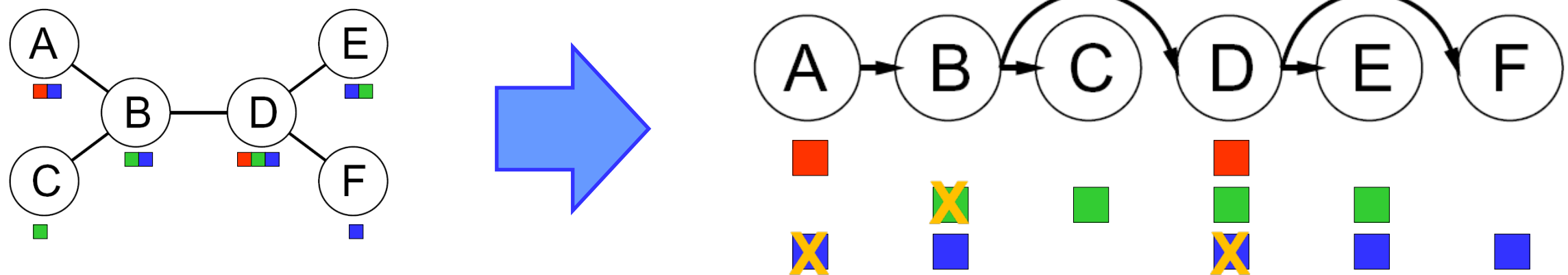

- ❖ Remove backward: For i = n : 2, apply RemoveInconsistent(Parent(X_i),X_i)
- ❖ Assign forward: For i = 1 : n, assign X_i consistently with Parent(X_i)

- ❖ Runtime: O(n d^2) (why?)

Tree-Structured CSPs

❖ Claim 1: After backward pass, all root-to-leaf arcs are consistent

❖ Proof: Each X→Y was made consistent at one point and Y's domain could not have been reduced thereafter (because Y's children were processed before Y)
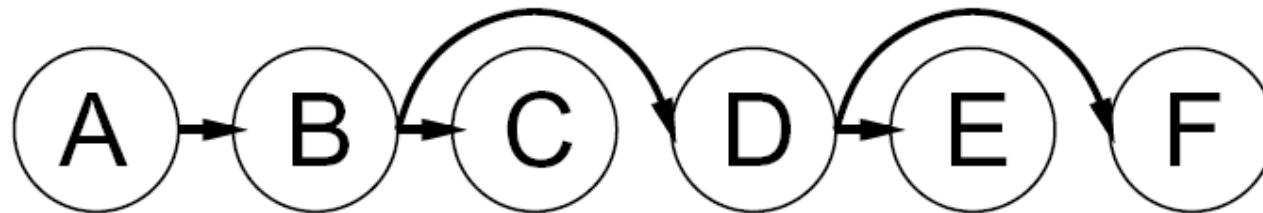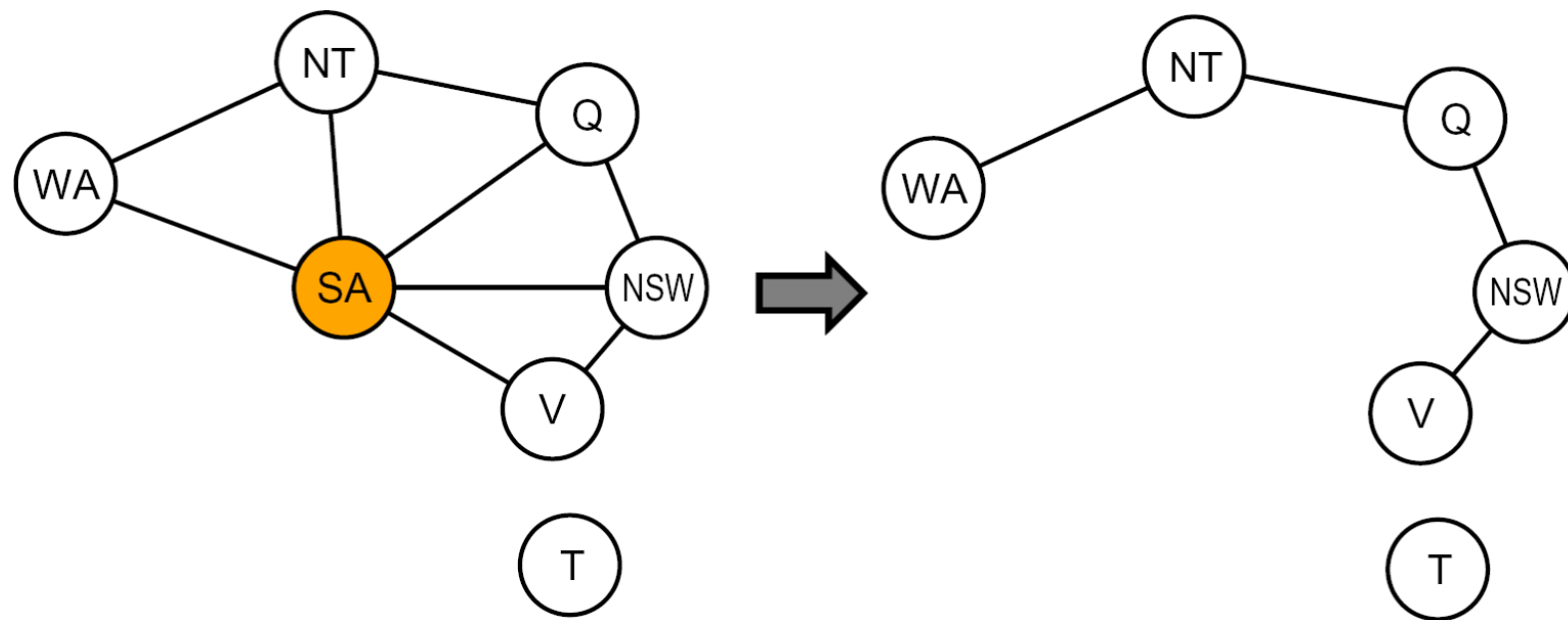

❖ Claim 2: If root-to-leaf arcs are consistent, forward assignment will not backtrack

❖ Proof: Induction on position

❖ Why doesn't this algorithm work with cycles in the constraint graph?

