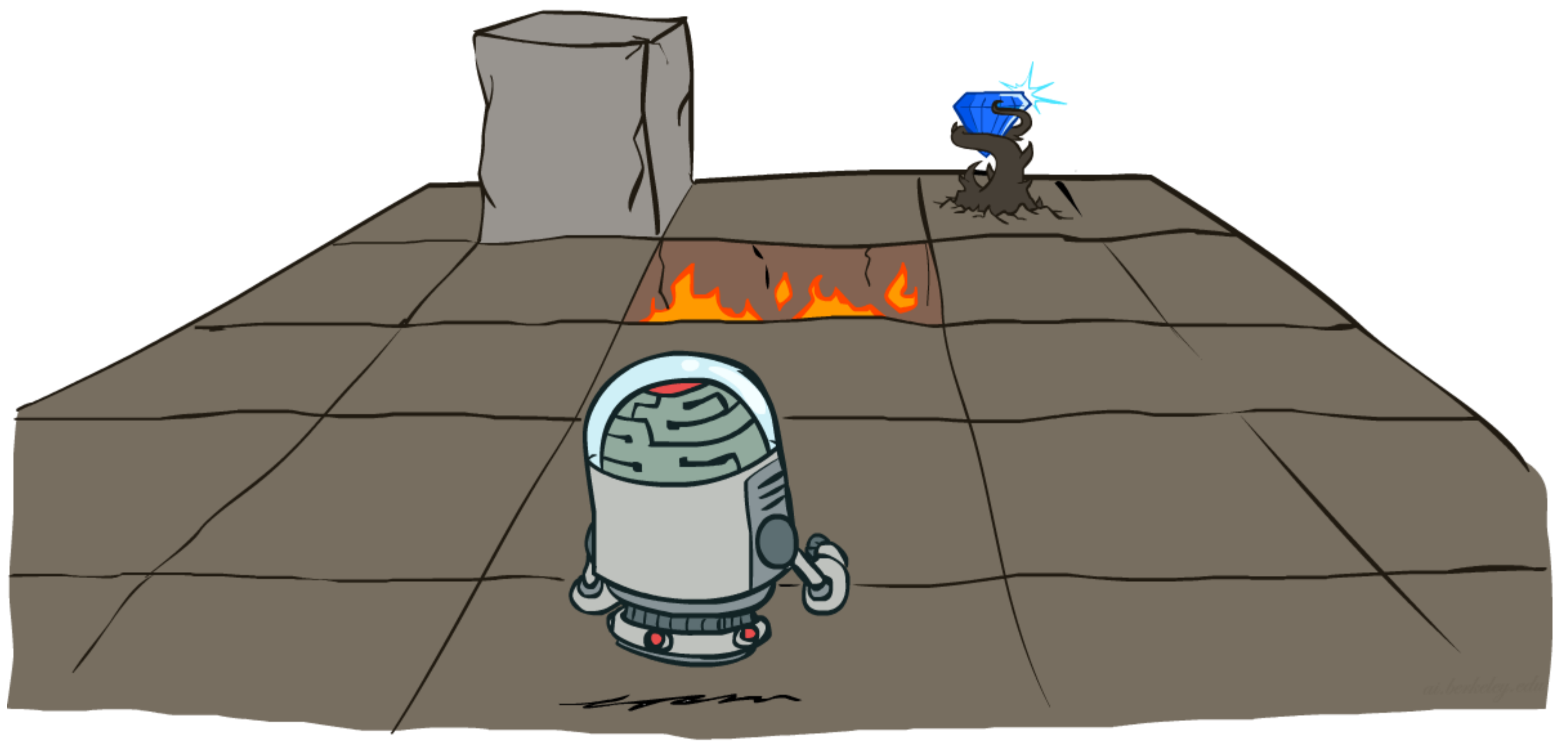
Announcements

- OH
  - Mondays and Wednesdays at 9am-10am
- HW2 (HW3 soon)
  - Due on Jun. 3 at 11:59pm
- Project 1 (Project 2 soon)
  - Due on Jun. 5 at 11:59pm (small extension)
- Mid-term Exam
  - Monday Jun. 22 at 4pm-5:40pm

Ve492: Introduction to Artificial Intelligence
Markov Decision Processes I

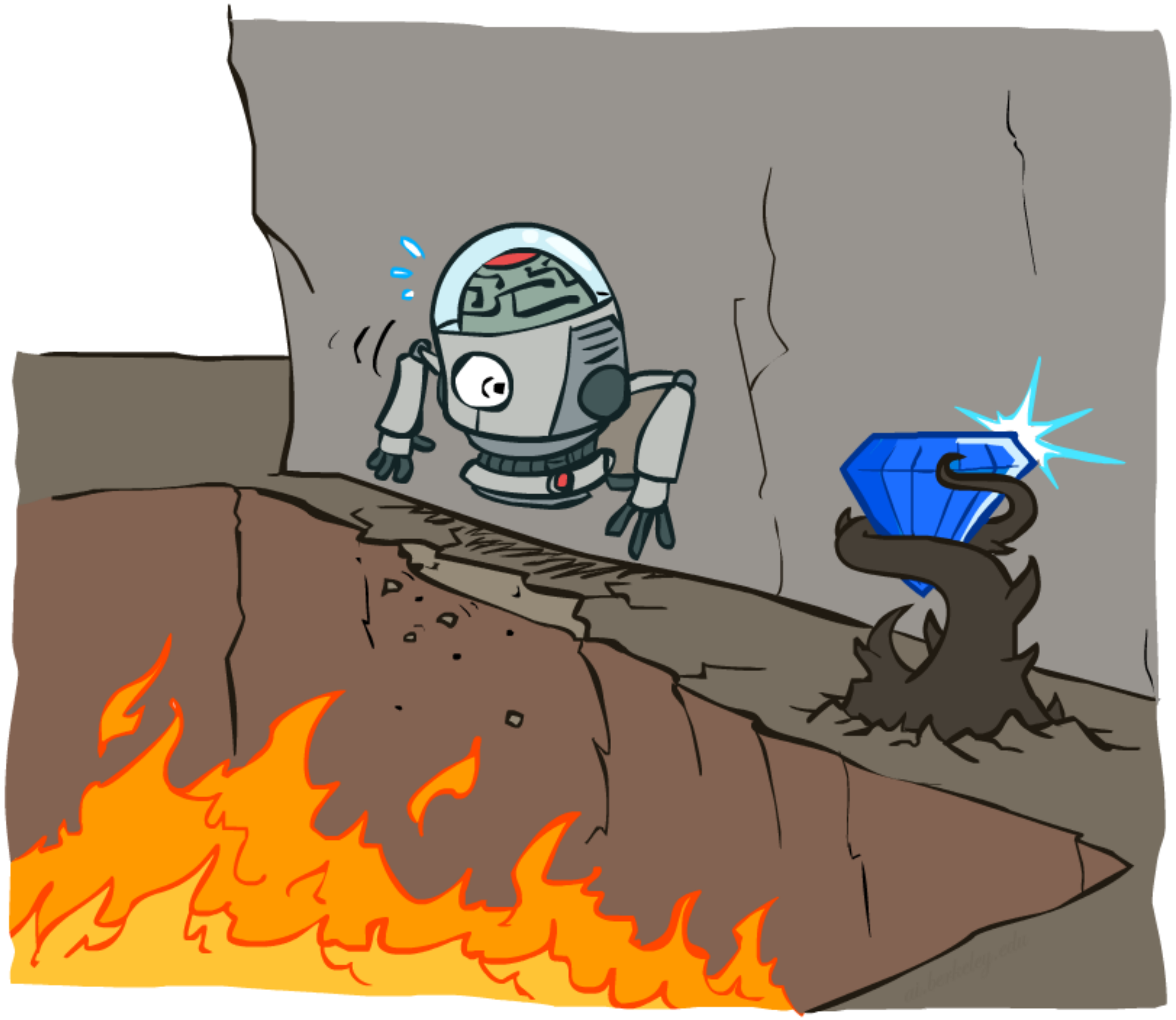


Paul Weng

UM-SJTU Joint Institute

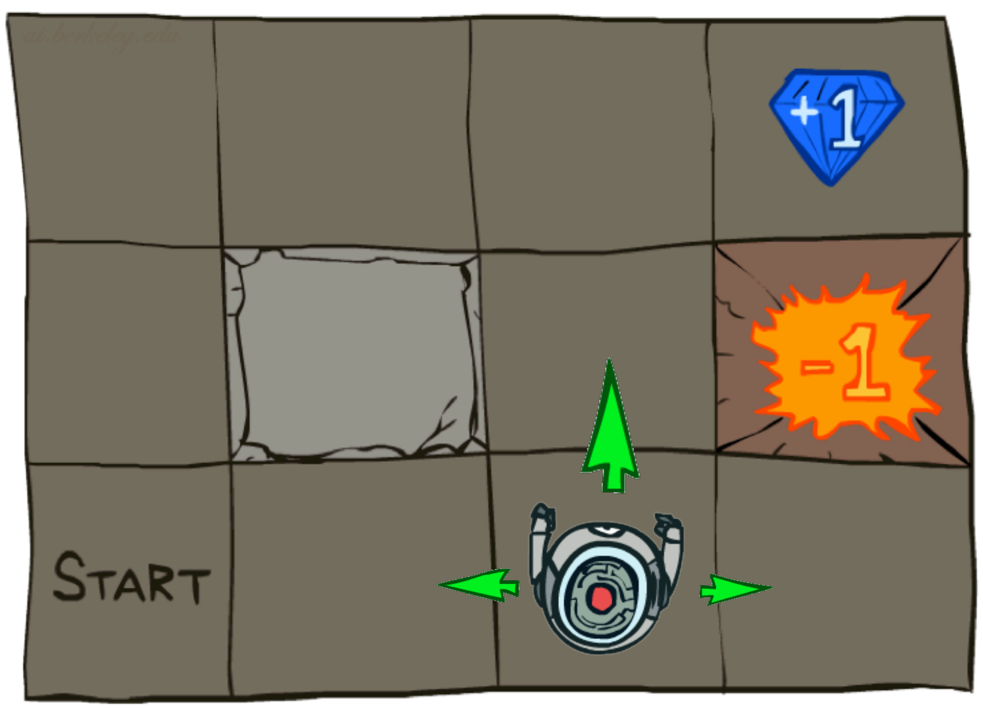
Slides adapted from http://ai.berkeley.edu, AIMA, UM, CMU

Non-Deterministic Search



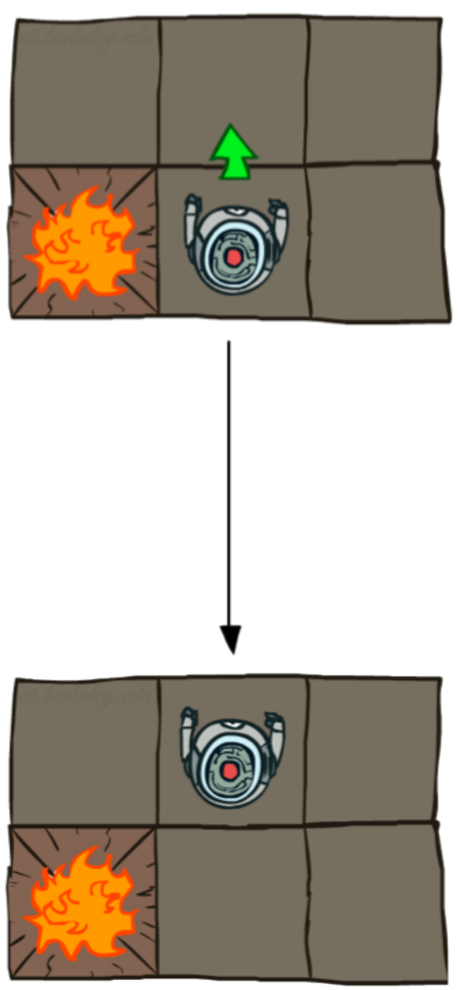
Example: Grid World

- A maze-like problem
  - The agent lives in a grid
  - Walls block the agent's path
- Noisy movement: actions do not always go as planned
  - 80% of the time, the action North takes the agent North (if there is no wall there)
  - 10% of the time, North takes the agent West; 10% East
  - If there is a wall in the direction the agent would have been taken, the agent stays put
- The agent receives rewards each time step
  - Small "living" reward each step (can be negative)
  - Big rewards come at the end (good or bad)
- Goal: maximize sum of rewards

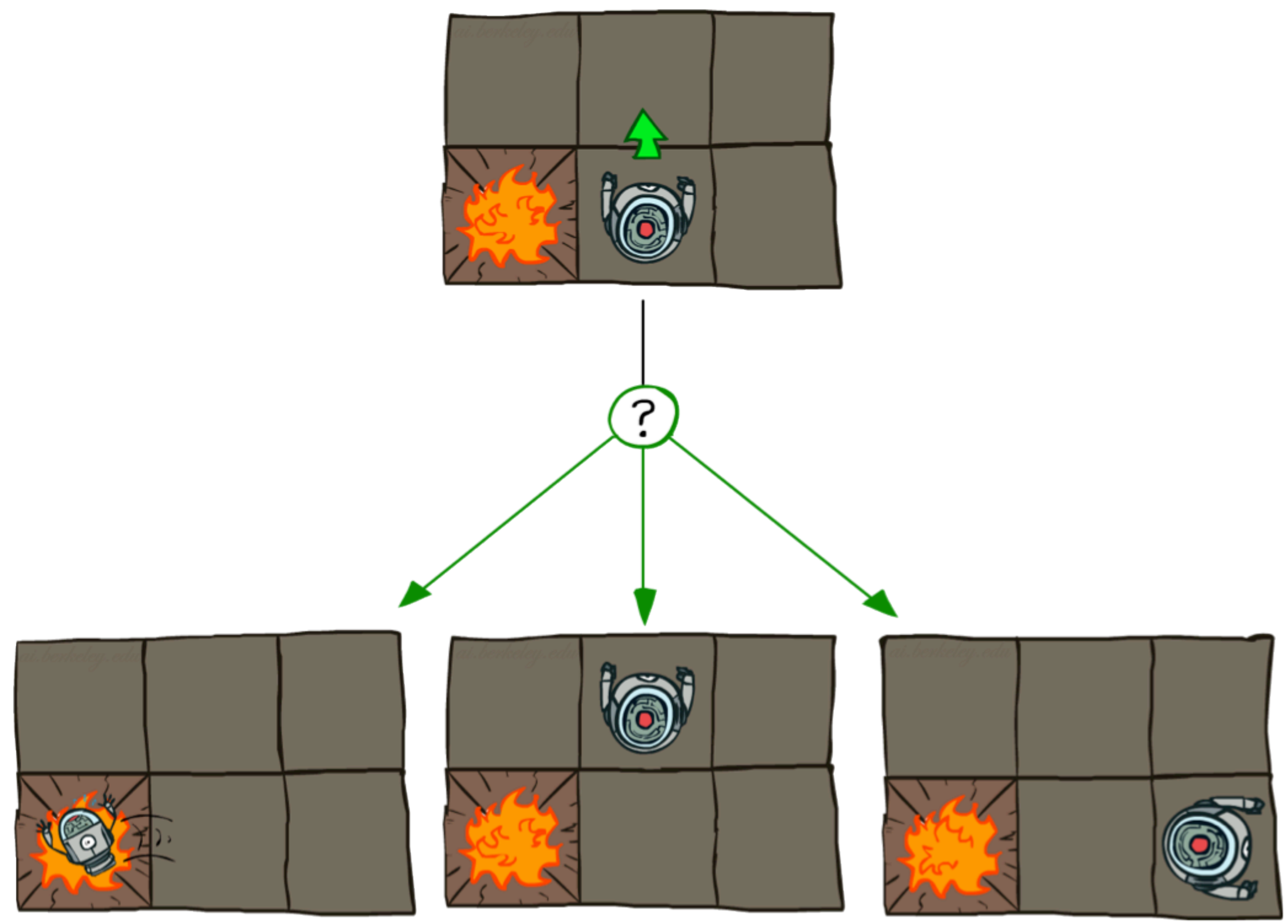


Grid World Actions

Deterministic Grid World



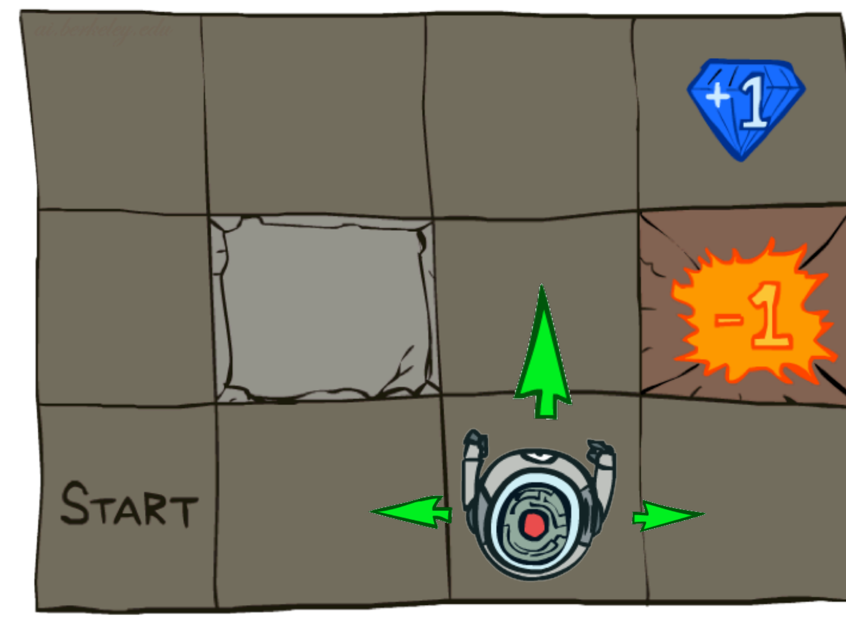
Stochastic Grid World



Markov Decision Processes

An MDP is defined by:

- A set of states $s \in S$
- A set of actions $a \in A$
- A transition function $T(s, a, s')$
  - Probability that a from s leads to s', i.e., $P(s' \mid s, a)$
  - Also called the model or the dynamics
- A reward function $R(s, a, s')$ R(s, a)
  - Sometimes just $R(s)$ or $R(s')$
- A start state
- Maybe a terminal state



MDPs are non-deterministic search problems

- One way to solve them is with expectimax search
- We'll have a new tool soon

What is Markov about MDPs?

