NYU

Steven Zhang <zz2589@nyu.edu>

Question to shortly extend 1-hour for HW3 submission

2 messages

Steven Zhang <zz2589@nyu.edu> Tue, Feb 15, 2022 at 9:19 PM
To: Michael Overton <mo1@nyu.edu>

Dear Professor Overton,

This is Steven from your Optimization course. I am wondering if I could request a precisely 1-hour extension for the HW3. I am currently working on my REU applications (which is due midnight today) and I am quite overwhelmed in managing all this stuff. 1 hour will definitely be enough for me.

Thank you very much for your consideration.

Best regards,
Zihan Zhang

Michael Overton <mo1@nyu.edu> Tue, Feb 15, 2022 at 9:54 PM
To: Steven Zhang <zz2589@nyu.edu>

Sure, but if you need a little longer that's also OK. Just print this email and include it in your submission.
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Convex HW3

Q1 (Inspired from online material from Stanford)

a) Since the range of x_i is expanded from $\{0,1\}$ to $[0,1]$, thus the feasible set of the relaxation includes the feasible set of Boolean LP. By logical induction, the Boolean LP will become infeasible if its relxation is infeasible and the optimal value of the relaxation is less than or equal to the optimal value of Boolean LP.

b) We could say the optimal solution of relaxation is also optimal for Boolean LP.

Q3 a) The feasible set is $[2,4]$. The unique optimal point is $x^*=2$, the optimal value is $p^*=5$.

b) The Lagrangian could be written as $\mathcal{L}(x,\lambda)=f_0(x)+\lambda f_1(x)$ $= x^2+1+\lambda(x^2-6x+8) = x^2(\lambda+1)-6\lambda x+(1+8\lambda)$. The plot shows the Lagrangian as a function of x for $\lambda=1,\overset{2,}{}5,9.$, while the blue curve is f_0. By observation, the minimum value of $L(x,\lambda)$ over x is always less than $p^*(\inf_x L(x,\lambda))$. It increases while λ increases from 0 to 2, and decreases after 2, and reaches a maximum at $\lambda=2$. We denote the 'low bound function' as $g(\lambda)$. $\lambda\geq 0$. Thus $p^*=g(\lambda)$ when $\lambda=2$. By property of quadratic function, when $\lambda>-1$, the minimum $x'=\frac{3\lambda}{1+\lambda}$. For $x\leq -1$ the function is unbounded below. Plug in, we have

$$g(\lambda)=\begin{cases}\frac{-9\lambda^2}{1+\lambda}+1+8\lambda & \lambda\geq -1\\ -\infty & \lambda\leq -1.\end{cases}$$

$\lambda=2$
$g(\lambda)=5$

$p^*\geq \inf_x L(x,\lambda)$

c) The lagrange dual problem will be

$$\text{maximize} \quad \frac{-9\lambda^2}{1+\lambda}+1+8\lambda=f$$
$$\text{subject to} \quad \lambda\geq 0.$$

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The dual optimum occurs at $\lambda = 2$, with $d^* = 5$. $\frac{\partial f}{\partial \lambda} = -\frac{\lambda^2+2\lambda-8}{(1+\lambda)^2}$.

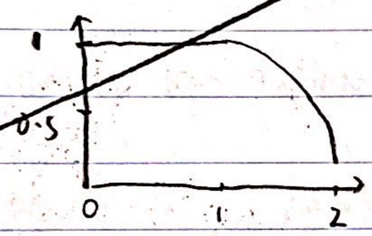
$\frac{\partial^2 f}{\partial \lambda^2} = -\frac{18}{(\lambda+1)^3}$. If the first derivative is 0, we have solution $\lambda = 2$ since $\lambda \geq 0$. The other -4 solution is dropped, but the second

~~Q2 a) The Lagran is $L(x,\lambda) = e^x - \lambda x$ subject to $x \geq 0$.~~

The dual function $g(\lambda) = \inf_{x \geq 0} (e^x - \lambda x)$

$\frac{d}{dx}(e^x + \lambda x) = e^x - \lambda$ to find the minimum point, $x = \log(\lambda)$

Thus $g(\lambda) = \begin{cases} \lambda - \lambda \log \lambda & \lambda \geq 1 \\ 1 & \text{o.w.} \end{cases}$



derivative of the dual problem $\frac{-18}{(\lambda+1)^3} < 0$ for $\lambda > 0$. Thus, the function is concave and thus it is a concave optimization problem. The strong duality holds since the dual gap is 0.

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Inspired from Stack Exchange.

