Applied Physics Letters

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Cite as: Appl. Phys. Lett. **98**, 051502 (2011); https://doi.org/10.1063/1.3549868
Submitted: 19 October 2010 • Accepted: 12 January 2011 • Published Online: 31 January 2011

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# Child–Langmuir law in the Coulomb blockade regime

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(Received 19 October 2010; accepted 12 January 2011; published online 31 January 2011)

The one-dimensional (1D) classical Child–Langmuir (CL) law has been extended to the Coulomb blockade (single to few electrons) regime, including the effect of single-electron charging. It is found that there is a threshold of voltage ($V_{\mathrm{th}}$) equals to one-half of the single-electron charging energy for electron injection assuming zero barrier at the interface. For voltage in the range of $1 < V/V_{\mathrm{th}} < 2$, there is only one electron inside the gap, and the time-averaged single-electron injected current is equal or higher than the 1D CL current. © *2011 American Institute of Physics*. [doi:10.1063/1.3549868]

Space-charge-limited (SCL) electron flow describes the maximum current density allowed for steady-state electron beam transport across a planar diode of spacing $D$ and applied voltage $V_g$. It has been an area of active research in the development of non-neutral plasma physics, high current diodes, high power microwave sources, vacuum microelectronics, and sheath physics in plasma processing. In the one-dimensional (1D) classical regime, it is known as the Child–Langmuir (CL) law[1,2] given by

$$J_{\mathrm{CL}} = \frac{4\epsilon_o}{9}\sqrt{\frac{2e}{m}}\frac{V_g^{3/2}}{D^2}, \tag{1}$$

where $e$ and $m$ are the charge and mass of the electron respectively, and $\epsilon_o$ is the permittivity of free space. Extensive studies had been done to extend the 1D classical CL law to multidimensional models,[3–7] quantum regime,[8–11] and ultrafast time scale.[12–14]

Various studies were conducted in using ultrafast lasers to excite localized photofield electron emission from metallic nanotips,[15–19] where a finite number of emitted electrons per pulse can be obtained.[19] In this paper, we are interested to study if the space charge effects of the electrons can be ignored when the electrons are emitted or injected into a gap at the limit of single-electron per injection, and thus provide a single-electron based model for the 1D classical CL law. In this case, the effects of single-electron charging or Coulomb blockade must be included. Our approach is based on a time of flight (TOF) model[20] formulated before, to study the SCL ion flow in solid.[21] Such TOF models have also been used intensively to measure the mobility of charge carriers in solids.[22,23]

In the model, we consider a capacitor with two large parallel plates separated at a distance $D$ with a constant applied voltage $V_g$. At time $t=0$, there are $N$ number of electrons located at the cathode ($x=0$), readily to be injected into the gap traveling toward the anode ($x=D$), and the order of the injection is labeled by $n=1,2,\ldots,N$. Here, we have assumed unlimited supply of electrons from the cathode, so $N$ is an input parameter, which is independent of the gap voltage and material. At low voltage closed to a threshold voltage (see below), we will able to reach the 1D CL law even with small $N<10$. The launch time of each $n$-th electron is determined by the electric field acting on the $n$-th electron calculated consistently by using Eq. (4) below. The electron will only be injected into the gap, when the acting electric field is nonzero.

The initial charge on the plate is $Q=CV_g$, where $C=\epsilon_o A/D$ is the capacitance, and $A$ is the emission area. When an electron of charge $q=e$ is injected from the cathode ($x=0$) to a location $x(t)\leq D$ inside the gap, the accelerating electric field at the front of the electron is[20]

$$E = \frac{V_g}{D} + \frac{e}{\epsilon_o A}\frac{x(t)}{D}. \tag{2}$$

Due to the time-varying electric fields at $x(t)$, there will be a redistribution of charge on the electrode. For anode, the charge will increase by an amount of $\Delta q = e\times x(t)/D$ [see Eq. (2)]. Including all the injected electrons ($n=1,2,\ldots N$), the charge collected at the anode as a function of time is[21]

$$q(t) = CV_g + \frac{e}{D}\sum_{n=1}^{N} x_n(t), \tag{3}$$

where $x_n(t)$ is the location of the $n$-th electron at time $t$.

