
Announcements

- ❖ OH

- ❖ Mondays and Wednesdays at 9am-10am

- ❖ HW2 (HW3 soon)

- ❖ Due on Jun. 3 at 11:59pm

- ❖ Project 1 (Project 2 soon)

- ❖ Due on Jun. 5 at 11:59pm (small extension)

- ❖ Mid-term Exam

- ❖ Monday Jun. 22 at 4pm-5:40pm

Ve492: Introduction to Artificial Intelligence

Markov Decision Processes I

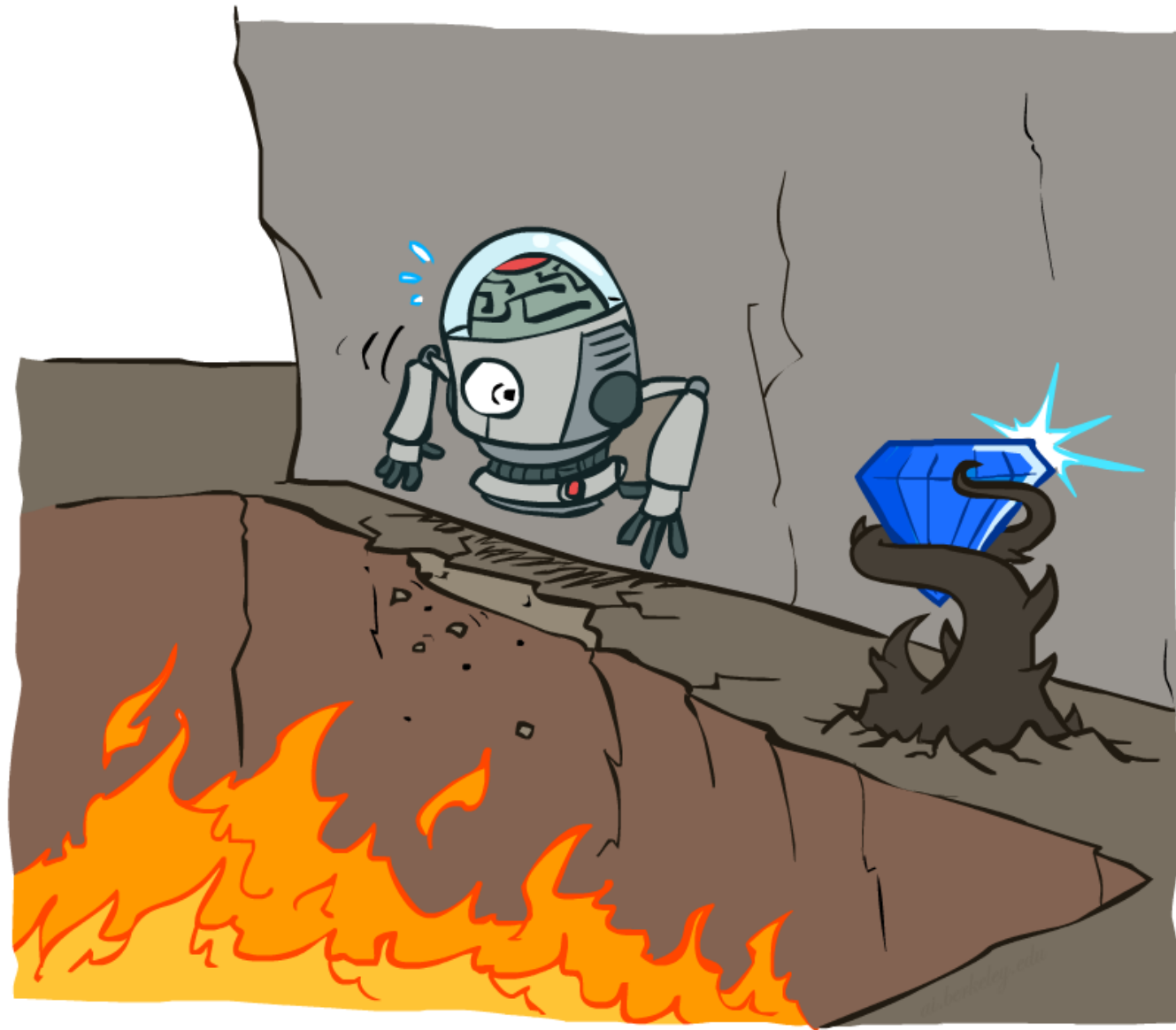


Paul Weng

UM-SJTU Joint Institute

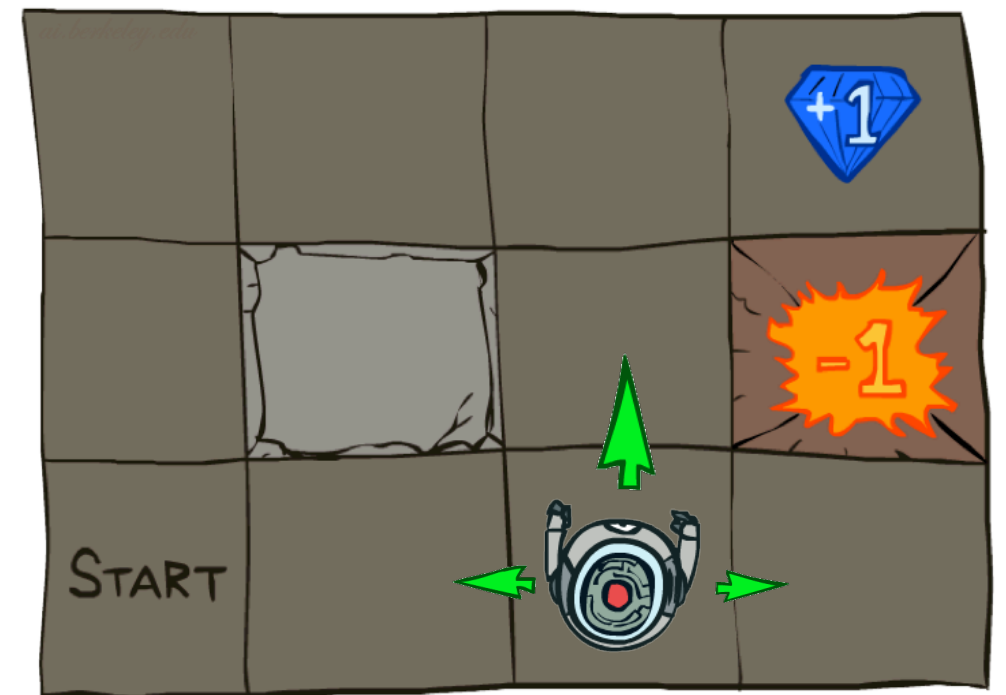
Slides adapted from <http://ai.berkeley.edu>, AIMA, UM, CMU

Non-Deterministic Search



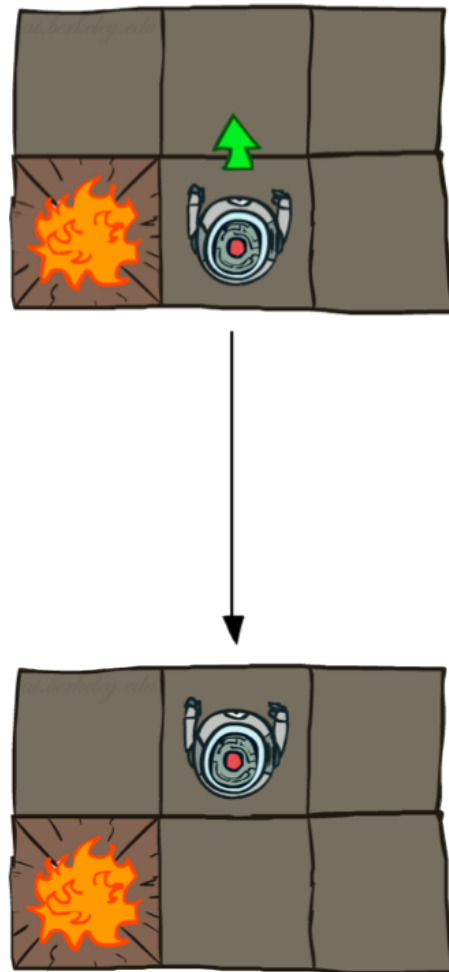
Example: Grid World

- ❖ A maze-like problem
 - ❖ The agent lives in a grid
 - ❖ Walls block the agent's path
- ❖ Noisy movement: actions do not always go as planned
 - ❖ 80% of the time, the action North takes the agent North (if there is no wall there)
 - ❖ 10% of the time, North takes the agent West; 10% East
 - ❖ If there is a wall in the direction the agent would have been taken, the agent stays put
- ❖ The agent receives rewards each time step
 - ❖ Small "living" reward each step (can be negative)
 - ❖ Big rewards come at the end (good or bad)
- ❖ Goal: maximize sum of rewards

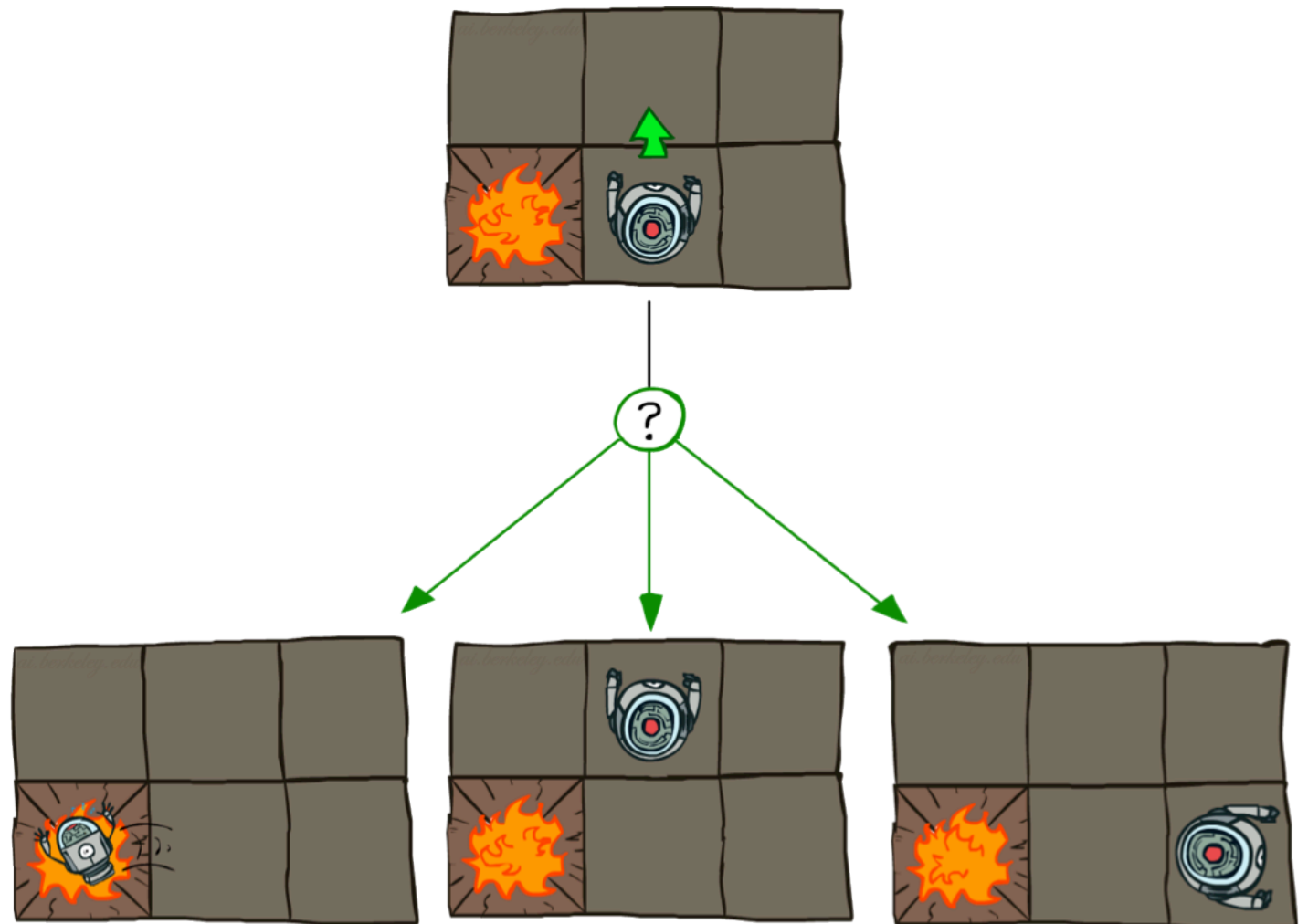


Grid World Actions

Deterministic Grid World



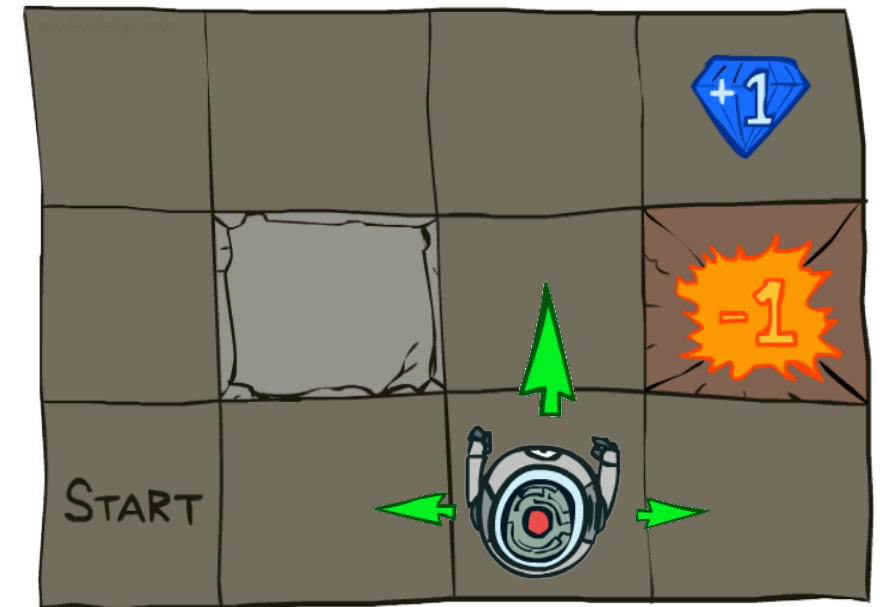
Stochastic Grid World



Markov Decision Processes

An MDP is defined by:

- ❖ A **set of states** $s \in S$
- ❖ A **set of actions** $a \in A$
- ❖ A **transition function** $T(s, a, s')$
 - ❖ Probability that a from s leads to s' , i.e., $P(s' | s, a)$
 - ❖ Also called the model or the dynamics
- ❖ A **reward function** $R(s, a, s')$
 - ❖ Sometimes just $R(s)$ or $R(s')$
- ❖ A **start state**
- ❖ Maybe a **terminal state**



MDPs are non-deterministic search problems

- ❖ One way to solve them is with expectimax search
- ❖ We'll have a new tool soon

What is Markov about MDPs?

