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Child-Langmuir flow with periodically varying anode voltage

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Using the Lagrangian technique, we study settled Child-Langmuir flows in a one dimensional planar diodes whose anode voltages periodically vary around given positive values. Our goal is to find analytically if the average currents in these systems can exceed the famous Child-Langmuir limit found for the stationary current a long time ago. The main result of our study is that in a periodic quasi-stationary regime the average current can be larger than the Child-Langmuir maximum even by 50% compared with its adiabatic average value. The cathode current in this case has the form of rectangular pulses which are formed by a very special triangular voltage modulation. This regime, i.e., periodicity, shape of pulses, and their amplitude, needs to be carefully chosen for the best performance. © *2015 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4913351]

I. INTRODUCTION

The Child-Langmuir (CL) theory[1–4] established a century ago the maximum current density of the space charge limited flow of charged particles in stationary conditions for idealized one-dimensional (1D) diodes. This maximum represents a challenge for physicists trying to exceed it in time dependent systems by periodically varying the boundary conditions (BC) of the diode. Obviously, if such a result is well substantiated, this would be practically important in applications because such idealized model provides a reasonable approximation for diodes with thermionic cathodes, gives very valuable current estimates for different emission laws, and can be implemented with corrections not only for one dimensional systems.

When the BC variations are slow and can be considered as quasi-adiabatic,[5,6] the mathematical structure of the CL limit readily allows to prove that generically it will be exceeded and can reach its adiabatic slightly higher average value.[5,6] We consider here the classical CL regime with zero initial velocity of electrons and zero cathode electric field. In this case, there is a well known statement[7,8] that the CL limit cannot be substantially exceeded by a periodical variations of BC, though in Ref. 7 the authors proved an increase by 17% using periodic manipulating the injected current and cathode electric field in the diode, i.e., in a different system. In quantum nano-systems, the current can be higher[9,10] due to tunneling, electron exchange-correlation interaction, and in Coulomb blockage regime, this result was confirmed experimentally.[11] The CL limit could be overcome by implying an intense laser irradiation of the tube bearing the current.[12] We limit ourselves to the widely used classical set up[1,2] applying a relatively fresh approach, initiated by Lomax[13] for modeling electron flow in Lagrangian variables. It has been successfully implemented in many publications, including Refs. 5, 6, and 14. The main part of our work can be performed analytically and its correctness can be validated.

Our rather modest goal is to find a quasi-stationary regime of the CL diode with periodically varying its anode voltage $V(t)$ which provides a time averaged current density $\bar{j}$ substantially exceeding the stationary current j_{CL} in the classical CL diode whose voltage is equal to the adiabatic average of $V(t)$. A more ambitious result when $\bar{j}$ is larger than j_{CL}, generated in the diode with the anode voltage $Max[V(t)]$, does not look reachable now after our many attempts with different shapes of $V(t)$ in diodes with classical boundary conditions for the cathode field and initial speed of electrons.

We start by listing some classical results for the stationary 1D CL flow in the planar diode, which will be needed for choosing parameters in the time dependent problem. When the anode voltage is V_a, the inter-electrode distance (in the x-direction) is x_a, the potential field and electron velocity are

$$\varphi(x) = V_a\left(\frac{x}{x_a}\right)^{4/3}, \quad v(x) = \left(\frac{x}{x_a}\right)^{2/3}\sqrt{\frac{2V_a}{m}}.$$

These equations imply formula for the current density and transition time

$$j_0 = \frac{4\epsilon_0}{9}\sqrt{\frac{2e}{m}\frac{V_a^{3/2}}{x_a^2}}, \; T_0 = \int_0^{x_a}\frac{dx}{v(x)} = 3x_a\sqrt{\frac{m}{2V_a}}, \qquad (1)$$

where ϵ_0 is the electric constant and m and e are the electron mass and absolute value of charge, respectively. An insignificant increase of the average current density when the BC (V and/or x_a) slowly vary periodically around their averages is a trivial result, see Refs. 5 and 6, which comes from the well known properties of generalized means in mathematics. When we are far from such slow variations, an exact study is needed with a more effective technique for the flow modeling.

A. Setting up method of solution in natural way

Let us use the Lagrange flow description initiated by Lomax[13] for modeling electron motion in a one-component plasma with independent variables τ and t, which represent the emission time of an electron and the current time. For our simple system, the location, velocity, and electric field, respectively, at the point of this electron are given by

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$$x(\tau,t),\ v(\tau,t),\ E(\tau,t). \tag{2}$$

To them should be added the cathode current density $j(t)$, and anode voltage $V(t)$ which are functions of time t only because the positions of the diode electrodes are assumed fixed, independent from flow.