To include the space charge effects, we solve Gauss's law to obtain the electric field acting on the $n$-th electron, which gives

$$E_n(t) = \frac{1}{CD}[q(t) - e(n-1/2)]. \tag{4}$$

The relation between $E_n(t)$ and $x_n(t)$ is then determined by the equation of motion given by

$$\frac{d^2x_n}{dt^2} = \frac{e}{m}E_n(t). \tag{5}$$

For the first electron ($n=1$), Eq. (4) gives $E_1(t)=V_g/D - e/(2A\epsilon_0)$, where the first term is the external applied field, and the second term is the electric field formulated from Gauss's law. At time $t=0$, the electric field $E_1(0)$ will be zero unless we have $q(0)>e/2$. By using $q(0)=e/2$ and $q(0)=CV_g$ [from Eq. (3)], the threshold voltage for the injection of electrons is

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$$V_{th} = \frac{e}{2C}, \tag{6}$$

which equals to one-half value of the single-electron charge potential $V_c = e/C$. The collected current is obtained by

$$I(t) = \frac{dq}{dt} = \frac{e}{D}\sum_{n=1}^{N} \dot{x}_n(t). \tag{7}$$

It is important to note that according to the traditional CL law, there is no such threshold voltage in the formulation. For any given voltage ($V_g > 0$), there is always a CL current assuming unlimited electron supply from the cathode. This threshold voltage is due to the effect of single-electron charging (included in our model), which was also not found in a recent capacitance model of CL law.[24] We will show later (in Figs. 2 and 3) that it is possible to have only one electron inside the gap (at an instant of time) with a time-average current close to (or higher than) the 1D CL law in the range of $1 < V_g/V_{\mathrm{th}} < 2$.

For convenience in numerical calculations, we introduce some normalized parameters: $\bar{x} = x/D$, $\bar{V}_g = V_g/V_{\mathrm{th}}$, $\bar{E}_n = E_n/E_c$, $\bar{q} = q/e$, $\bar{t} = t/t_c$, and $\bar{I} = I/I_c$. The normalized constants $V_{\mathrm{th}} = 0.5 \times V_c$ is the threshold voltage [see Eq. (6)], $V_c = e/C$ is the single-electron charging potential, $E_c = V_c/D$, $t_c^{-1} = e/\sqrt{m\epsilon_o AD}$ is the plasma frequency for one electron per unit volume of $A \times D$, and $I_c = e/t_c$. Using the normalized parameters, the governing equations become

$$\bar{q}(\bar{t}) = \bar{V}_g/2 + \sum_{n}^{N} \bar{x}_n(\bar{t}), \tag{8}$$

$$\bar{E}_n(\bar{t}) = \bar{q}(\bar{t}) - (n - 1/2), \tag{9}$$

$$\frac{d^2\bar{x}_n}{d\bar{t}^2} = \bar{E}_n(\bar{t}),$$

$$\bar{I}(\bar{t}) = \sum_{n=1}^{N} \dot{\bar{x}}_n(\bar{t}). \tag{10}$$

Using the normalized equations, the time dependent $\bar{x}_n$ and $\bar{I}$ are solved by the standard fourth-order Runge-Kutta method with the initial conditions of $\bar{x}_n(0) = 0$ and $\dot{\bar{x}}_n(0) = 0$.

To compare with the steady-state 1D CL law, our results will be presented in terms of the 1D CL current and its transit time: $I_{\mathrm{CL}} = A \times J_{\mathrm{CL}}$, and $T_{\mathrm{CL}} = 3D/\sqrt{2eV_g/m}$. For a gap of $D = 100$ nm with an area $A = D^2$, the threshold voltage is about $V_{\mathrm{th}} = 0.09$ V. At $V_g/V_{\mathrm{th}} = 1.02$, we have $I_{\mathrm{CL}} = 62.91$ nA, and $T_{\mathrm{CL}} = 1.68$ ps. While we have assumed $D = 100$ nm and $A = D^2$ to illustrate some practical values, the normalized results ($I/I_{\mathrm{CL}}$ at different $V_g/V_{\mathrm{th}}$) shown below are independent of the exact values of $D$.

In Fig. 1, we plot the normalized position $\bar{x} = x_n(t)/D$ for the first five electrons ($n = 1$ to 5, from left to right) across the gap with $D = 100$ nm as a function of normalized time $t/T_{\mathrm{CL}}$ at $V_g/V_{\mathrm{th}} =$ (a) 1.02 and (b) 3. As mentioned before, when the electrical field acting on the $n$-th electron is larger than zero, the electron will enter the gap, before that time, the electron is punched on the cathode. At low voltage of $V_g/V_{\mathrm{th}} = 1.02$ shown in Fig. 1(a) (slightly higher above the threshold value), there is only one electron inside the gap during its transit time between $0 < t/T_{\mathrm{CL}} < 1.7$ for the first electron, $1.7 < t/T_{\mathrm{CL}} < 3.5$ for the second electron and so on. The $(n+1)$-th electron will only enter the gap, when the preceeding $n$-th electron is about to reach the anode. At higher $V_g/V_{\mathrm{th}} = 3$, the subsequent electron can enter the gap earlier, and there will be two electrons inside the gap with a transit time of about $t/T_{\mathrm{CL}} = 0.75$, which is shown in Fig. 1(b).