- ❖ “Markov” generally means that given the present state, the future and the past are independent
- ❖ For Markov decision processes, “Markov” means action outcomes depend only on the current state

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \dots, S_0 = s_0)$$
$$=$$

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$



Andrey Markov
(1856-1922)

- ❖ This is just like search, where the successor function could only depend on the current state (not the history)

Policies

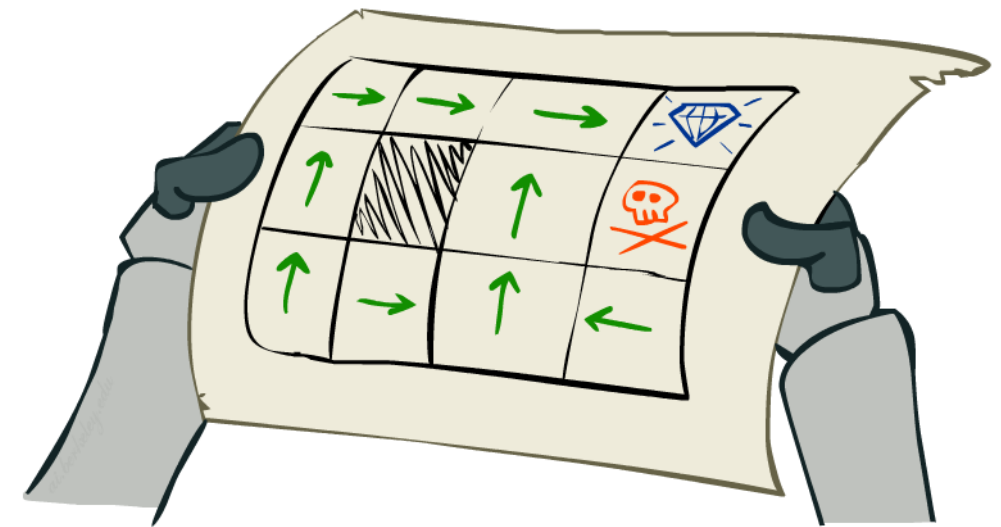
In deterministic single-agent search problems, we wanted an optimal **plan**, or sequence of actions, from start to a goal

For MDPs, we want an optimal policy $\pi^*: S \rightarrow A$

- ❖ A policy π gives an action for each state
- ❖ An optimal policy is one that maximizes expected utility if followed
- ❖ An explicit policy defines a reflex agent

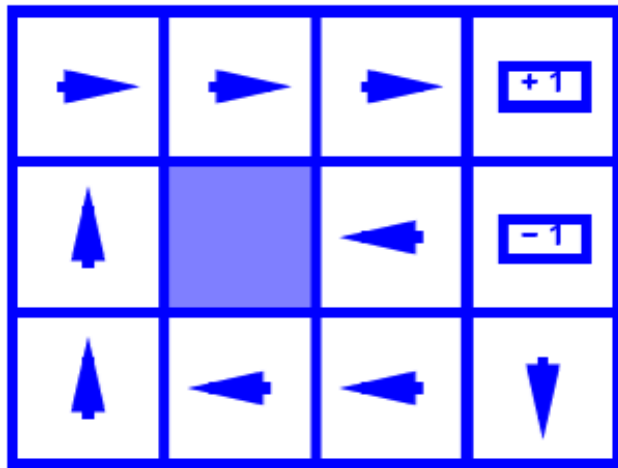
Expectimax didn't compute entire policies

- ❖ It computed the action for a single state only

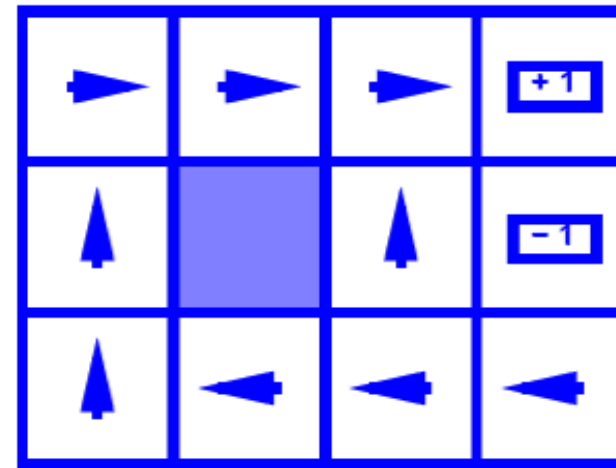


Optimal policy when $R(s, a, s') = -0.03$
for all non-terminals s

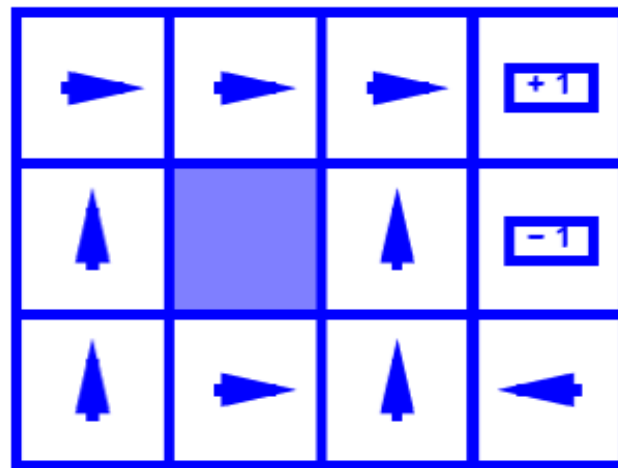
Optimal Policies



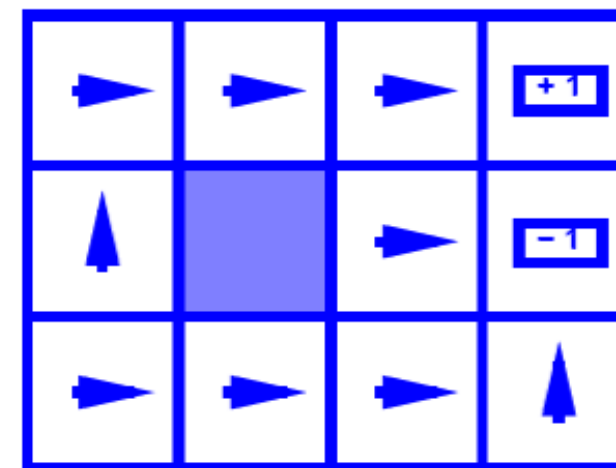
$$R(s) = -0.01$$



$$R(s) = -0.03$$

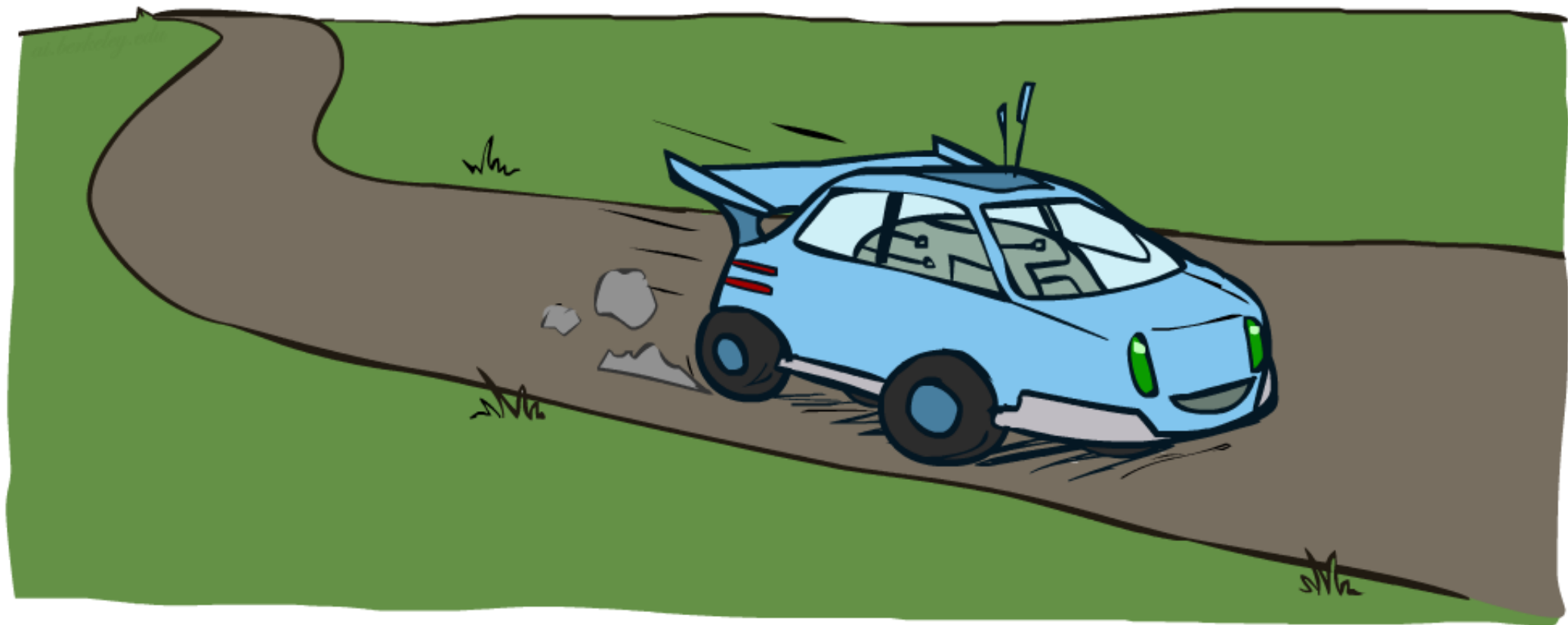


$$R(s) = -0.4$$



$$R(s) = -2.0$$

Example: Racing



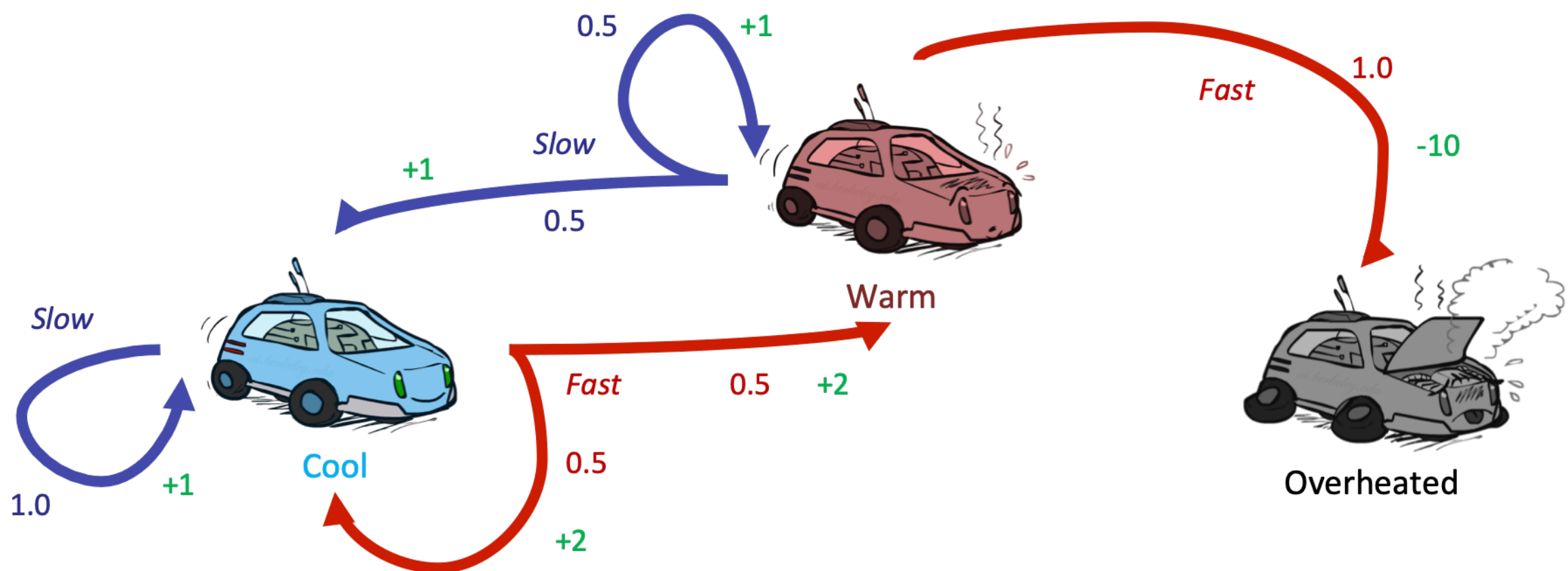
Example: Racing

A robot car wants to travel far, quickly

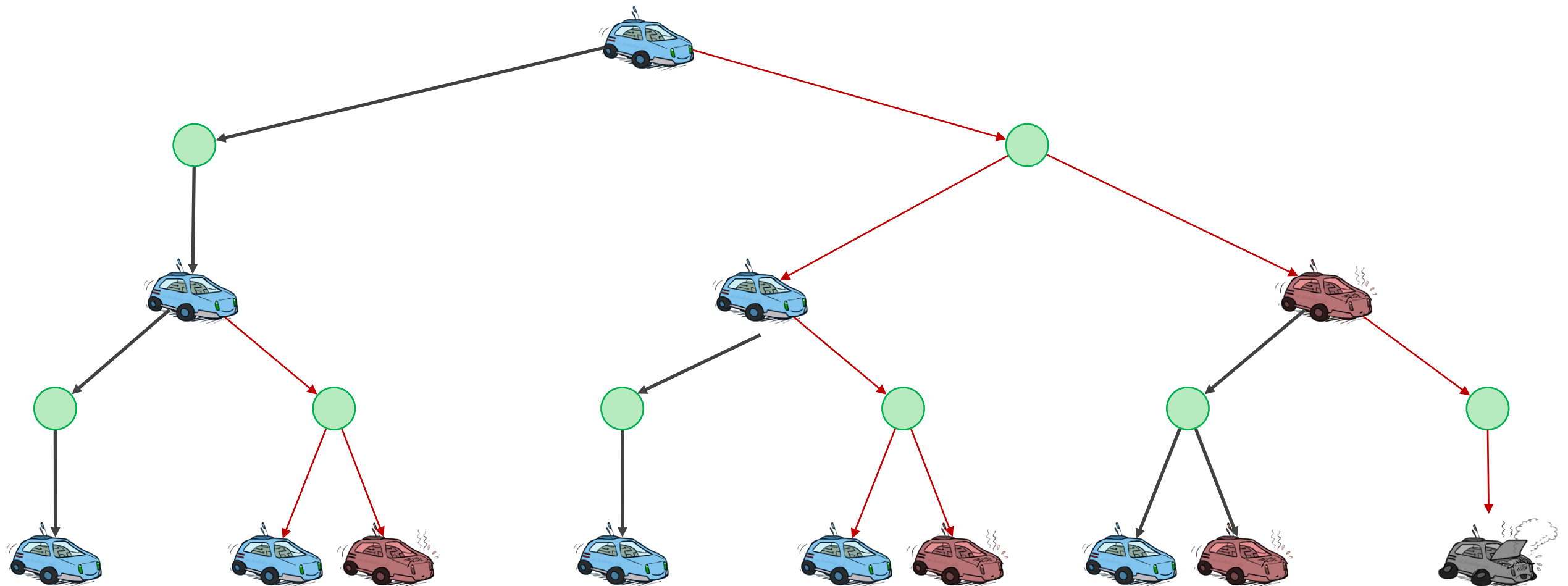
Three states: **Cool**, **Warm**, Overheated