❖ Note: we'll see this basic idea again with Bayes' nets

Improving Structure
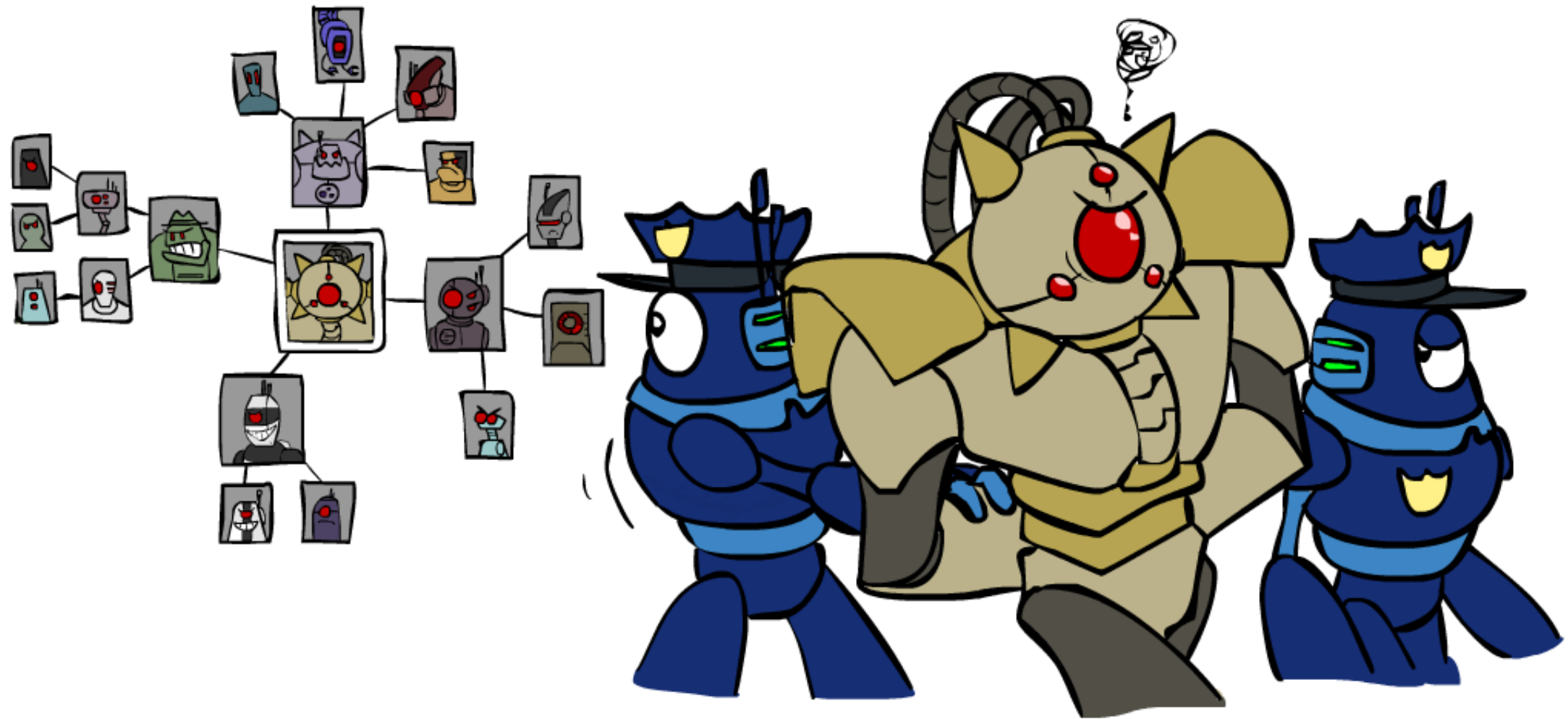
Nearly Tree-Structured CSPs

- ❖ Conditioning: instantiate a variable, prune its neighbors' domains

- ❖ Cutset conditioning: instantiate (in all ways) a set of variables such that the remaining constraint graph is a tree

- ❖ Cutset size c gives runtime $O((d^c) (n-c) d^2)$, very fast for small c
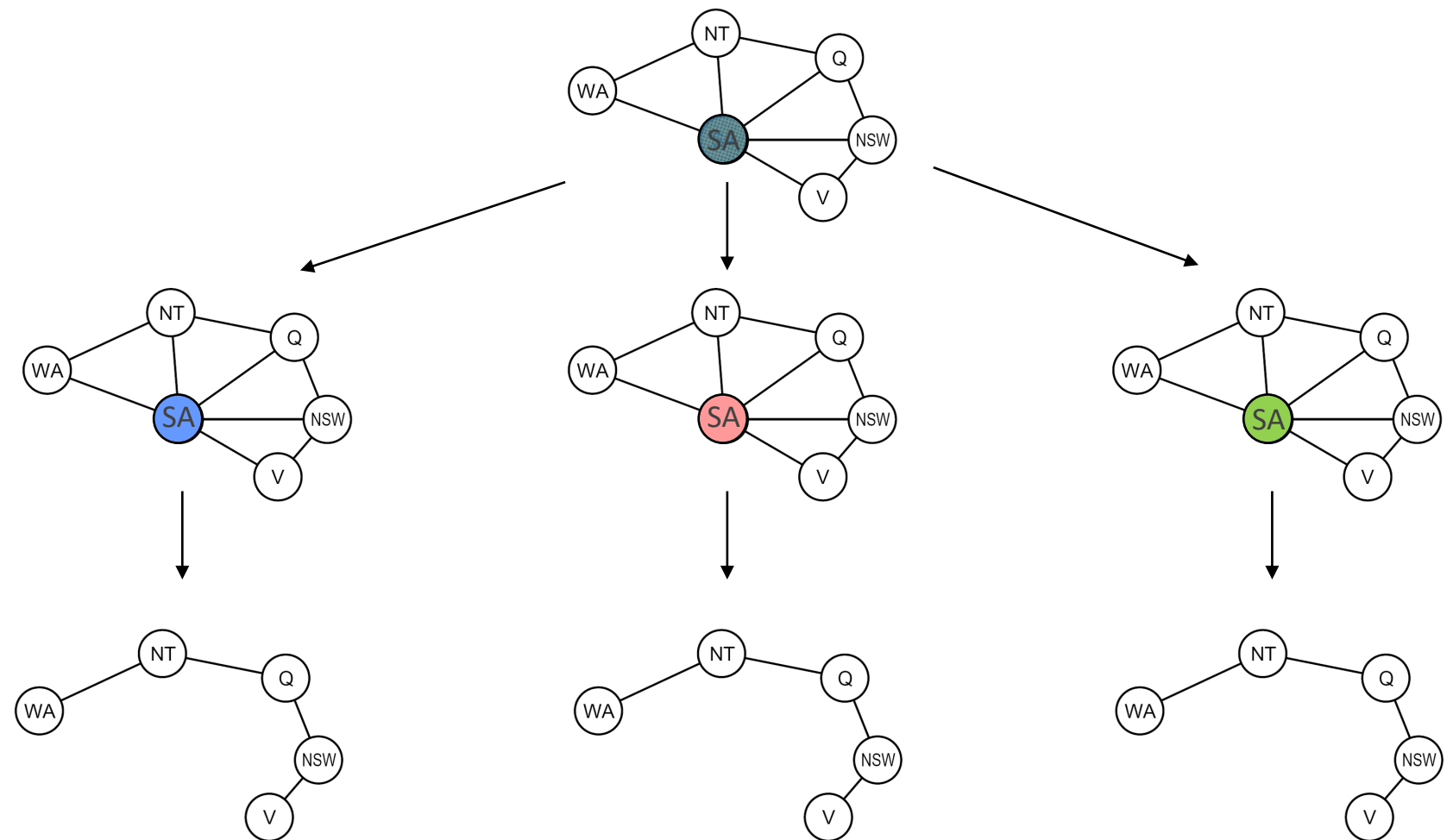
Cutset Conditioning

