HW3

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1. (a) Note that if $x_i \in \{0, 1\}$, then $0 \leq x_i \leq 1$. Thus if the constraints of the Boolean LP are satisfied, then so are the constraints of the LP relaxation. This means that the feasible set of the Boolean LP is a subset of the feasible set of the LP relaxation. Since the LP relaxation minimizes over a larger set, its optimal value is a lower bound of the optimal value of the Boolean LP. Moreover, if the LP relaxation is infeasible, then the Boolean LP is infeasible.

   (b) If the solution to the LP relaxation is $x_i \in \{0.1\}$, then it is also the solution to the Boolean LP since the feasible set of the Boolean LP is a subset of the feasible set of the LP relaxation.

2. (a) The Lagrangian is $L(x, \lambda) = e^x - \lambda x$. If $\lambda < 0$, then $L(x, \lambda)$ is unbounded below. If $\lambda = 0$, then $L(x, \lambda) = e^x$ whose infimum is 0. If $\lambda > 0$, the minimum of $L(x, \lambda)$ is achieved when $\frac{\partial L}{\partial x} = 0$, i.e. $x = \log \lambda$. Thus the minimal value of $L(x, \lambda)$ is $\lambda - \lambda \log \lambda$. Hence, the Lagrange dual function is

$$g(\lambda) = \begin{cases} \lambda - \lambda \log \lambda & \lambda > 0 \\ 0 & \lambda = 0 \\ -\infty & \lambda < 0 \end{cases}$$

   Figure 1 shows the plot of the dual function $g(\lambda)$ for $\lambda \in [0, 2]$.

   (b) The Lagrange dual problem is $\max_{\lambda \geq 0} g(\lambda)$. If $\lambda > 0$, then $g(\lambda) = \lambda - \lambda \log \lambda$ is concave since $g''(\lambda) - \frac{1}{\lambda} < 0$. Thus, $g(\lambda)$ is maximized when $g'(\lambda) = -\log \lambda = 0$, i.e. $\lambda = 1$. In this case, $g(\lambda) = 1$. Therefore, the dual optimal solution is $\lambda^* = 1$ and the dual optimal value is $d^* = 1$. Since $p^* = d^*$, strong duality holds.

3. (a) The feasible set is $\{x \mid (x-2)(x-4) \leq 0\} = [2, 4]$. Since $x^2 + 1$ is an increasing function on $[0, +\infty)$, the optimal solution is $x^* = 2$ and the optimal value is $p^* = 5$.

   (b) The Lagrangian is $L(x, \lambda) = x^2 + 1 + \lambda(x-2)(x-4) = (\lambda+1)x^2 - 6\lambda x + (8\lambda + 1)$. Note that when $\lambda \leq -1$, the Lagrangian is unbounded below. When $\lambda > -1$, the minimizer of the Lagrangian is $x = \frac{3\lambda}{\lambda+1}$ and the minimal value of the Lagrangian is $g(\lambda) = -\frac{9\lambda^2}{\lambda+1} + 8\lambda + 1$. Therefore the Lagrange dual function is

$$g(\lambda) = \begin{cases} -\frac{9\lambda^2}{\lambda+1} + 8\lambda + 1 & \lambda > -1 \\ -\infty & \lambda \leq -1 \end{cases}$$

   Figure 2 shows the plot of the objective function, the optimal point, and the Lagrangian for $\lambda = 1, 2, 3$ on the feasible set $[2, 4]$. We can see from the plot that indeed $p^* \geq \inf_x L(x, \lambda)$. Figure 3 shows the plot of the dual function $g(\lambda)$.

   (c) The Lagrange dual problem is $\max_{\lambda \geq 0} -\frac{9\lambda^2}{\lambda+1} + 8\lambda + 1$. It is a concave maximization problem since $g''(\lambda) = -\frac{18}{\lambda^3+1} < 0, \forall \lambda \geq 0$. To find the dual optimal solution, we thus find λ such that $g'(\lambda) = -\frac{9\lambda^2+18\lambda}{(\lambda+1)^2} + 8 = 0$. Hence the dual optimal solution is $\lambda^* = 2$ and the dual optimal value is $d^* = 5$. In this case, $p^* = d^*$, so strong duality holds.