From the Newton and Maxwell equations[13] for fixed τ, we have

$$v=\left(\frac{\partial x}{\partial t}\right)_\tau,\ \left(\frac{\partial v}{\partial t}\right)_\tau=\frac{eE}{m},\ \left(\frac{\partial^3 x}{\partial t^3}\right)_\tau=\frac{e}{m\epsilon_0}j(t). \tag{3}$$

We use the CL assumptions that electrons leave the cathode with zero velocity and their density at the cathode is infinite making a vanishing cathode electric field. Integrating Eq. (3) and using BC, we obtain

$$v(\tau,t)=\frac{e}{m\epsilon_0}\int_\tau^t (t-t')j(t')dt',$$
$$x(\tau,t)=\frac{e}{2m\epsilon_0}\int_\tau^t (t-t')^2 j(t')dt'. \tag{4}$$

Equations (3) and (4) are valid for electrons inside the diode, i.e., in the interval $(\tau,\tau+T(t))$, where $T(t)$ is the transition time of electrons, which hit the anode at time t. The second Eq. (4) we rewrite now by keeping track of the current time only

$$x_a=\frac{e}{2m\epsilon_0}\int_{t-T(t)}^t (t-t')^2 j(t')dt', \tag{5}$$

and taking into account that the electron meets the anode at t; therefore, the left side of Eq. (5) is constant. In the customary Eulerian variables (t, x) function $x(t)$ measures the distance of the electron from the cathode, thus, x_a is the anode coordinate and (5) can be viewed as a BC for our problem valid for all t.

There is one more BC for the anode voltage $V_a=\int E(\tau,t)(\partial x/\partial\tau)d\tau$, which with the help of equations above can be written as

$$V_a(t)=\frac{e}{2m\epsilon_0^2}\int_{t-T(t)}^t dt'(t-t\prime)^2 j(t')\int_{t'}^t j(\tau)d\tau. \tag{6}$$

Thus, we have two Eqs. (5) and (6) valid for any time $t \geq T(t)$ and therefore their derivatives can be used too when $V(t)$ is given and the current $j(t)$ is the subject of computations. As the transit time is unknown, the derivative of Eq. (5) will be used for expressing $T'(t)$ in terms of original functions and then constructing new equations

$$\frac{dT}{dt}(t)=1-\frac{2}{T^2(t)j(t-T)}\int_{t-T(t)}^t (t-t')j(t')dt'. \tag{7}$$

Then applying Eqs. (5)–(7), the first derivative of $V(t)$ has the form

$$\frac{dV_a}{dt}(t)=\frac{x_a}{\epsilon_0}j(t)-\frac{e}{m\epsilon_0^2}\int_{t-T(t)}^t dt'(t-t')j(t')\int_{t-T}^{t'} j(\tau)d\tau. \tag{8}$$

In the same way, two more exact equation can be found by differentiation in the following form:

$$\frac{d^2V_a}{dt^2}(t)=\frac{x_a}{\epsilon_0}\frac{dj}{dt}(t)+\frac{e}{m\epsilon_0^2}\left[\frac{2}{T^2(t)}J_1^2(t)-\frac{1}{2}J_0^2(t)\right], \tag{9}$$

$$\frac{d^3V_a}{dt^3}(t)=\frac{x_a}{\epsilon_0}\frac{d^2j}{dt^2}(t)-\frac{e}{m\epsilon_0^2}\left[j(t)J_0(t)-\frac{6}{T^2(t)}J_0(t)J_1(t)+4\frac{T'(t)+2}{T^3(t)}J_1^2(t)\right], \tag{10}$$

where temporarily we denote $J_n=\int_{t-T(t)}^t (t-t')^n j(t')dt'$ and for $T'(t)$, Eq. (7) should be used. The higher derivatives of $V(t)$ can be calculated too but their expressions are more cumbersome.

In our previous works[5,14] and others, we proceeded by a proper modeling the unknown function $j(t)$ in terms of polynomials with coefficients which could be evaluated with help of equations similar to (5)–(10). This can be called a natural way. In the present situation, the left sides of Eqs. (6) and (8)–(10) for $V_a(t)$ and its derivatives are of the second order for these coefficients. Though they can be solved anyway with some difficulties, we remember that our goal is a principal question of finding the current density when the diode parameters are periodically varying (the anode voltage in this work) and offer a different simpler method to answer this question. It is very important to note that the described above procedure using the set of Eqs. (5), (6), and (8)–(10) is necessary for studying transition processes in plasma flow. But when we have a time dependent flow with an already established periodic pattern, only two equations, namely, Eqs. (5) and (6), are needed because the process is cyclic: one has to find the functions $j(t)$ and $T(t)$ on an interval of only one period.