Choose a cutset

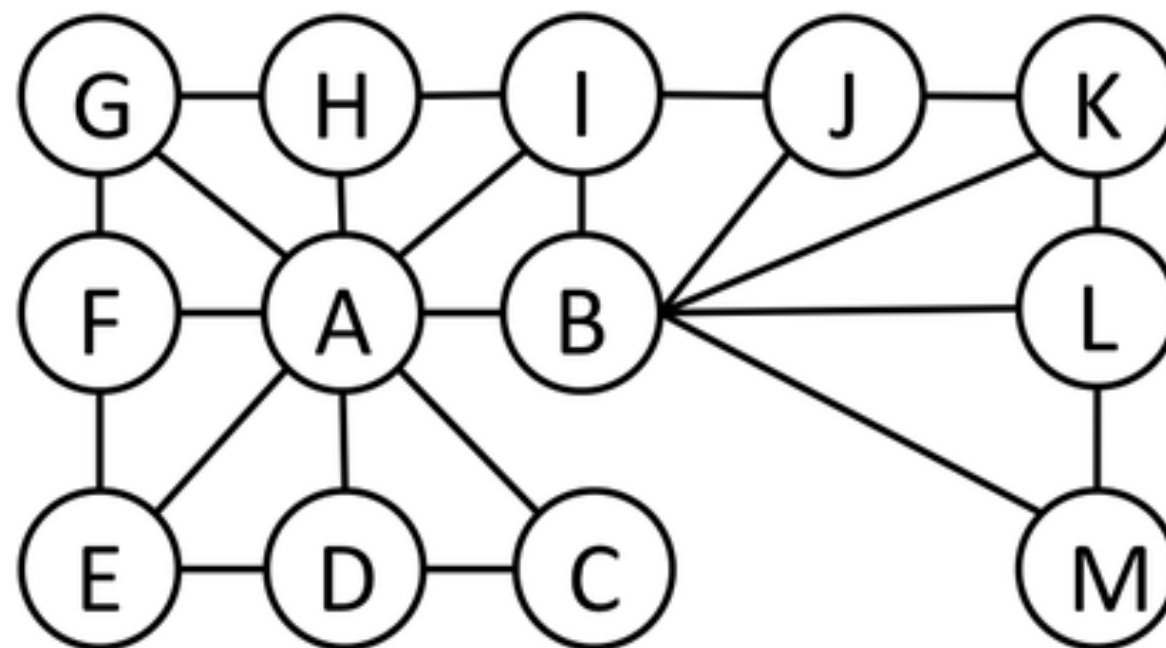Instantiate the cutset
(all possible ways)

Compute residual CSP
for each assignment

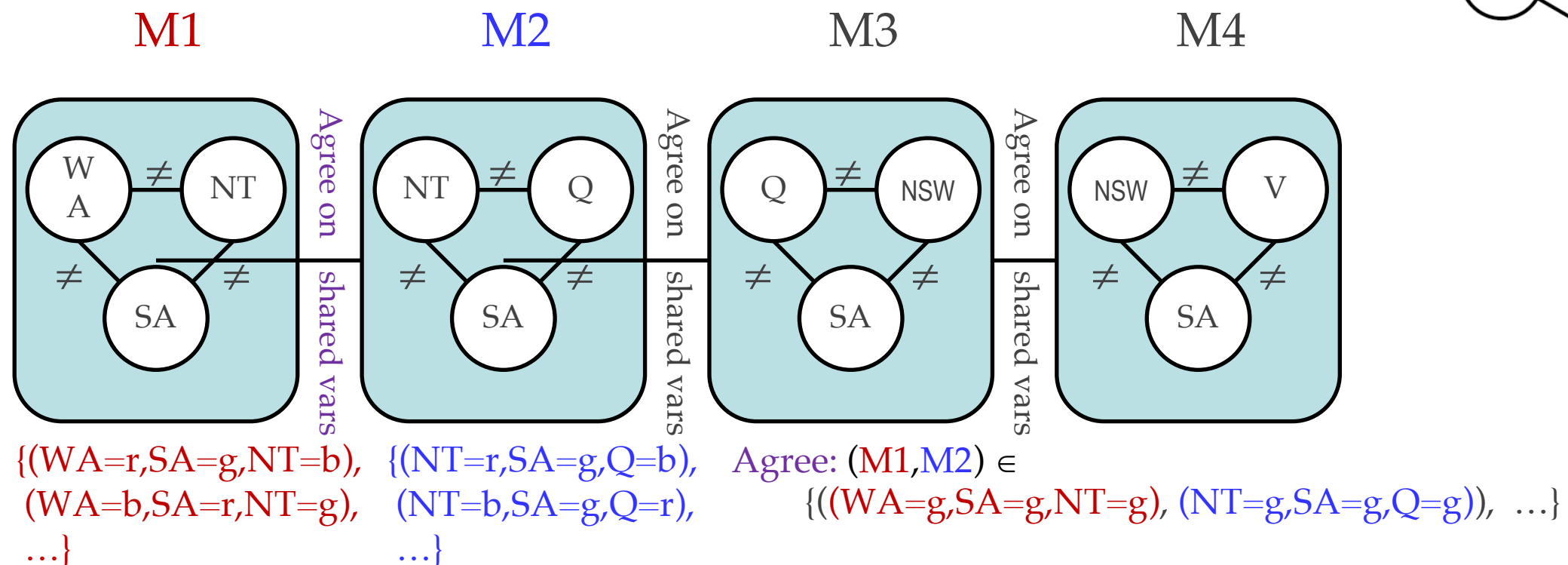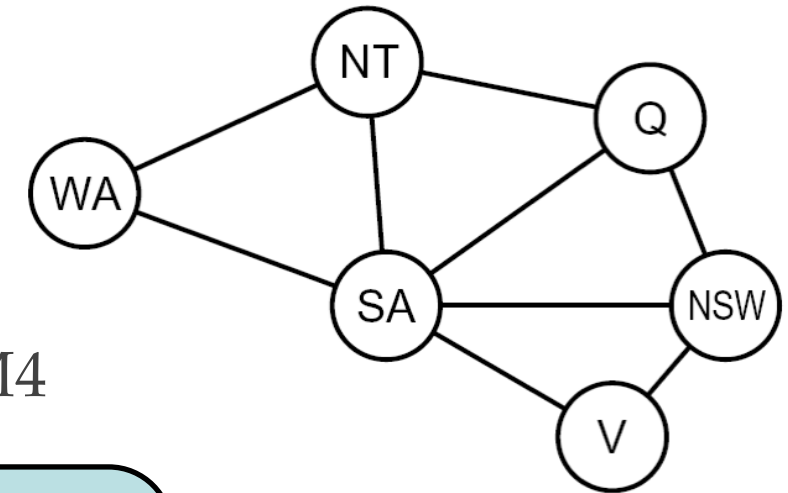Solve the residual
CSPs (tree structured)

Quiz: Cutset

❖ Find the smallest cutset for the graph below.
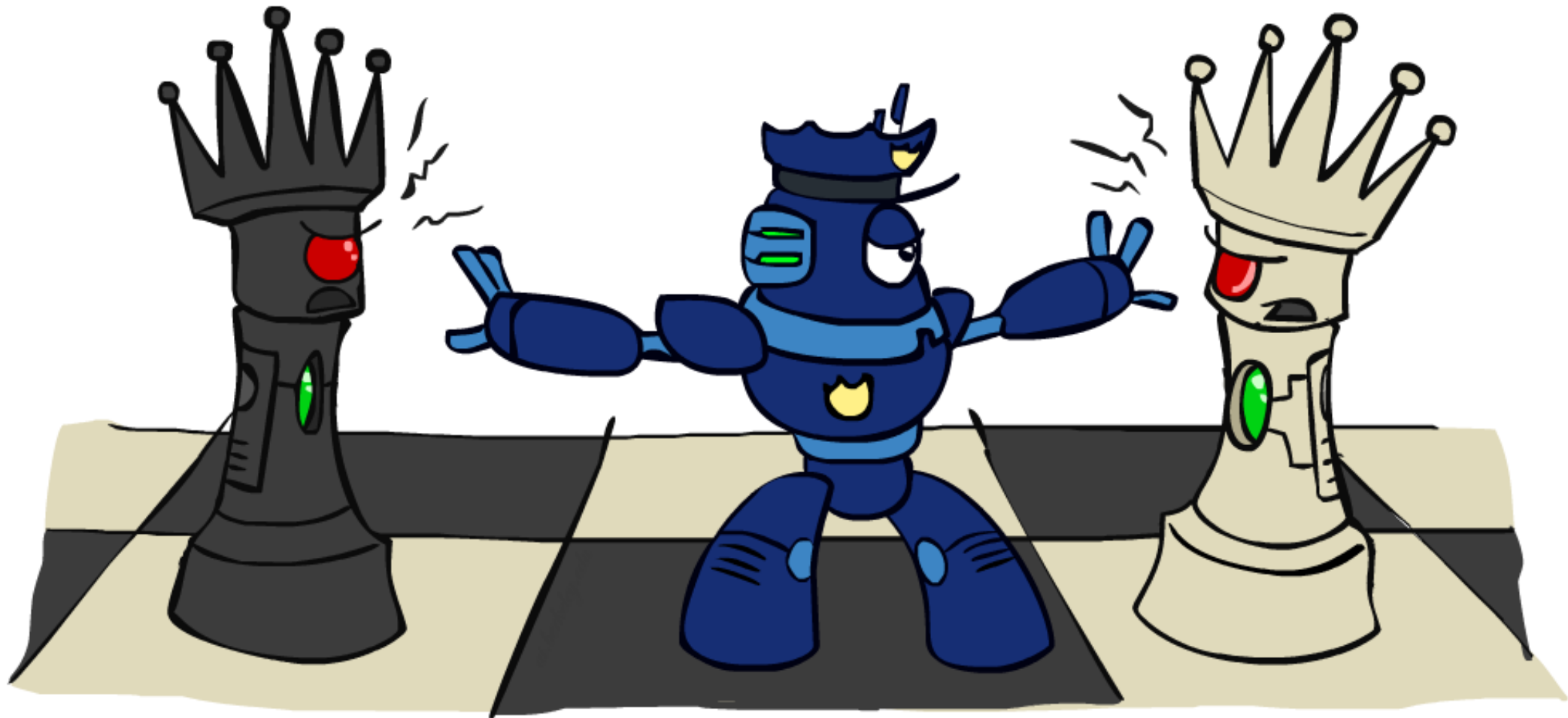
Tree Decomposition*