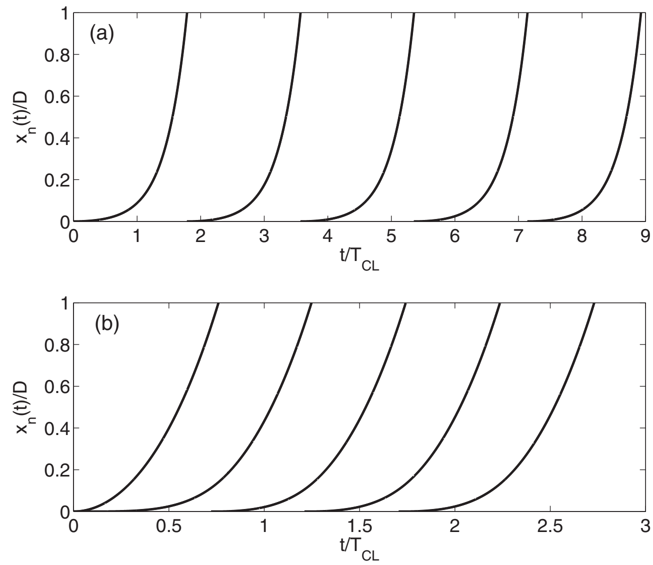


FIG. 1. The time-dependence of the normalized position of the first five injected electrons: $x_n/D$ for $n = 1$, 2, 3, 4, and 5 (left to right) at $V_g/V_{\mathrm{th}} =$ (a) 1.02 and (b) 3.

In terms of the 1D CL current, the time evolutions of the injected single-electron based current $I(t)$ are shown in Fig. 2 for various voltages $V_g/V_{\mathrm{th}} =$ (a) 1.02, (b) 2, and (c) 500. To study the time to reach the CL current, the time-average of the currents $I_{\mathrm{ave}} = \int_0^t I(\tau)d\tau / \int_0^t d\tau$ are also plotted (dashed lines) in the figures. From Fig. 2(a), it is clear that the current $I(t)$ will increase (and decrease) when one electron is accelerated across the gap (and collected at the anode). Note there is only one electron at this low voltage case of $V_g/V_{\mathrm{th}} = 1.02$, and we see that it is approaching the CL current (dashed line) after 1 or 2 transit time of about $t/T_{\mathrm{CL}} = 1.7$.

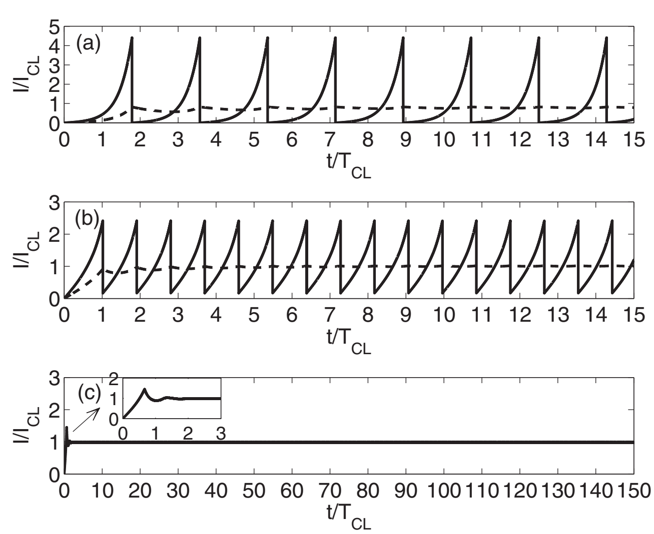


FIG. 2. The time-dependence of the normalized injected current $I(t)/I_{\mathrm{CL}}$ at $V_g/V_{\mathrm{th}} =$ (a) 1.02, (b) 2, and (c) 500. The dashed lines are the time-average calculation of $I(t)$.