B. Finding solution by alternative method

While the natural technique outlined above deals with a set of nonlinear equations whose number is to be as large as it is needed for computing $j(t)$ with a sufficient precision, one can see how simple is to solve an inverted problem of finding the potential of the anode when the current density is given. As the diode regime is assumed periodic, i.e., $V_a(t)$, $T(t)$, and $j(t)$ are periodic functions with the same period, we may take $j(t)$ in a form of an arbitrary periodic function, substitute it into Eq. (5) for a chosen a set of values t_k within one period and satisfy (5) for each t_k. Thus, $T(t)$ will be determined on a discrete set of t_k. Then by inserting these $T(t_k)$ into Eq. (6), $V_a(t)$ will be found as a periodic function in the same set of points t_k. This will provide a technique to consider the computed periodic function $V_a(t)$ as a source of the already given $j(t)$. If one wishes the shape of $V_a(t)$ can be manipulated by modifying $j(t)$ even to make $V_a(t)$ approaching to a convenient for practice form. This is a tedious work but quite feasible and rather simple. Note, however, that this method is based on the assumption about the unique correspondence between functions $j(t)$ and $V_a(t)$ in their proper domains for the case when the cathode electric field is always zero.

II. HARMONIC CURRENT VARIATIONS

We start by dropping the subscript in $V_a(t)$ and considering the simple model with the harmonic current oscillations whose treatment can be done analytically almost completely.

$$j(t) = A + b\cos\omega t,\ \ A > b > 0. \tag{11}$$

Here, $\theta = 2\pi/\omega > 0$ is the period of current oscillations. After substituting (11) into (5) and (6) for $t \geq T$ we have, respectively,

$$\frac{2m\epsilon_0}{e}x_a = \frac{AT^3}{3} + \frac{b}{\omega^3}\left\{\left[2-(\omega T)^2\right]\sin\omega(t-T) + 2\omega T\cos\omega(t-T) - 2\sin\omega t\right\}, \tag{12}$$

$$\begin{aligned}\frac{2m\epsilon_0^2}{e}V(t) = {} & A^2\frac{T^4}{4} + \frac{2m\epsilon_0}{\omega e}bx_a\sin\omega t - Ab\frac{T^3}{\omega}\sin\omega(t-T) \\ & + \frac{2Ab}{\omega^4}\left[(T^2\omega^2-2)\cos\omega(t-T) + 2\omega T\sin\omega(t-T) + 2\cos\omega t\right] \\ & - \frac{b^2}{8\omega^4}\left[(2T^2\omega^2-1)\cos 2\omega(t-T) + 2T\omega\sin 2\omega(t-T) + \cos 2\omega t\right].\end{aligned} \tag{13}$$

Equations (12) and (13), written in physical variables, will be exploited in dimensionless ones for simplifying the numerical part of analysis outlined above. For this, we take $e = m = x_a = 1$, and the current density j is replaced with j/ϵ_0. Note that the stationary current density and transition time, see Eq. (1), in these units are $j = 4\sqrt{2}V_a^{3/2}/9$ and $T_0 = 3/\sqrt{2V_a}$.

We study numerically seven cases of $j(t)$ given by Eq.(11) with $\omega = 0.3$, 0.35, 0.4, 0.45, 0.5, 1.5, and $\omega = 3$ with $b = 0.4$ and 0.8 and $A = 1$. The period of oscillations θ varies from 2.1 to 21, while the electron transition time in our calculations was between 1.6 and 5, i.e., comparable or much smaller than θ, but even when $\omega = 0.3$ the regime is rather far from adiabatic. It is exhibited in Fig. 1.

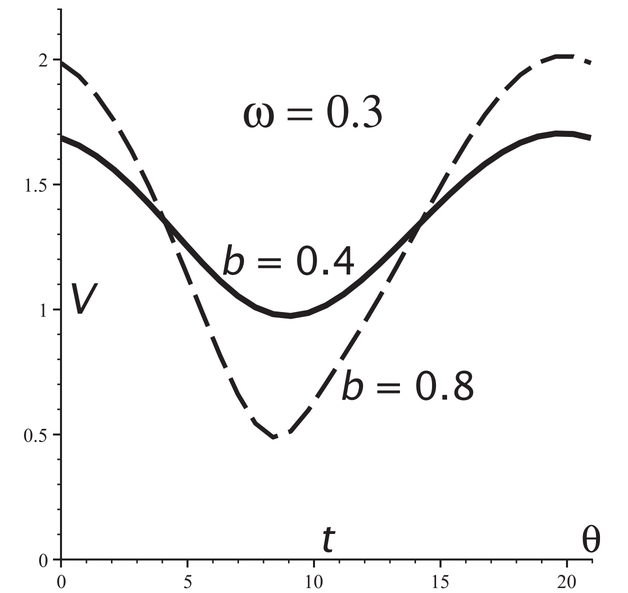

FIG. 1. Plots of $V(t)$ when $b = 0.4$ and $b = 0.8$ and cathode current given by Eq. (11).