Two actions: **Slow**, **Fast**

Going faster gets double reward

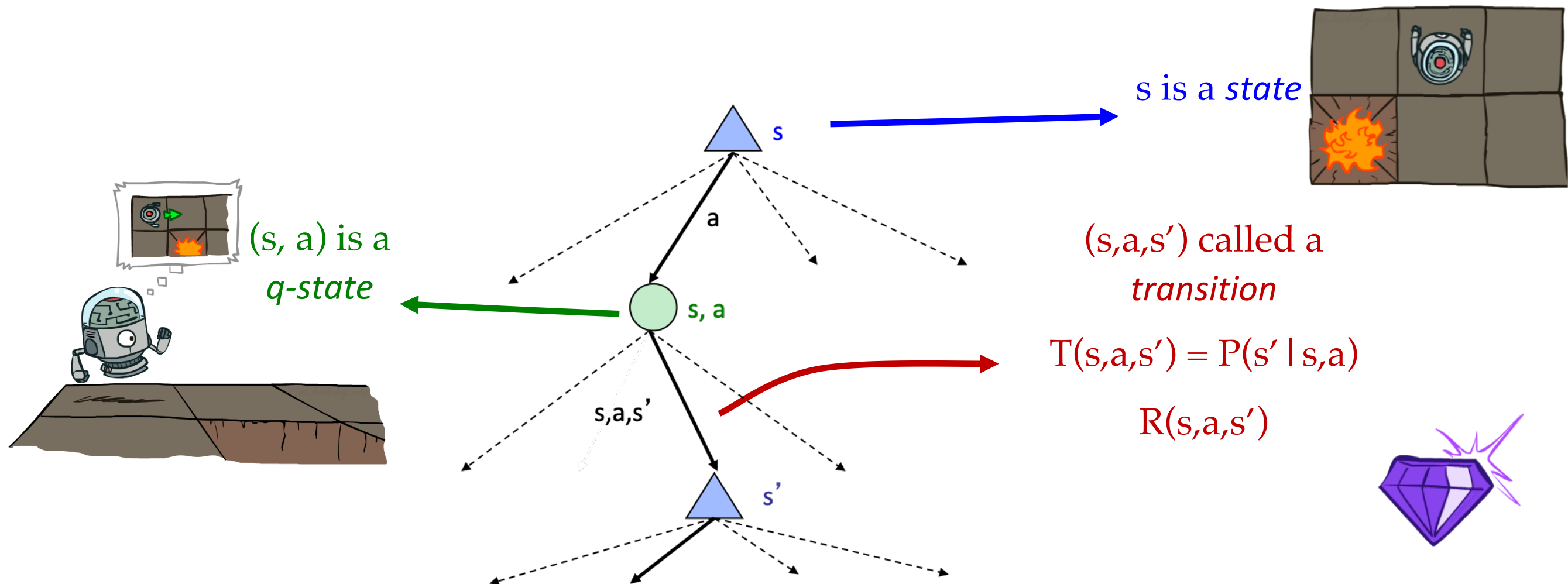


Racing Search Tree

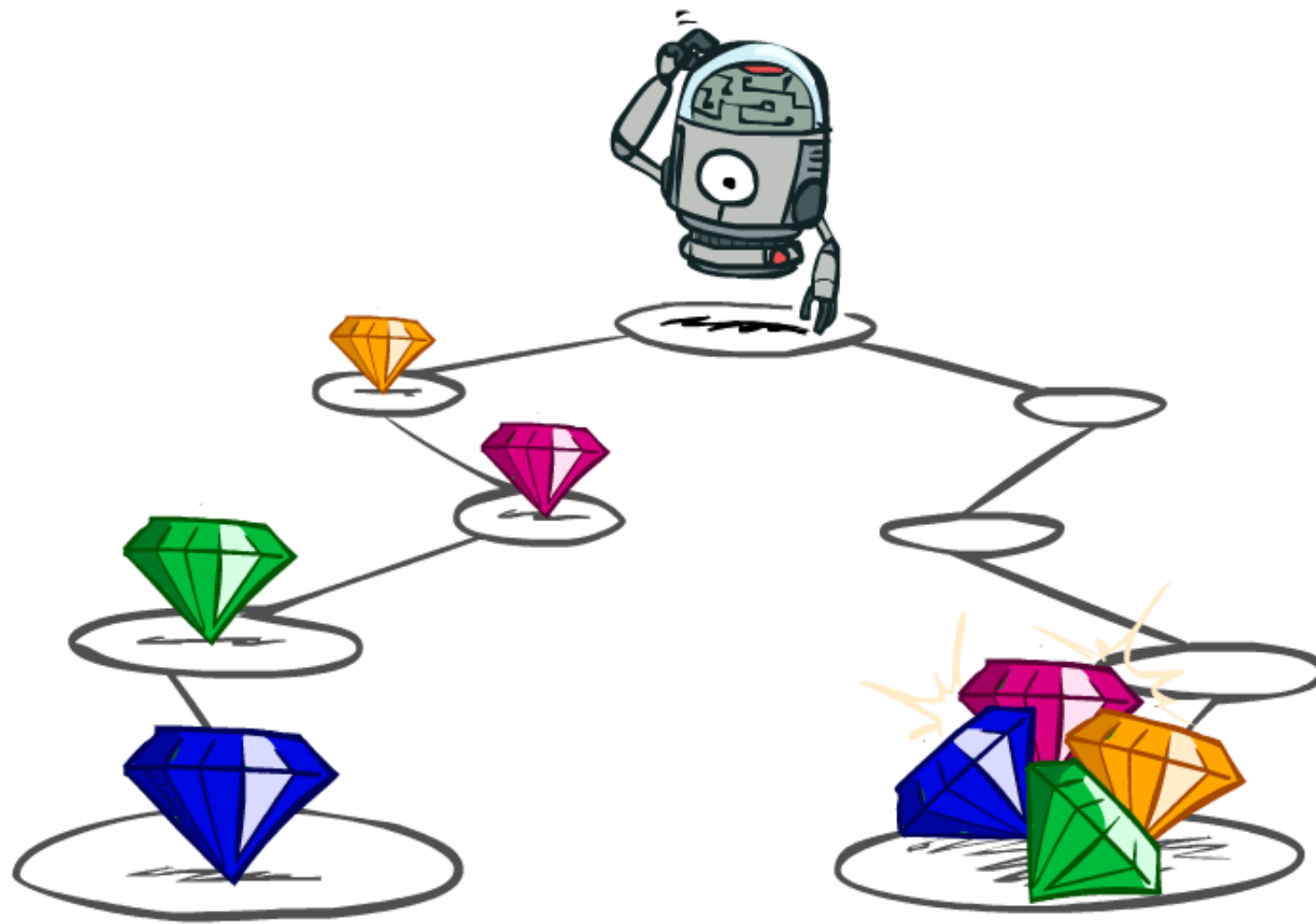


MDP Search Trees

Each MDP state projects an expectimax-like search tree



Utilities of Sequences



Utilities of Sequences

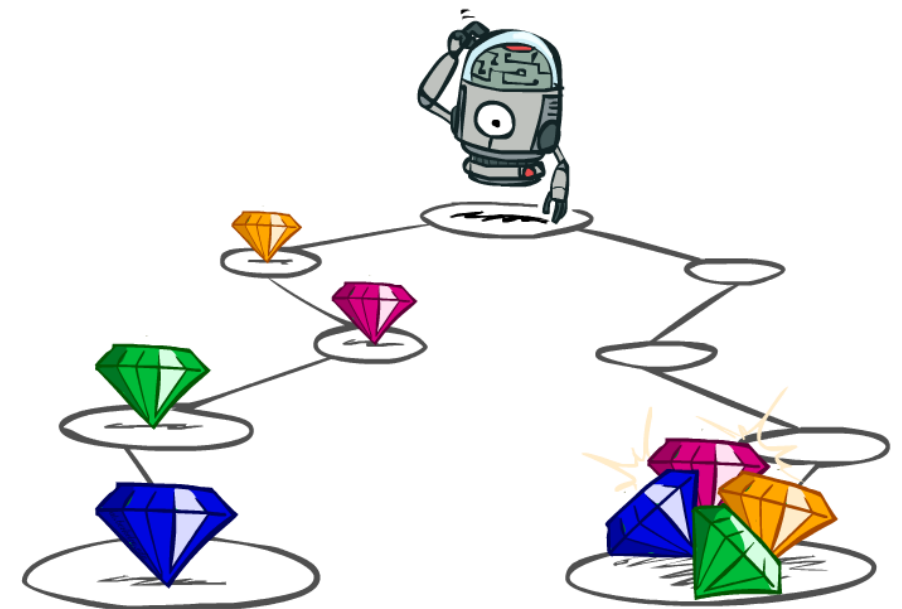
What preferences should an agent have over reward sequences?

More or less?

$[1, 2, 2]$ or $[2, 3, 4]$

Now or later?

$[0, 0, 1]$ or $[1, 0, 0]$



Discounting

It's reasonable to maximize the sum of rewards

It's also reasonable to prefer rewards now to rewards later

One solution: values of rewards decay exponentially



1

Worth
Now



γ

Worth Next
Step

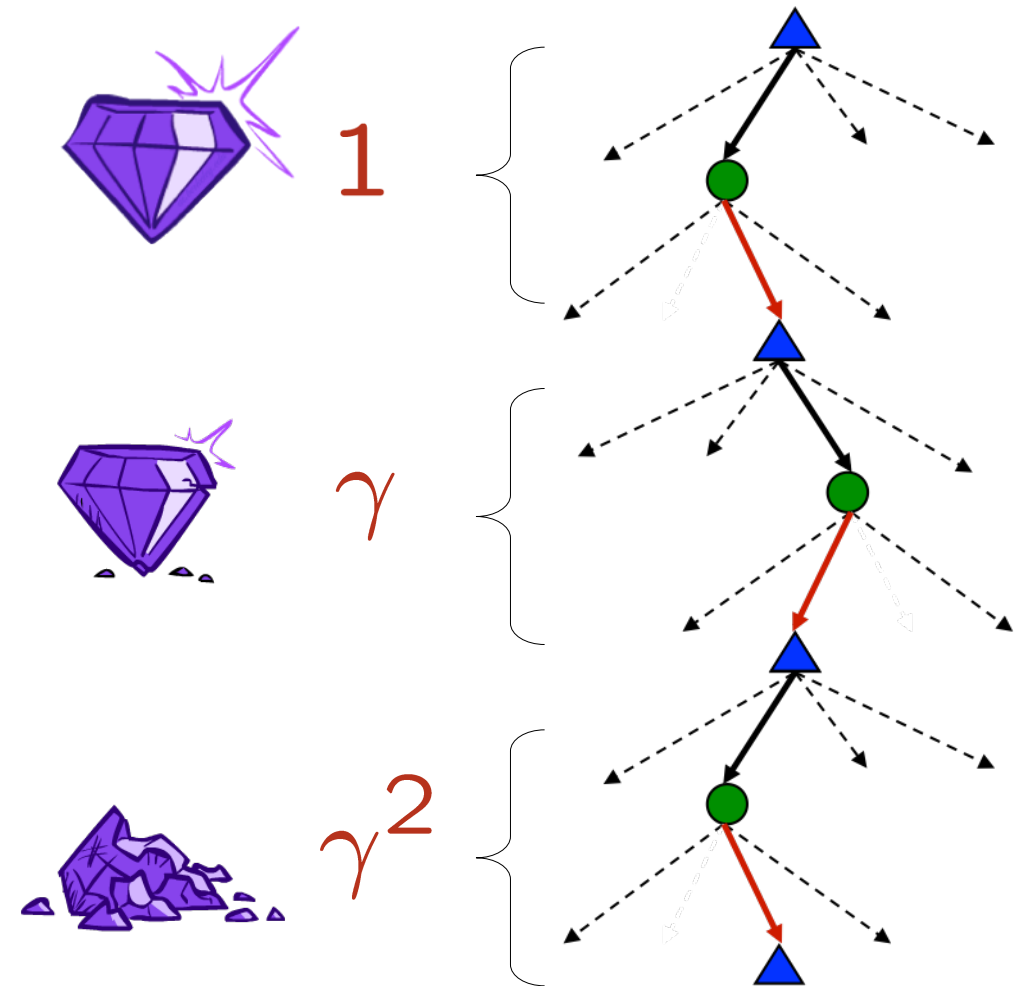


γ^2

Worth In Two
Steps

Discounting

- ❖ How to discount?
 - ❖ Each time we descend a level, we multiply in the discount once
- ❖ Why discount?
 - ❖ Sooner rewards probably do have higher utility than later rewards
 - ❖ Also helps our algorithms converge
- ❖ Example: discount of 0.5
 - ❖ $U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3$
 - ❖ $U([1,2,3]) < U([3,2,1])$