- Idea: create a tree-structured graph of mega-variables
  - Each mega-variable encodes part of the original CSP
 - Subproblems overlap to ensure consistent solutions

M1   M2   M3   M4

{(WA=r,SA=g,NT=b),   {(NT=r,SA=g,Q=b),   Agree: (M1,M2) ∈
 (WA=b,SA=r,NT=g),   (NT=b,SA=g,Q=r),   {((WA=g,SA=g,NT=g), (NT=g,SA=g,Q=g)), …}
 …}   …}

Iterative Improvement

Iterative Algorithms for CSPs

- Local search methods typically work with "complete" states, i.e., all variables assigned

- To apply to CSPs:
  - Take an assignment with unsatisfied constraints
  - Operators *reassign* variable values
  - No fringe! Live on the edge.

- Algorithm: While not solved,
  - Variable selection: randomly select any conflicted variable
  - Value selection: min-conflicts heuristic:
    - Choose a value that violates the fewest constraints
    - I.e., hill climb with h(n) = total number of violated constraints
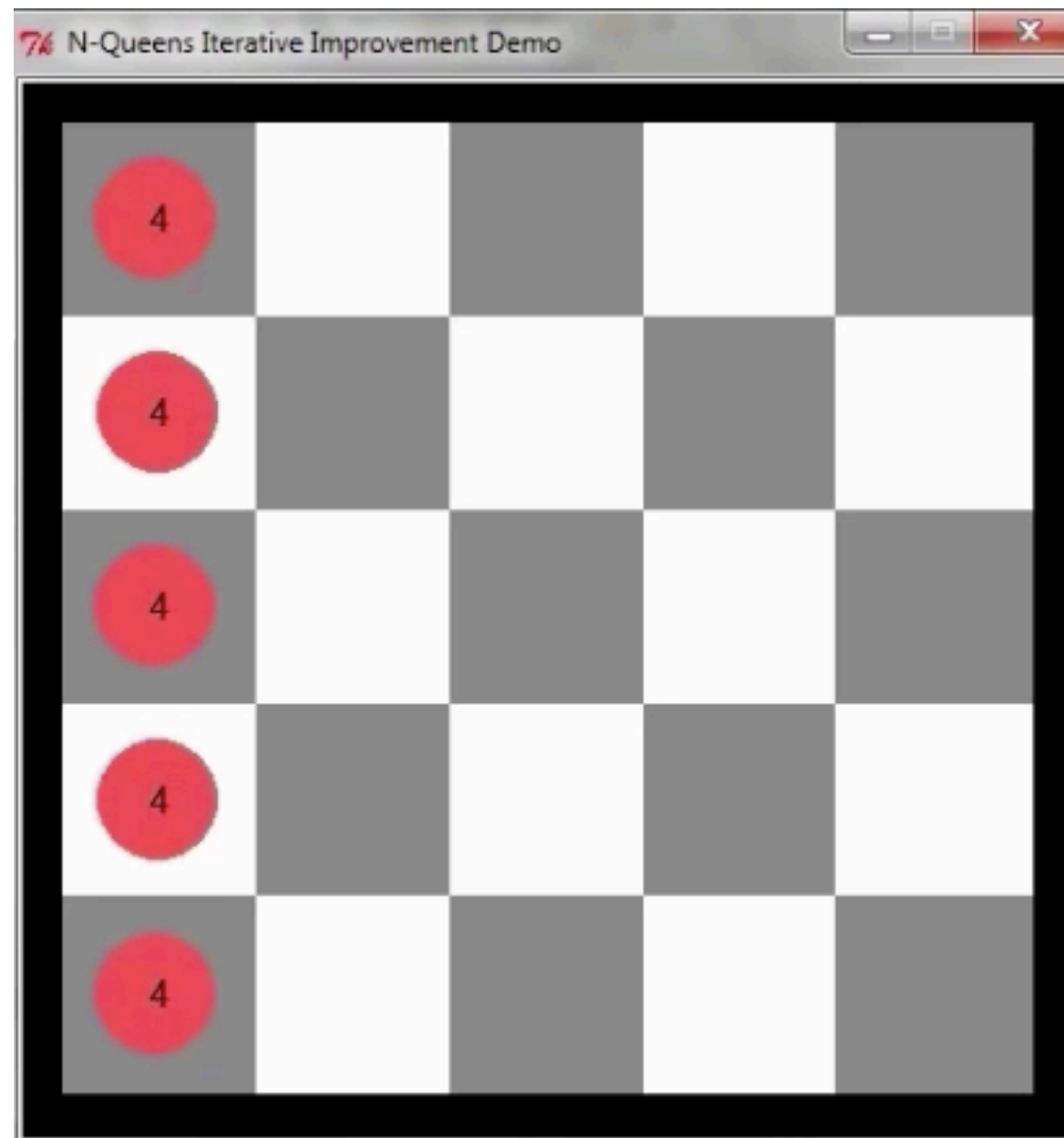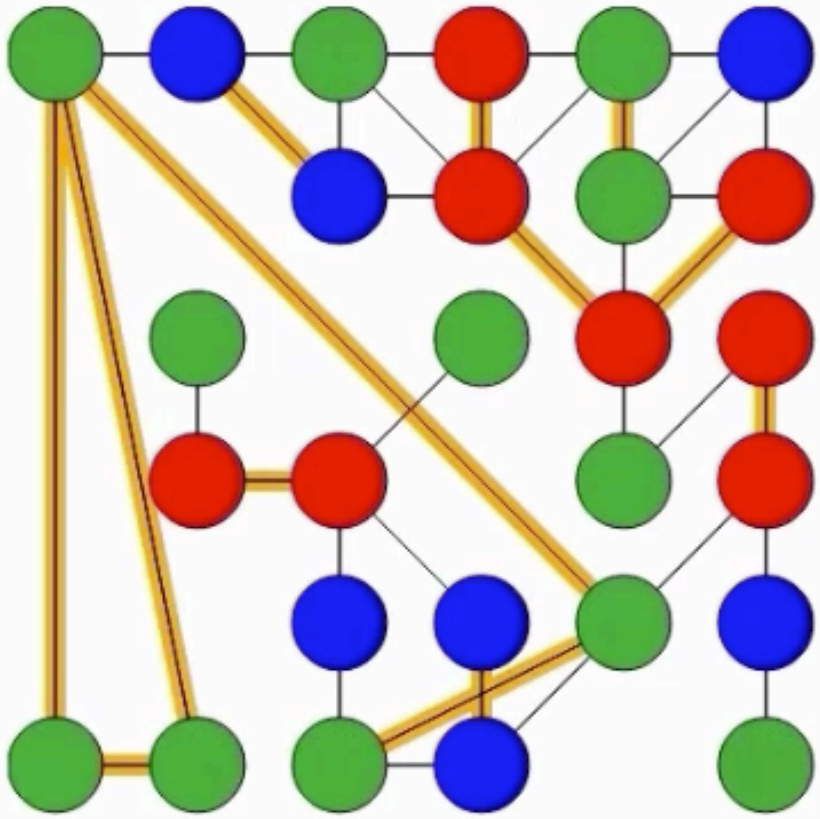
Example: 4-Queens