The $V(t)$ dependences are not far from harmonic and compared with $j(t)$, they are almost synchronous in time. The result of computations for $\omega = 0.5$ is shown in Fig. 2 together with corresponding curves of $j(t)$.

The transition T time for $b = 0.4$ in the accepted units varies within one period from 1.626 to 2.133 and thus the chosen interval of periodicity $2\pi/\omega \approx 12.57$ is 6–8 times longer. The average voltage on this interval with $b = 0.4$ is $V \approx 1.316$, when $b = 0.8$ it is 1.326. The $V(t)$ variations are between 0.982 and 1.700, i.e., near $\pm 25\%$ about the average, less than $\pm 40\%$ of the current variations. In the case $b = 0.8$ $V_{min} = 0.519$, $V_{max} = 2.006$, i.e., $\pm 59\%$ around 1.263, i.e., less than $\pm 80\%$ for current.

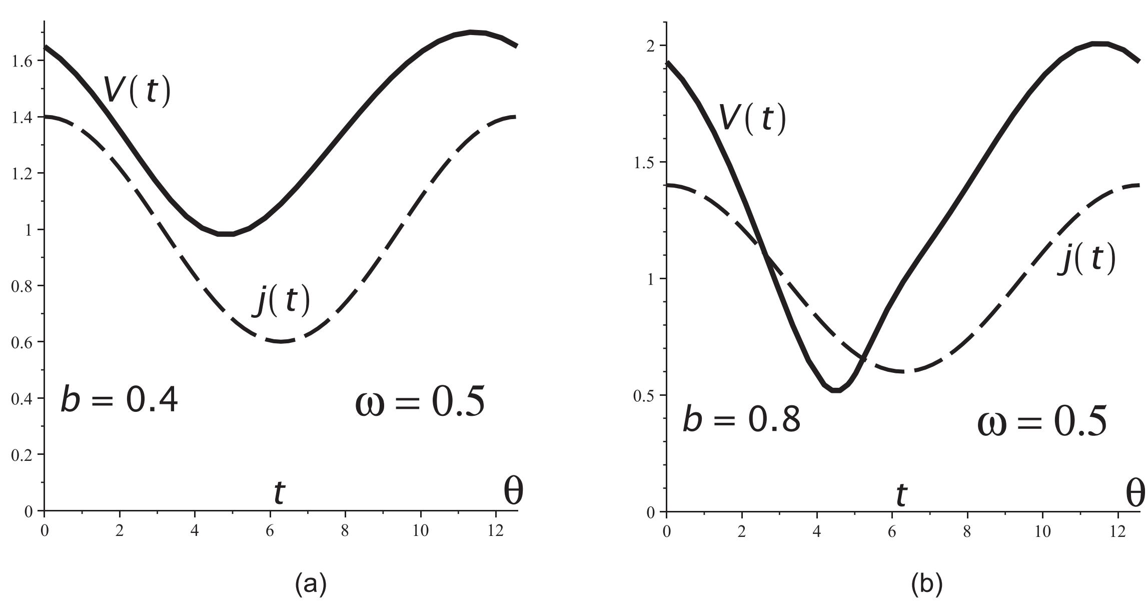

FIG. 2. Plots of the $j(t) = 1 + b\cos\omega t$ and corresponding $V(t)$ on the interval $0 < t < \frac{2\pi}{\omega}$.