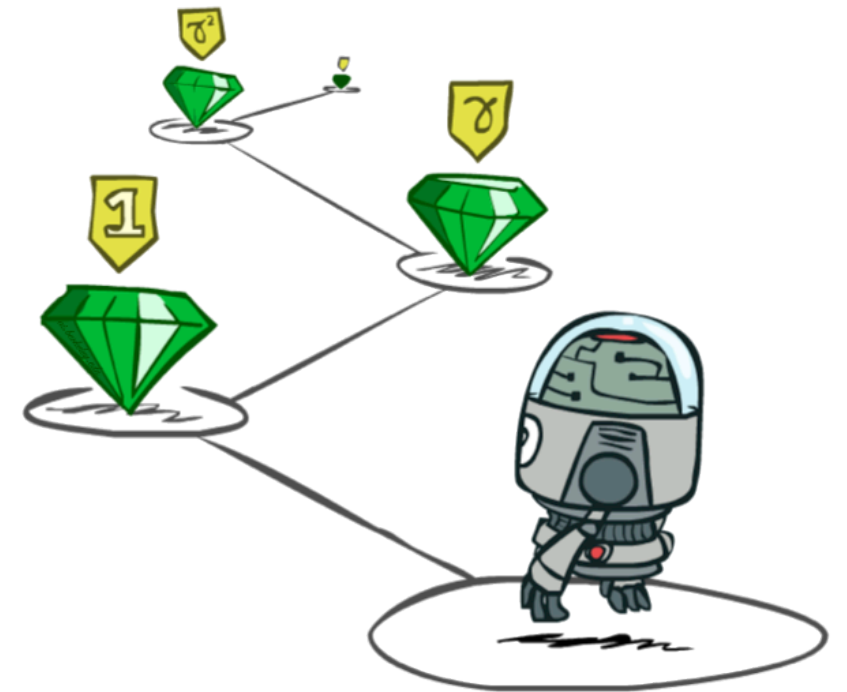
Quiz: Discounted Sum of Rewards

- ❖ What is the value of $U[2,4,8]$ with $\gamma = 0.5$?
- ❖ What is the value of $U[8,4,2]$ with $\gamma = 0.5$?

Stationary Preferences

Theorem: if we assume stationary preferences:

$$\begin{aligned} [a_1, a_2, \dots] &\succ [b_1, b_2, \dots] \\ \Leftrightarrow \\ [r, a_1, a_2, \dots] &\succ [r, b_1, b_2, \dots] \end{aligned}$$



Then: there are only two ways to define utilities

❖ Additive utility:

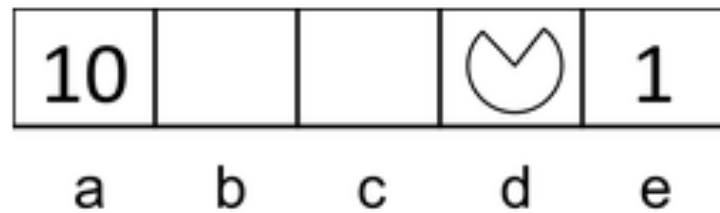
$$U([r_0, r_1, r_2, \dots]) = r_0 + r_1 + r_2 + \dots$$

❖ Discounted utility:

$$U([r_0, r_1, r_2, \dots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \dots$$

Discounting

Given:



- ❖ Actions: East, West, and Exit (only available in exit states a, e)
- ❖ Transitions: deterministic

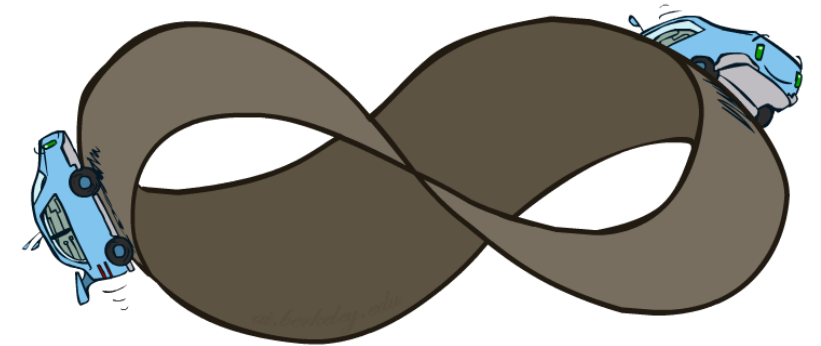
For $\gamma = 1$, what is the optimal policy?

For $\gamma = 0.1$, what is the optimal policy?

For which γ are West and East equally good when in state d?

Infinite Utilities?!

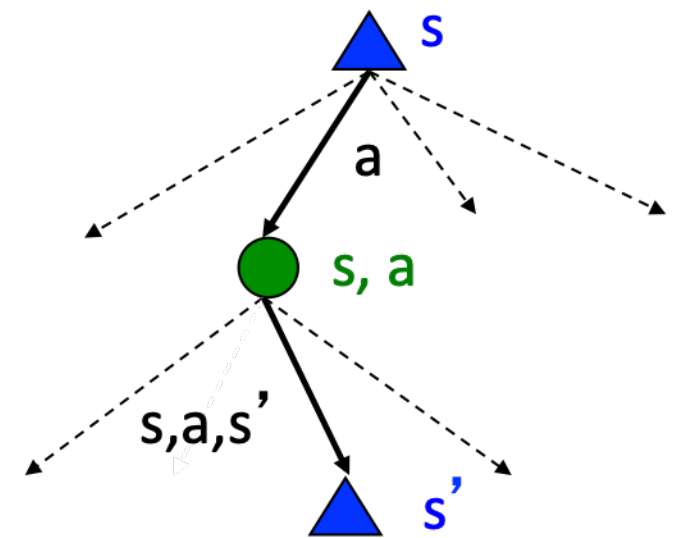
- ❖ Problem: What if the game lasts forever? Do we get infinite rewards?
- ❖ Solutions:
 - ❖ **Finite horizon**: (similar to depth-limited search)
 - ❖ Terminate episodes after a fixed T steps (e.g. life)
 - ❖ Gives nonstationary policies (π depends on time left)
 - ❖ **Discounting**: use $0 < \gamma < 1$
$$U([r_0, \dots, r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \leq R_{\max}/(1 - \gamma)$$
 - ❖ Smaller γ means smaller “horizon” – shorter term focus
 - ❖ **Absorbing state**: guarantee that for every policy, a terminal state will eventually be reached (like “overheated” for racing)



Recap: Defining MDPs

Markov decision processes:

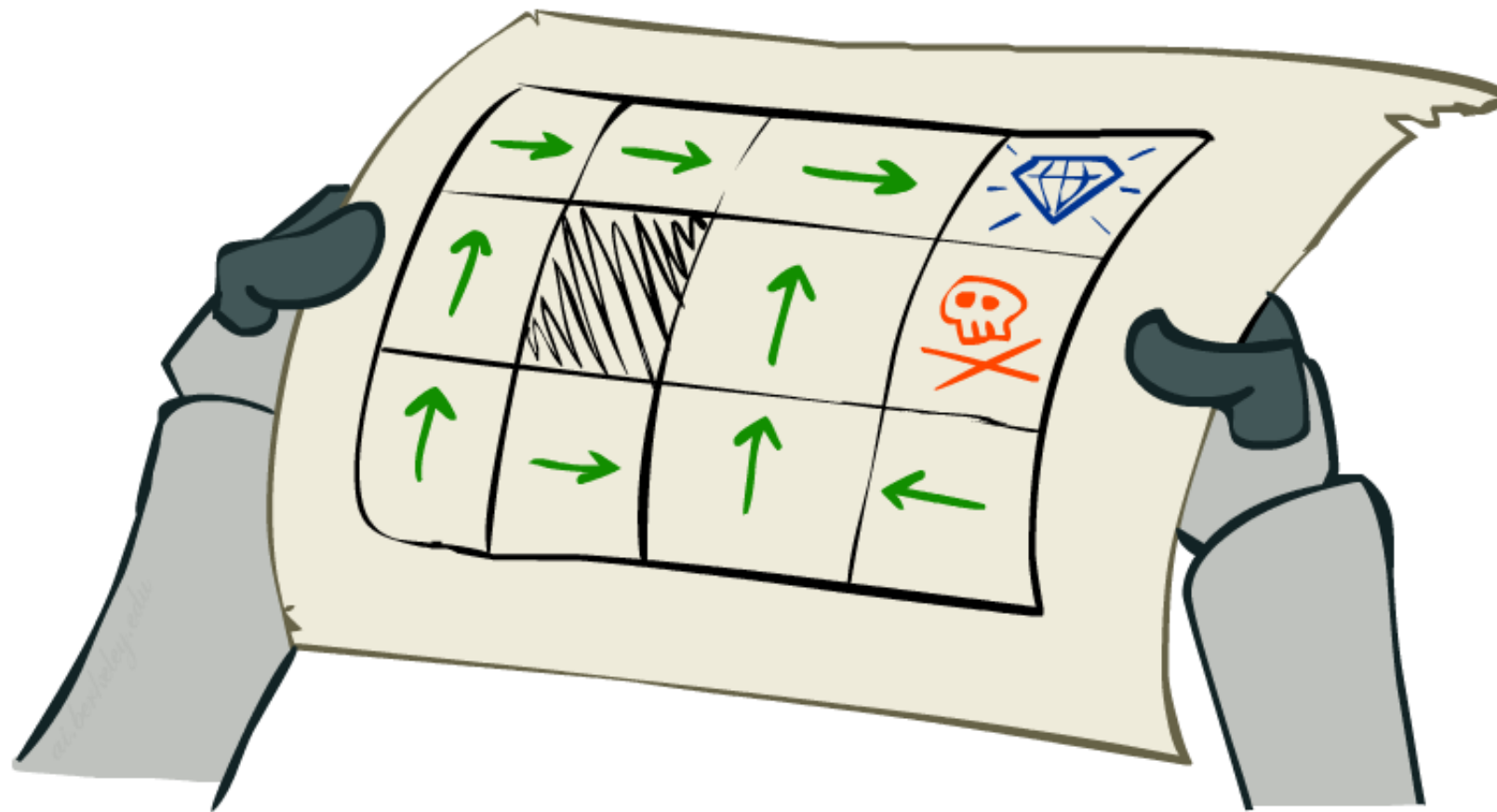
- ❖ Set of states S
- ❖ Start state s_0
- ❖ Set of actions A
- ❖ Transitions $P(s' | s, a)$ (or $T(s, a, s')$)
- ❖ Rewards $R(s, a, s')$ (and discount γ)



MDP quantities so far:

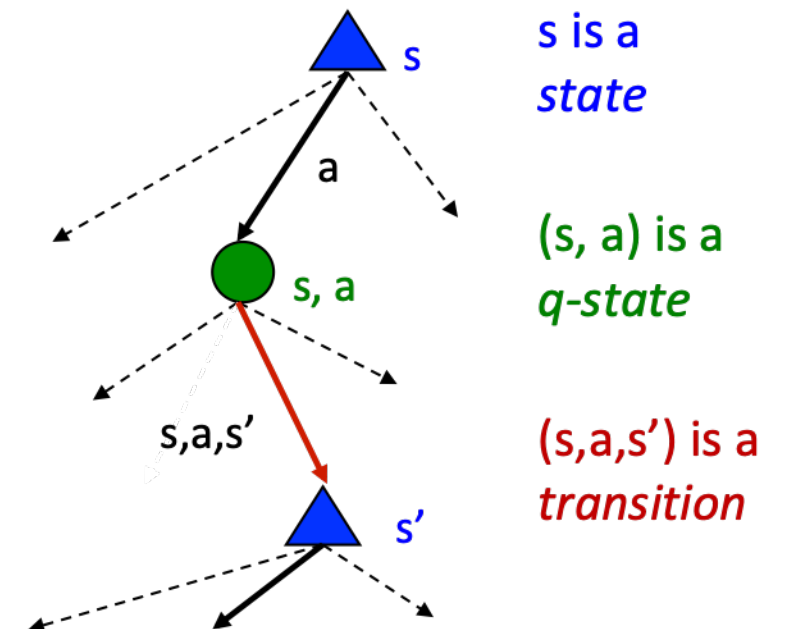
- ❖ Policy = Choice of action for each state
- ❖ Utility = sum of (discounted) rewards