- "Markov" generally means that given the present state, the future and the past are independent

- For Markov decision processes, "Markov" means action outcomes depend only on the current state

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t, S_{t-1} = s_{t-1}, A_{t-1}, \ldots S_0 = s_0)$$

$$=$$

$$P(S_{t+1} = s' | S_t = s_t, A_t = a_t)$$

Andrey Markov
(1856-1922)

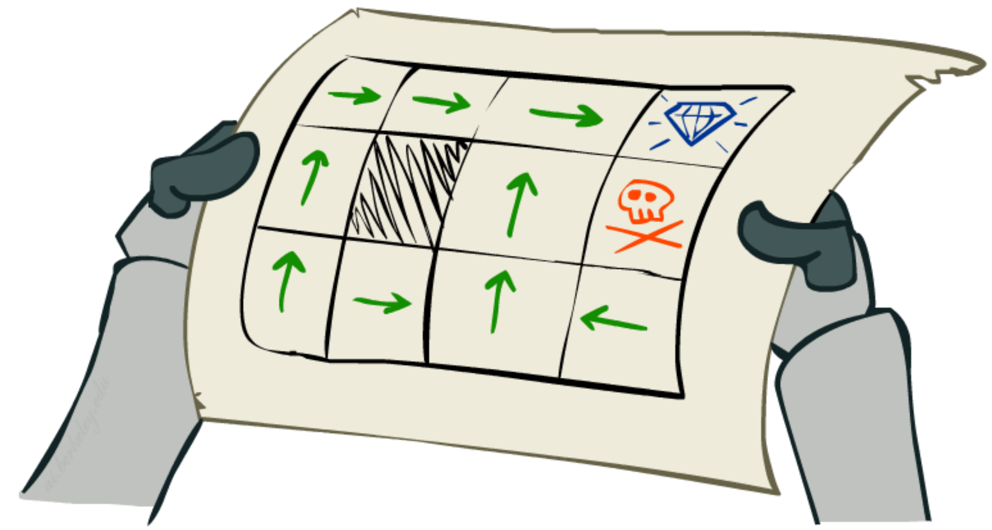
- This is just like search, where the successor function could only depend on the current state (not the history)

Policies

In deterministic single-agent search problems, we wanted an optimal plan, or sequence of actions, from start to a goal

For MDPs, we want an optimal policy $\pi^*: S \rightarrow A$

- A policy π gives an action for each state
- An optimal policy is one that maximizes expected utility if followed
- An explicit policy defines a reflex agent

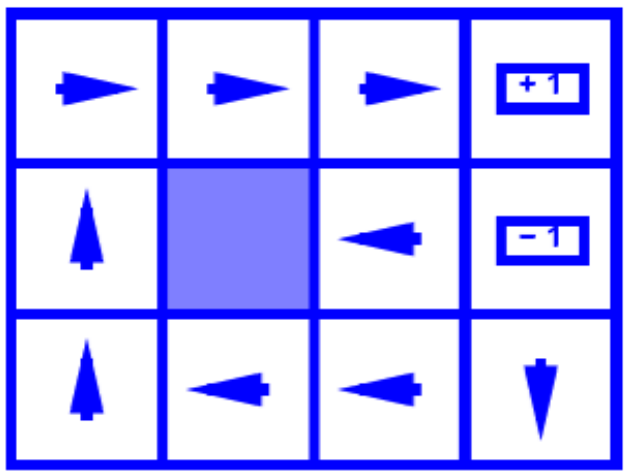


Optimal policy when R(s, a, s') = -0.03 for all non-terminals s

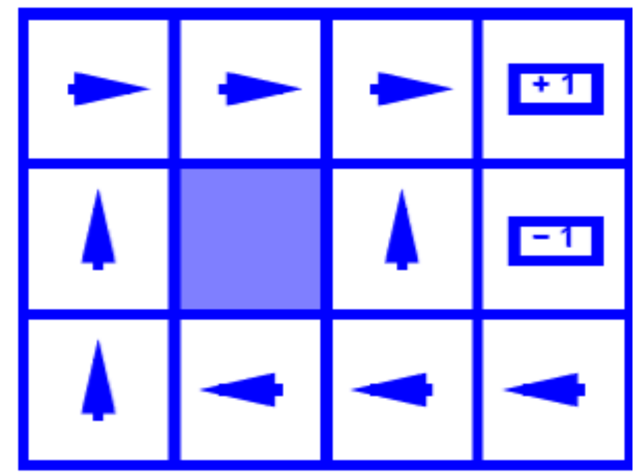
Expectimax didn't compute entire policies