Q4 a) $H = \begin{bmatrix} \frac{2}{y} & -\frac{2x}{y^2} \\ -\frac{2x}{y^2} & \frac{2x^2}{y^3} \end{bmatrix}$ $\quad \frac{1}{2}y^3 H = \begin{bmatrix} y^2 & -xy \\ -xy & x^2 \end{bmatrix}$

$$(s,t)\begin{pmatrix} y^2 & -xy \\ -xy & x^2 \end{pmatrix}\begin{pmatrix} s \\ t \end{pmatrix} = (sy-tx)^2 \geq 0 \quad \forall s,t \in \mathbb{R}^2.$$

Or from leading minors perspective.

$$\Delta_{(1)} = \frac{\partial^2 f(x,y)}{\partial x^2} = \frac{2}{y} > 0 \quad \Delta_{(2)} = \frac{\partial^2 f(x,y)}{\partial y^2} = \frac{2x^2}{y^3} > 0.$$

$$\Delta_{(1,2)} = \begin{vmatrix} \frac{2}{y} & -\frac{2x}{y^2} \\ -\frac{2x}{y^2} & \frac{2x^2}{y^3} \end{vmatrix} = 0.$$ Thus, all leading minors are nonnegative on $D = \{(x,y) \mid y > 0\}$.

Thus, the Hessian matrix is semidefinite and $\frac{x^2}{y}$ is convex.

Thus that is a convex optimization problem.

The optimal value is $p^* = 1$.

b) The Lagrangian is $L(x,y,\lambda) = e^{-x} + \frac{\lambda x^2}{y}$

The dual function is

$$g(\lambda) = \inf_{x,y>0} \left(e^{-x} + \frac{\lambda x^2}{y}\right) = \begin{cases} 0 & \lambda \geq 0 \\ -\infty & \lambda < 0 \end{cases}$$

Thus the dual problem is

$$\begin{aligned} &\text{maximize} && 0 \\ &\text{subject to} && \lambda \geq 0 \end{aligned}$$

The optimal value $d^* = 0$. The optimal duality gap is $p^* - d^* = 1$.

c) Slater's condition is not satisfied since $x = 0$ for any feasible pair (x,y), which is not a strictly feasible set.

Q5 (Discuss with Hanchao, Zhang).

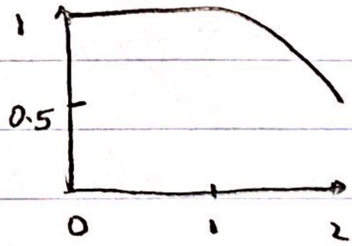
By definition, we could write $\mathcal{A} = \{(u, v, t) \mid \exists x \in D, f_i(x) \le u_i,$
$i \in [1, m], h_i(x) = v_i, i \in [1, p],$
$f_0(x) \le t\}$

Assume $f_0 \sim f_m$ are convex and $h_1 \sim h_p$ affine, we could then know $epi(f_0) \sim epi(f_m)$, $epi(h_1) \sim epi(h_p)$ are all convex sets. The intersection of convex sets are also convex, we deduce that $\mathcal{A}$ is convex, as desired.

Q2. a) $L(x, \lambda) = e^x - \lambda x$.

$(e^x - \lambda x)' = e^x - \lambda$, $x = \log \lambda$. Plug in.

We have $g(\lambda) = \begin{cases} \lambda - \log \lambda & \lambda \ge 1 \\ 1 & o.w. \end{cases}$



For $0 < \lambda < 1$, we have $g(\lambda) = 1$.

b) The dual problem is

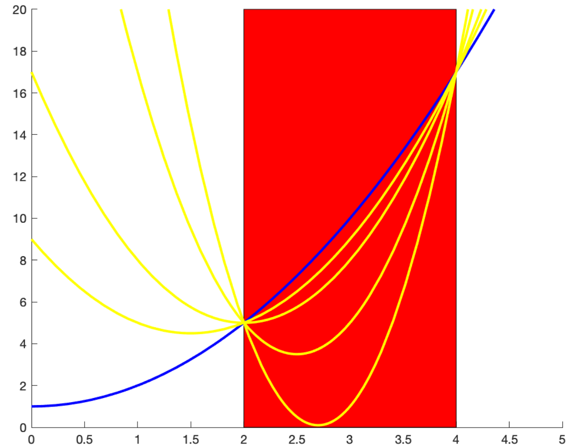
$$\max \ \lambda - \lambda \log(\lambda)$$
$$\text{s.t.} \ \lambda \ge 1.$$

$\frac{\partial g(\lambda)}{\partial \lambda} = -\log(\lambda) = 0$, so $d^* = 1$ when $\lambda \in [0, 1]$.

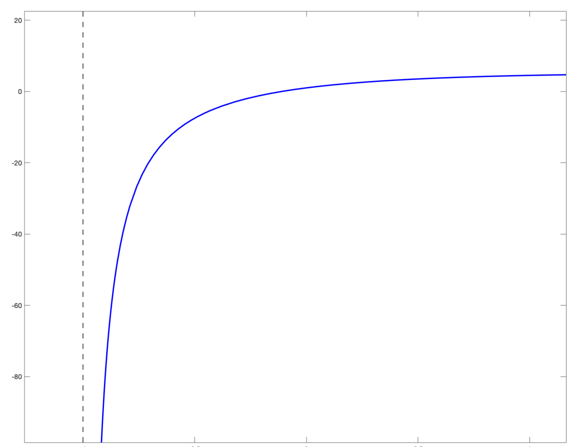
The strong duality holds since the dual gap is 0.

Attached Graphs

Question 3



Plot the Lagrangian L(x, λ) versus x for a few positive values of λ



Lagrange dual function g