h = 5   h = 2   h = 0
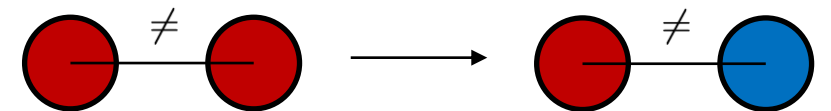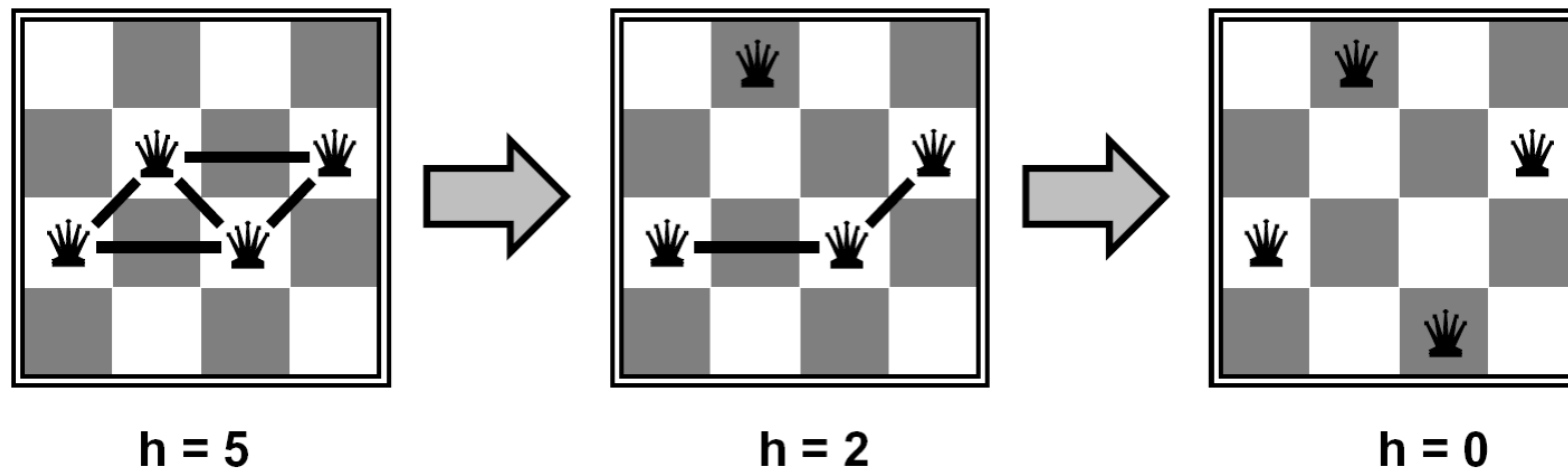
- ❖ States: 4 queens in 4 columns ($4^4 = 256$ states)
- ❖ Operators: move queen in column
- ❖ Goal test: no attacks
- ❖ Evaluation: $h(n)$ = number of attacks

Video of Demo Iterative Improvement – n Queens

Video of Demo Iterative Improvement – Coloring

Performance of Min-Conflicts

- ❖ Given random initial state, can solve n-queens in almost constant time for arbitrary n with high probability (e.g., n = 10,000,000)!
- ❖ The same appears to be true for any randomly-generated CSP *except* in a narrow range of the ratio

$$R = \frac{\text{number of constraints}}{\text{number of variables}}$$
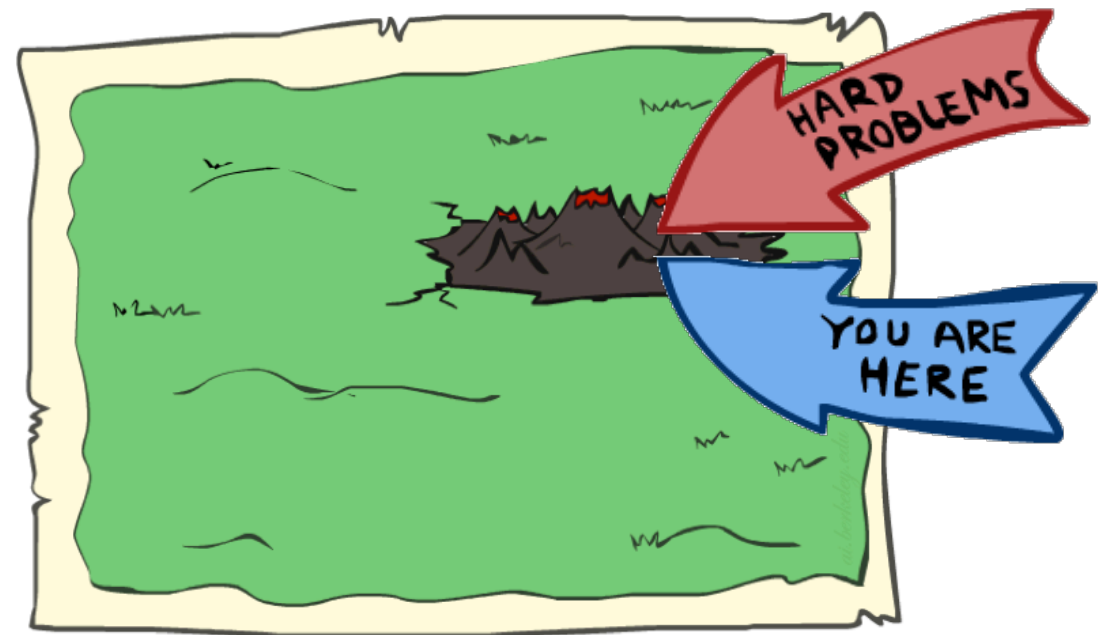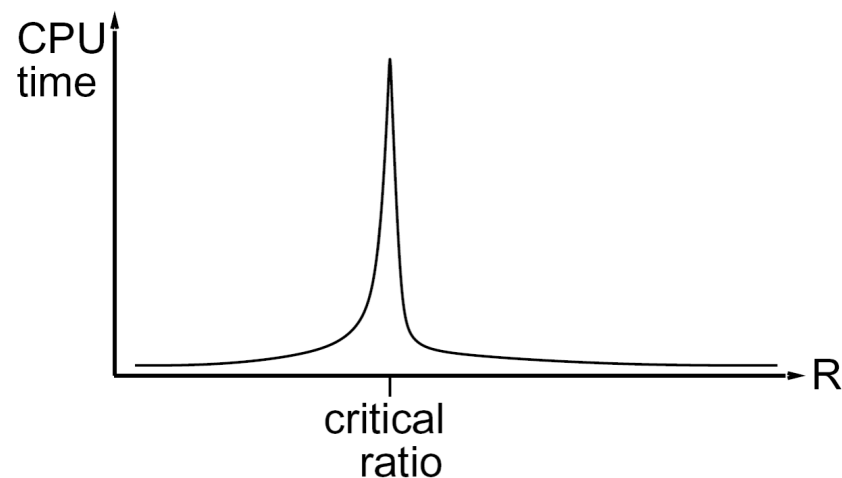
Summary: CSPs