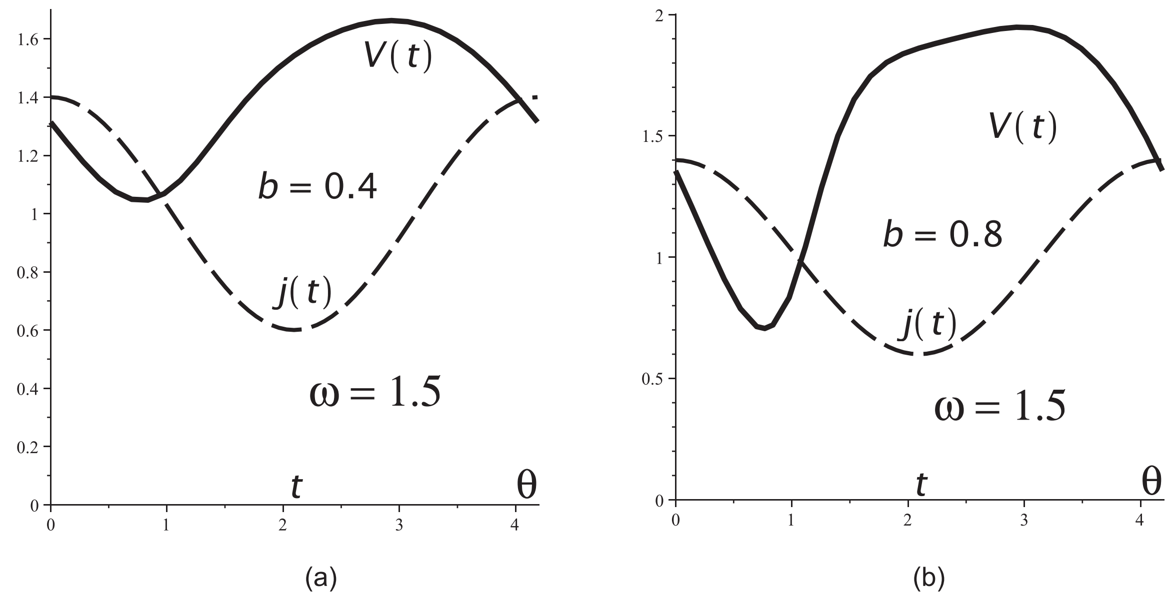

FIG. 3. Plots of the $j(t)$ together with $V(t)$ for $\omega = 1.5$ on the interval $0 < t < \frac{2\pi}{\omega}$.

Let us consider two average voltages

$$V_1 = \frac{1}{\theta}\int_0^\theta V(t)dt,\ V_2 = \left[\frac{1}{\theta}\int_0^\theta V^{3/2}(t)dt\right]^{2/3}, \tag{14}$$

and, respectively, two current densities $j_1 = 4\sqrt{2}V_1^{3/2}/9$ and $j_2 = 4\sqrt{2}V_2^{3/2}/9$. For non-negative $V(t)$ always $j_2 \geq j_1$. j_2 represents the average current in the adiabatic regime when the electron flow can attain the stationary state at any time. We introduce three parameters for describing the ratios of the average of the time dependent current density j_{av} in the diode to j_1, j_2, and also the ratio of j_{av} to j_3 which is the current density in a stationary diode with anode voltage $V_a = Max[V(t)]$, where $0 < t < \theta$

$$k_1 = \frac{j_{av}}{j_1} - 1, \quad k_2 = \frac{j_{av}}{j_2} - 1, \quad k_3 = \frac{j_{av}}{j_3} - 1,\ k_3 < k_2 < k_1. \tag{15}$$

j_1, j_2 can be viewed too as currents in the stationary CL diodes with $V_a = V_1$ and V_2, respectively. When k_1 is positive, our time dependent system looks like it gives a larger current density, but this might be incorrect in our situation. A more reliable would be a positive k_2, but we do not expect to have $k_3 > 0$.

In Fig.3, we see results of similar to $\omega = 0.5$ computations when $\omega = 1.5$. Though the shape of $V(t)$ time dependence is very different, the transit time varies the limits quite close to ones for $\omega = 0.5$.

The coefficients k in this case with $\omega = 1.5$ for $b = 0.8$ are both negative and not small: $k_1 = -0.122$, $k_2 = -0.161$, while for $b = 0.4$ they are $k_1 = 0.008$, $k_2 = -0.018$, while $k_3 = -0.414$ and -0.258, respectively.

Fig. 4 illustrates the case $\omega = 3$.

The results of evaluating averages of $V(t)$ strongly depend on shapes of curves for V in Figs. 2–4. They are for $b = 0.4$: $V_{0.5} = 1.348$, $V_{1.5} = 1.401$, $V_3 = 1.367$, while for