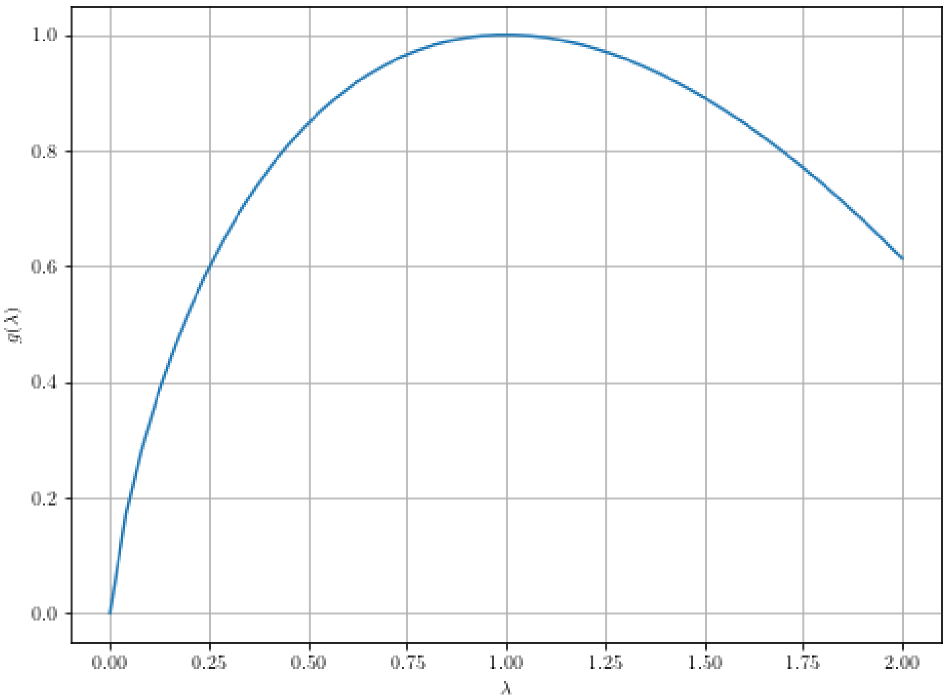

Figure 1: Lagrange dual function

4. (a) To show that this is a convex optimization problem, we need to verify that $f_0(x, y) = e^{-x}, f_1(x, y) = \frac{x^2}{y}$ are convex. Since $\nabla\nabla f_0(x, y) = \begin{bmatrix} e^{-x} & 0 \\ 0 & 0 \end{bmatrix}$ is clearly positive semi-definite, $f_0(x, y)$ is convex. Furthermore, $\nabla\nabla f_1(x, y) = \begin{bmatrix} \frac{2}{y} & -\frac{2x}{y^2} \\ -\frac{2x}{y^2} & \frac{2x^2}{y^2} \end{bmatrix}$ and we can show that it is positive semi-definite by showing that both of its eigenvalues are non-negative. Recall that the product of the two eigenvalues equals the determinant, which is 0 for $\nabla\nabla f_1(x, y)$, and so one of the eigenvalue is 0. Also recall that the sum of the two eigenvalues equals the trace, which is $\frac{2}{y} + \frac{x^2}{y^2}$, and it is non-negative on the domain $\mathcal{D} = \{(x, y) \mid y > 0\}$. Therefore, the other eigenvalue is non-negative. Hence, both eigenvalues of $\nabla\nabla f_1(x, y)$ are non-negative, so $f_1(x, y)$ is convex.
To derive the optimal value and solution, note that $x^2 \geq 0, y > 0$, so we need $x = 0$ to satisfy the constraint $\frac{x^2}{y} \leq 0$. Therefore the optimal value is $p^* = 1$ and the optimal solution is $\{(x, y) \mid x = 0, y > 0\}$.

   (b) The Lagrangian is $L(x, y, \lambda) = e^{-x} + \frac{\lambda x^2}{y}$. If $\lambda < 0$, then $L(x, y, \lambda)$ is unbounded below. If $\lambda \geq 0$, then $L(x, y, \lambda) > 0$ and the infimum is 0, e.g. by taking $y = x^4, x \to +\infty$. Thus the Lagrange dual function is

$$g(\lambda) = \begin{cases} 0 & \lambda \geq 0 \\ -\infty & \lambda < 0 \end{cases}$$

   Thus, the Lagrange dual problem is $\max_{\lambda \geq 0} 0$. So, the optimal value is $d^* = 0$ and the optimal solution is $\{\lambda \mid \lambda \geq 0\}$. The optimal duality gap is $p^* - d^* = 1$.

   (c) Slater's condition does not hold since no point in $\mathcal{D} = \{(x, y) \mid y > 0\}$ satisfies $\frac{x^2}{y} < 0$.

5. Note that $\mathcal{D}$ is convex since it is the intersection of the domains of convex functions f_i, h_i. Let $(u_1, v_1, t_1), (u_2, v_2, t_2) \in \mathcal{A}$ and $\theta \in [0, 1]$. Then by definition of $\mathcal{A}$, $\exists x_1, x_2 \in \mathcal{D}$

such that $f_i(x_1) \leq u_1, h_i(x_1) = v_1, f_i(x_2) \leq u_2, h_i(x_2) = v_2$. Then by convexity of $\mathcal{D}$, $\theta x_1 + (1-\theta)x_2 \in \mathcal{D}$, and $\forall i$,

$$\begin{aligned} f_i(\theta x_1 + (1-\theta)x_2) &\leq \theta f_i(x_1) + (1-\theta)f_i(x_2) \\ &\leq \theta u_1 + (1-\theta)u_2 \end{aligned}$$

Furthermore, since h_i are affine, we can write $h_i(x) = A_i x + b_i, \forall i$. Then,

$$\begin{aligned} h_i(\theta x_1 + (1-\theta)x_2) &= A_i(\theta x_1 + (1-\theta)x_2) + b_i \\ &= \theta(A_i x_1 + b_i) + (1-\theta)(A_i x_2 + b_i) \\ &= \theta h(x_1) + (1-\theta)h(x_2) \\ &= \theta v_1 + (1-\theta)v_2 \end{aligned}$$

Moreover, by convexity of f_0,

$$\begin{aligned} f_0(\theta x_1 + (1-\theta)x_2) &\leq \theta f_0(x_1) + (1-\theta)f_0(x_2) \\ &\leq \theta t_1 + (1-\theta)t_2 \end{aligned}$$

Therefore, $\theta(u_1, v_1, t_1) + (1-\theta)(u_2, v_2, t_2) \in \mathcal{A}$, which means $\mathcal{A}$ is convex.