- CSPs are a special kind of search problem:
    - States are partial assignments
    - Goal test defined by constraints

- Basic solution: backtracking search

- Speed-ups:
    - Ordering
    - Filtering
    - Structure
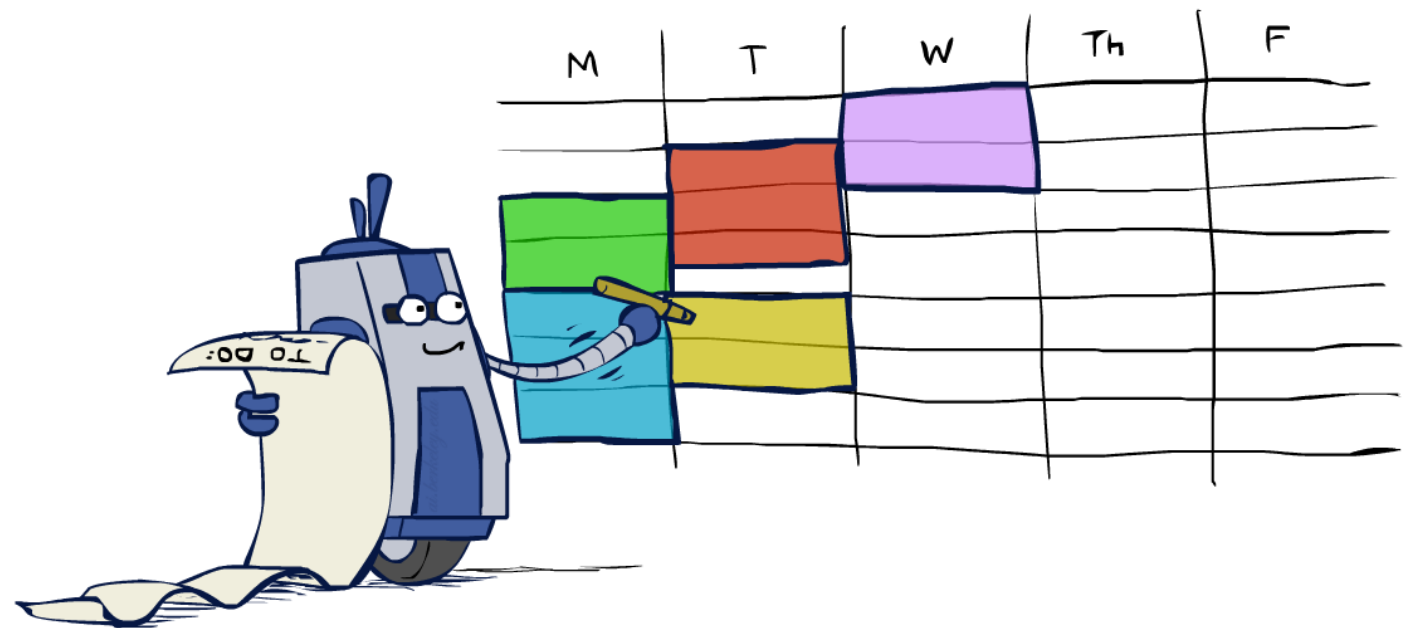
- Iterative min-conflicts is often effective in practice

Local Search

Local Search

- Tree search keeps unexplored alternatives on the fringe (ensures completeness)
- Local search: improve a single option until you can't make it better (no fringe!)
- New successor function: local changes
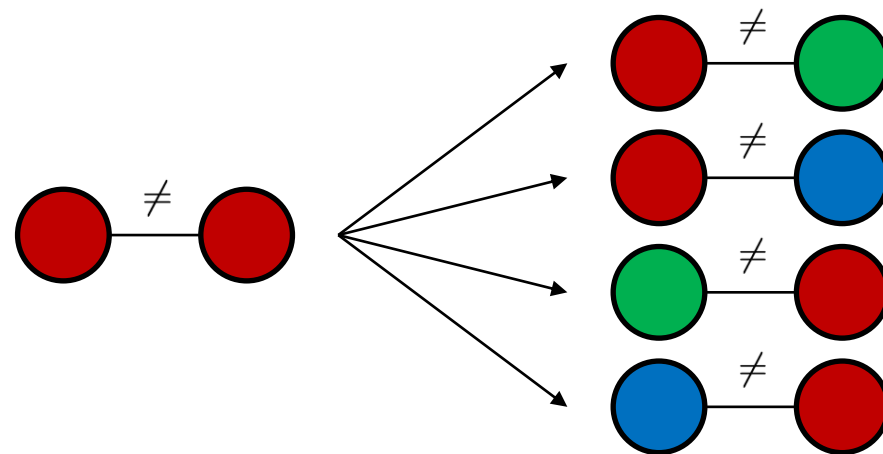
- Generally much faster and more memory efficient (but incomplete and suboptimal)

Hill Climbing

- Simple, general idea:
  - Start wherever
  - Repeat: move to the best neighboring state
  - If no neighbors better than current, quit

- What's bad about this approach?
  - Complete?
  - Optimal?

- What's good about it?

Hill Climbing Algorithm

```
function HILL-CLIMBING( problem) returns a state that is a local maximum
    inputs: problem, a problem
    local variables: current, a node
                     neighbor, a node

    current ← MAKE-NODE(INITIAL-STATE[problem])
    loop do
        neighbor ← a highest-valued successor of current
        if VALUE[neighbor] ≤ VALUE[current] then return STATE[current]
        current ← neighbor
    end
```