- It computed the action for a single state only

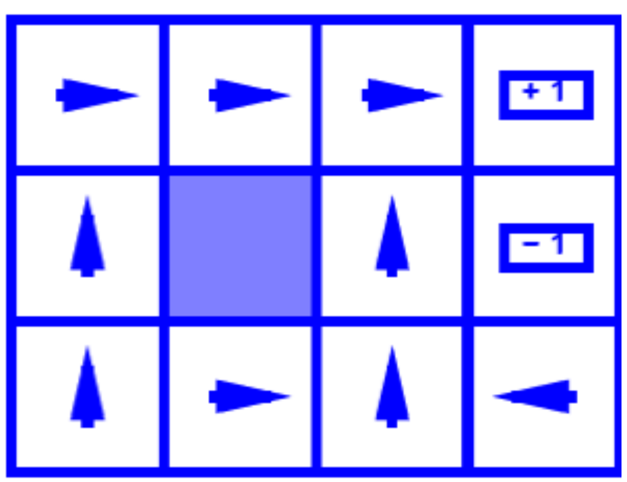
Optimal Policies



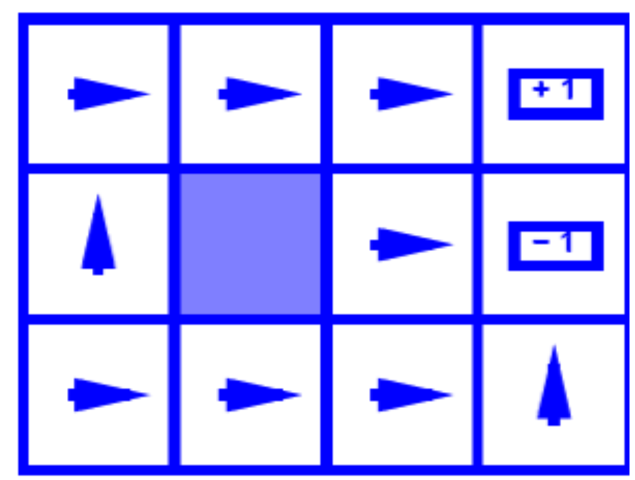
R(s) = -0.01



R(s) = -0.03

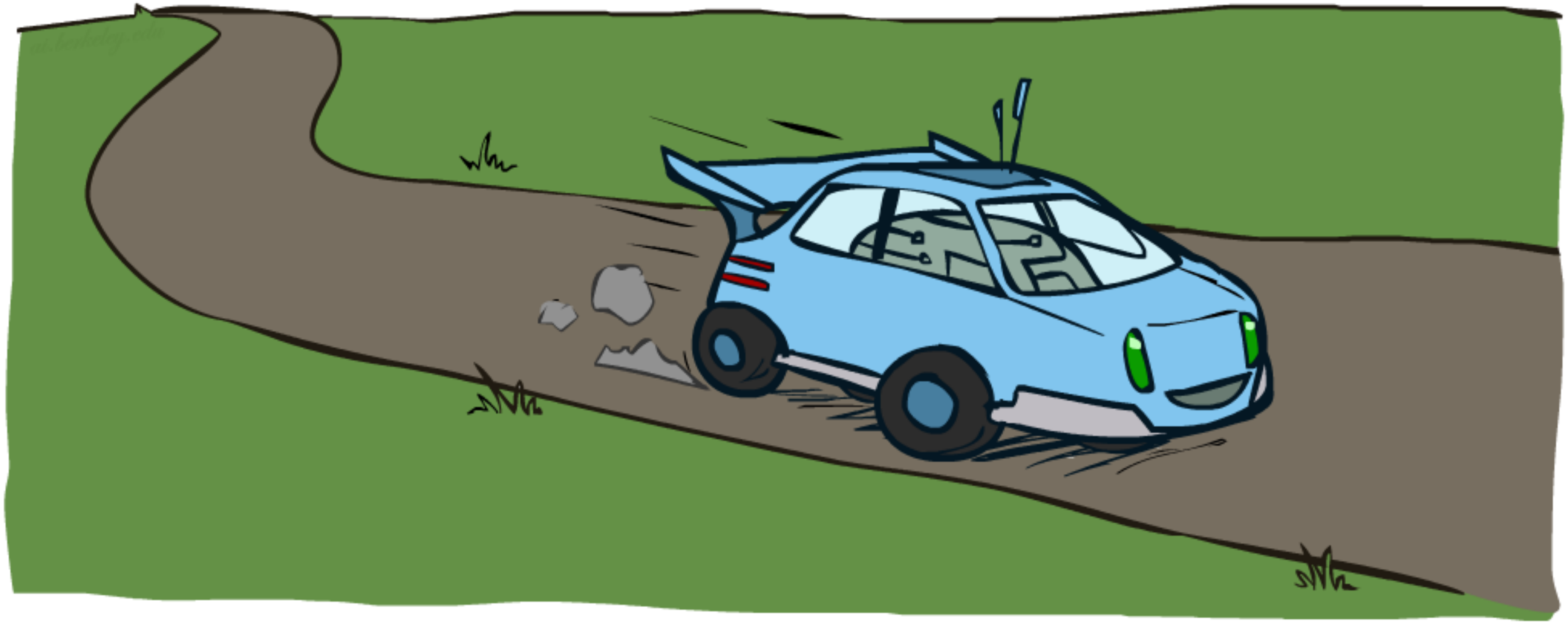


R(s) = -0.4



R(s) = -2.0

Example: Racing



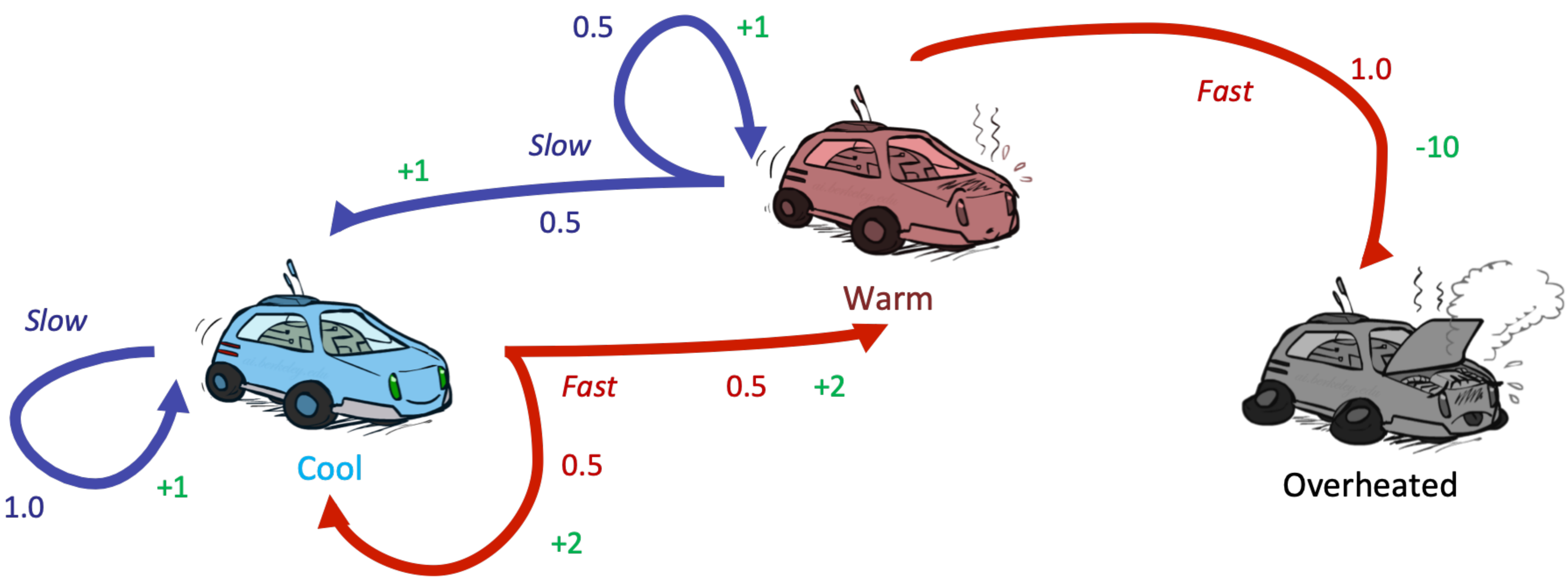
Example: Racing

A robot car wants to travel far, quickly

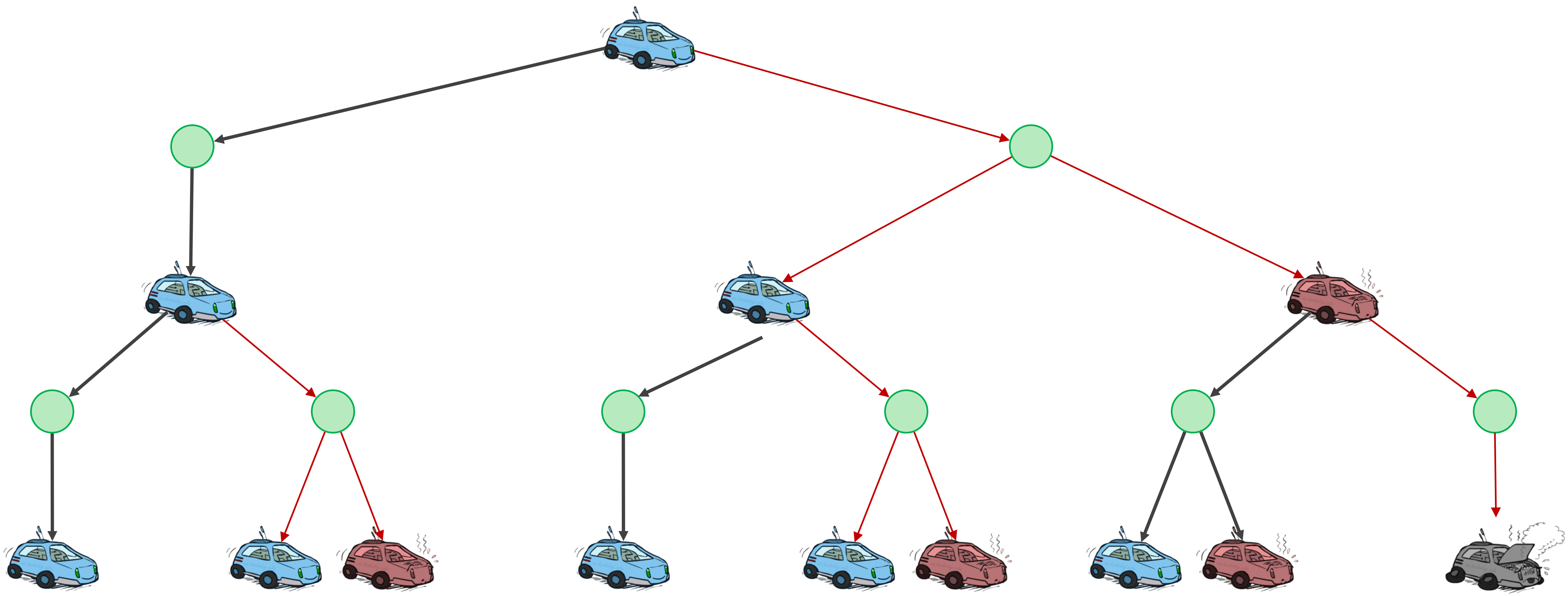
Three states: Cool, Warm, Overheated

Two actions: *Slow*, *Fast*

Going faster gets double reward

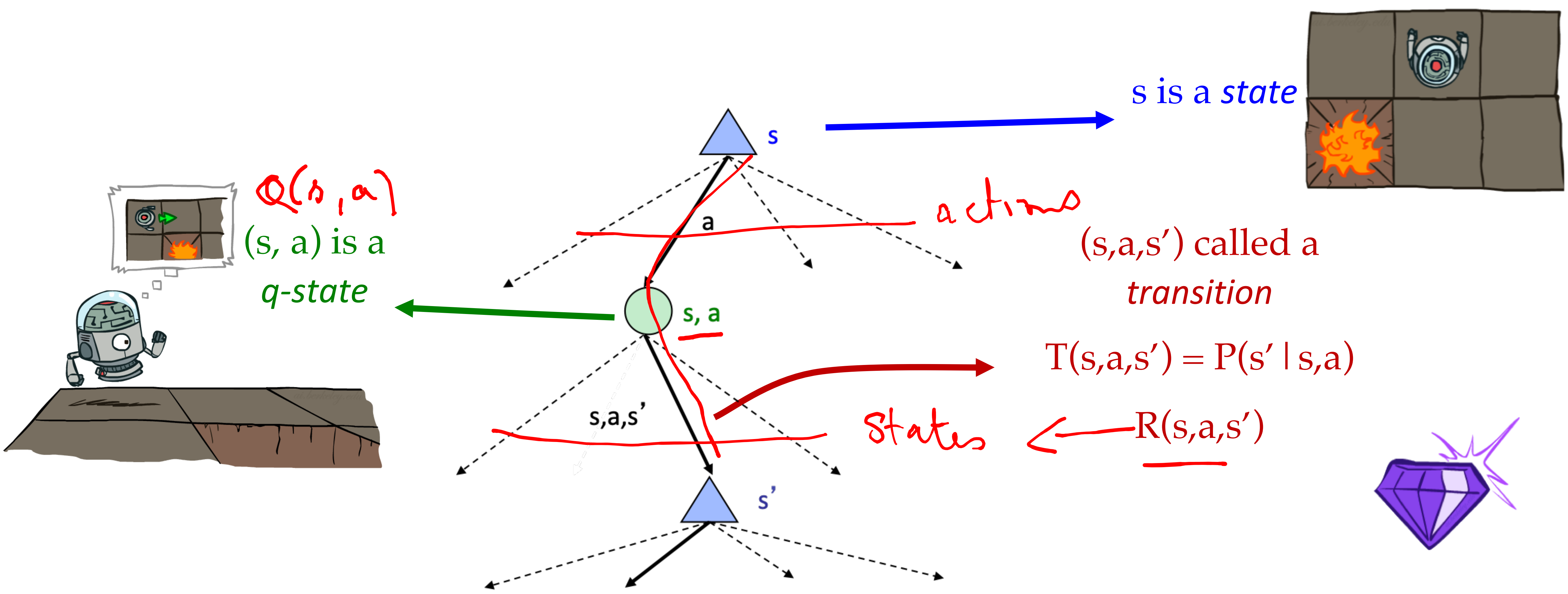


Racing Search Tree

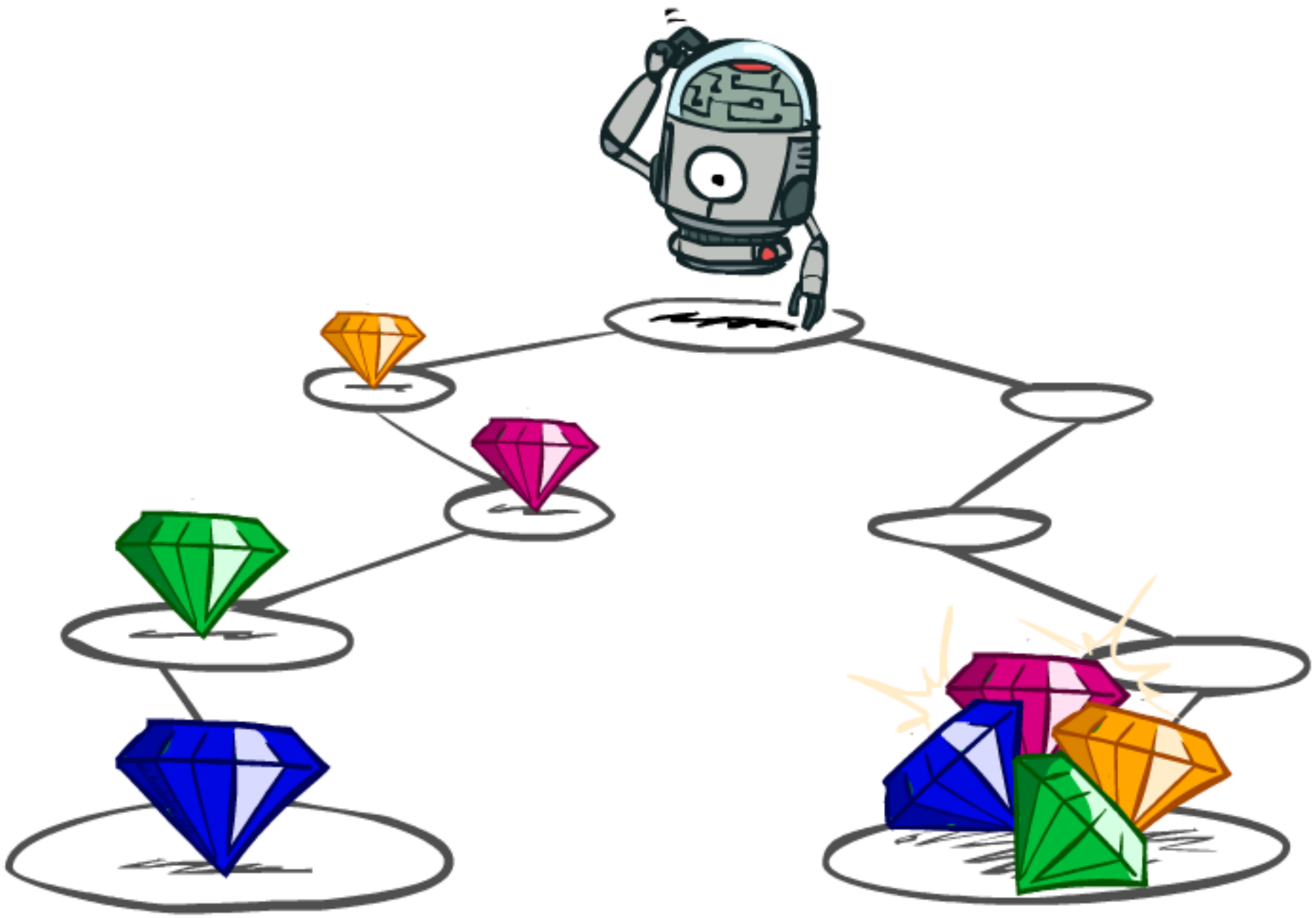


MDP Search Trees

Each MDP state projects an expectimax-like search tree



Utilities of Sequences



Utilities of Sequences

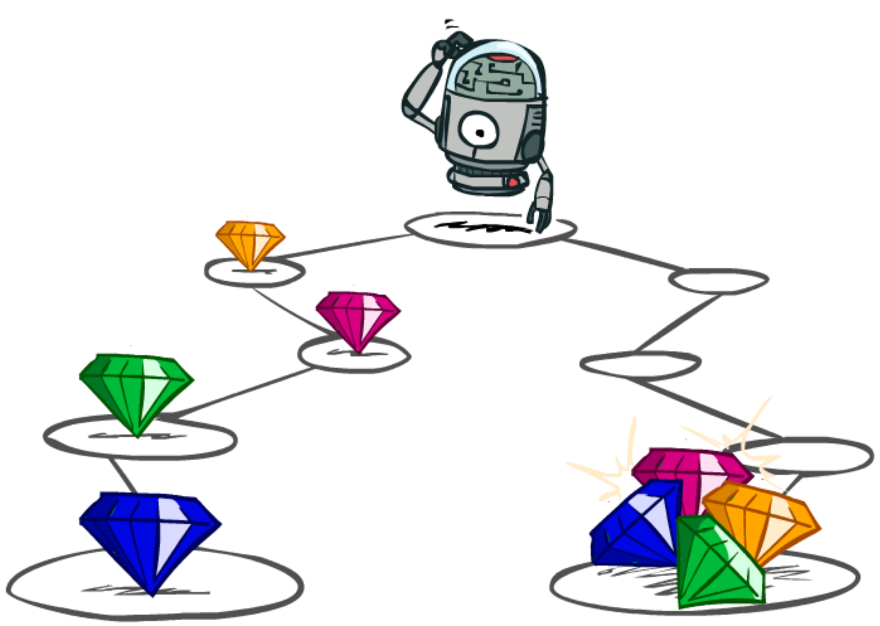
What preferences should an agent have over reward sequences?

More or less?

[1, 2, 2] or [2, 3, 4]

Now or later?

[0, 0, 1] or [1, 0, 0]



Discounting

It's reasonable to maximize the sum of rewards

It's also reasonable to prefer rewards now to rewards later

One solution: values of rewards decay exponentially

1

Worth Now

$0 \leq \gamma \leq 1$

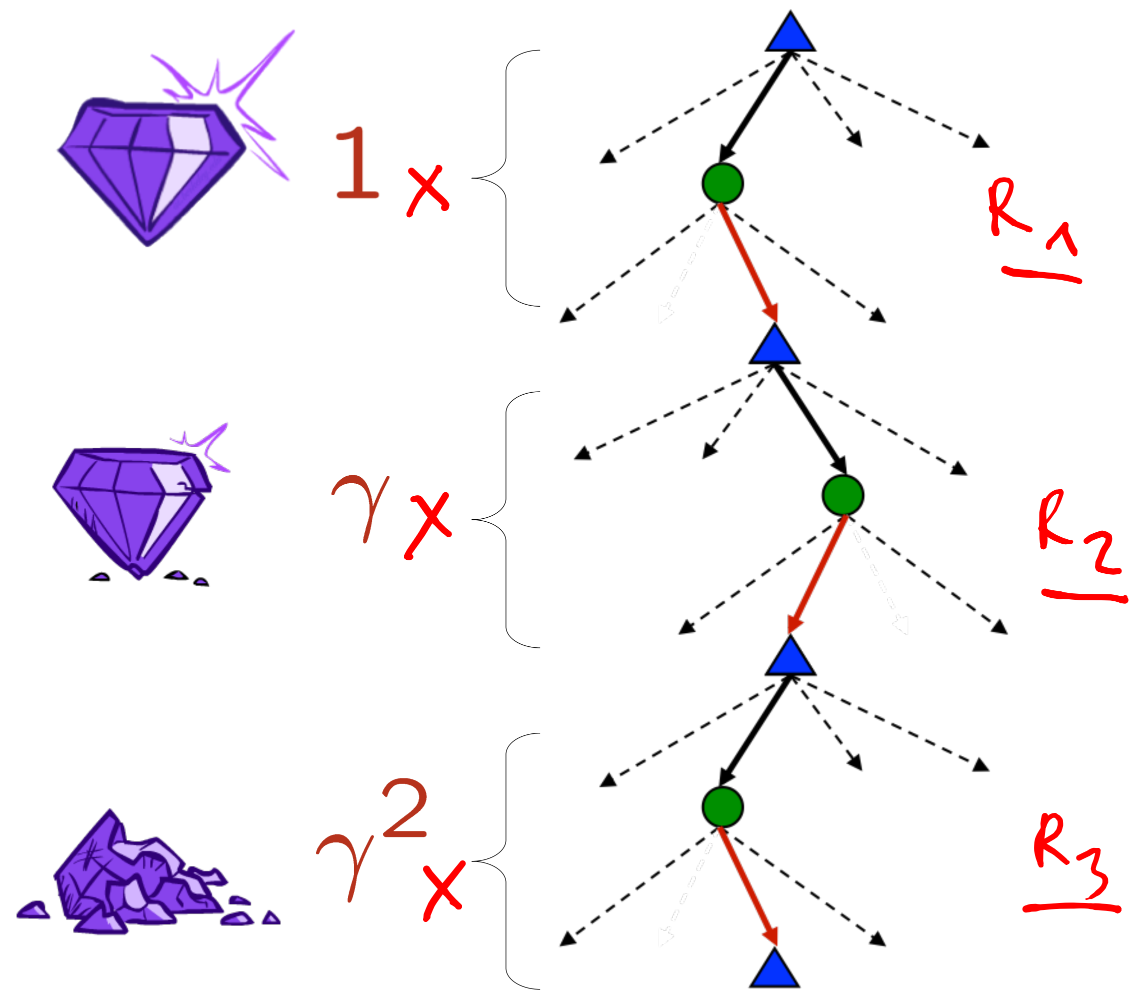
Worth Next Step

γ^2

Worth In Two Steps

Discounting

- How to discount?
  - Each time we descend a level, we multiply in the discount once
- Why discount?
  - Sooner rewards probably do have higher utility than later rewards
  - Also helps our algorithms converge
- Example: discount of 0.5
  - U([1,2,3]) = 1*1 + 0.5*2 + 0.25*3
  - U([1,2,3]) < U([3,2,1])



Quiz: Discounted Sum of Rewards

- What is the value of U[2,4,8] with $\gamma = 0.5$?
- What is the value of U[8,4,2] with $\gamma = 0.5$?

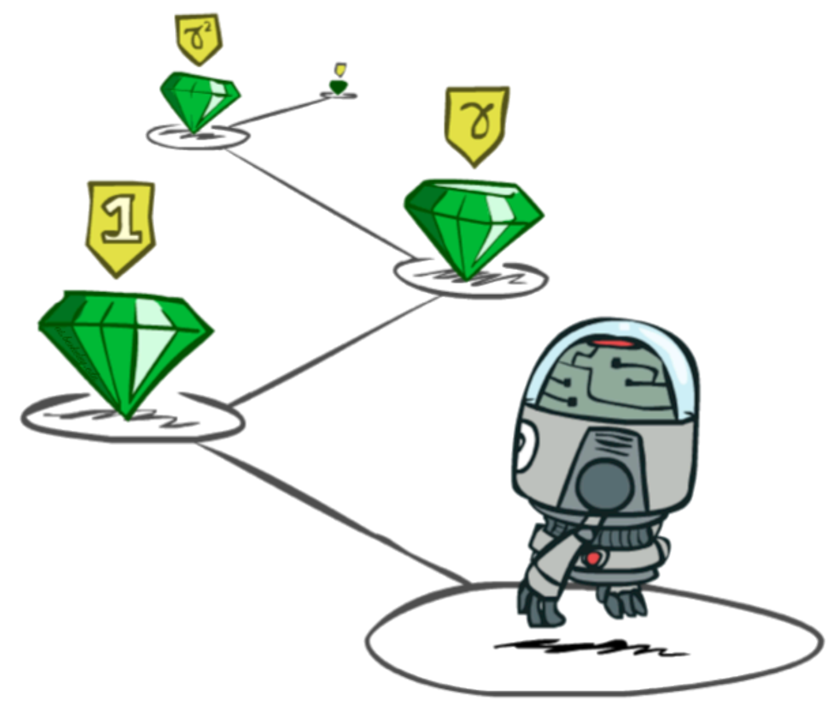
Stationary Preferences

Theorem: if we assume stationary preferences:

$$[a_1, a_2, \ldots] \succ [b_1, b_2, \ldots]$$

$$\Updownarrow$$

$$[r, a_1, a_2, \ldots] \succ [r, b_1, b_2, \ldots]$$



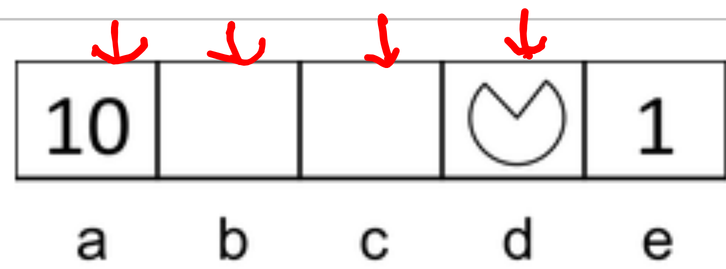
Then: there are only two ways to define utilities

- Additive utility: $U([r_0, r_1, r_2, \ldots]) = r_0 + r_1 + r_2 + \cdots$
- Discounted utility: $U([r_0, r_1, r_2, \ldots]) = r_0 + \gamma r_1 + \gamma^2 r_2 \cdots$

$\gamma = 1$

Discounting

Given:



R(s)

- Actions: East, West, and Exit (only available in exit states a, e)
- Transitions: deterministic

For $\gamma = 1$, what is the optimal policy?

≥ 10 for $\gamma = 1$

0.1 for $\gamma = 0.1$

For $\gamma = 0.1$, what is the optimal policy?

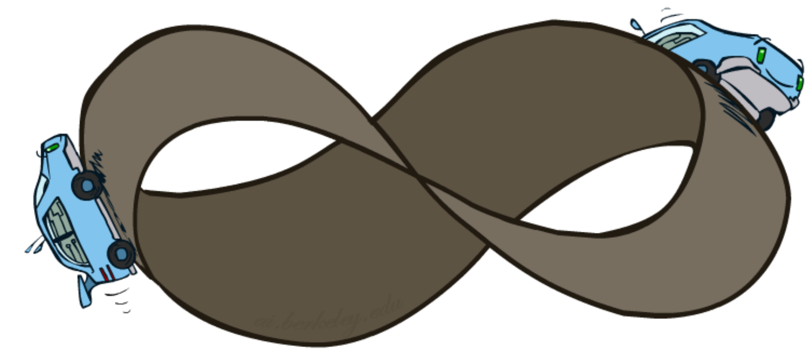
$10 \times \gamma^3 = 0.1$

For which γ are West and East equally good when in state d?

$10 \times \gamma^3 = \gamma$

Infinite Utilities?!

- Problem: What if the game lasts forever? Do we get infinite rewards?



- Solutions:

  = finite # of decision.

  - Finite horizon: (similar to depth-limited search)
    - Terminate episodes after a fixed T steps (e.g. life)
    - Gives nonstationary policies (π depends on time left)

  - Discounting: use $0 < \gamma < 1$

    $\frac{1}{1-\gamma}$ effective horizon.

    $$U([r_0, \ldots r_\infty]) = \sum_{t=0}^{\infty} \gamma^t r_t \le R_{\max}/(1-\gamma)$$

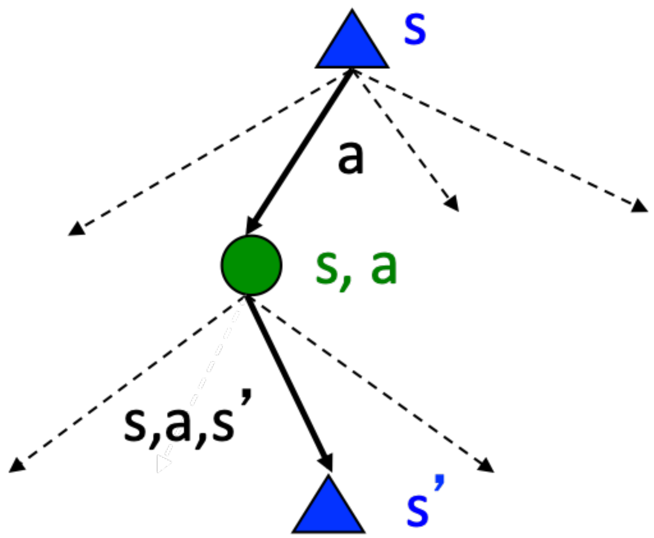
    - Smaller γ means smaller "horizon" – shorter term focus

  - Absorbing state: guarantee that for every policy, a terminal state will eventually be reached (like "overheated" for racing)

Recap: Defining MDPs

Markov decision processes:

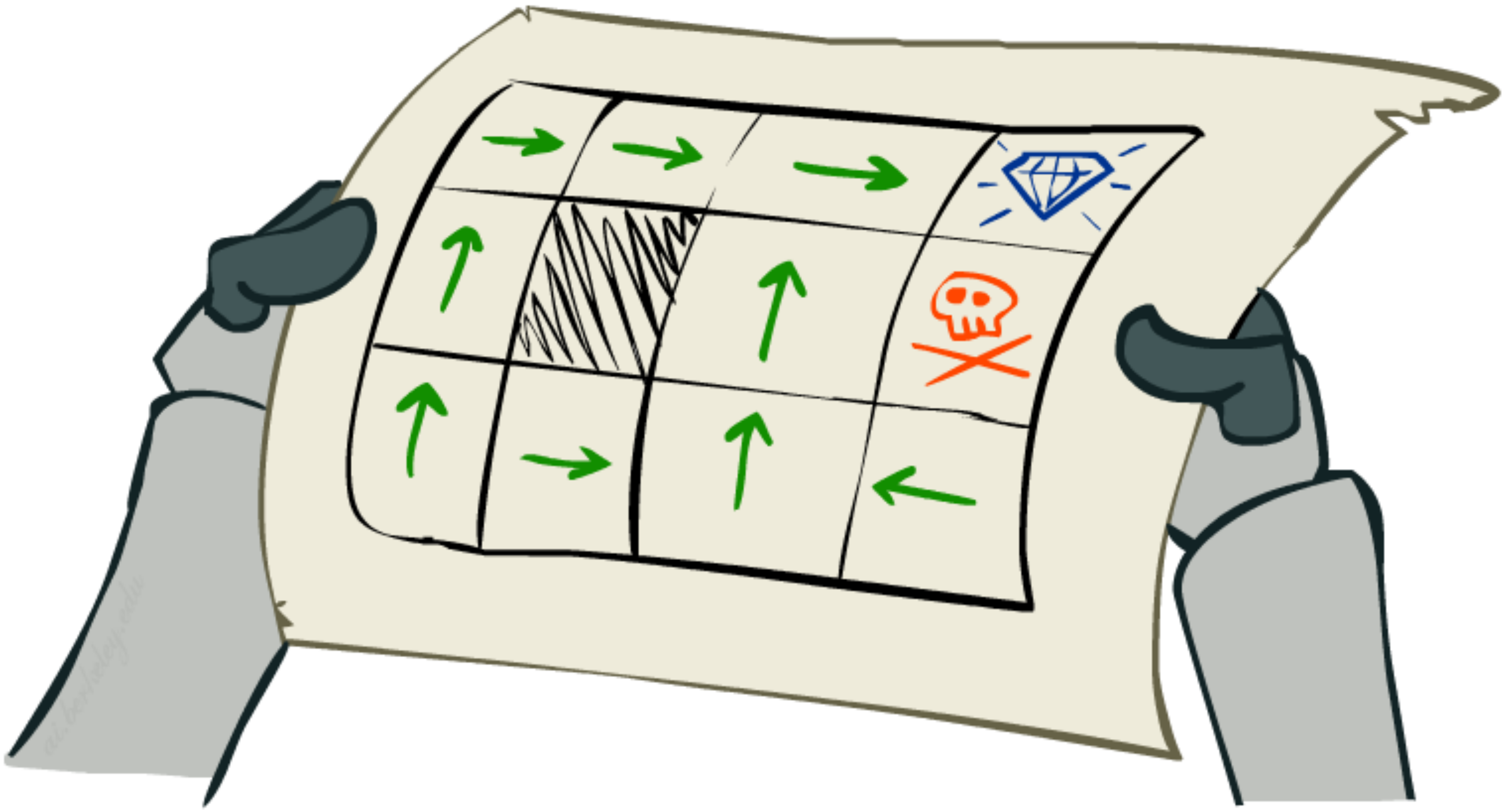
- Set of states S
- Start state s_0
- Set of actions A
- Transitions P(s' | s,a) (or T(s,a,s'))
- Rewards R(s,a,s') (and discount γ)



MDP quantities so far:

- Policy = Choice of action for each state
- Utility = sum of (discounted) rewards

Solving MDPs

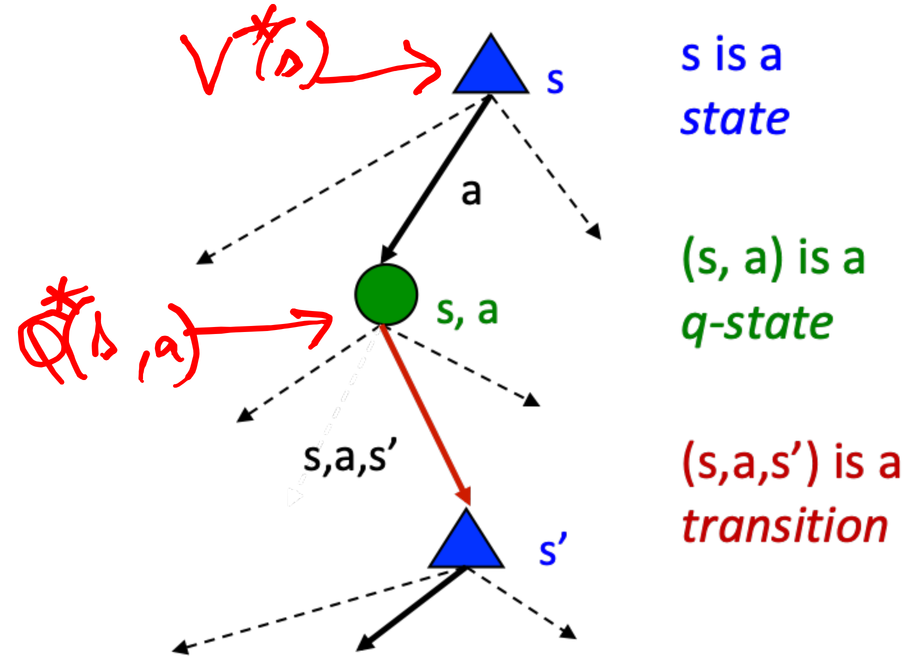


Optimal Quantities

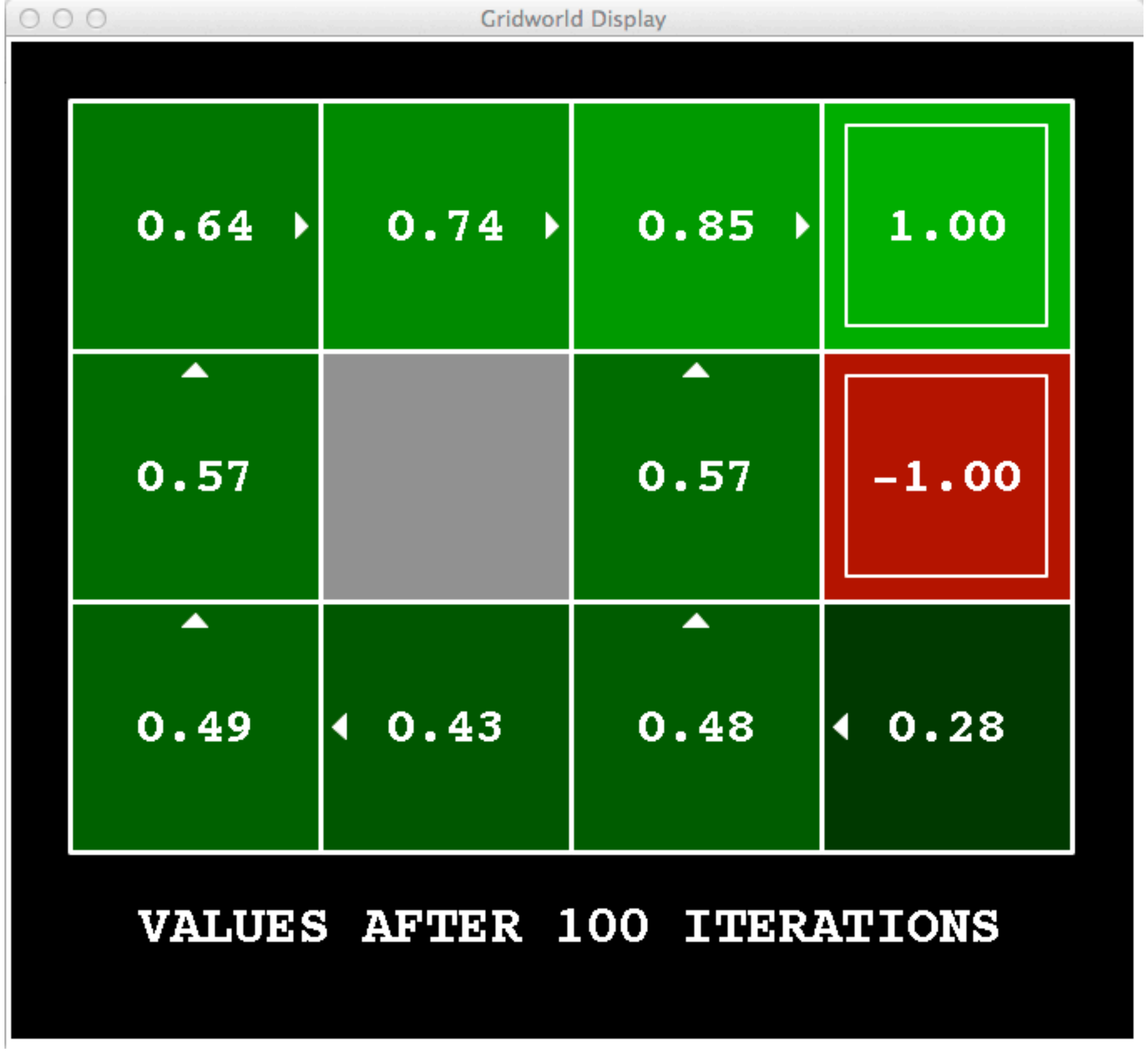
- The value (utility) of a state s:
  $V^*(s)$ = expected utility starting in s and acting optimally

- The value (utility) of a q-state (s,a):
  $Q^*(s,a)$ = expected utility starting out having taken action a from state s and (thereafter) acting optimally

- The optimal policy:
  $\pi^*(s)$ = optimal action from state s

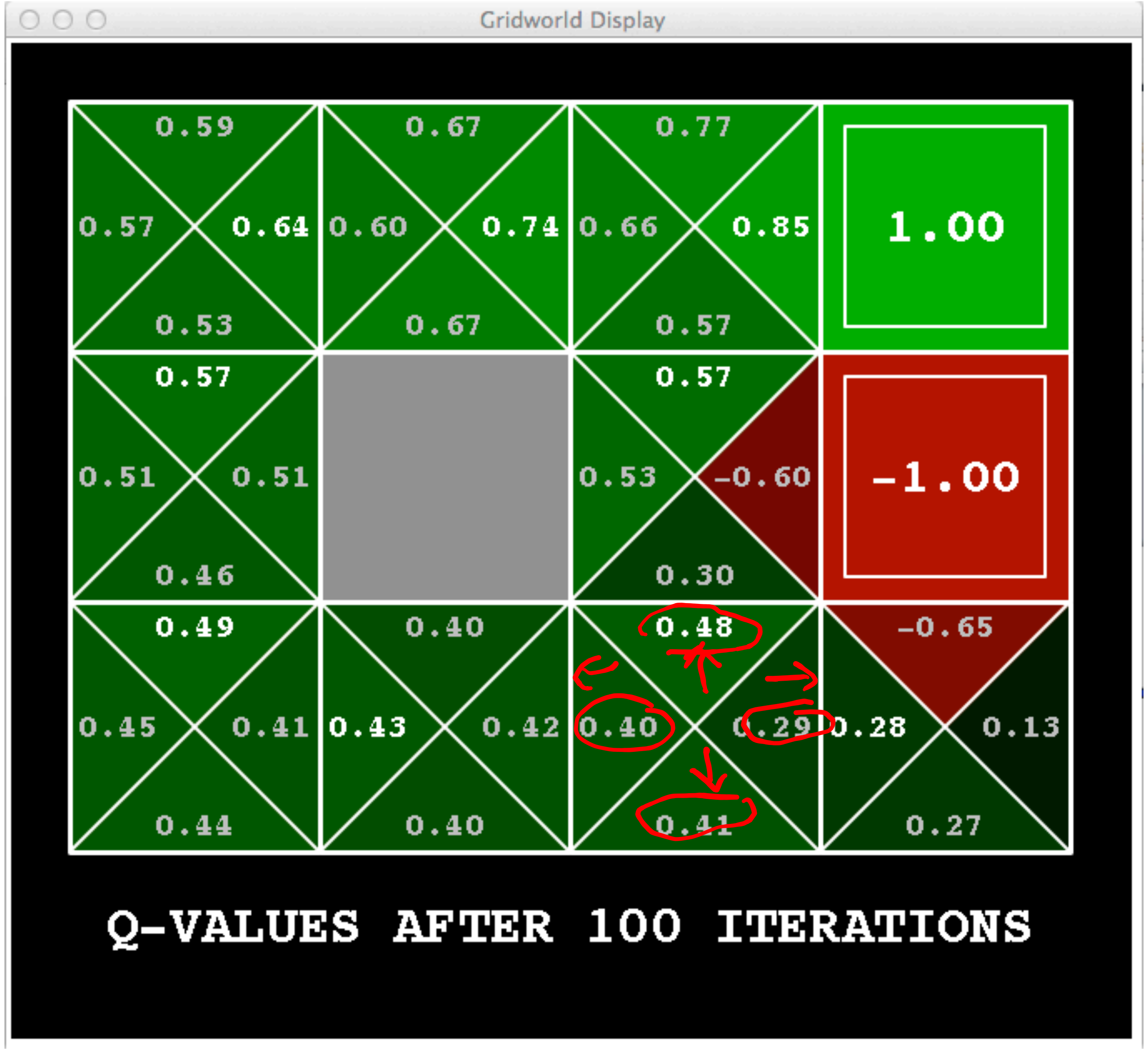


Snapshot of Demo – Gridworld V Values



Noise = 0.2
Discount = 0.9
Living reward = 0

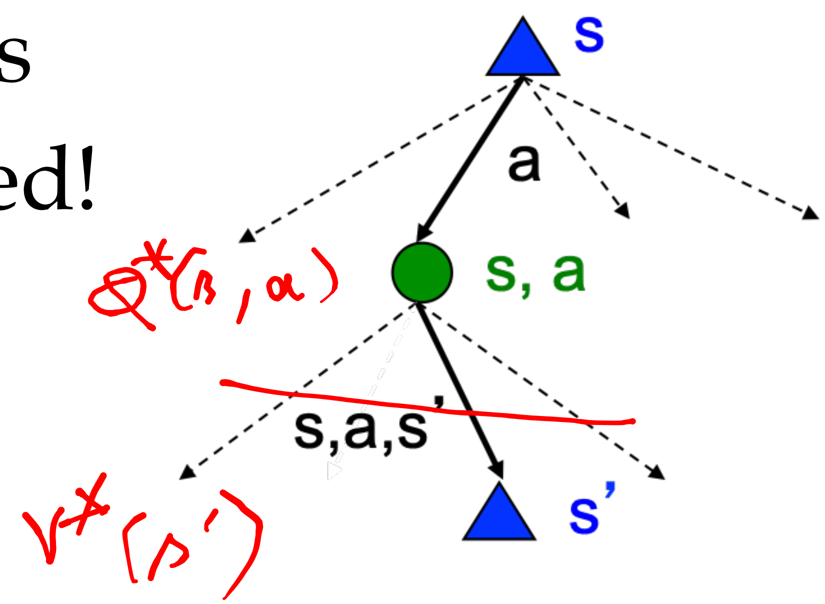
Snapshot of Demo – Gridworld Q Values



Noise = 0.2
Discount = 0.9
Living reward = 0

Values of States

- Fundamental operation: compute the (expectimax) value of a state
  - Expected utility under optimal action
  - Average sum of (discounted) rewards
  - This is just what expectimax computed!



- Recursive definition of value:

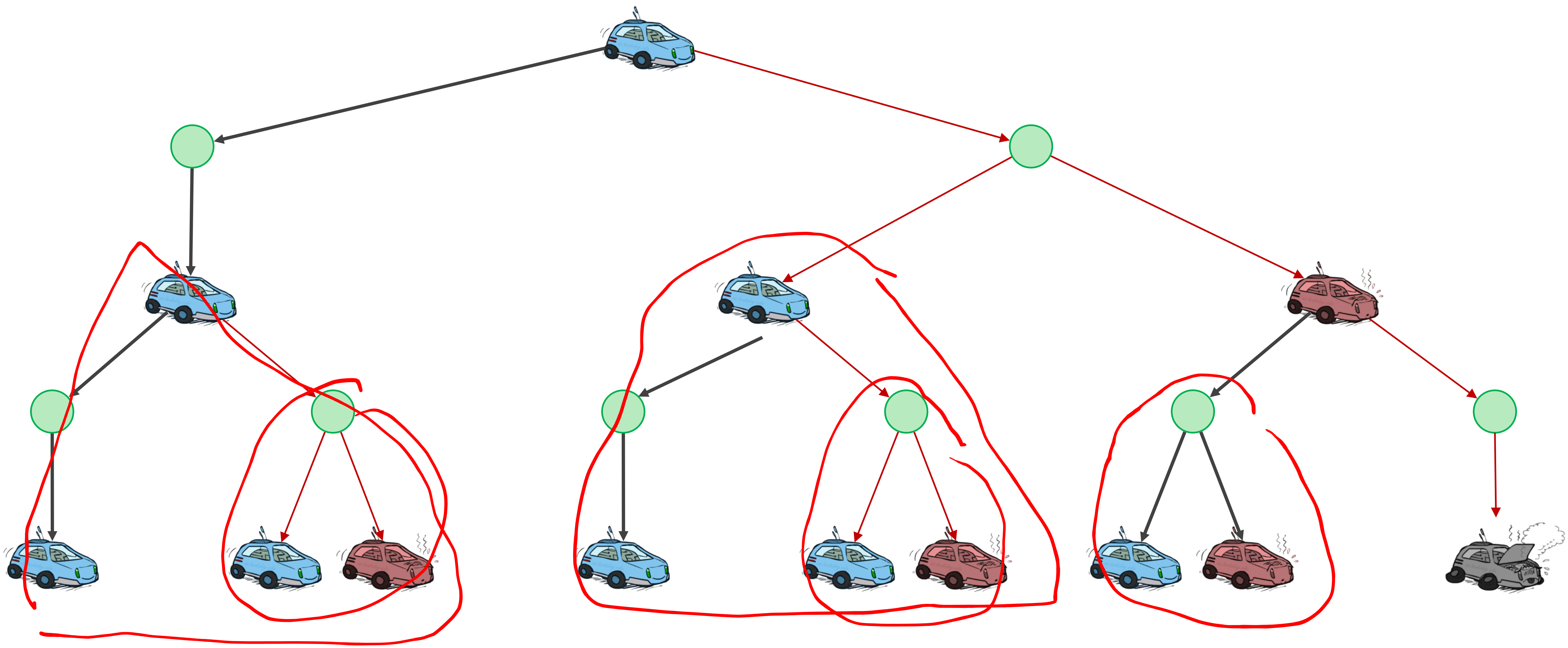
$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$

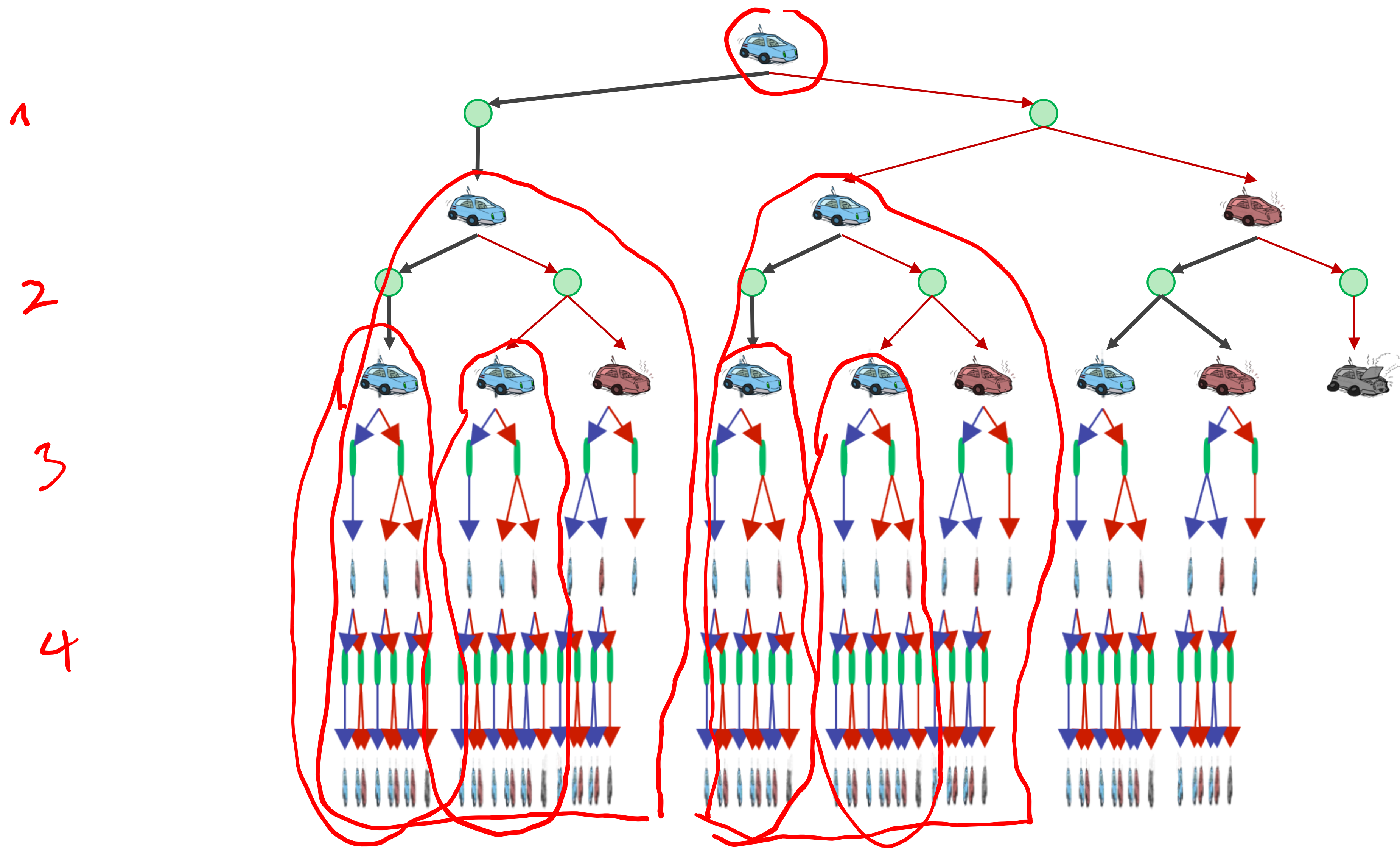
$$V^*(s) = \max_a \boxed{\sum_{s'} T(s,a,s') \left[R(s,a,s') \right.} \left. + \gamma V^*(s') \right]$$