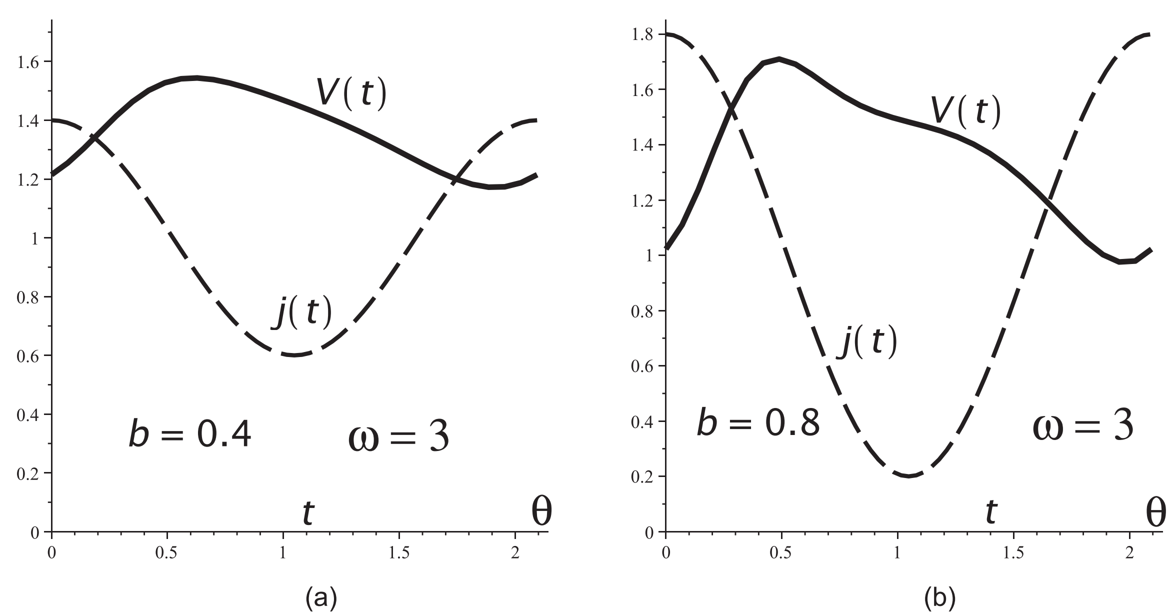

FIG. 4. Plots of the $j(t)$ together with $V(t)$ for $\omega = 3$ on the interval $0 < t < \frac{2\pi}{\omega}$.

TABLE I. Data for sine-wave cathode current.

k_1	$\omega=0.3$	$\omega=0.4$	$\omega=0.5$	$\omega=0.6$	$\omega=1.5$	$\omega=3$
b = 0.4	0.0607	0.0575	0.0535	0.0490	0.0076	0.0448
b = 0.8	0.0874	0.0660	0.0421	0.0175	−0.0122	0.0415
k_2						
b = 0.4	0.0295	0.0266	0.0231	0.0191	−0.0176	0.0245
b = 0.8	0.0104	−0.0055	−0.0237	−0.0431	−0.1612	0.0136

$b=0.8$, we get $V_{0.5}=1.305$, $V_{1.5}=1.586$, $V_3=1.370$, where subscripts indicate the values of ω. In all cases, the average current given by Eq. (11) is 1. In the stationary CL systems with these anode voltages, the current densities would be for $b=0.4$ $j_{0.5}=0.984$, $j_{1.5}=1.042$, $j_3=1.01$ and $j_{0.5}=0.937$, $j_{1.5}=1.255$, $j_3=1.01$ for $b=0.8$. Thus, the voltage variations here do not change substantially the current density except for ω near 0.5 when the time dependent system provides a slightly higher current.

The values of k_1 and k_2 are shown in Table I.

The largest k_2 here is about 3% when $b=0.4$ and $\omega=0.3$ though usually k_2 is much smaller, $k_3<0$ always. This means that slow almost harmonic variations of anode voltage, see Fig. 2, with the amplitude about ±27% about the average give a small increase 3% above the adiabatic CL limit (the current density amplitude varies is ±40%). Thus, the general opinion that varying the anode voltage actually does not help to overcome the CL maximum is correct when the diode current is modulated by harmonic functions. Even the current, produced by the average voltage in the stationary system, is exceeded only by 8.7% when $\omega=3$, $b=0.8$, see Table I. We will study also different forms of current modulation.

III. EFFECTIVE FORMS OF CURRENT MODULATION

Our experimentation using the same method as above with the cathode current modulated by periodic rectangular, trapezoidal, and triangular pulses shows that one can have a substantial gain in the average current compared with the

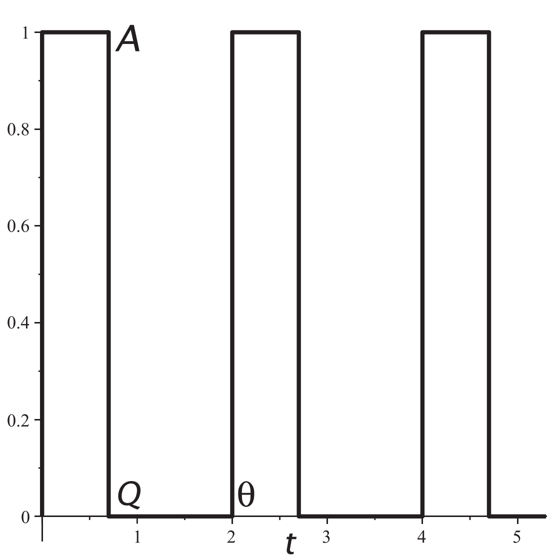

FIG. 5. General form of modulated current.

TABLE II. Results for different pulse amplitudes.

A	k_1	k_2	k_3	I_{av}
0.5	0.104	0.072	−0.230	0.25
1.0	0.361	0.321	−0.321	0.5
2.0	0.122	0.104	−0.346	1

stationary cases. It turned out that to have this result the amplitude, period, and width of pulses should be carefully tuned in our dimensionless units, but clearly this can be realized in wide variety of combinations of the physical units.

A. Rectangular pulses

The cathode current is taken in the form shown in Fig. 5 where θ is the period, Q is the pulse width, and A is the pulse amplitude. Here, they are 2, 0.7, and 1, respectively, but we play with them in the search for good results shown in Table II for different amplitudes A, while the period $\theta=2$ and pulse width $Q=1$ are fixed.