Hill Climbing Diagram
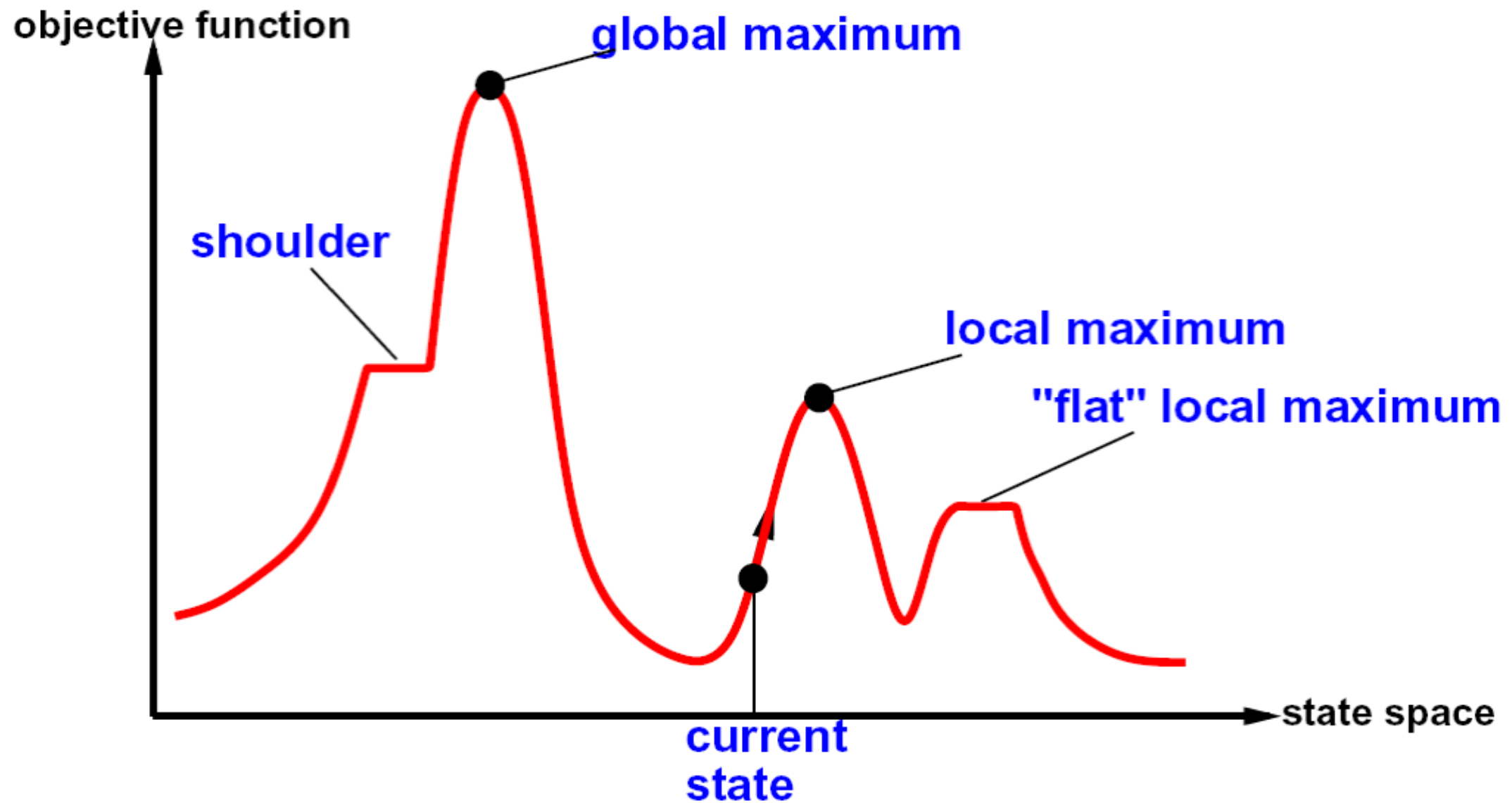
Quiz: Hill Climbing
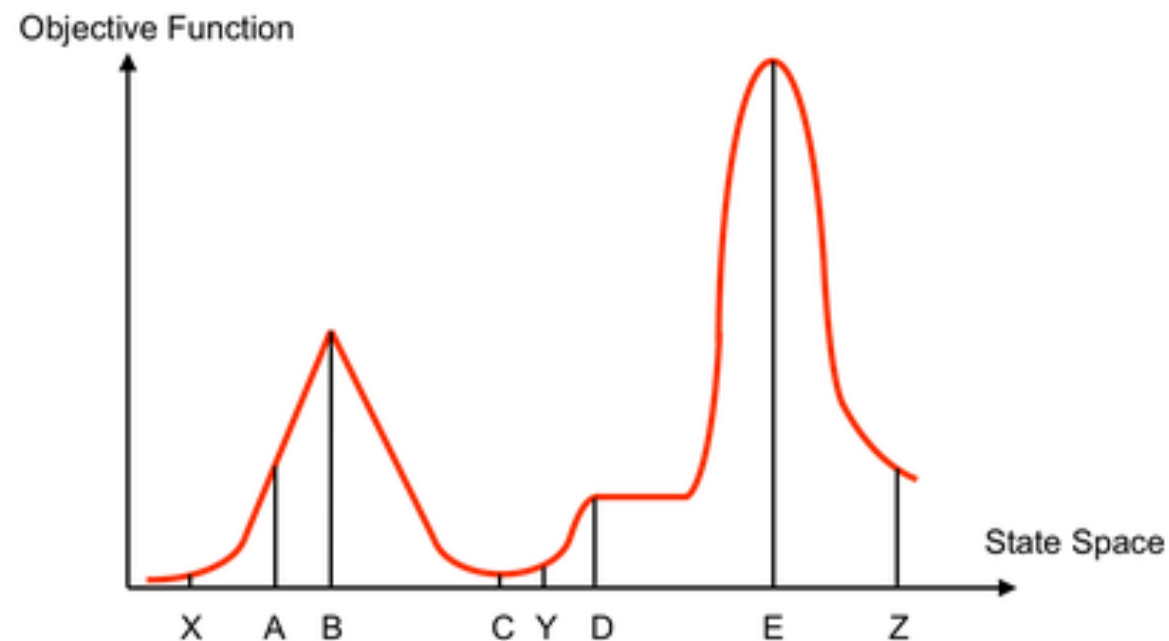
Starting from X, where do you end up ?

Starting from Y, where do you end up ?

Starting from Z, where do you end up ?