Solving MDPs

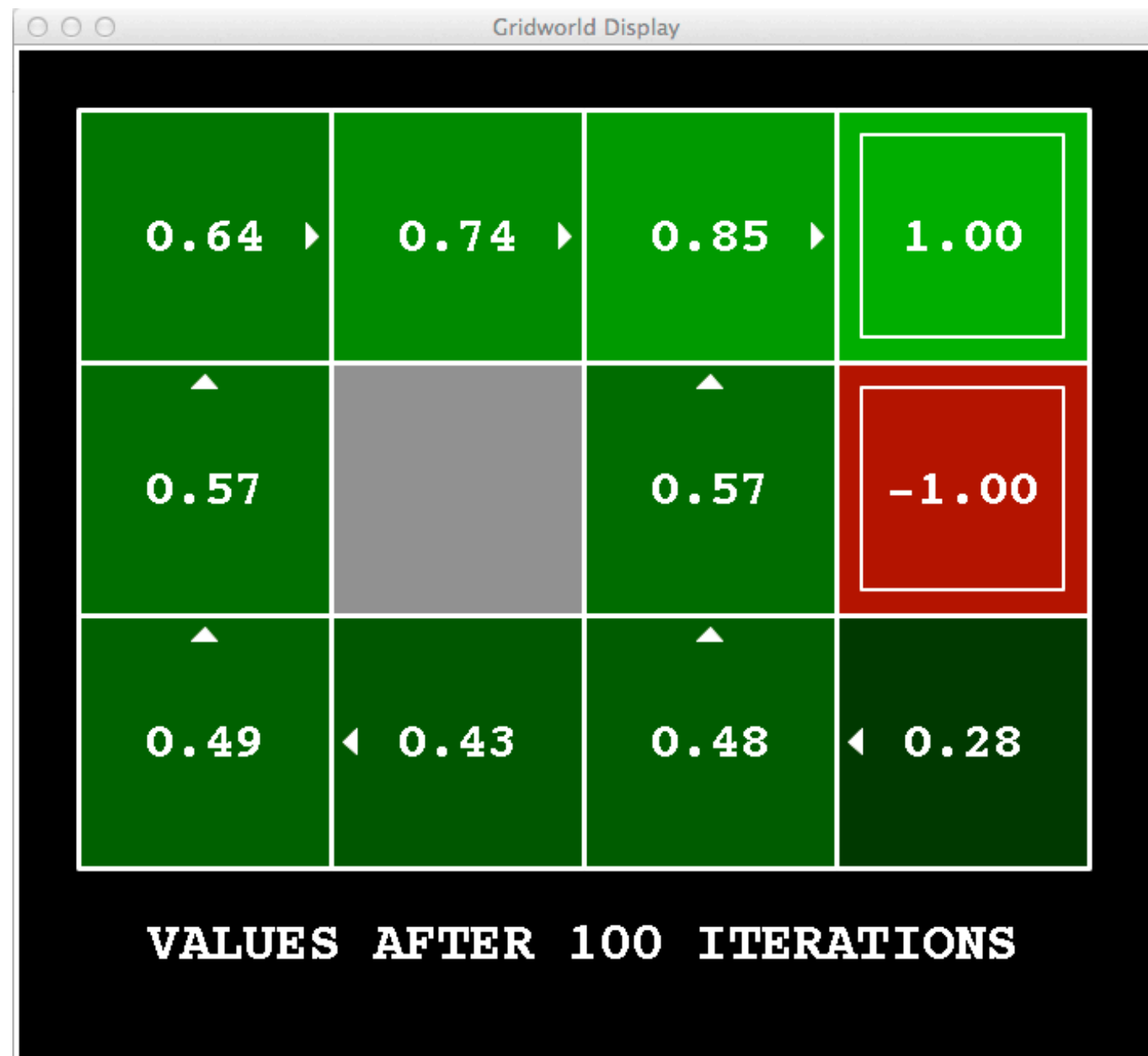


Optimal Quantities

- The value (utility) of a state s :
 $V^*(s)$ = expected utility starting in s and acting optimally
- The value (utility) of a q-state (s,a) :
 $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally
- The optimal policy:
 $\pi^*(s)$ = optimal action from state s

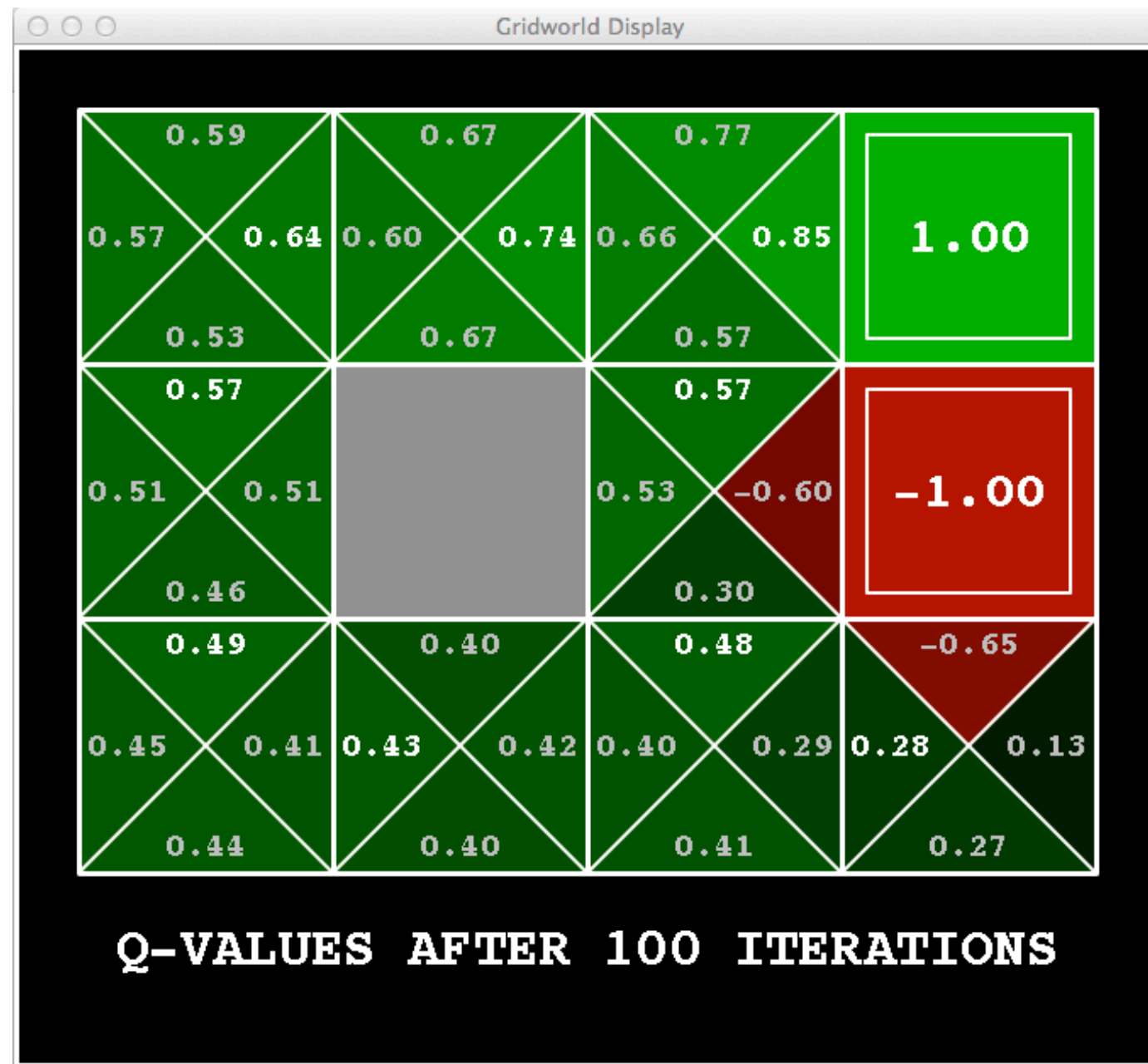


Snapshot of Demo – Gridworld V Values



Noise = 0.2
Discount = 0.9
Living reward = 0

Snapshot of Demo – Gridworld Q Values



Noise = 0.2
Discount = 0.9
Living reward = 0

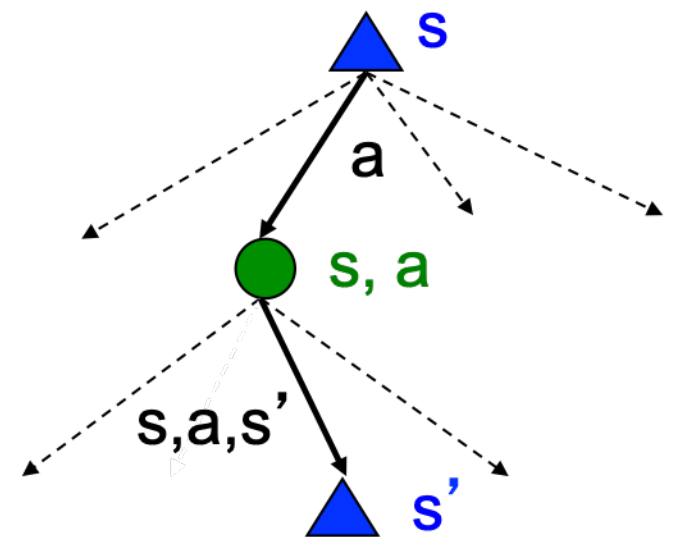
Values of States

- ❖ Fundamental operation: compute the (expectimax) value of a state
 - ❖ Expected utility under optimal action
 - ❖ Average sum of (discounted) rewards
 - ❖ This is just what expectimax computed!
- ❖ Recursive definition of value:

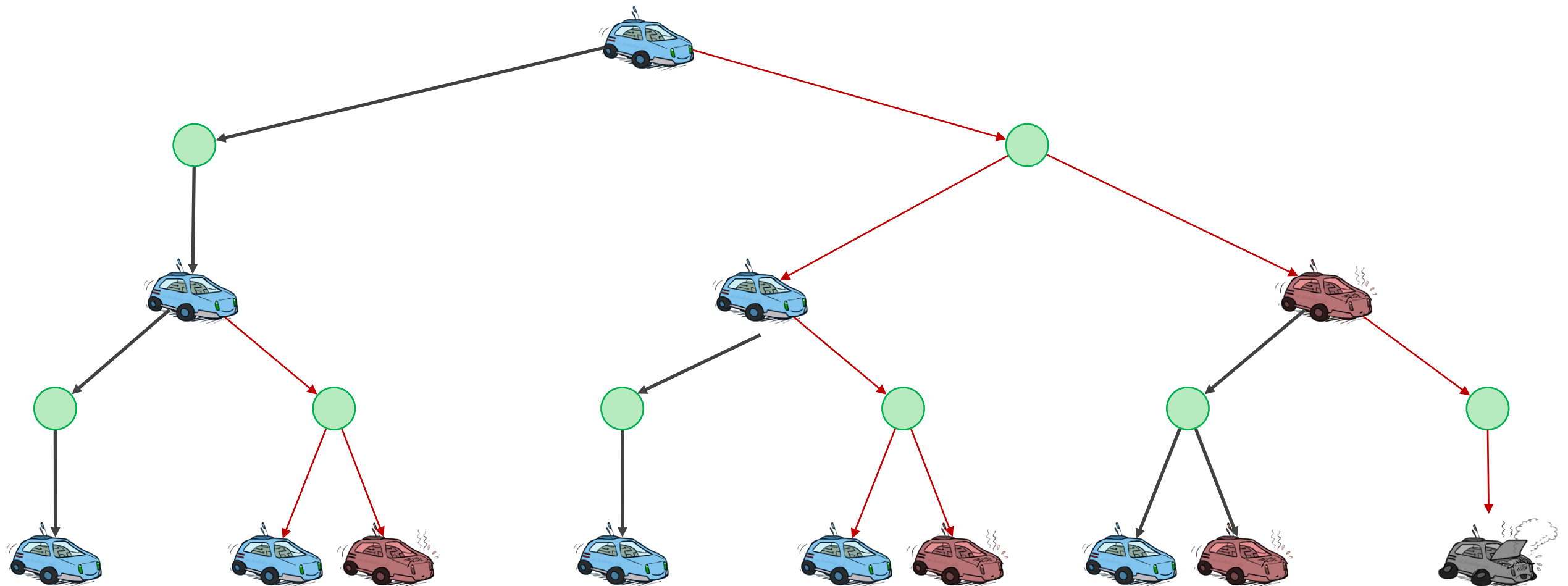
$$V^*(s) = \max_a Q^*(s, a)$$

$$Q^*(s, a) = \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$

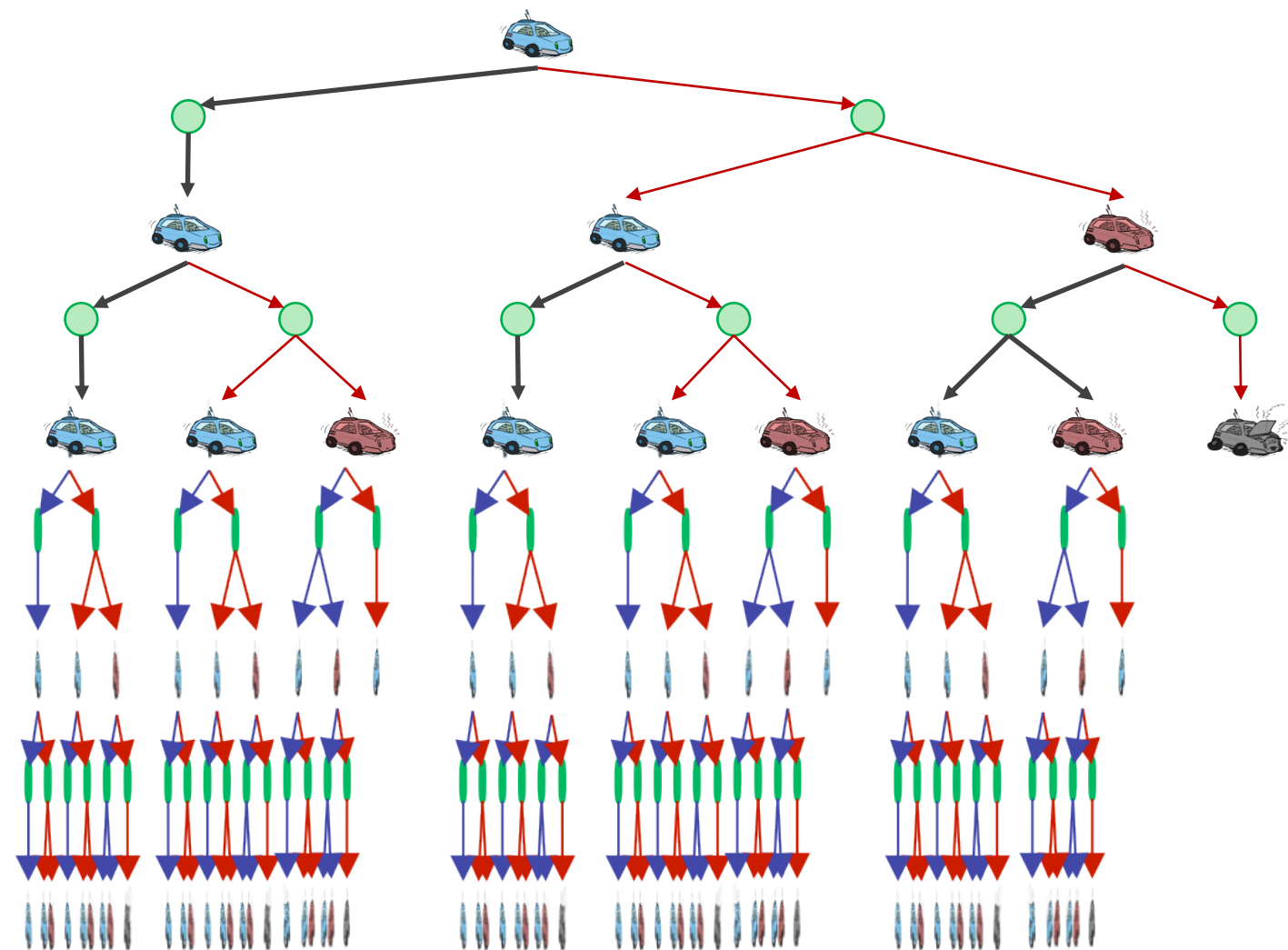
$$V^*(s) = \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V^*(s')]$$