$$= \bar{R}(s,a) + \gamma \sum_{s'} T(s,a,s') V^*(s')$$

Racing Search Tree

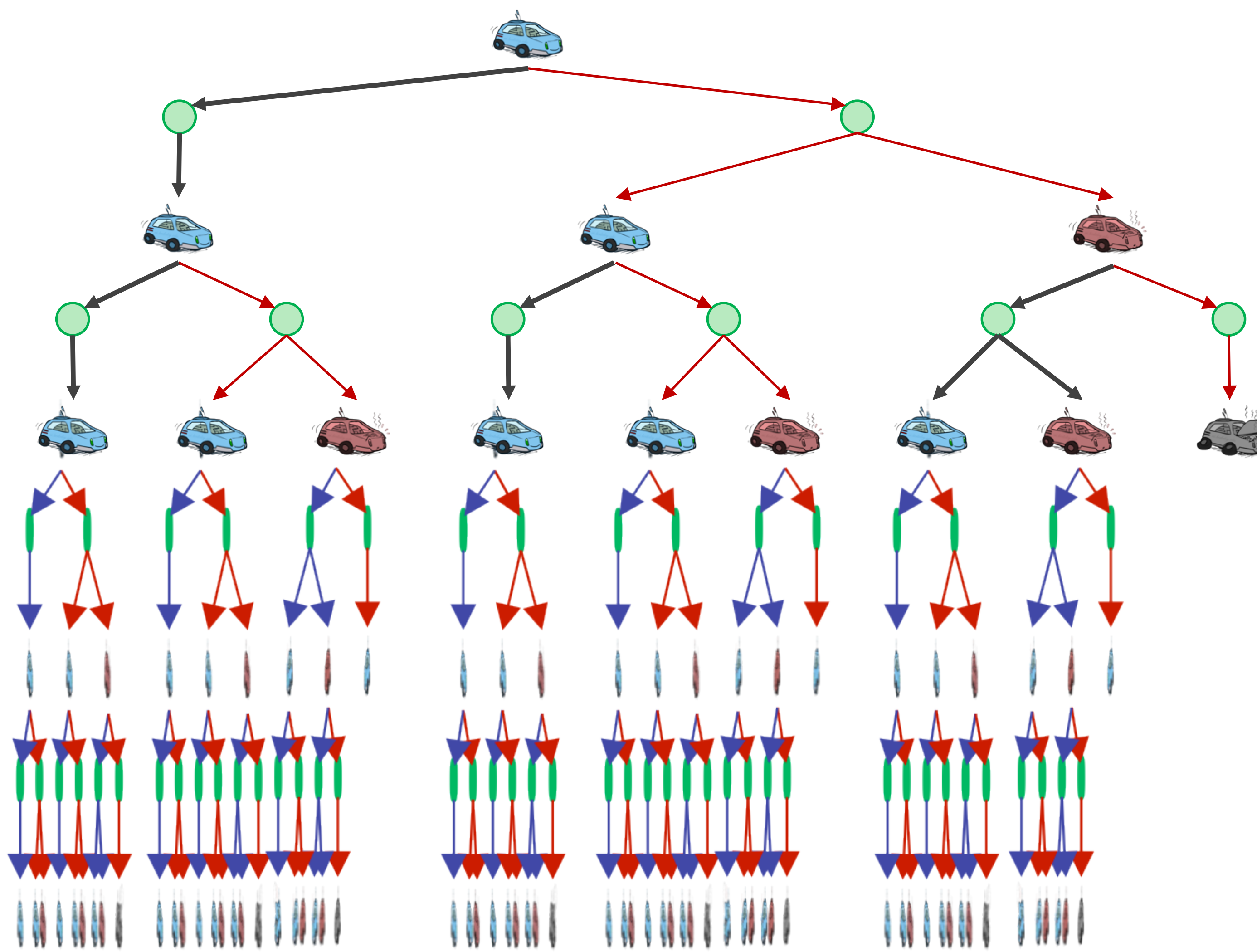


Racing Search Tree



Racing Search Tree

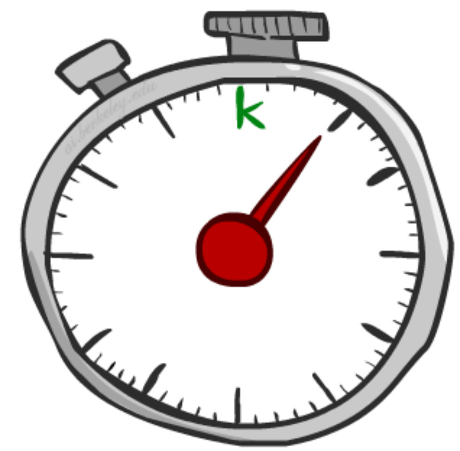
- We're doing way too much work with expectimax!
- Problem: States are repeated
  - Idea: Only compute needed quantities once
- Problem: Tree goes on forever
  - Idea: Do a depth-limited computation, but with increasing depths until change is small
  - Note: deep parts of the tree eventually don't matter if $\gamma < 1$

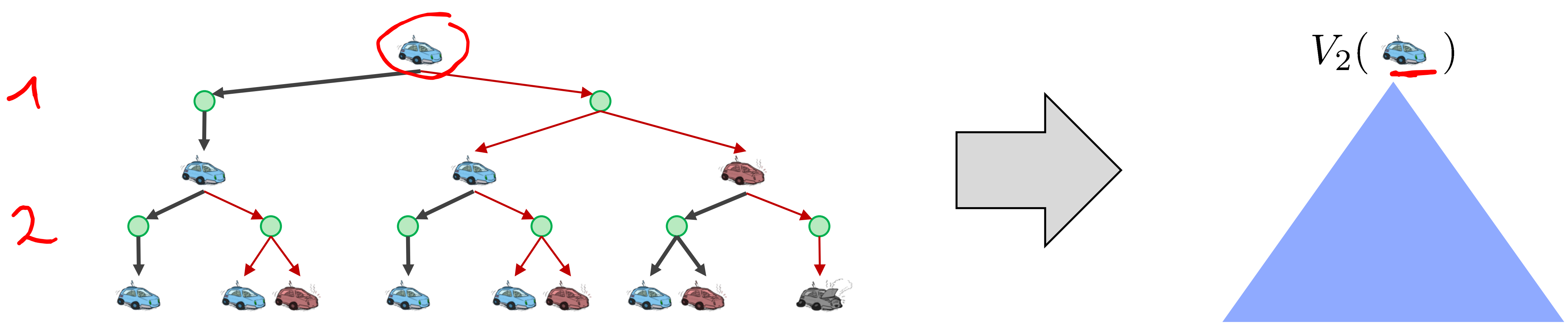


$\gamma^{k-1}($ $)$

Time-Limited Values

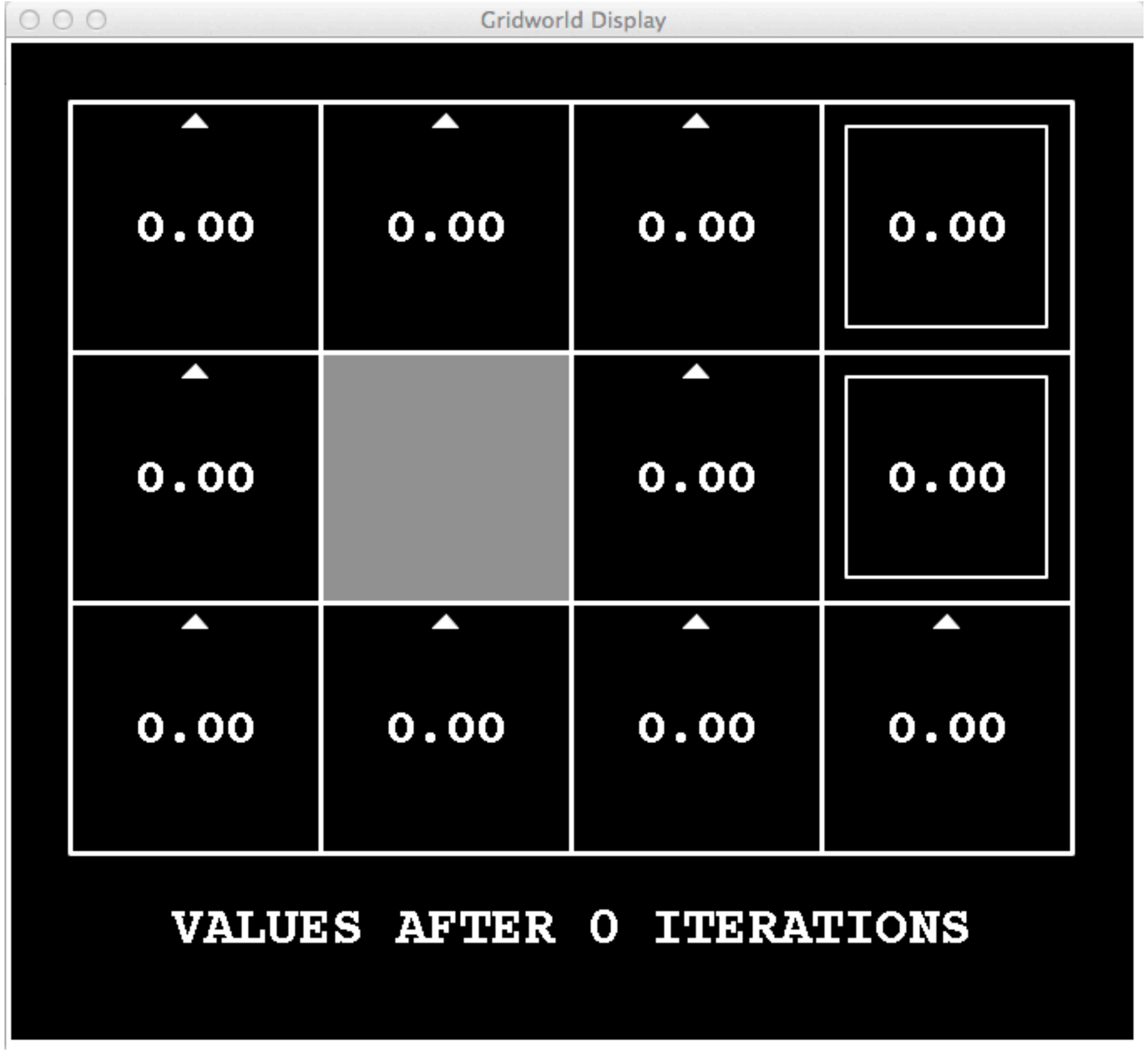
- Key idea: time-limited values
- Define $V_k(s)$ to be the optimal value of s if the game ends in k more time steps
  - Equivalently, it's what a depth-k expectimax would give from s