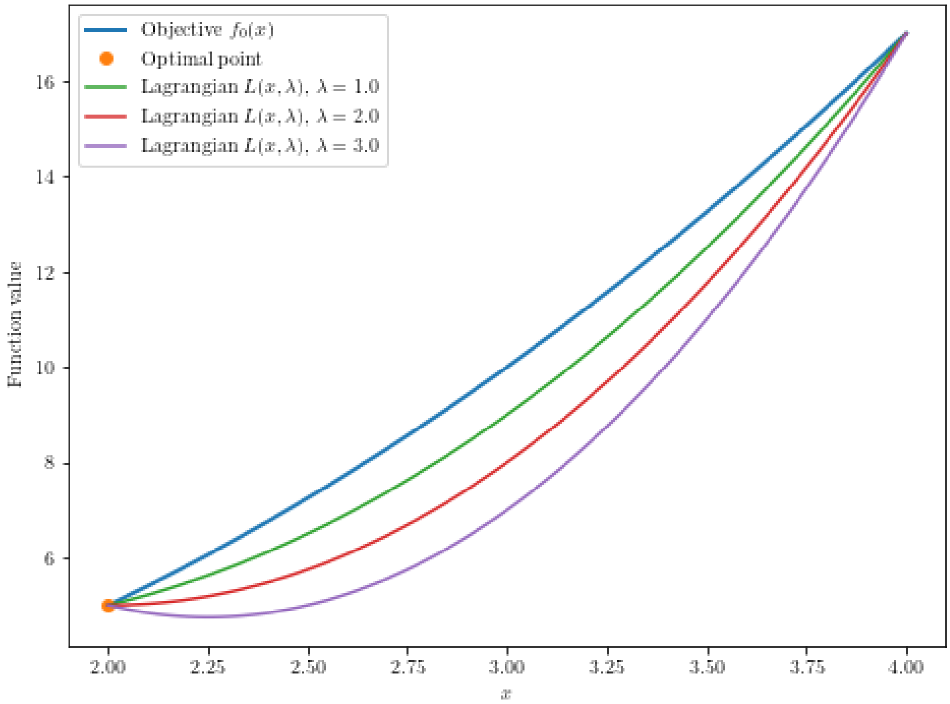

Figure 2: Objective function and the Lagrangian on the feasible set

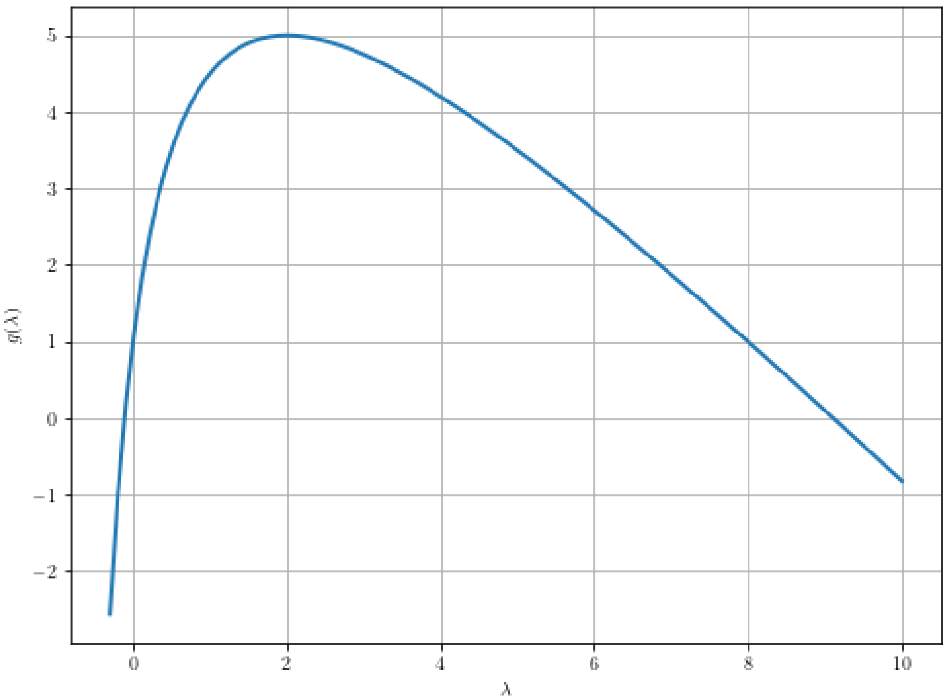

Figure 3: Lagrange dual function