Racing Search Tree

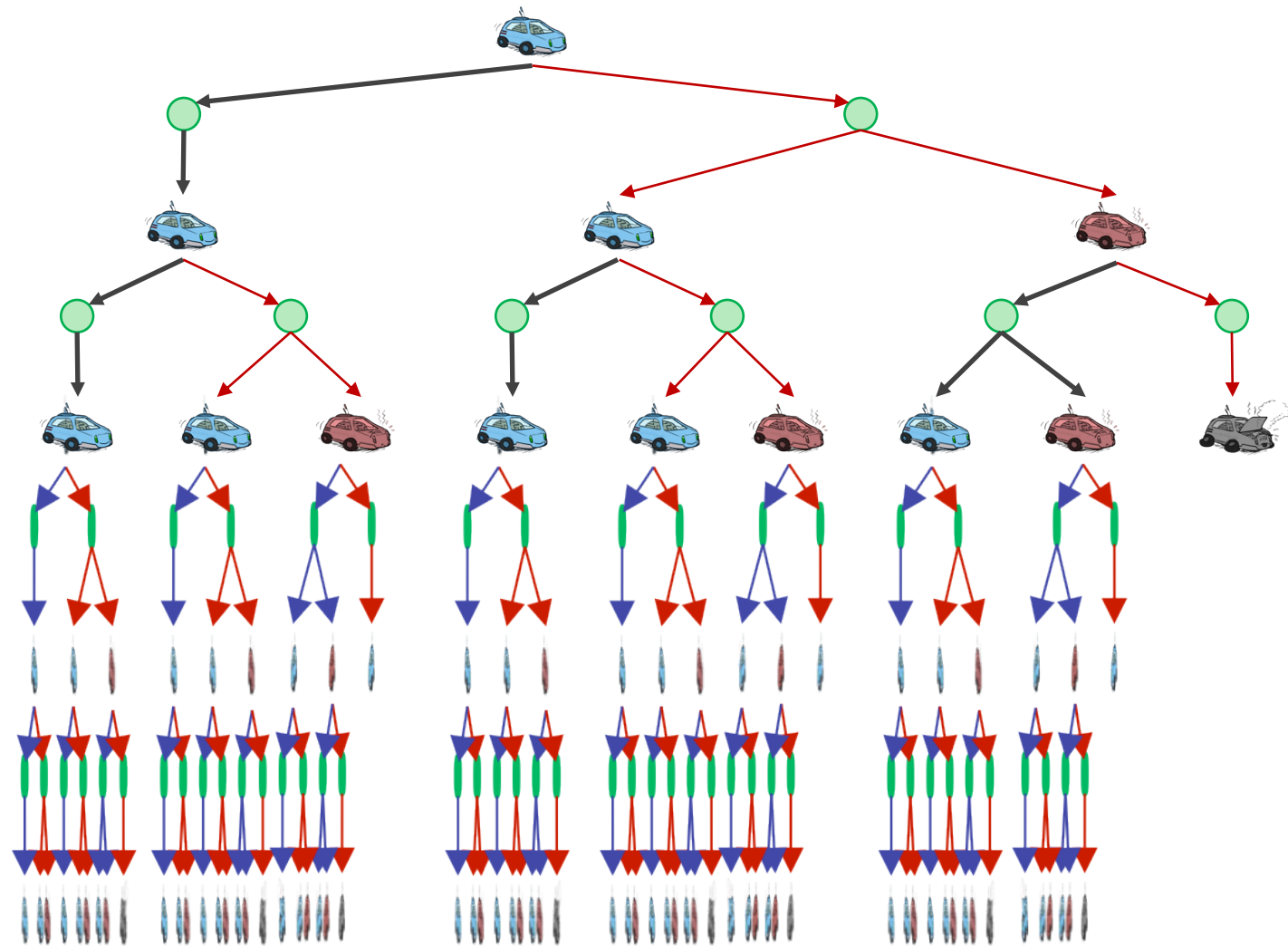


Racing Search Tree



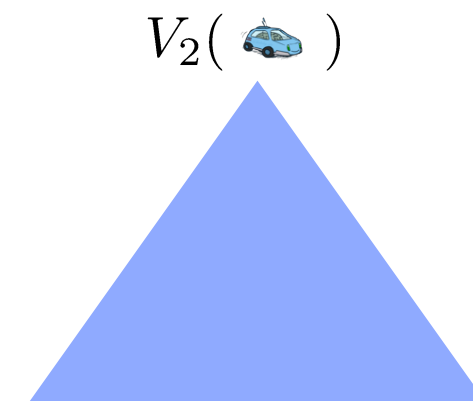
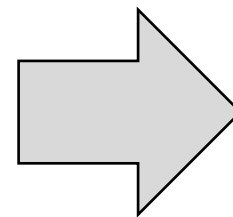
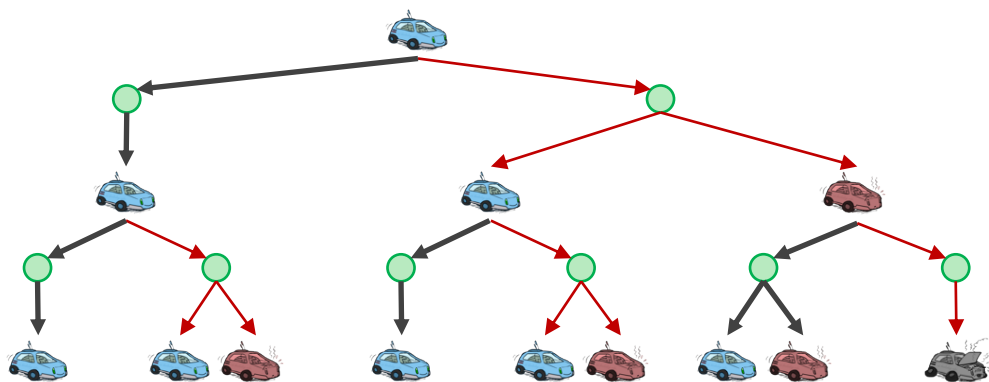
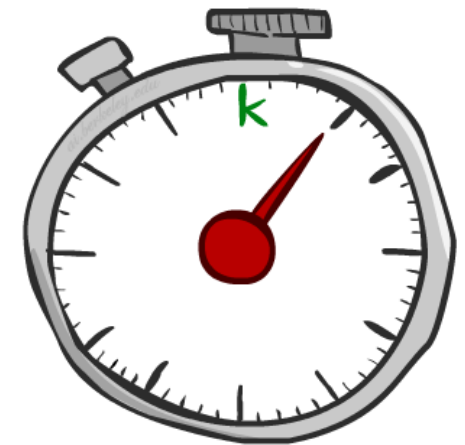
Racing Search Tree

- ❖ We're doing way too much work with expectimax!
- ❖ Problem: States are repeated
 - ❖ Idea: Only compute needed quantities once
- ❖ Problem: Tree goes on forever
 - ❖ Idea: Do a depth-limited computation, but with increasing depths until change is small
 - ❖ Note: deep parts of the tree eventually don't matter if $\gamma < 1$

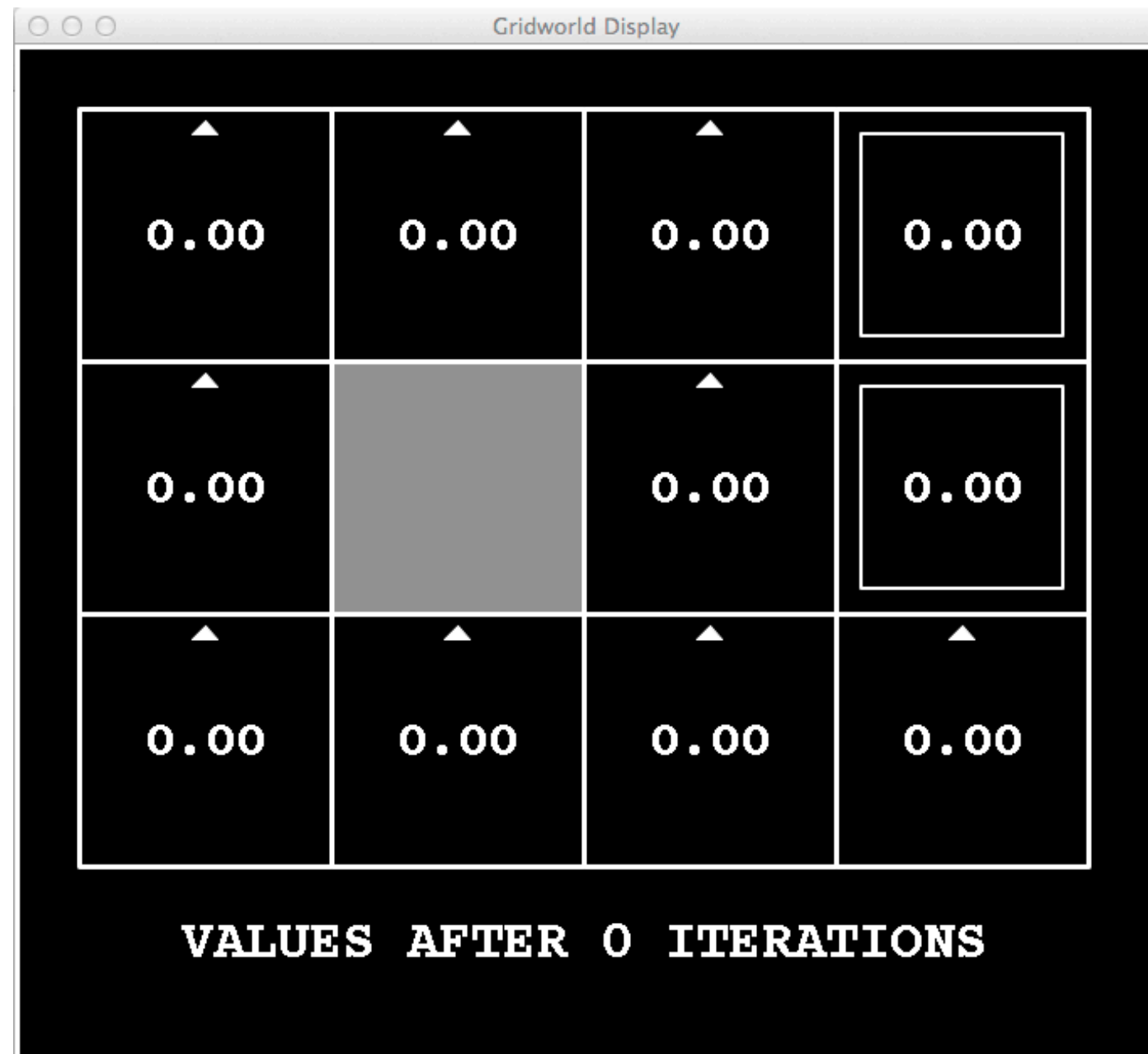


Time-Limited Values

- ❖ Key idea: time-limited values
- ❖ Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
 - ❖ Equivalently, it's what a depth- k expectimax would give from s