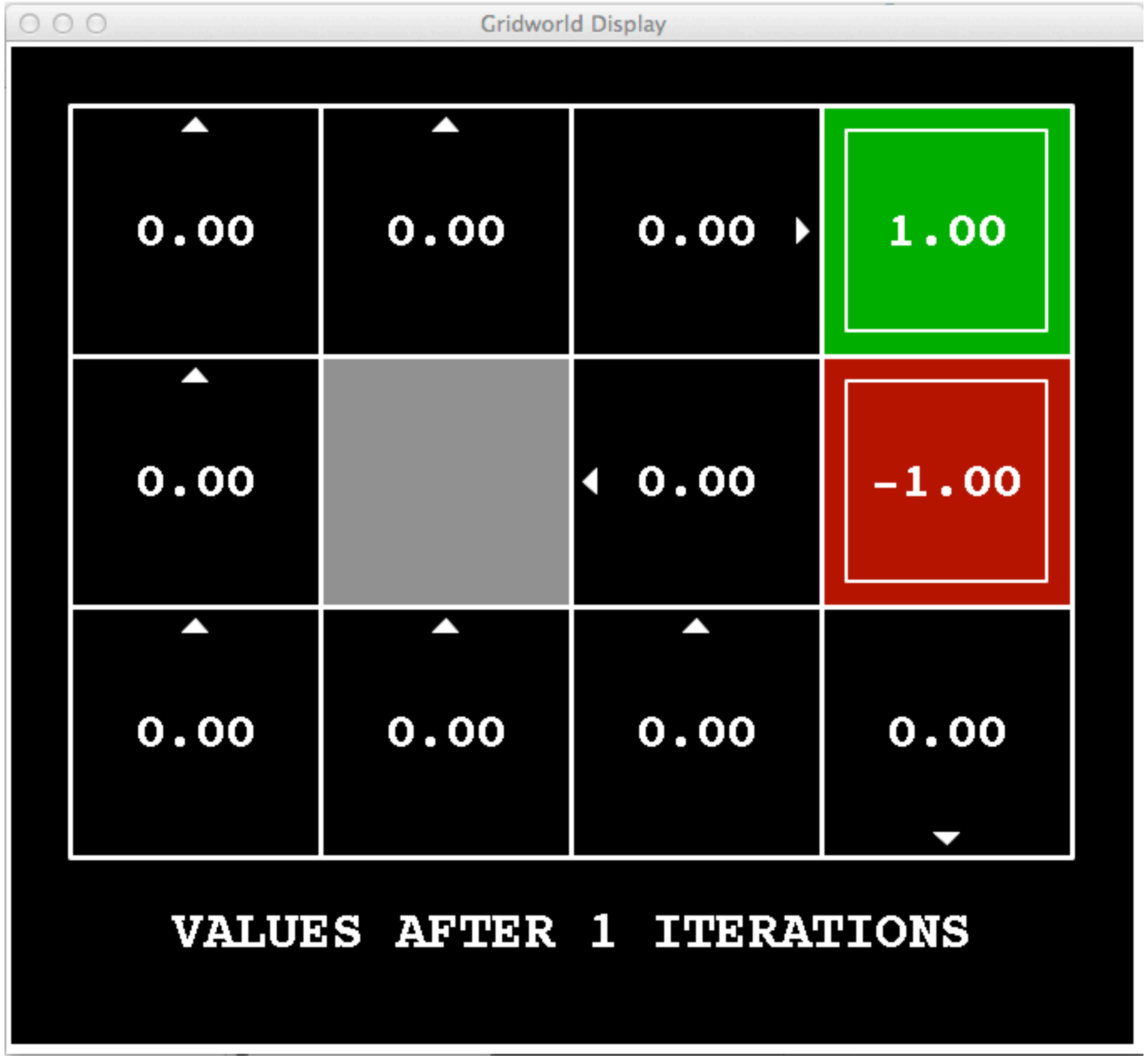
k=0

$V_k(s)$



Noise = 0.2
Discount = 0.9
Living reward = 0

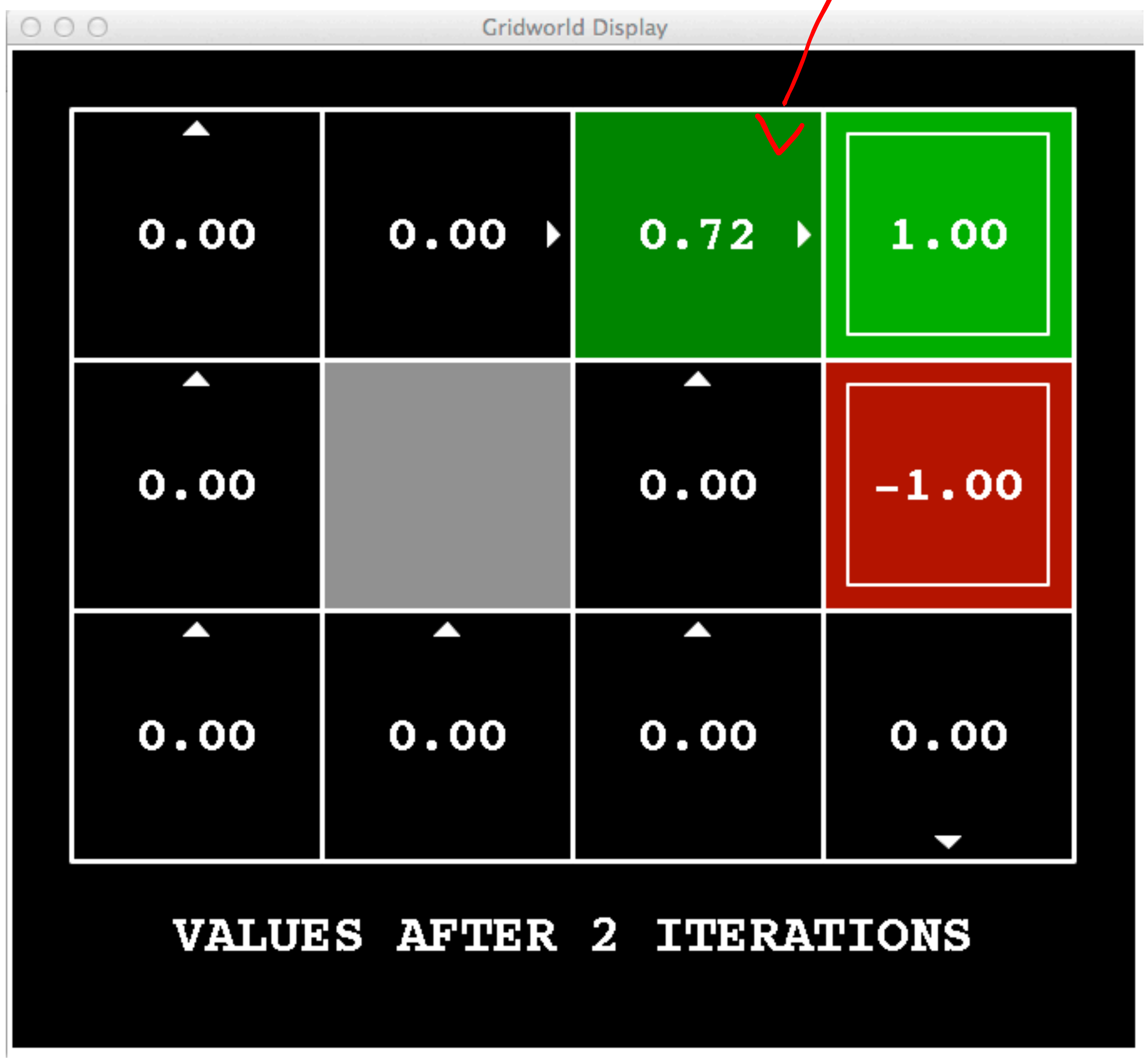
k=1



Noise = 0.2
Discount = 0.9
Living reward = 0

k=2

$0 + \overset{0.9}{\gamma} \times 0.8 \times 1$

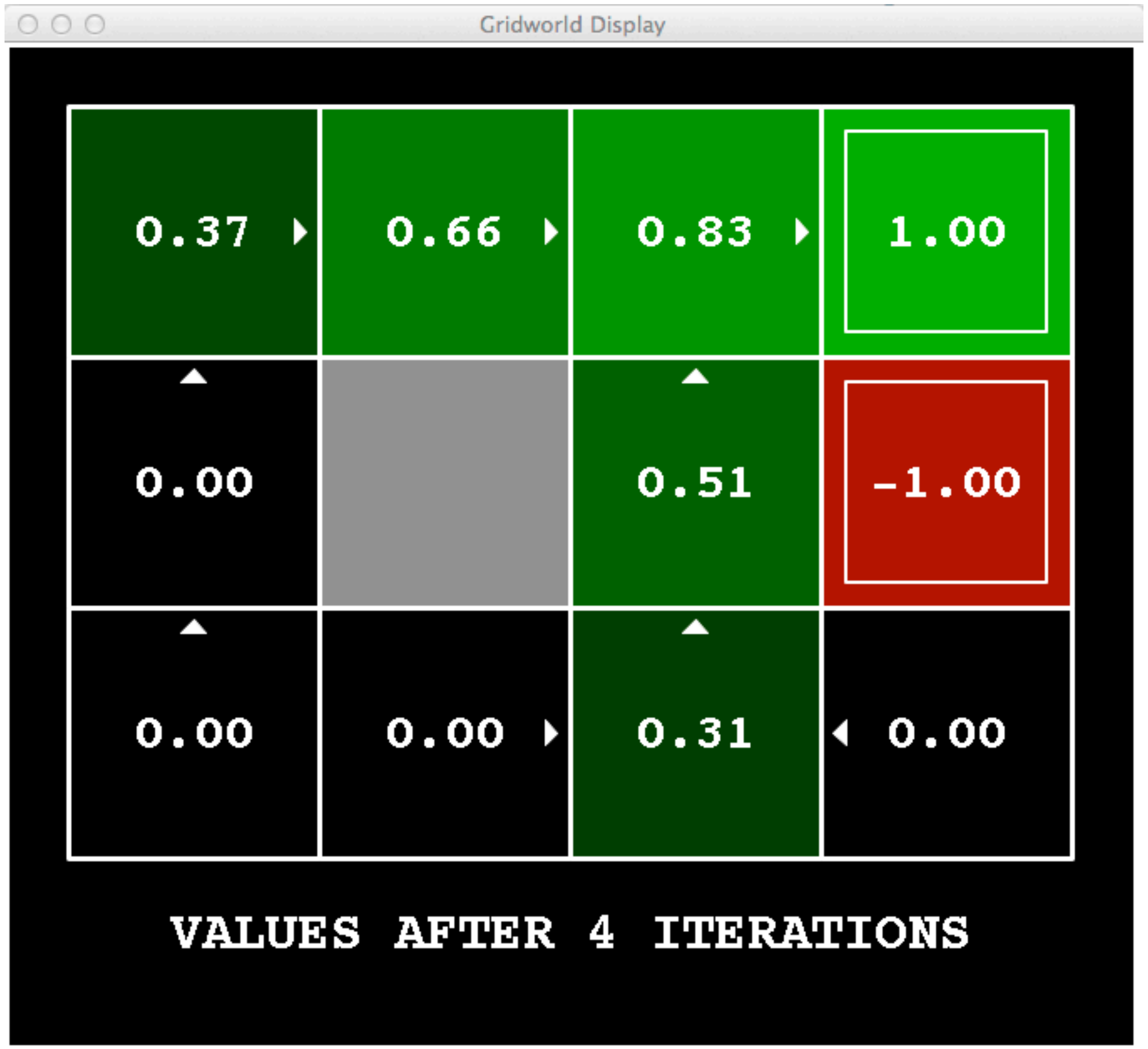


Noise = 0.2
Discount = 0.9
Living reward = 0

k=3

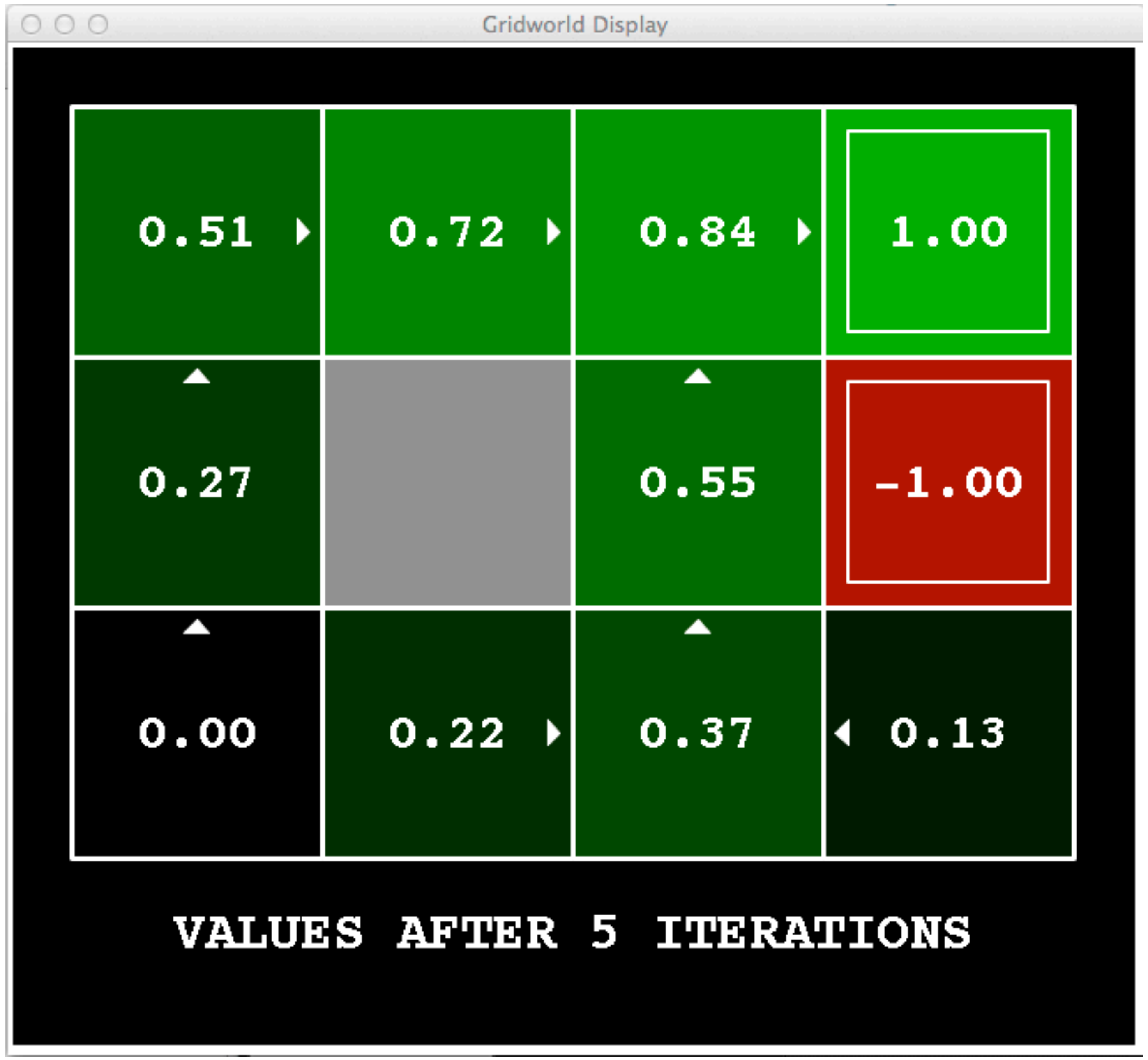
Noise = 0.2
Discount = 0.9
Living reward = 0

k=4



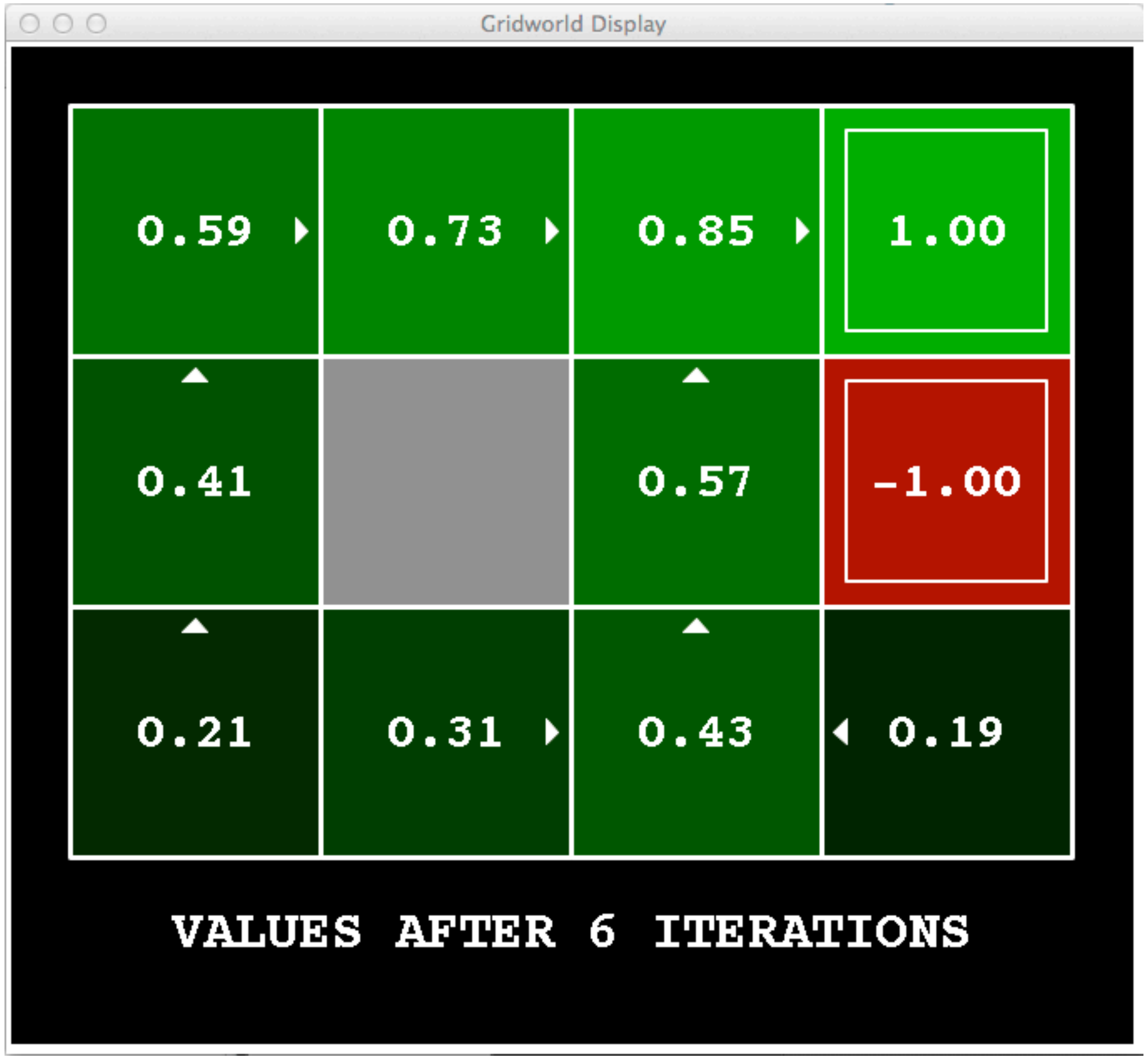
Noise = 0.2
Discount = 0.9
Living reward = 0

k=5



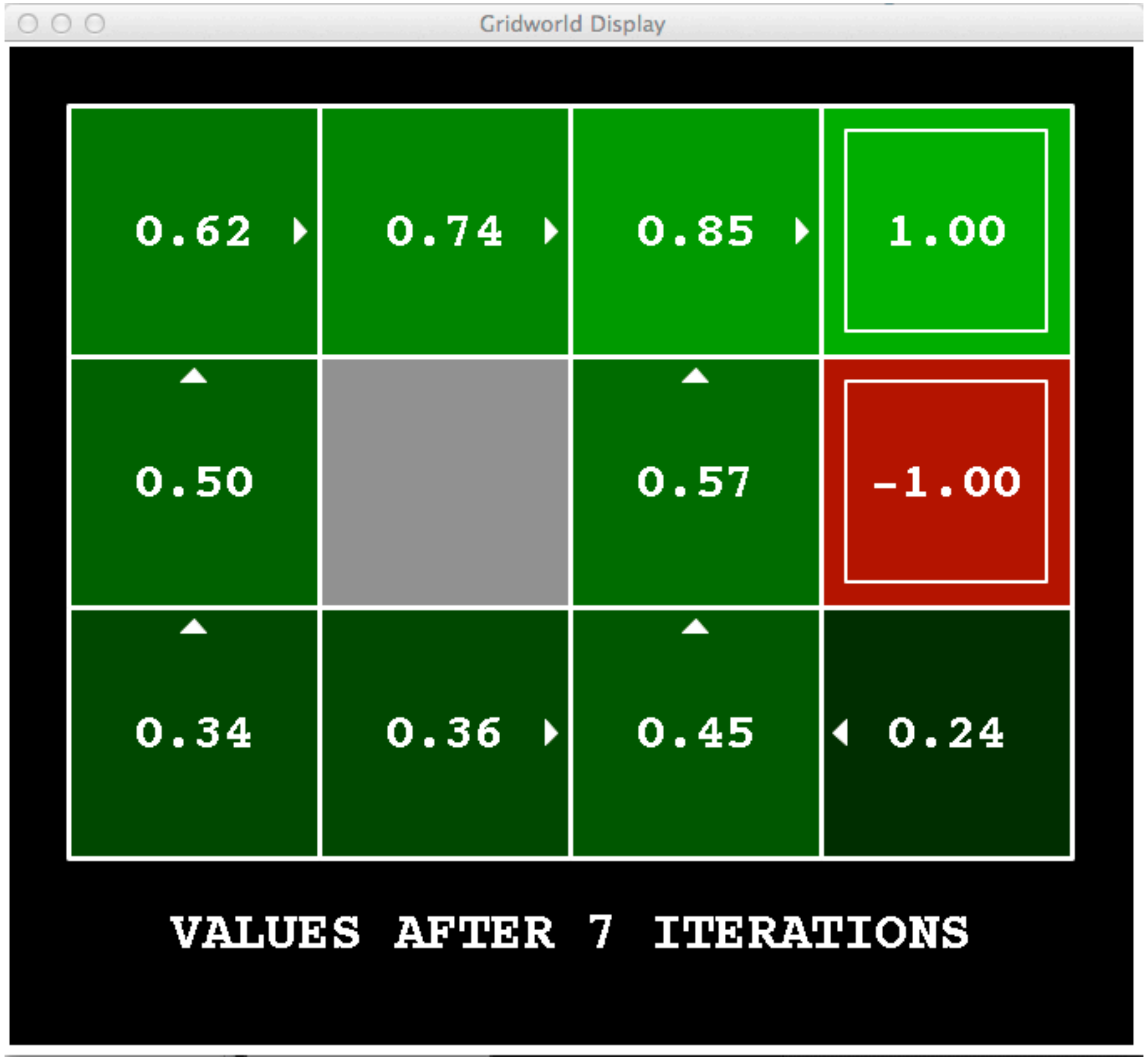
Noise = 0.2
Discount = 0.9
Living reward = 0

k=6



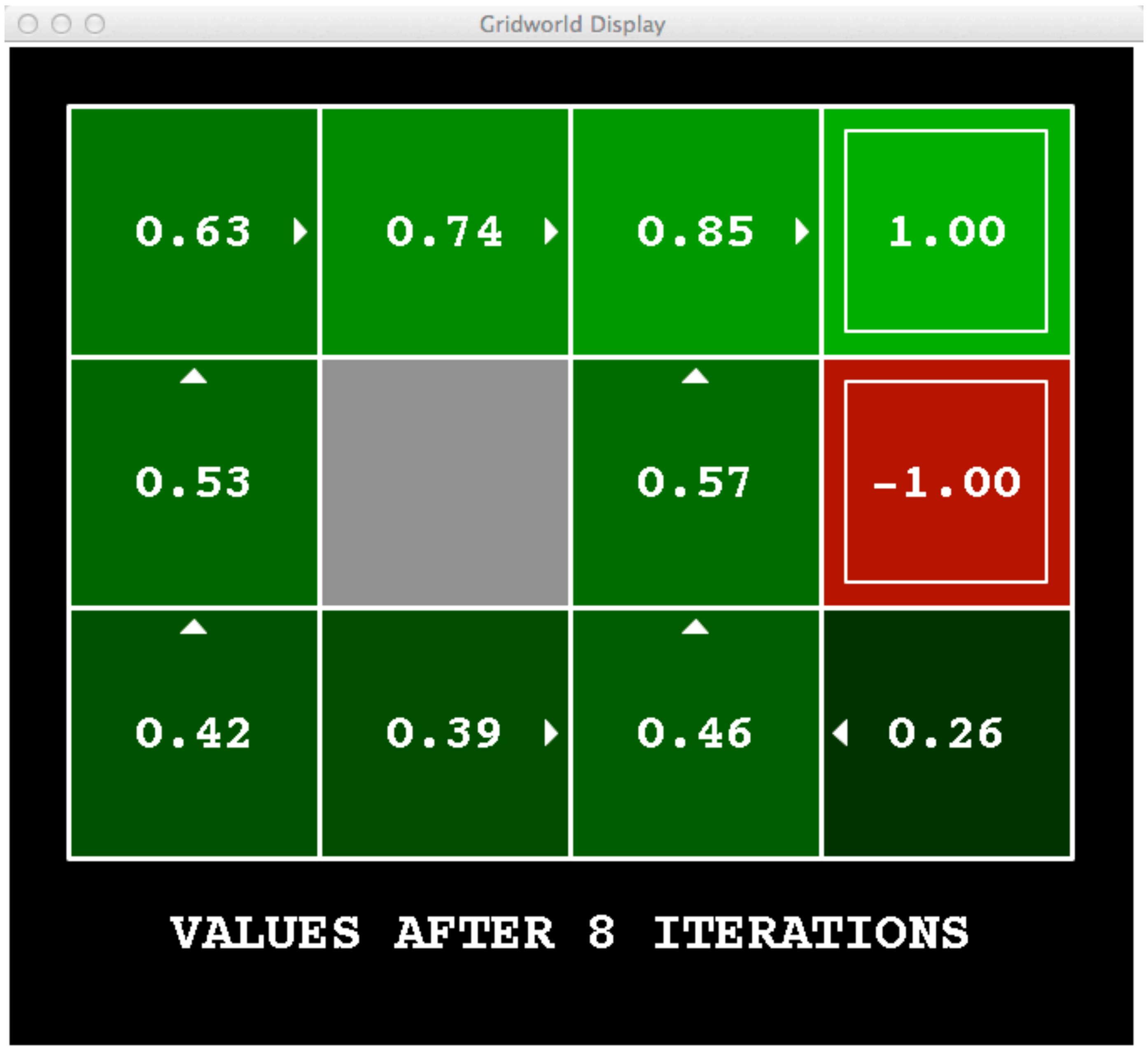
Noise = 0.2
Discount = 0.9
Living reward = 0