Simulated Annealing

❖ Idea: Escape local maxima by allowing downhill moves
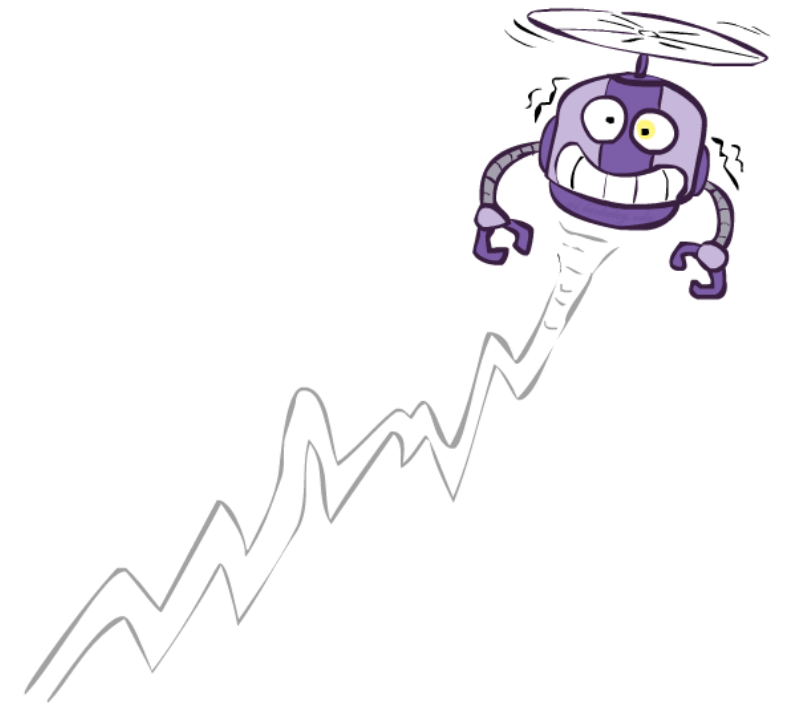 ❖ But make them rarer as time goes on

```
function SIMULATED-ANNEALING( problem, schedule) returns a solution state
    inputs: problem, a problem
            schedule, a mapping from time to "temperature"
    local variables: current, a node
                     next, a node
                     T, a "temperature" controlling prob. of downward steps

    current ← MAKE-NODE(INITIAL-STATE[problem])
    for t ← 1 to ∞ do
        T ← schedule[t]
        if T = 0 then return current
        next ← a randomly selected successor of current
        ΔE ← VALUE[next] – VALUE[current]
        if ΔE > 0 then current ← next
        else current ← next only with probability e^{ΔE/T}
```
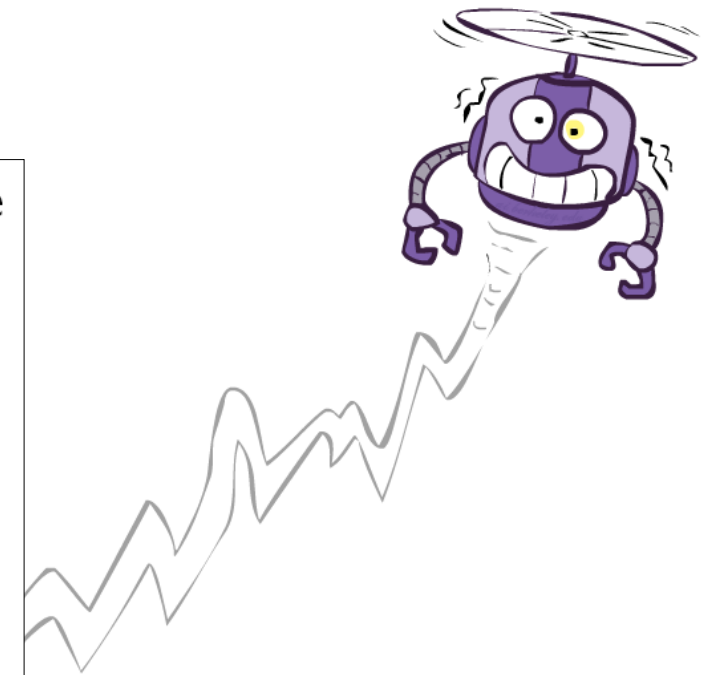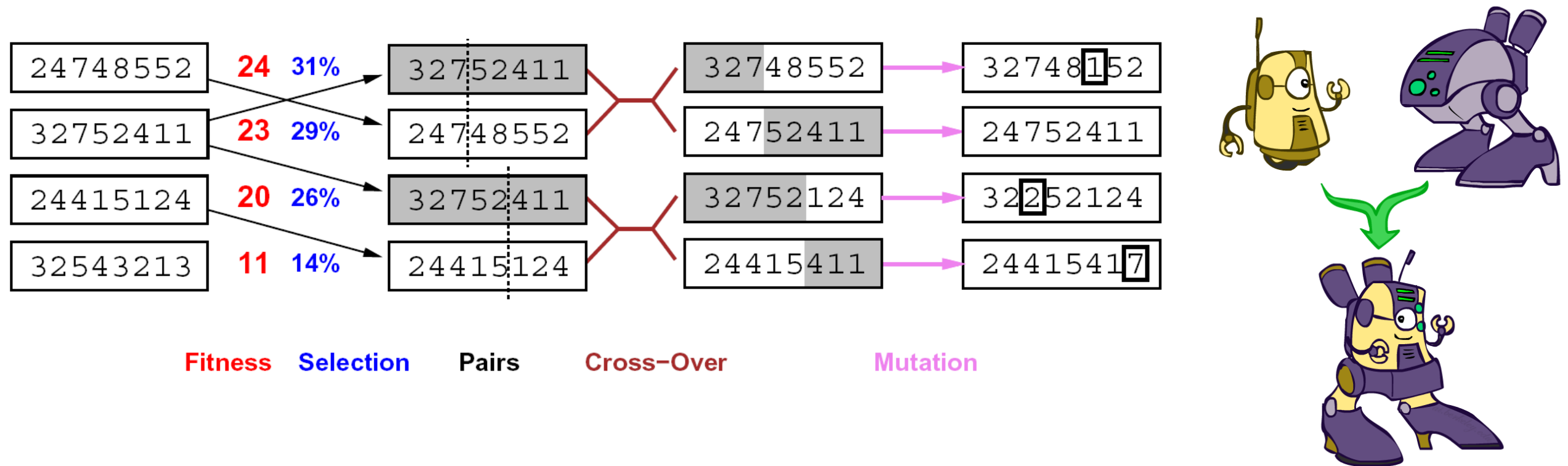
Simulated Annealing

- ❖ Theoretical guarantee:
  - ❖ Stationary distribution: $p(x) \propto e^{\frac{E(x)}{kT}}$

  - ❖ If T decreased slowly enough, will converge to optimal state!

- ❖ Is this an interesting guarantee?

- ❖ Sounds like magic, but reality is reality:
  - ❖ The more downhill steps you need to escape a local optimum, the less likely you are to ever make them all in a row
  - ❖ "Slowly enough" may mean exponentially slowly
 - ❖ Random restart hillclimbing also converges to optimal state…

Genetic Algorithms

24748552	**24** **31%**	32752411	32748552	32748152
32752411	**23** **29%**	24748552	24752411	24752411
24415124	**20** **26%**	32752411	32752124	32252124
32543213	**11** **14%**	24415124	24415411	24415417

Fitness   **Selection**   **Pairs**   **Cross−Over**   **Mutation**

- ❖ Genetic algorithms use a natural selection metaphor
  - ❖ Keep best N hypotheses at each step (selection) based on a fitness function
  - ❖ Also have pairwise crossover operators, with optional mutation to give variety

- ❖ Possibly the most misunderstood, misapplied (and even maligned) technique around

Example: N-Queens
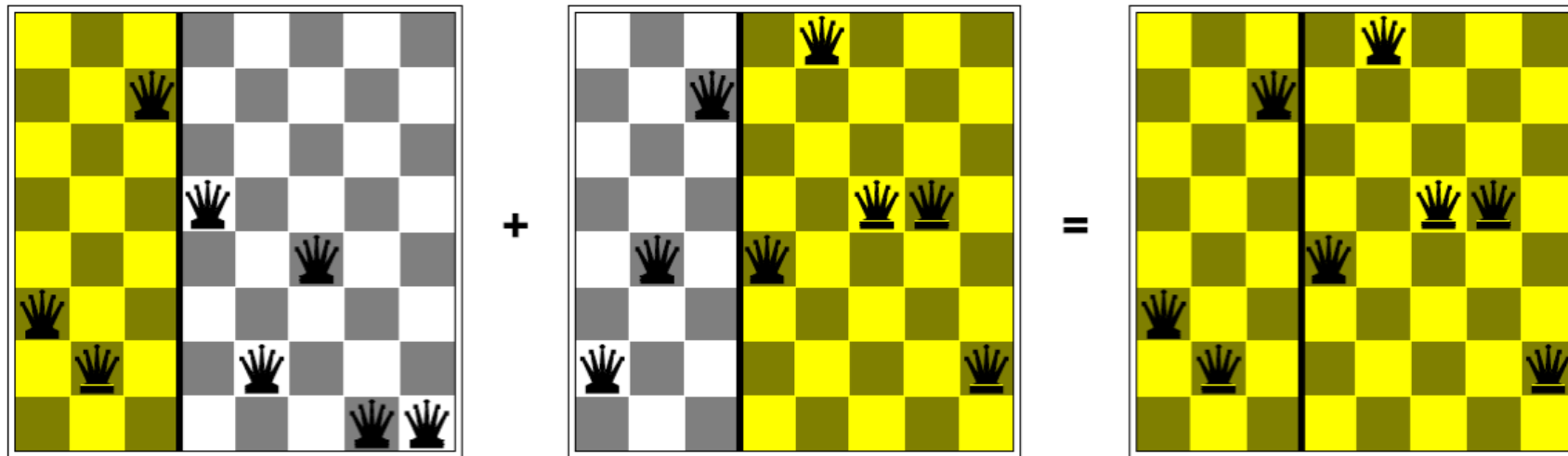
- ❖ Why does crossover make sense here?
- ❖ When wouldn't it make sense?
- ❖ What would mutation be?
- ❖ What would a good fitness function be?