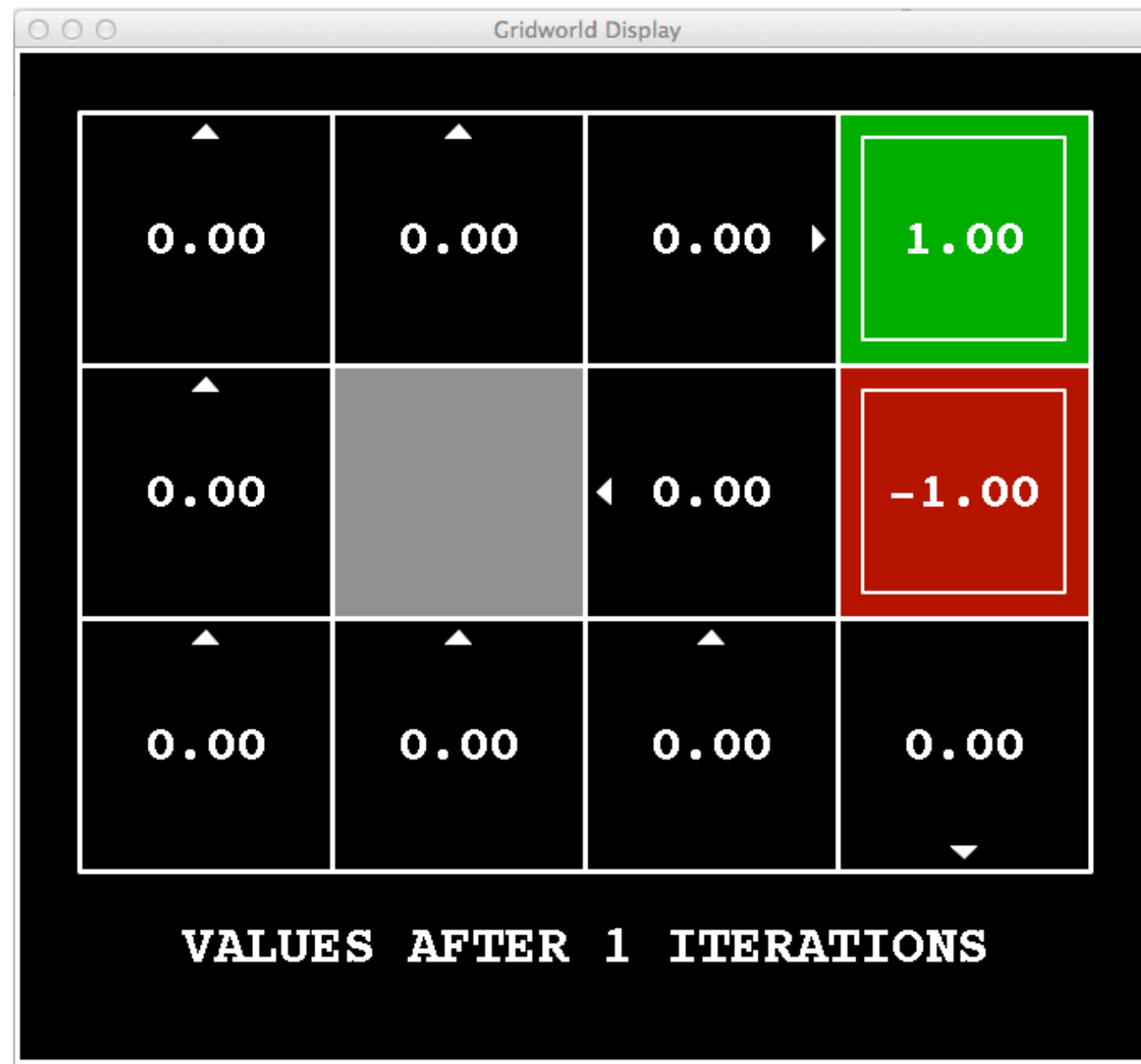


$k=0$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=1



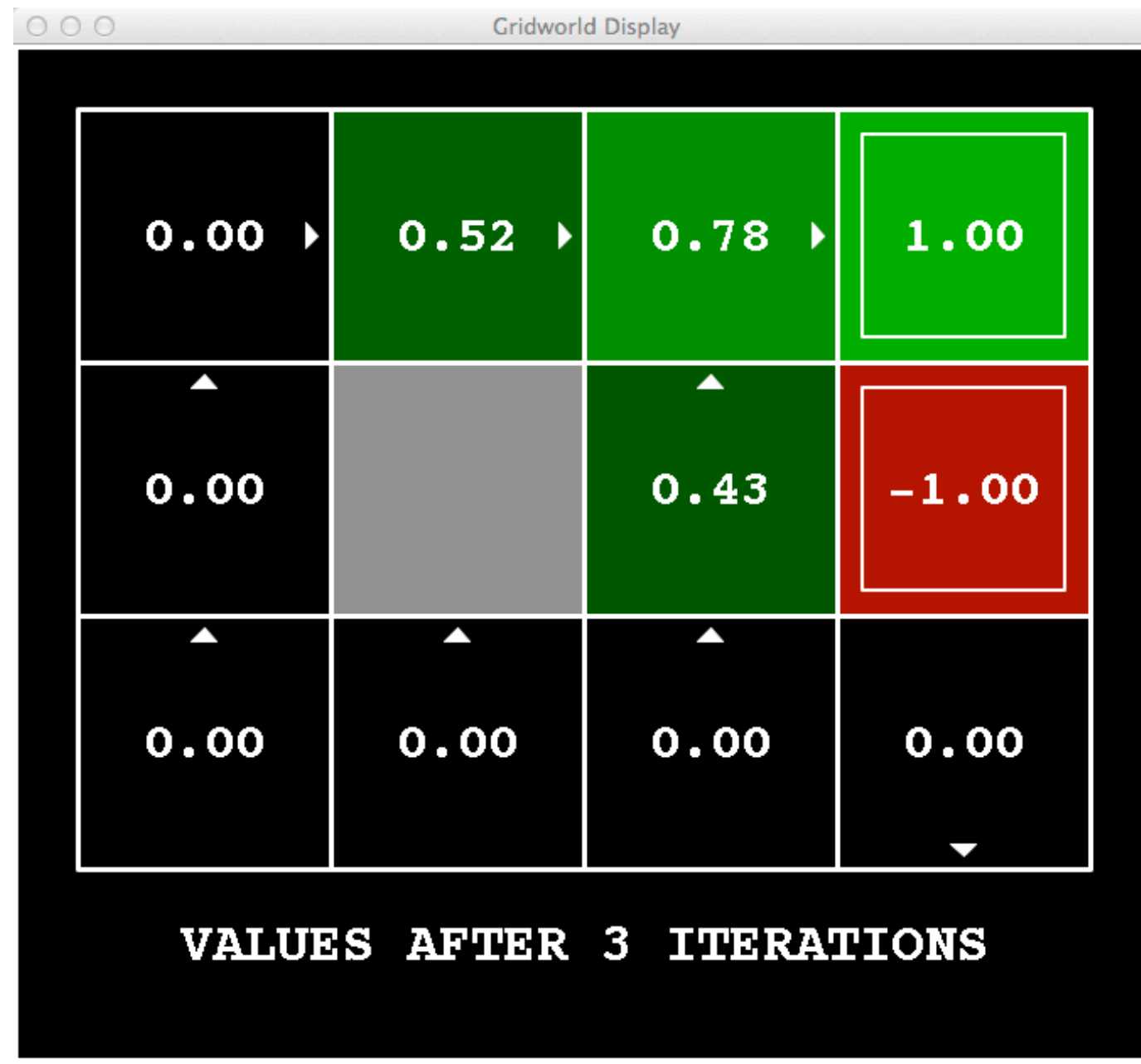
Noise = 0.2
Discount = 0.9
Living reward = 0

$k=2$



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=3$



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=4$



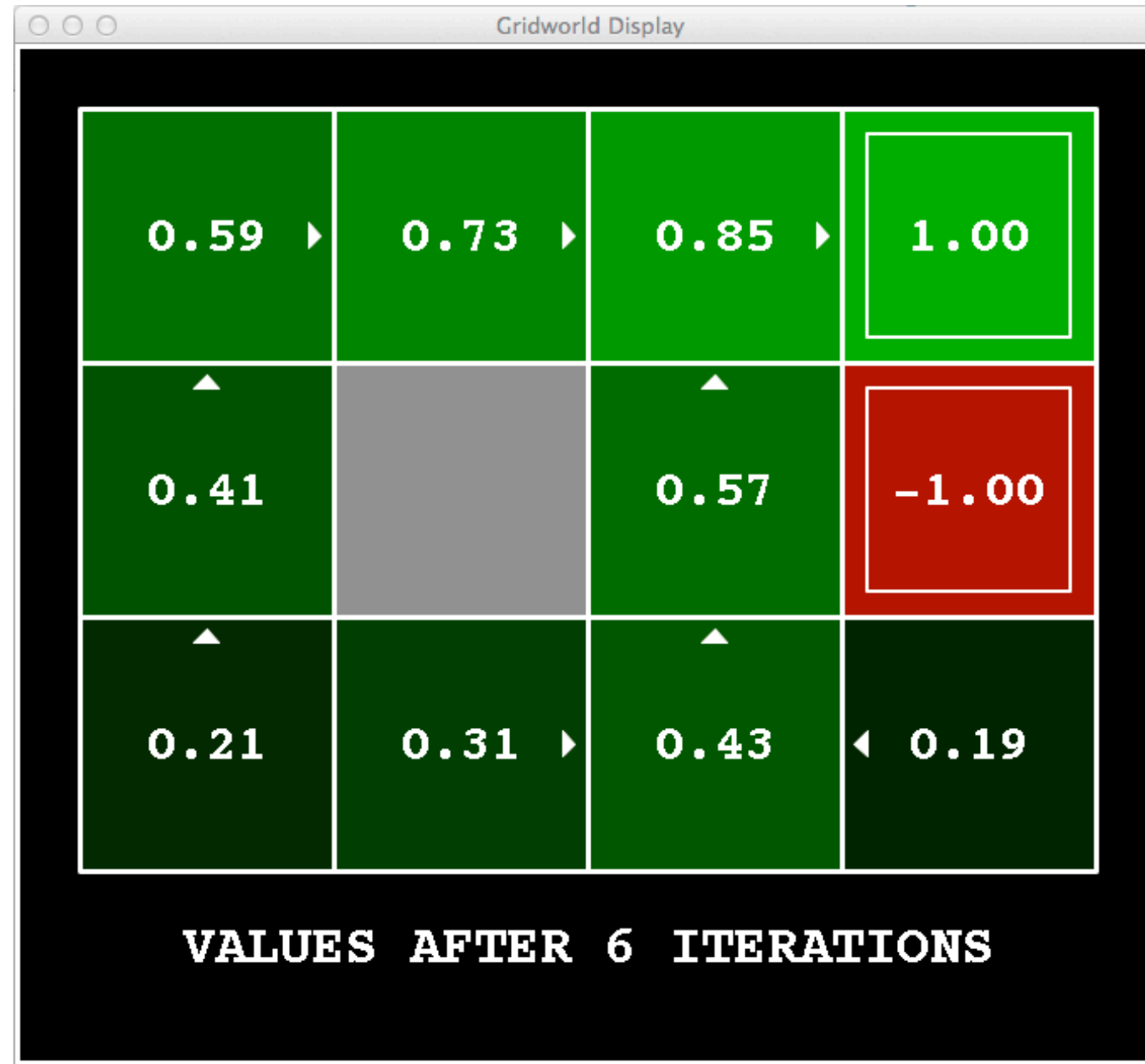
Noise = 0.2
Discount = 0.9
Living reward = 0

k=5



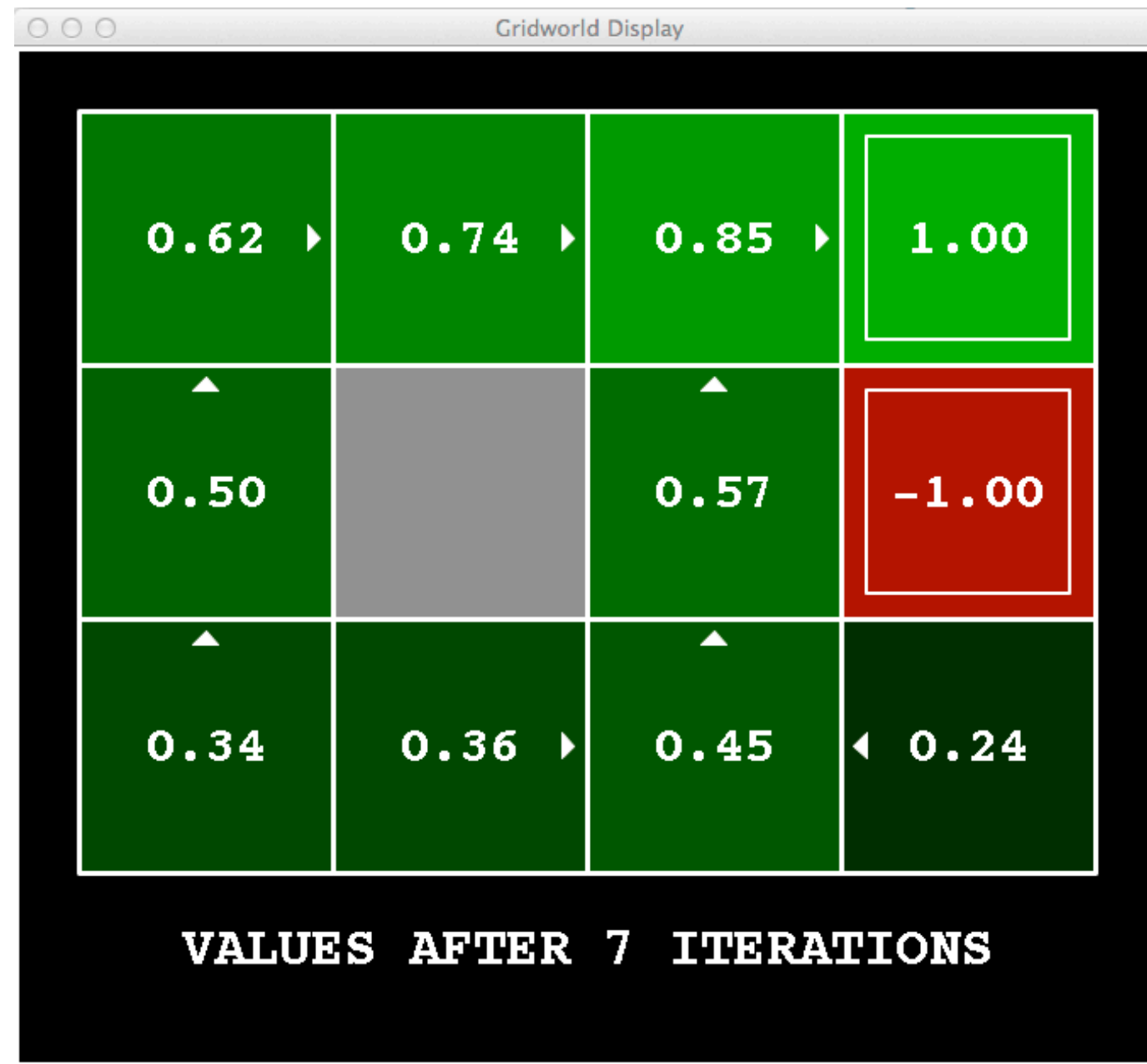
Noise = 0.2
Discount = 0.9
Living reward = 0

k=6



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=7$



Noise = 0.2
Discount = 0.9
Living reward = 0

k=8



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=9$



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=10$



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=11$



Noise = 0.2
Discount = 0.9
Living reward = 0

$k=12$



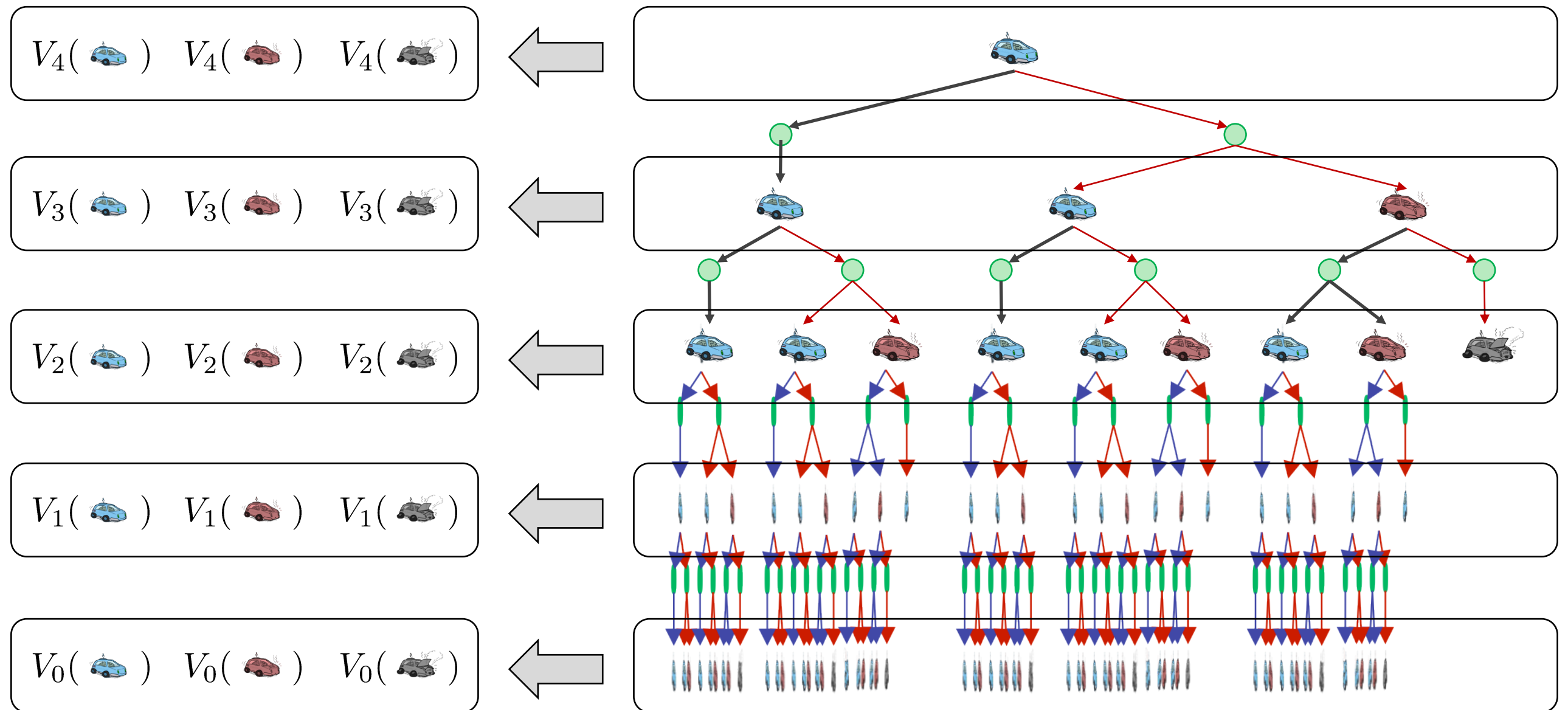
Noise = 0.2
Discount = 0.9
Living reward = 0