k=7



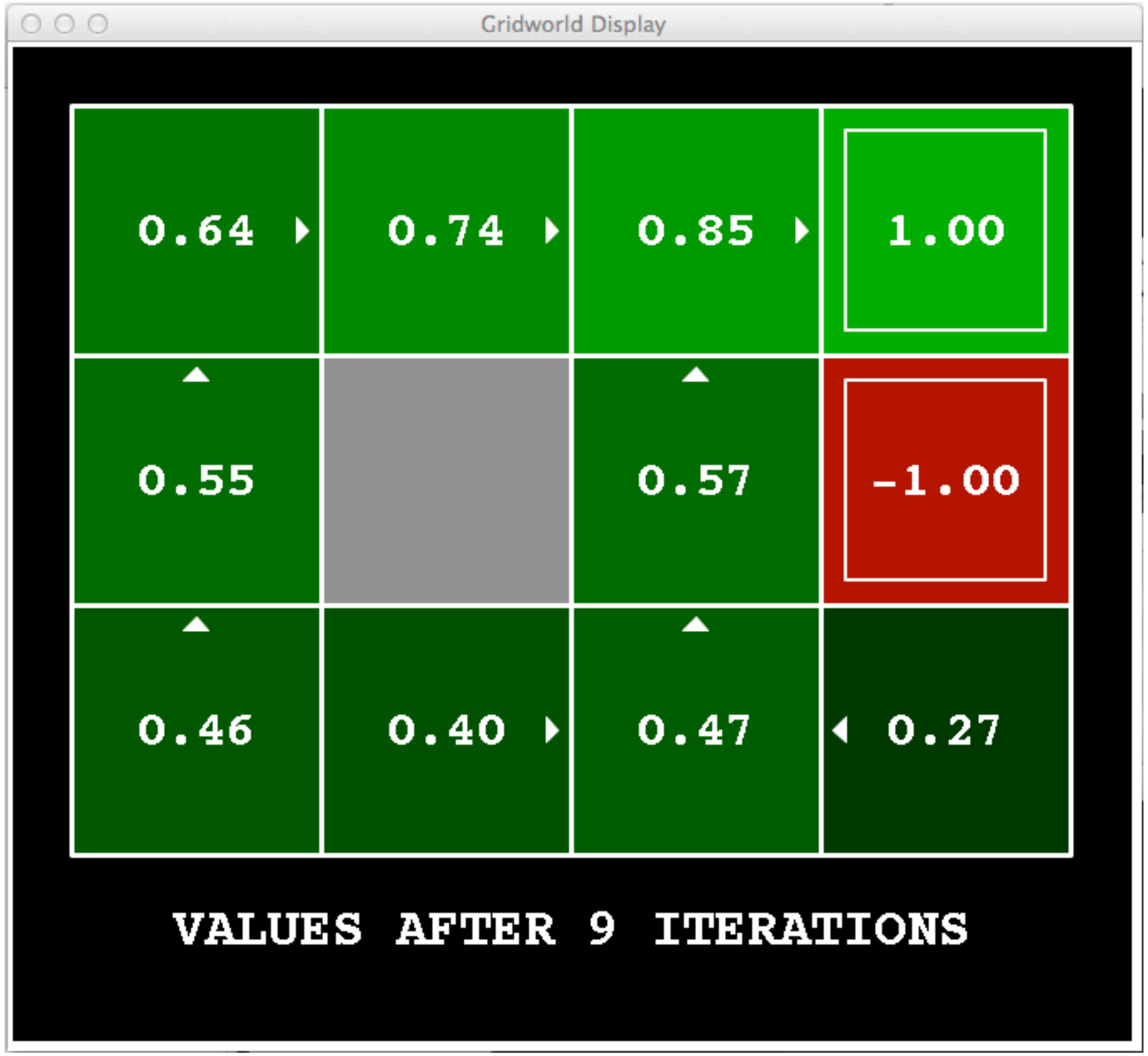
Noise = 0.2
Discount = 0.9
Living reward = 0

k=8



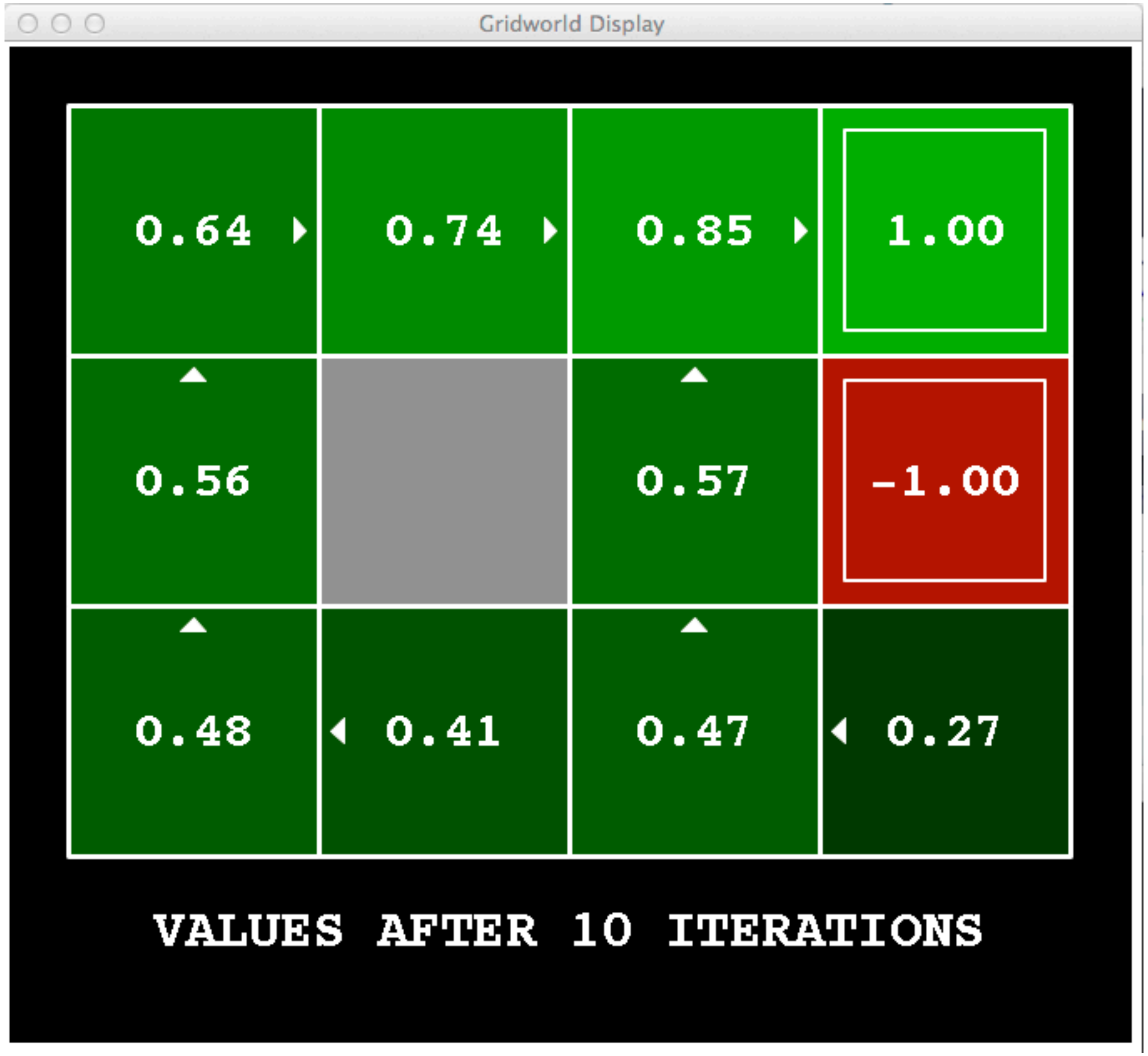
Noise = 0.2
Discount = 0.9
Living reward = 0

k=9



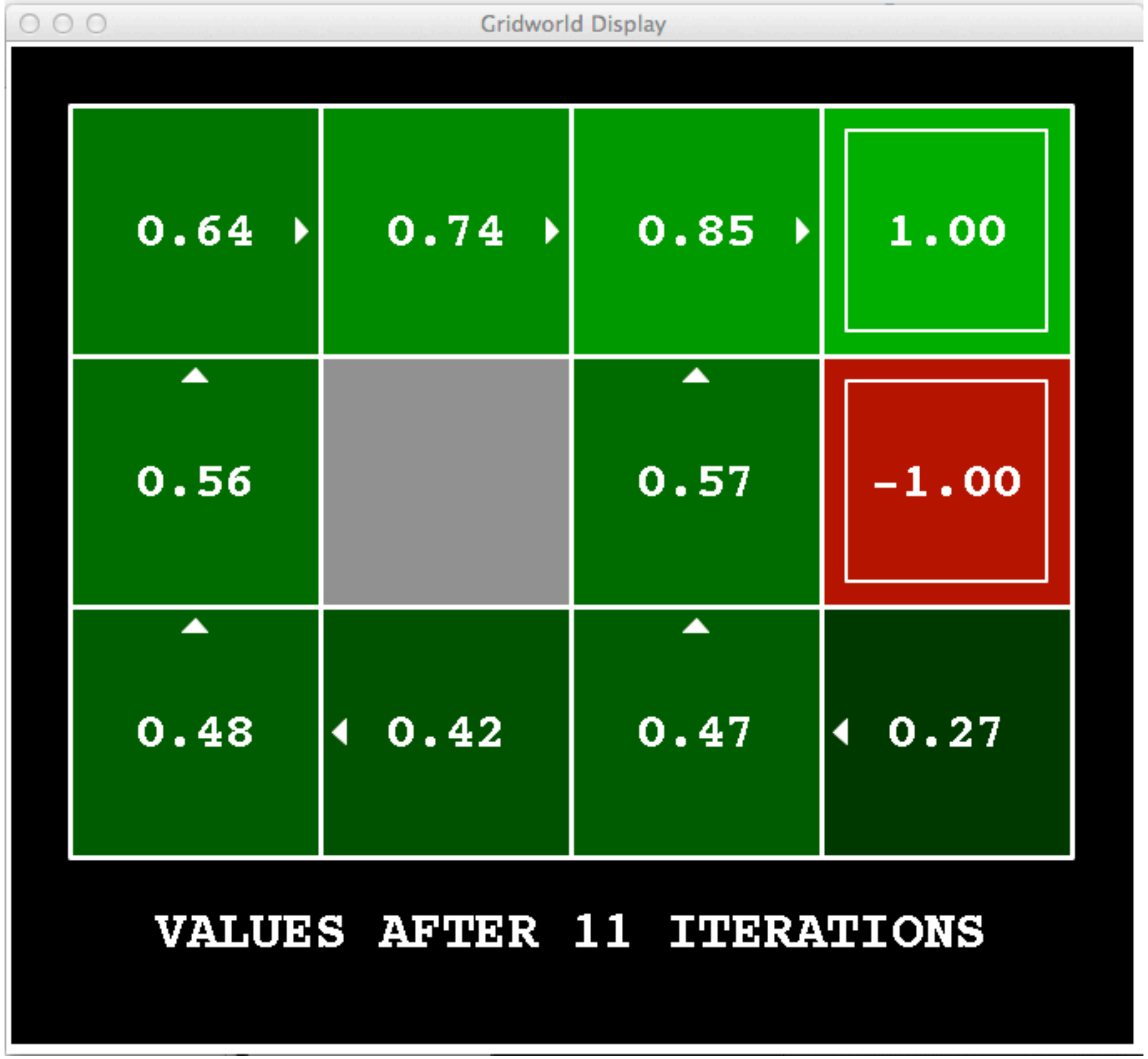
Noise = 0.2
Discount = 0.9
Living reward = 0

k=10



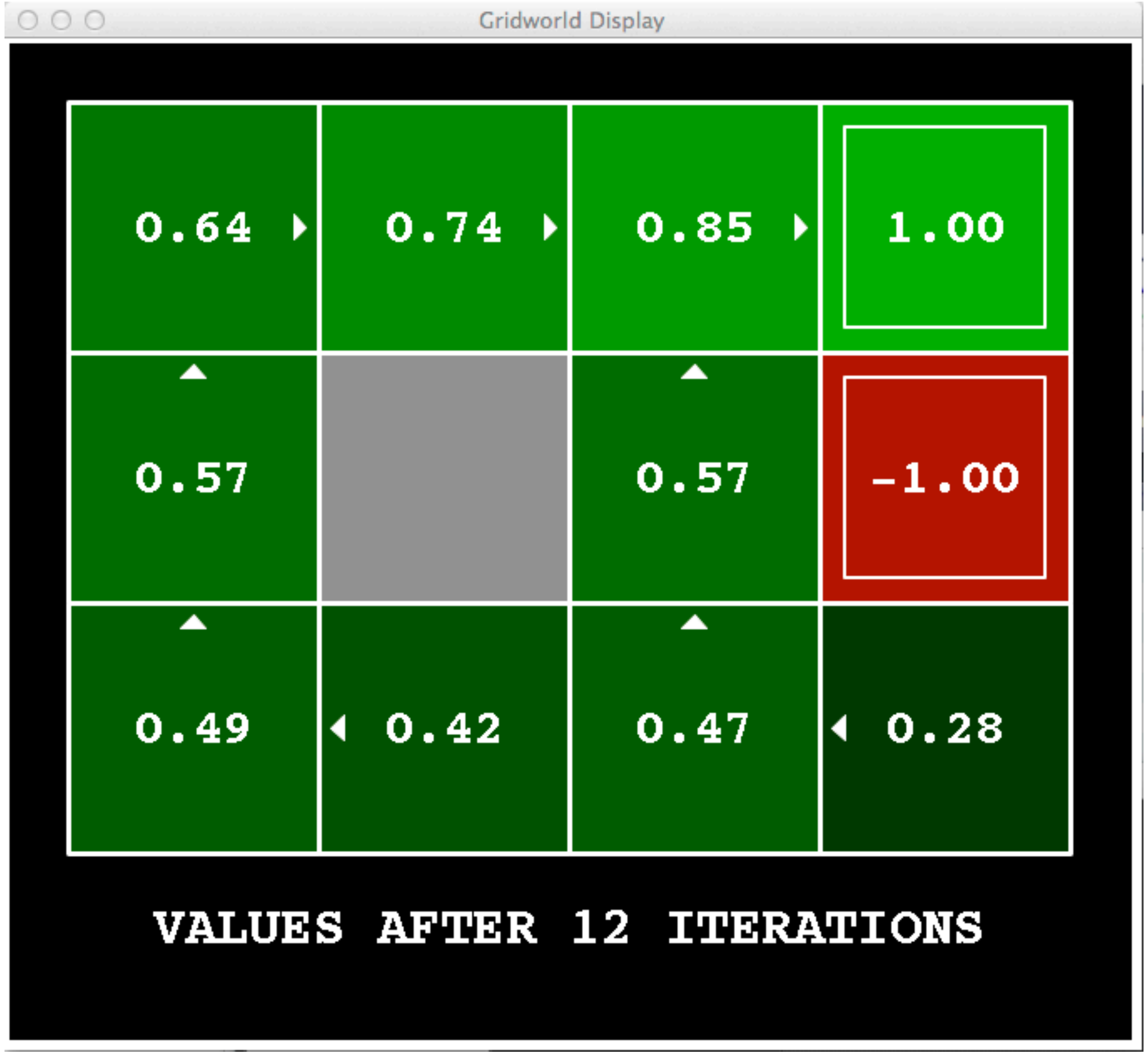
Noise = 0.2
Discount = 0.9
Living reward = 0