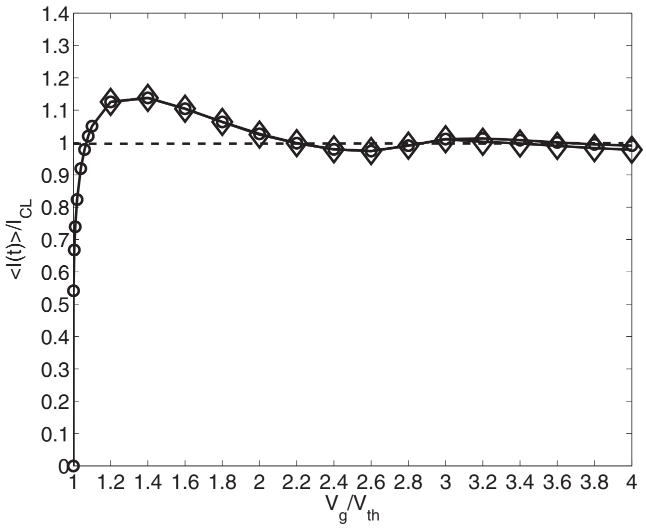


FIG. 3. The normalized calculations of $\langle I\rangle/I_{CL}$ (in symbols) as a function of $1<V_g/V_{th}<4$ with $N=500$ (circles) and 50 (diamonds).

Similar behavior is found at higher voltage of $V_g/V_{th}=2$ [Fig. 2(b)] but with a shorter transit time of about $t/T_{CL}=1$. By increasing the voltage to values much larger than $V_{th}[V_g/V_{th}=500]$ as show in Fig. 2(c), the current $I(t)$ will reach a steady-state value equal to the 1D CL law. At this very high voltage limit, it is very similar to the traditional CL model, the transient behavior at $t/T_{CL}<0.6$ shows that the current is increasing until the first electron is collected at the anode, for which the current will reach a steady-state value ($I(t)/I_{CL}=1$) after a few oscillations.

Another way to compare with the 1D CL law is by taking the average value of $N$ number of single-electron injection, given by $\langle I(t)\rangle = \lim_{N\to\infty} \frac{Ne}{t_N}$, where $t_N$ is the accumulative transit time of all the $N$ injected electrons. The normalized results ($\langle I\rangle/I_{CL}$) are shown in Fig. 3 in the range of $1<V_g/V_{th}<4$ by using $N=500$ (circles) and $N=50$ (diamonds). As mentioned earlier, the value of $N$ is not important as long as $N$ is large enough to reach a numerical stable result, where $N$ can be as small as $N<10$ for small gap voltage close to the threshold voltage [see Fig. 2(a) also]. When there is only 1 electron passing across the gap in the range of $1<V_g/V_{th}<2$ (or $e/2C<V_g<e/C$), it is still possible to reach the 1D CL law. In this single-electron limit, the space charge field is induced through the modification of charges on the electrodes during its transit time. From the figure, it is interesting to see that we may have a current higher than the 1D CL current ($\langle I\rangle/I_{CL}>1$) in the range of about $V_g/V_{th}=1.06$ to 2. For example, the maximum value is about $\langle I\rangle/I_{CL}\approx 1.13$ at $V_g/V_{th}\approx 1.4$. We confirm that this maximum current is not due to numerical error, and the results are consistent from using different values of $N$ (=10 to 500) at $V_g/V_{th}=1.4$. The increment is probably due to the nature of time-evolution of the current, which may have a higher upper bound of time-average CL law as suggested in a recent paper.[25]

To observe the proposed 1D classical CL law in the Coulomb blockade (or single-electron) regime, it is assumed that there is unlimited account of electrons to be injected into the gap without having any potential barrier at the cathode interface. Assuming a planar gap with an emission size ($\sqrt{A}$) and gap spacing ($D$) in submicrometer scale, the magnitude of the threshold voltage ($V_{th}$) or the single-electron charging potential is less than 1 V. At this low voltage regime, the applied electric field $V_{th}/D$ is less than $10^6$ V/m, and it is not sufficient for significant emission of electrons at room temperature (to reach the SCL regime), due to the existence of work function (or potential barrier). Some possible methods to increase electron emission at low voltage regime include using laser excitation of electron photofield emission,[19] or using sharp field emitters to enhance the tip's local field.

In summary, we have extended the 1D classical CL law to the Coulomb blockade (or single-electron) regime, where the applied voltage is comparable to the single-electron charging potential. It is found that there is a threshold voltage in order for the single-electron injected current to reach the 1D CL law. The single-electron limit of the 1D classical CL law is demonstrated by having only one electron being transport across a gap when the applied voltage is slightly above the threshold value. Smooth transition to the traditional CL model is demonstrated.

This work was supported by a Singapore MOE grant (Grant No. 2008-T2-01-033), USA AFOSR AOARD grant (Grant No. 10-4110), and USA ONRG grant (Grant No. N62909-10-1-7135). Careful readings of the revised manuscript by S.H. Chen, W.S. Koh, and A.L. Garner is acknowledged.

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