$k=100$

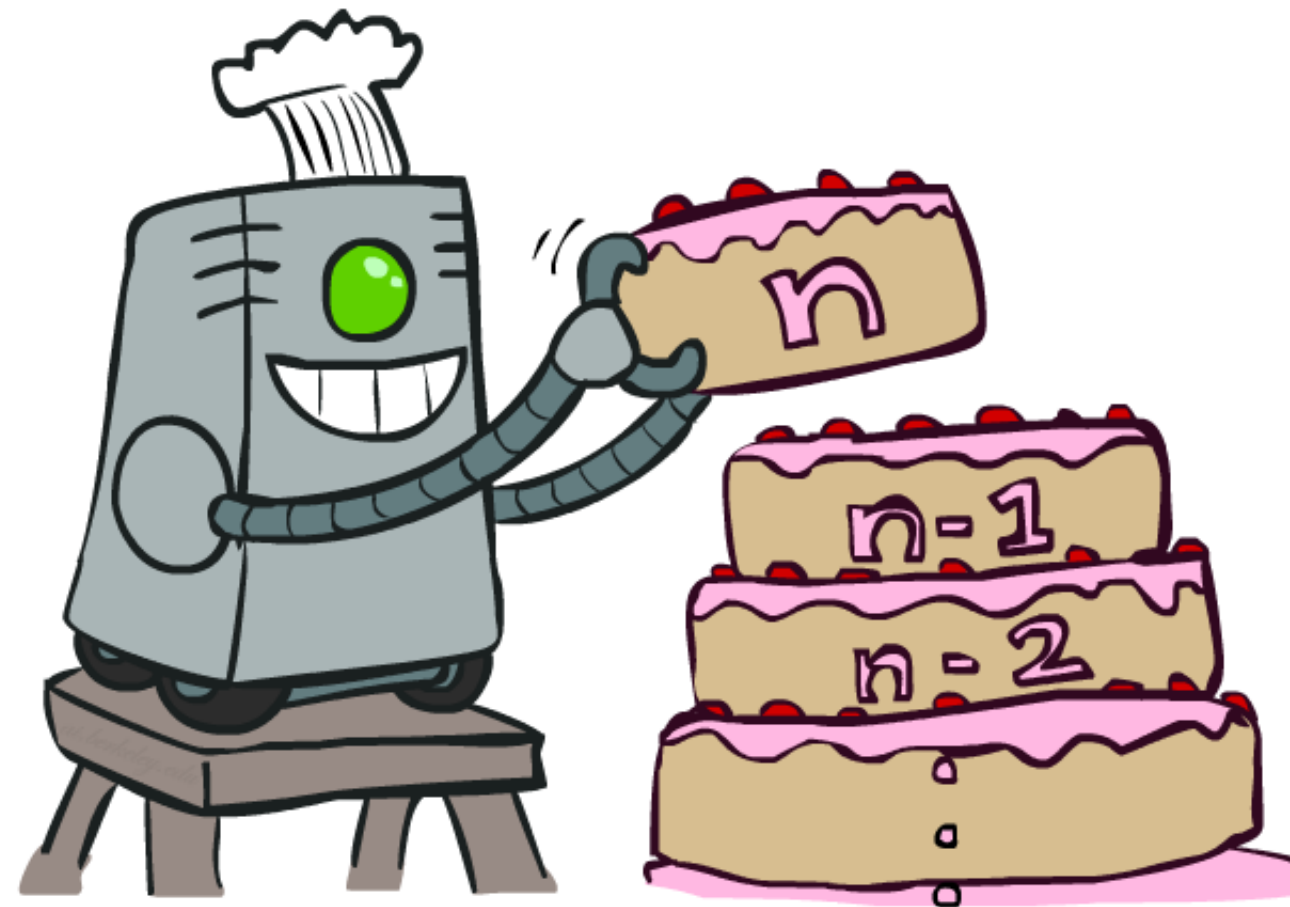


Noise = 0.2
Discount = 0.9
Living reward = 0

Computing Time-Limited Values



Value Iteration

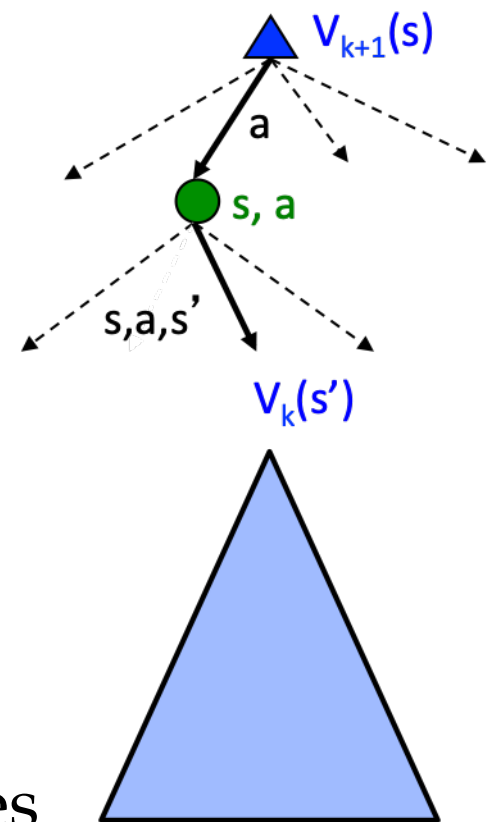


Value Iteration



- ❖ Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero
- ❖ Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

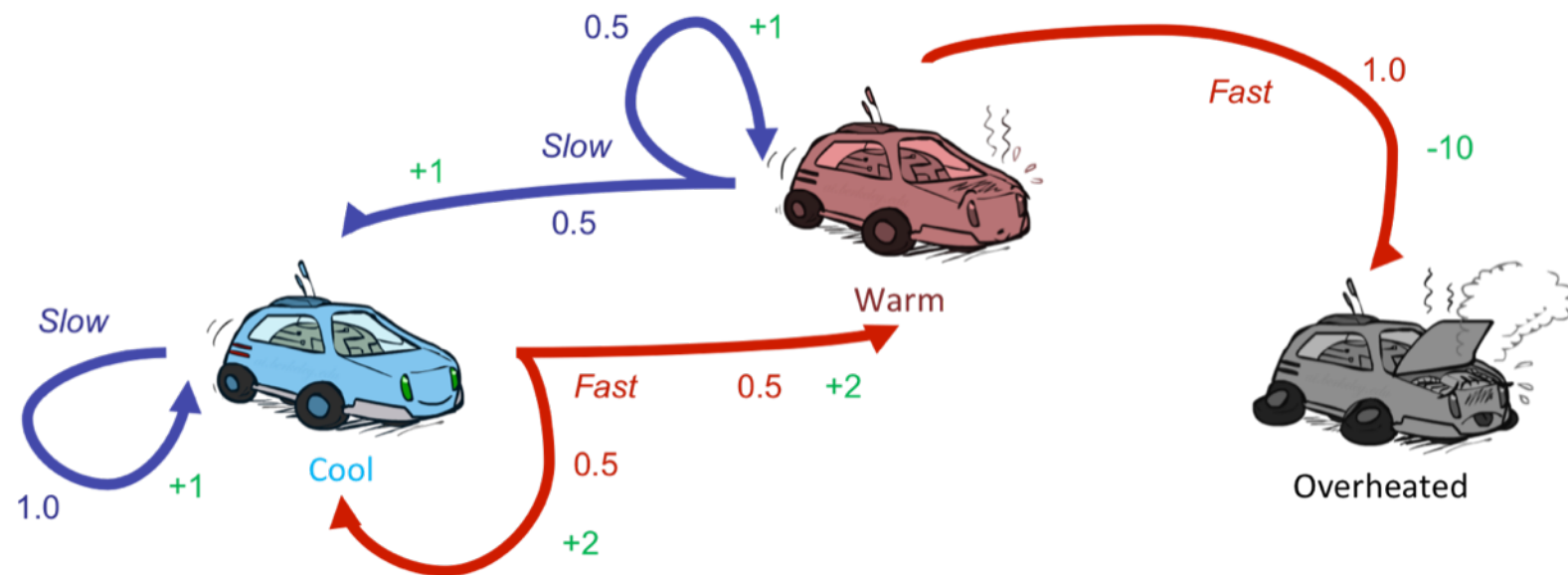
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

- ❖ Repeat until convergence
- ❖ Complexity of each iteration: $O(S^2A)$
- ❖ Theorem: will converge to unique optimal values
 - ❖ **Basic idea**: approximations get refined towards optimal values
 - ❖ Policy may converge long before values do



Example: Value Iteration




			
V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0

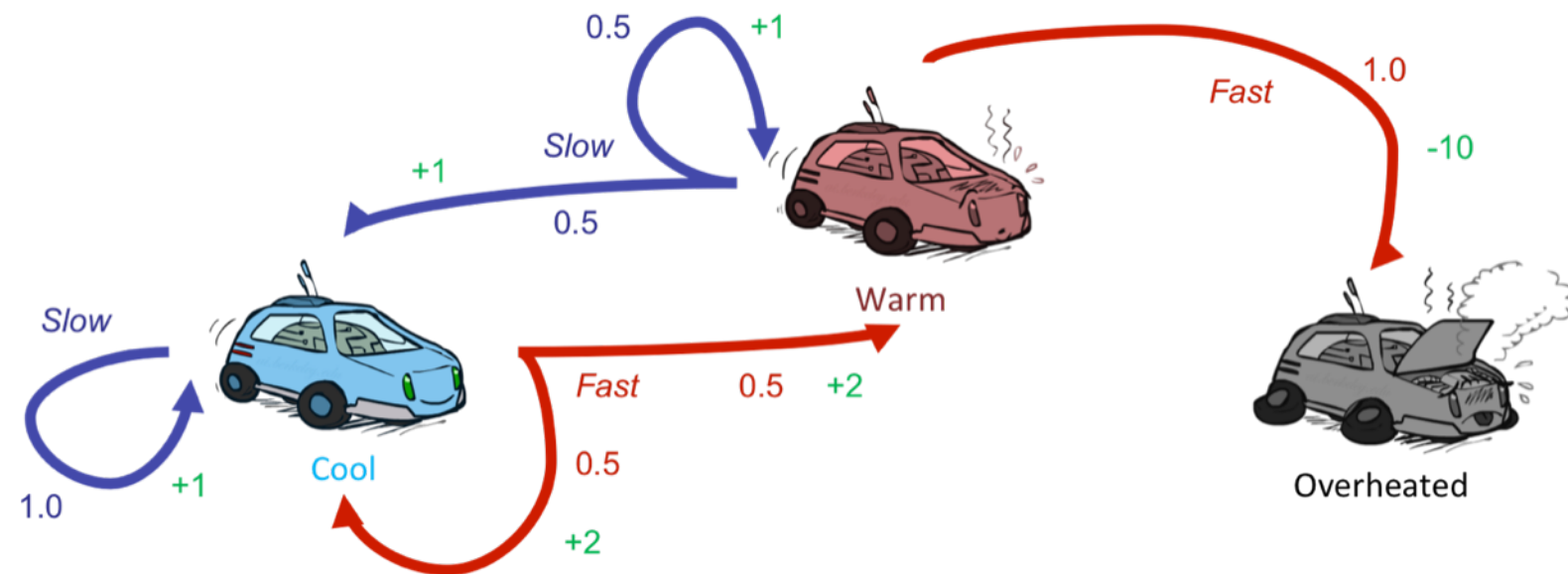


Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

Quiz: Value Iteration

			
V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0



Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s, a, s') [R(s, a, s') + \gamma V_k(s')]$$

What are the values for V_3 ?

Convergence

- ❖ How do we know the V_k vectors are going to converge?
- ❖ Case 1: If the tree has maximum depth M , then V_M holds the actual untruncated values
- ❖ Case 2: If the discount is less than 1
 - ❖ **Sketch:** For any state V_k and V_{k+1} can be viewed as depth $k+1$ expectimax results in nearly identical search trees
 - ❖ The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
 - ❖ That last layer is at best all R_{MAX}
 - ❖ It is at worst R_{MIN}
 - ❖ But everything is discounted by γ^k that far out
 - ❖ So V_k and V_{k+1} are at most $\gamma^k \max |R|$ different
 - ❖ So as k increases, the values converge