k=11



Noise = 0.2
Discount = 0.9
Living reward = 0

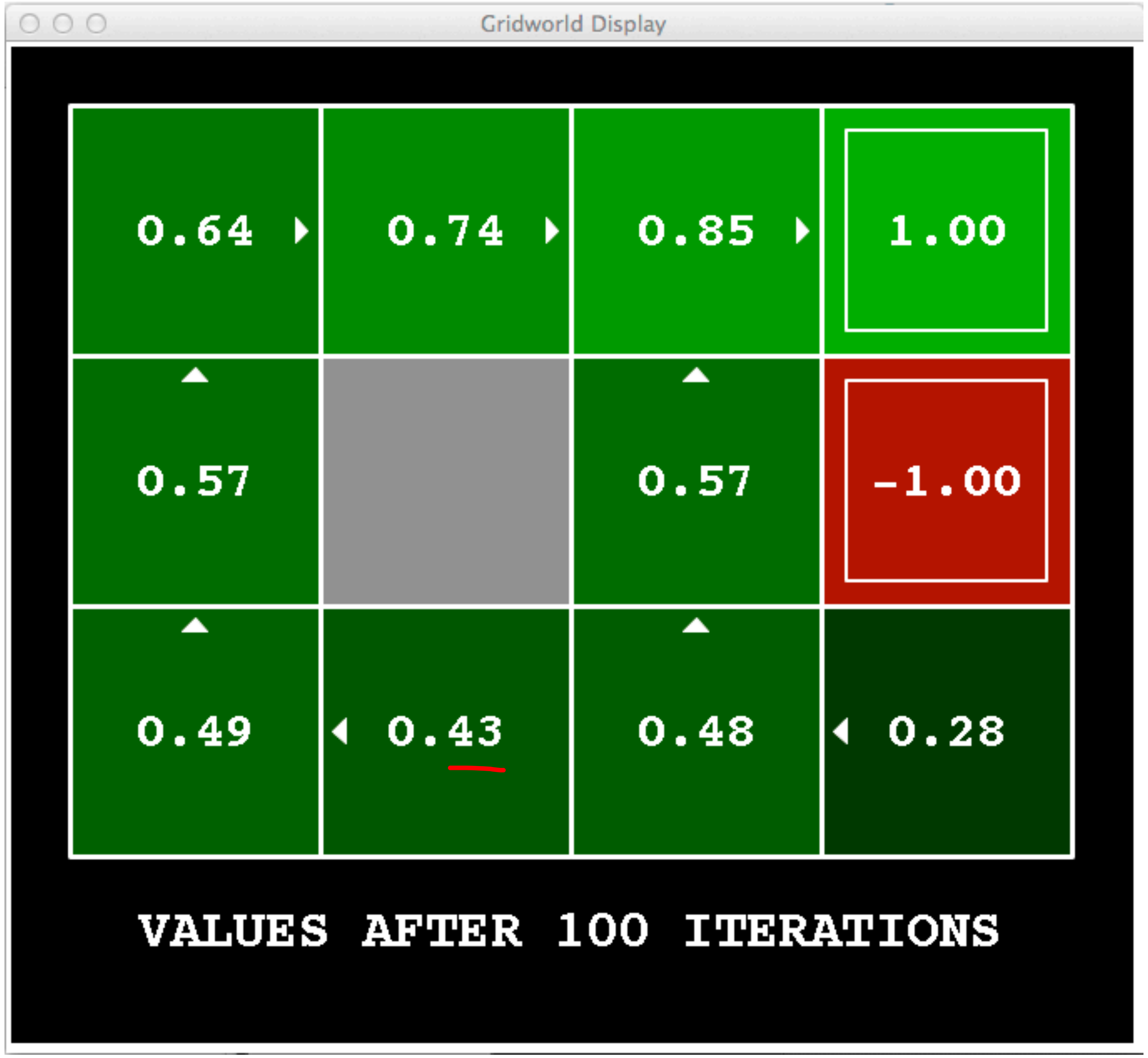
k=12



Noise = 0.2
Discount = 0.9
Living reward = 0

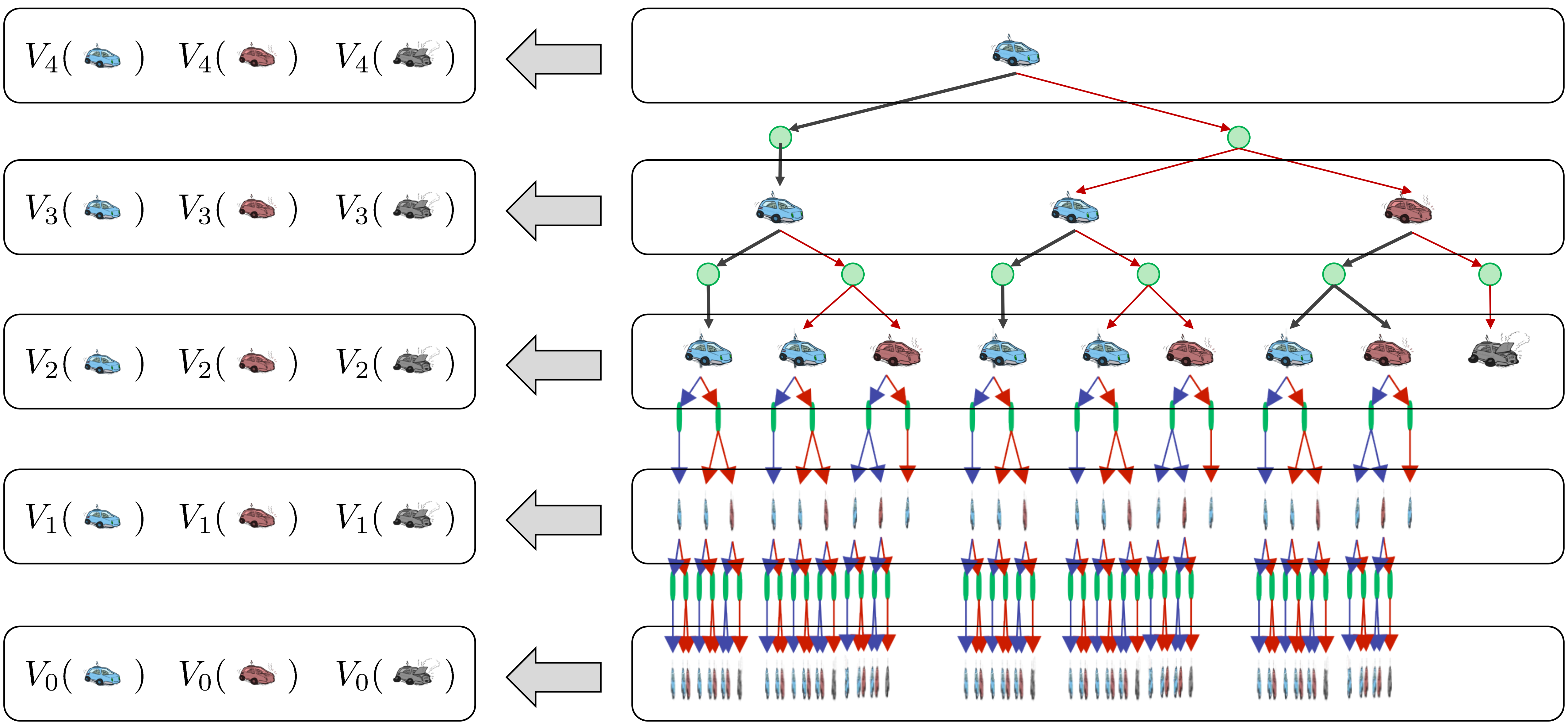
k=100

V^*

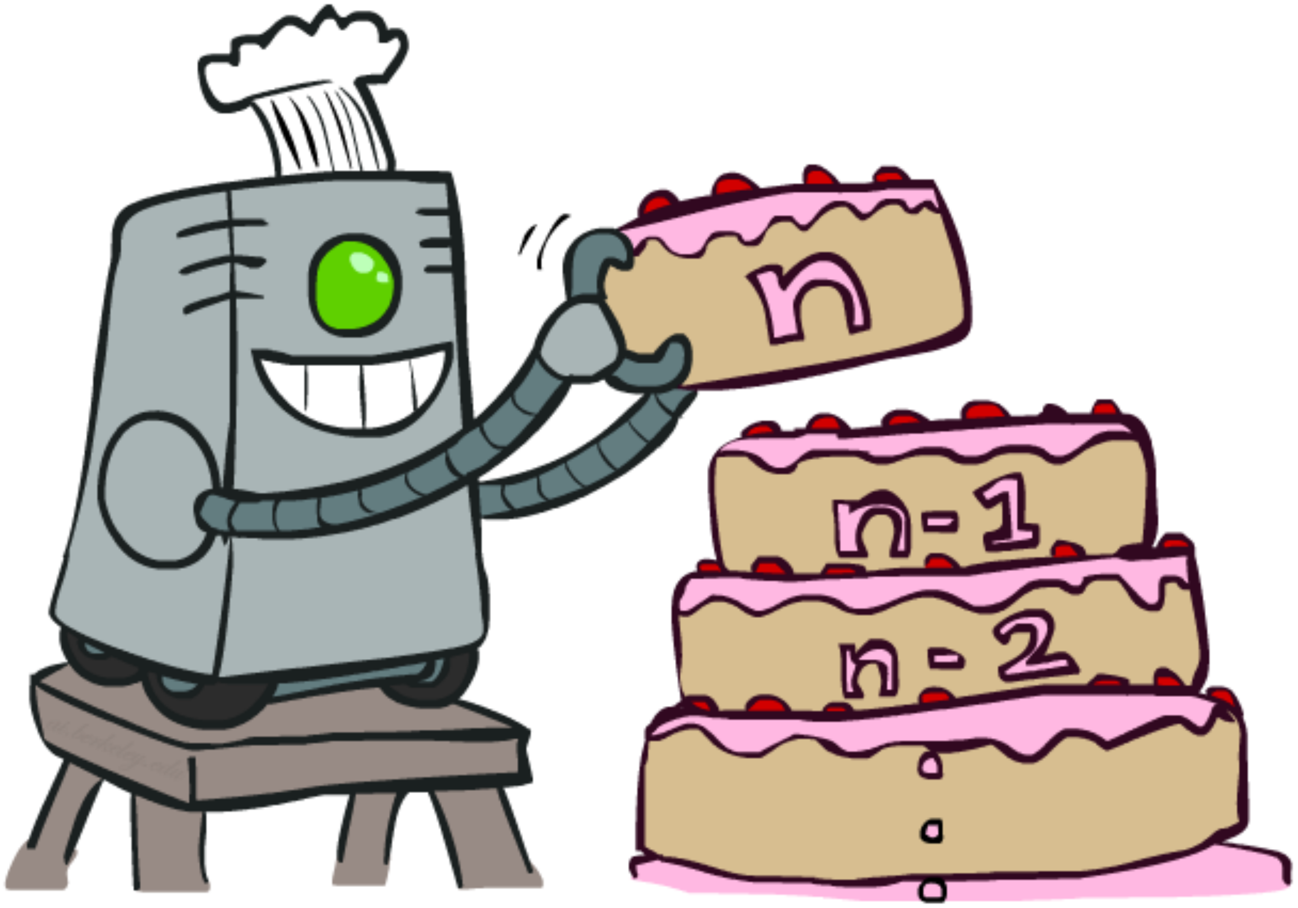


Noise = 0.2
Discount = 0.9
Living reward = 0

Computing Time-Limited Values



Value Iteration

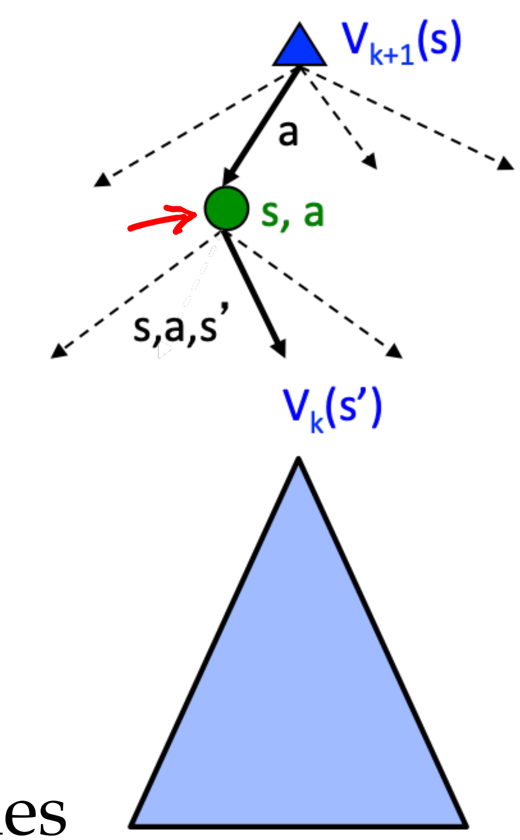


Value Iteration

- Start with $V_0(s) = 0$: no time steps left means an expected reward sum of zero

- Given vector of $V_k(s)$ values, do one ply of expectimax from each state:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$



- Repeat until convergence

max iterations.

$\|V_{k+1} - V_k\|_\infty < \epsilon$

$|S| \times |A| \times |S|$

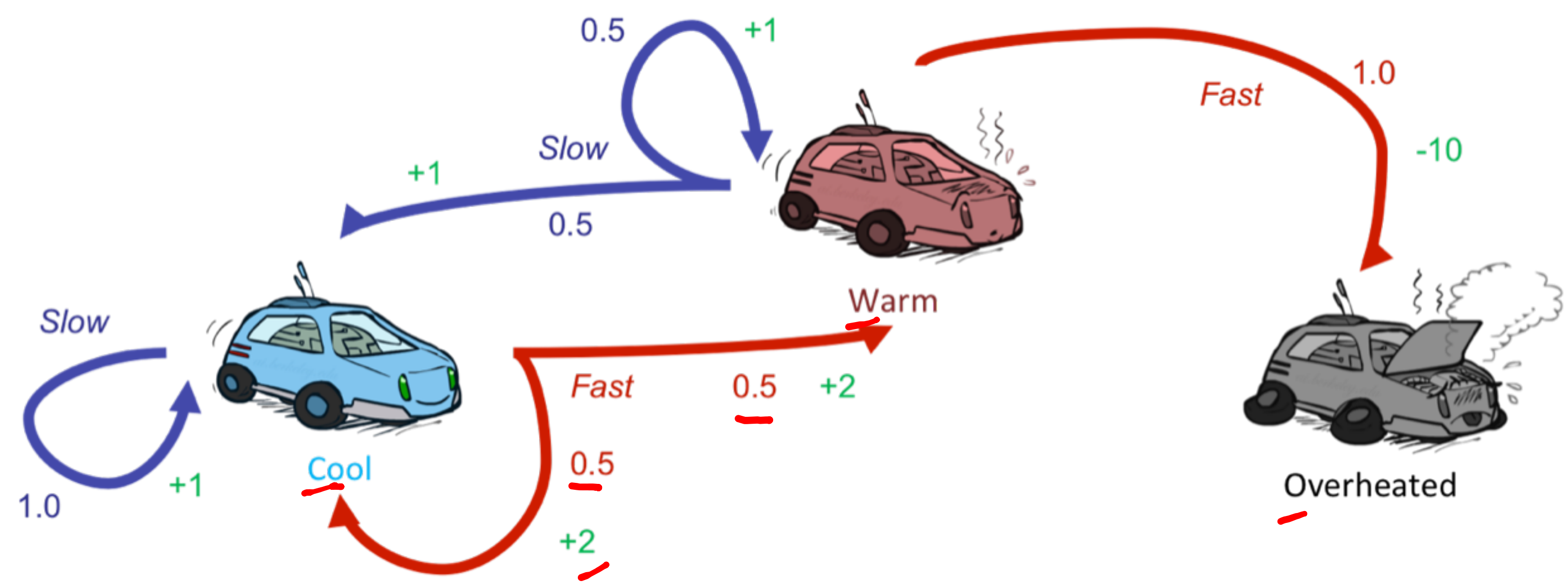
- Complexity of each iteration: $O(S^2A)$

$\gamma < 1$

- Theorem: will converge to unique optimal values
  - Basic idea: approximations get refined towards optimal values
  - Policy may converge long before values do

Example: Value Iteration

V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0



Assume no discount! $\gamma = 1$

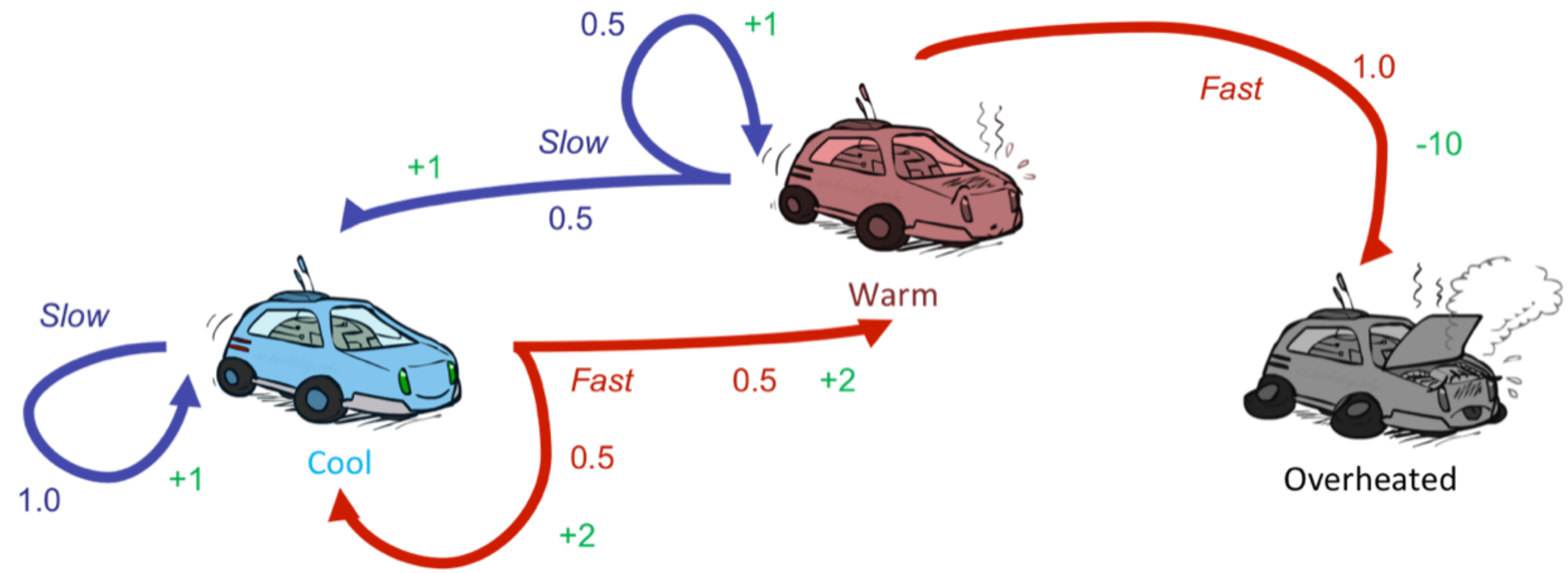
$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$

$$V_1(c) = \max\left(1 + V_0(c),\ \frac{1}{2}\left(2 + V_0(c)\right) + \frac{1}{2}\left(2 + V_0(w)\right)\right) = 2$$

$$3 \qquad \frac{4}{2} + \frac{3}{2} = \frac{7}{2}$$

Quiz: Value Iteration

V_2	3.5	2.5	0
V_1	2	1	0
V_0	0	0	0



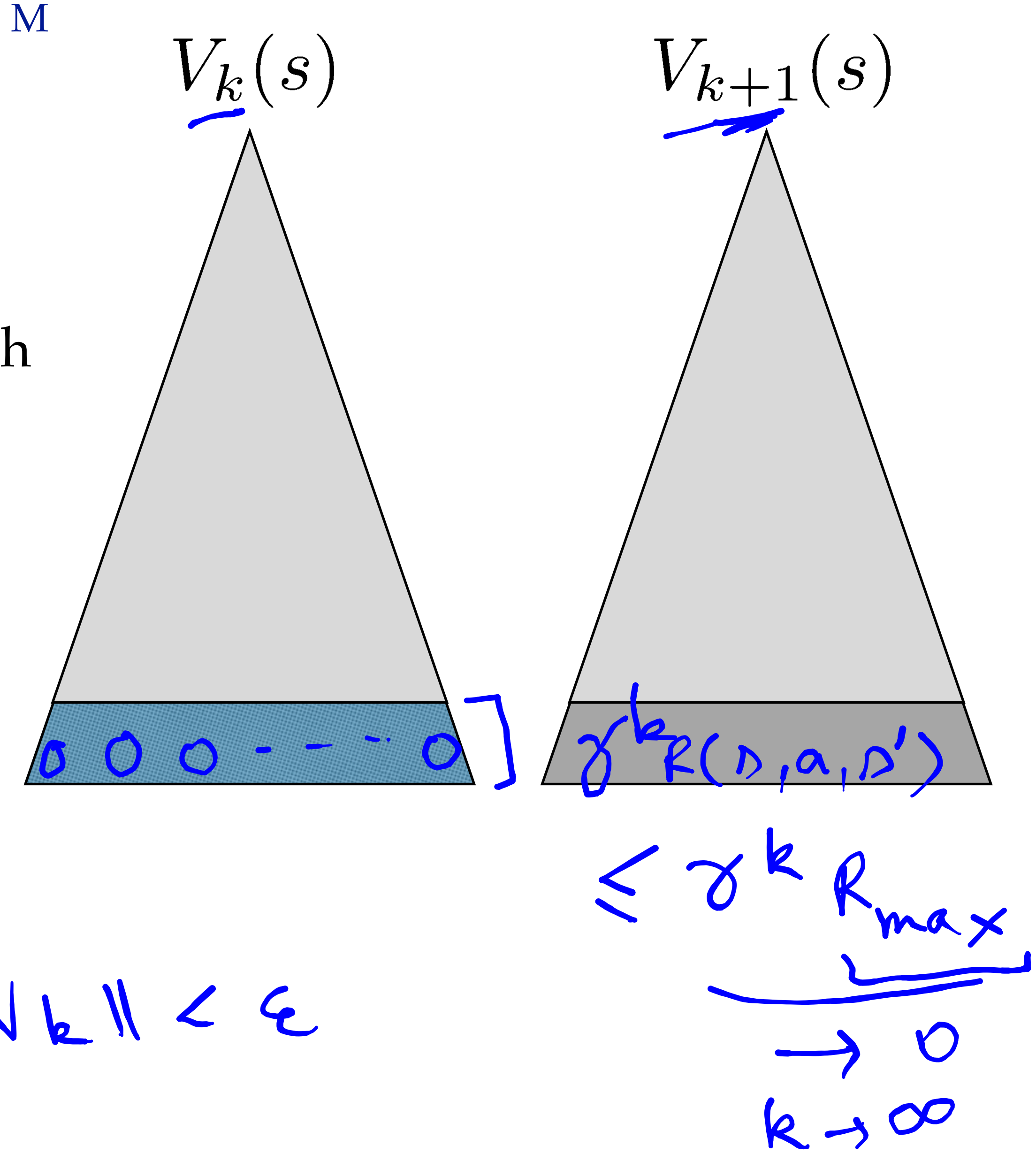
Assume no discount!

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$

What are the values for V_3? (5 4 0)

Convergence

- How do we know the V_k vectors are going to converge?
- Case 1: If the tree has maximum depth M, then V_M holds the actual untruncated values
- Case 2: If the discount is less than 1
  - Sketch: For any state V_k and V_{k+1} can be viewed as depth k+1 expectimax results in nearly identical search trees
  - The difference is that on the bottom layer, V_{k+1} has actual rewards while V_k has zeros
  - That last layer is at best all R_{MAX}
  - It is at worst R_{MIN}
  - But everything is discounted by γ^k that far out
  - So V_k and V_{k+1} are at most $\gamma^k \max|R|$ different
  - So as k increases, the values converge



$\|V_{k+1} - V_k\| < \epsilon$