One can see the gain 32% above the adiabatic average for the amplitude $A=1$. Our following attempts will be with $\theta=2$, $A=1$, and different values of the pulse duration Q.

Now in the case $Q=0.7$, the average current exceeds the adiabatic average by 50.4%. All our next work did not give a better gain, but we will show it below because sometimes it is important to have a larger current even when the gain is smaller a little.

Both Tables II and III clearly show that the stationary CL diode with the anode voltage $V_{max}(t)$ generates the current density $\sim 30\%$ higher than our periodic system.

The best result of our study is as follows: rectangular current pulses with the periodicity $\theta=2$, amplitude of 1, and pulse width of 0.35θ. In this case, $k_2=0.504$. Plots of the cathode j, anode j_a current densities, and corresponding anode voltage $V(t)$ are shown in Fig. 6.

The average current density of 0.35 in Fig. 6 corresponds to the current, produced by the stationary CL diode with the anode voltage $V_0=0.677$, while in our diode $V_1=0.507$, $V_2=0.516$, and the maximum $V_3=0.868$, see Eq. (14). Note that $V_3>V_0$ on the time interval ~ 0.33 which is only 16% of the period θ.

Then keeping the same pulse amplitude and width of 0.35θ, we considered $\theta=1.5$ and 2.5. The results are $k_1=-0.138$, $k_2=-0.142$ and $k_1=0.391$, $k_2=0.365$, respectively. This means that tuning is quite critical and might be the reason that the effective regime exhibited in Fig. 5 was not found earlier. Surprisingly, the voltage curve in this

TABLE III. Current efficiency vs relative pulse ratio.

Q	k_1	k_2	k_3	I_{av}
0.6	0.362	0.344	−0.324	0.30
0.7	0.543	0.504	−0.311	0.35
0.8	0.507	0.469	−0.301	0.40
0.9	0.406	0.376	−0.283	0.45
1.0	0.361	0.321	−0.321	0.50
1.2	0.183	0.170	−0.292	0.60

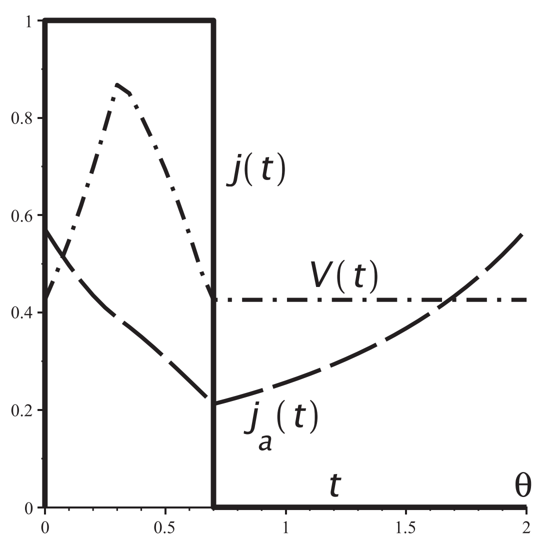

FIG. 6. Current densities and anode voltage when gain is 50%.

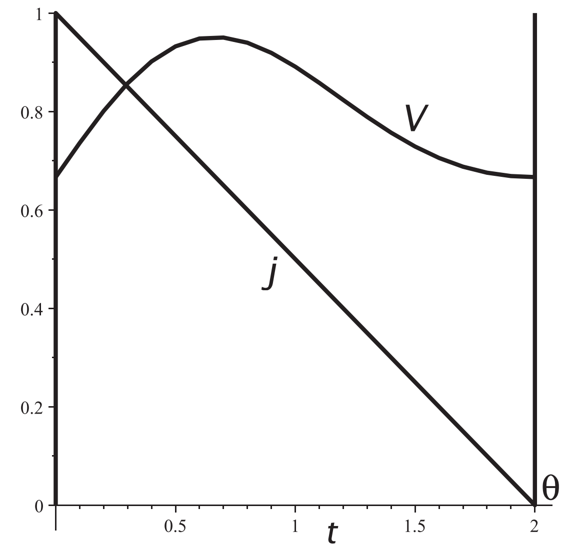

FIG. 7. Triangular form of modulated current.

regime is of a very simple form and can be easily implemented in practice for the modulation of the anode voltage. Even a rough approximation of $V(t)$ with three line segments between 4 points (0,0.426), (0.31,0.868), (0.7,0.426), (2,0.426) is very close to the curve in Fig. 6, the average standard difference is only 0.015, i.e., ∼0.3%.

The anode current density at any time t in Fig. 6 is evaluated by a following computation. As

$$\rho(\tau,t) = \frac{\partial E}{\partial \tau} \Big/ \frac{\partial x}{\partial \tau}(\tau,t)$$

one has using Eq. (3) for $\tau = t - T(t)$ in our problem

$$j_a(t) = j_a(\tau,t) = \rho(\tau,t)v(\tau,t) = \frac{2}{T^2(t)}\int_{t-T(t)}^{t}(t-t')j(t')dt'. \tag{16}$$

The anode current $j_a(t)$, shown in Fig. 6, never becomes zero unlike the cathode current. The electron transition time in this regime varies from $T(0) = 2.03$ to the maximum $T(0.7) = 3.33$.

We performed three additional calculations with the pulse period of 2, width of 0.7 (as in Fig. 5), and the amplitudes $A = 0.9$, 1.0, 1.1, but with the constant current component $j_{min} = 0.1$. The results are $k_1 = 0.286$, 0.309, 0.310 and $k_2 = 0.271$, 0.291, 0.293, respectively, i.e., lower than in the regime shown in Fig. 6.

B. Trapezoidal pulses

In the degenerated case of a triangular pulse with $\theta = Q = 2$ and $A = 1$ shown in Fig. 7, we plotted the cathode current density and calculated anode voltage which creates this current. The average current is 0.5 and the current gain is low $k_1 = 0.087$, $k_2 = 0.081$. When we keep the same triangular modulation with $A = 1$, period of 1, and the average current of 0.5, the current gain is even lower $k_1 = 0.0296$, $k_2 = 0.0288$ and the current variations are only ±7.3%.

A trapezoidal pulse determined by its apexes (0,0), (0.7,1), (1.3,0), (2,0) with period 2 yields $k_1 = 0.261$, $k_2 = 0.243$, $k_3 = -0.243$, and the average current of 0.5. We studied also pulses with oblique upper sides and $\theta = 2$. They gave k_2 from $0.185\,(I_{av} = 0.54)$ to $0.401\,(I_{av} = 0.36)$ depending on the slope angle, but a corresponding rectangular modulation (with the same θ and Q) is more effective. Nevertheless, we expect that there is a possibility to get even larger current gain than 50% by a proper modulation.

IV. SOLUTION IN PHYSICAL UNITS

All our results can be converted to physical units by quite a simple way. Equations (5) and (6) were rescaled before by temporary assuming $e = m = \epsilon_0 = 1$, and in the following computations we used them. One can easy come back by noticing that this technique for Eq. (5) was equivalent to taking $x_a m\epsilon_0/e = 1$, i.e., using a special value for x_a. When $x_a = d$, the current density $j(t)$ is to be replaced as follows:

$$j_d(t) = \frac{d}{x_a}j(t) = \frac{dm\epsilon_0}{e}j(t). \tag{17}$$

This j_d will enter in Eq. (6) and modify the anode voltage

$$V_d(t) = \frac{d^2 m}{e}V(t). \tag{18}$$

Thus, all our solutions above can be used in physical units by multiplying them by the coefficients given in Eqs. (17) and (18). The period of modulation should be closely correlated with the electron transition time roughly estimated by Eq. (1).

V. CONCLUSION

The maximum stationary current in a planar one dimensional diode with negligible initial electron velocity v_0 and cathode electric field was found a century ago by Child and

Langmuir.[1,2] This current is determined by the diode geometry, anode voltage, and limited by the space charge, created by moving electrons. This limit comes from the assumption of infinite emissivity of the cathode which in many cases is a good model for the thermionic emission. It is obvious that currents produced by the field emission and photo emission cannot exceed this limit (when v_0 can be neglected). The CL maximum continues to be very important and useful in two- and three-dimensional systems.

The question if this limit can be exceeded, when the boundary conditions of the diode (like the anode voltage V_a in our study) are time dependent, attracts a long lasting interest of researchers. Clearly, a much larger current is quite possible in transition regimes but this question relates mainly to an established quasi-stationary periodic process which desirably might produce a systematically higher average current with the same average voltage. The conventional answer, obtained by numerical analysis in Refs. 7 and 8 and others with modulated cathode current, electric field, and anode voltage, is discouraging: no substantial current gain.

Implementing serious, though quite plausible, assumptions about zero (or at least negligible) cathode electric field and the one-to-one correspondence $V(t) \leftrightarrows j(t)$ (anode voltage and cathode current) in a quasi-stationary regime, we found here by a predominantly analytical technique that the CL maximum can be overcome even by 50% in a carefully tuned periodic process. Our analysis always produced only one solution $V(t_k)$ as soon as transit time T_k was found for a chosen current modulation. This provides some level of confidence for the assumed one-to-one correspondence, though the cathode current modulation for having a larger current can be useful by itself in different systems. The CL diode whose anode voltage V_a is $Max[V(t)]$ has always a higher current density than in our periodic regime with $V_a = V(t)$. We believe that our work opens a promising window for new calculations and experiments.

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