

Guaranteed Minimum-Rank Solutions of Linear Matrix Equations via Nuclear Norm Minimization*

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Abstract. The affine rank minimization problem consists of finding a matrix of minimum rank that satisfies a given system of linear equality constraints. Such problems have appeared in the literature of a diverse set of fields including system identification and control, Euclidean embedding, and collaborative filtering. Although specific instances can often be solved with specialized algorithms, the general affine rank minimization problem is NP-hard because it contains vector cardinality minimization as a special case. In this paper, we show that if a certain restricted isometry property holds for the linear transformation defining the constraints, the minimum-rank solution can be recovered by solving a convex optimization problem, namely, the minimization of the nuclear norm over the given affine space. We present several random ensembles of equations where the restricted isometry property holds with overwhelming probability, provided the codimension of the subspace is sufficiently large. The techniques used in our analysis have strong parallels in the compressed sensing framework. We discuss how affine rank minimization generalizes this preexisting concept and outline a dictionary relating concepts from cardinality minimization to those of rank minimization. We also discuss several algorithmic approaches to minimizing the nuclear norm and illustrate our results with numerical examples.

Key words. rank, convex optimization, matrix norms, random matrices, compressed sensing, semidefinite programming

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I. Introduction. Notions such as order, complexity, or dimensionality can often be expressed by means of the rank of an appropriate matrix. For example, a low-rank matrix could correspond to a low-degree statistical model for a random process (e.g., factor analysis), a low-order realization of a linear system [38], a low-order controller for a plant [30], or a low-dimensional embedding of data in Euclidean space [48]. If

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Proof. Consider a full singular value decomposition of A ,

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V',$$

and let $\hat{B} := U' BV$. Partition \hat{B} as

$$\hat{B} = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{bmatrix}.$$

Defining now

$$B_1 := U \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & 0 \end{bmatrix} V', \quad B_2 := U \begin{bmatrix} 0 & 0 \\ 0 & \hat{B}_{22} \end{bmatrix} V',$$

it can be easily verified that B_1 and B_2 satisfy the conditions (1)–(4). \square

We now proceed to a proof of Theorem 3.3.

Proof of Theorem 3.3. By optimality of X^* , we have $\|X_0\|_* \geq \|X^*\|_*$. Let $R := X^* - X_0$. Applying Lemma 3.4 to the matrices X_0 and R , there exist matrices R_0 and R_c such that $R = R_0 + R_c$, $\text{rank } R_0 \leq 2 \text{ rank } X_0$, and $X_0 R'_c = 0$ and $X'_0 R_c = 0$. Then

$$(3.3) \quad \|X_0\|_* \geq \|X_0 + R\|_* \geq \|X_0 + R_c\|_* - \|R_0\|_* = \|X_0\|_* + \|R_c\|_* - \|R_0\|_*,$$

where the middle assertion follows from the triangle inequality and the last one from Lemma 2.3. Rearranging terms, we can conclude that

$$(3.4) \quad \|R_0\|_* \geq \|R_c\|_*.$$

Next we partition R_c into a sum of matrices R_1, R_2, \dots , each of rank at most $3r$. Let $R_c = U \text{diag}(\sigma)V'$ be the SVD of R_c . For each $i \geq 1$, define the index set $I_i = \{3r(i-1) + 1, \dots, 3ri\}$, and let $R_i := U_{I_i} \text{diag}(\sigma_{I_i})V'_{I_i}$ (notice that $\langle R_i, R_j \rangle = 0$ if $i \neq j$). By construction, we have

$$(3.5) \quad \sigma_k \leq \frac{1}{3r} \sum_{j \in I_i} \sigma_j \quad \forall k \in I_{i+1},$$

which implies $\|R_{i+1}\|_F^2 \leq \frac{1}{3r} \|R_i\|_*^2$. We can then compute the following bound:

$$(3.6) \quad \sum_{j \geq 2} \|R_j\|_F \leq \frac{1}{\sqrt{3r}} \sum_{j \geq 1} \|R_j\|_* = \frac{1}{\sqrt{3r}} \|R_c\|_* \leq \frac{1}{\sqrt{3r}} \|R_0\|_* \leq \frac{\sqrt{2r}}{\sqrt{3r}} \|R_0\|_F,$$

where the last inequality follows from (2.1) and the fact that $\text{rank } R_0 \leq 2r$. Finally, note that the rank of $R_0 + R_1$ is at most $5r$, so we may put all this together as

$$\begin{aligned} \|\mathcal{A}(R)\| &\geq \|\mathcal{A}(R_0 + R_1)\| - \sum_{j \geq 2} \|\mathcal{A}(R_j)\| \\ &\geq (1 - \delta_{5r}) \|R_0 + R_1\|_F - (1 + \delta_{3r}) \sum_{j \geq 2} \|R_j\|_F \\ &\geq \left((1 - \delta_{5r}) - \sqrt{\frac{2}{3}}(1 + \delta_{3r}) \right) \|R_0\|_F \\ &\geq \left((1 - \delta_{5r}) - \frac{9}{11}(1 + \delta_{3r}) \right) \|R_0\|_F. \end{aligned} \tag{3.7}$$

the set of models that satisfy the desired constraints is convex, then choosing the simplest model can be cast as a *rank minimization problem*,

$$(1.1) \quad \begin{cases} \text{minimize} & \text{rank } X \\ \text{subject to} & X \in \mathcal{C}, \end{cases}$$

where $X \in \mathbb{R}^{m \times n}$ is the decision variable and \mathcal{C} is some given convex constraint set. This problem arises in various application areas; see, for example, [37, 52]. In certain instances with very special structure, the rank minimization problem can be solved by using the singular value decomposition (SVD), or can be exactly reduced to the solution of linear systems [54, 60]. In general, however, the problem (1.1) is a challenging nonconvex optimization problem, and even when \mathcal{C} is an affine subspace it seems to require worst-case exponential running time in both theory and practice. For the general case, a variety of heuristic algorithms based on local optimization, including alternating projections and its variations [42, 58], alternating matrix inequalities [68], linearization [31], and augmented Lagrangian methods [34], have been proposed.

A recent heuristic introduced by Fazel et al. in [37, 38] minimizes the *nuclear norm*, or the sum of the singular values of the matrix, over the constraint set. The nuclear norm is a convex function, can be optimized efficiently, and is the best convex lower approximation of the rank function over the set of matrices with spectral norm less than or equal to one; see Theorem 2.2 below. When the matrix variable is symmetric and positive semidefinite, this heuristic is equivalent to the trace heuristic sometimes used by the systems and control community (e.g., [6, 54]). The nuclear norm heuristic has been observed to produce very low-rank solutions in practice, but a theoretical characterization of when it produces the minimum-rank solution has not been previously available. This paper provides the first such mathematical characterization.

In this paper, we focus on the scenario where the set of feasible models or designs is affine in the matrix variable, and we consider the *affine rank minimization problem*,

$$(1.2) \quad \begin{cases} \text{minimize} & \text{rank } X \\ \text{subject to} & \mathcal{A}(X) = b, \end{cases}$$

where $X \in \mathbb{R}^{m \times n}$ is the decision variable and the linear map $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ and vector $b \in \mathbb{R}^p$ are given. Our work is built upon a large body of literature on a related optimization problem. When the matrix variable is constrained to be diagonal, the affine rank minimization problem reduces to finding the *sparsest vector* in an affine subspace. This problem is commonly referred to as *cardinality minimization*, since one seeks the vector whose support has the smallest cardinality, and is known to be NP-hard [56]. Just as in the case of rank minimization, a variety of heuristic algorithms have been proposed to solve cardinality minimization problems including projection pursuit [40, 51, 62] and orthogonal matching pursuit [19, 24, 61].

For diagonal matrices, the sum of the singular values is equal to the sum of the absolute values (i.e., the ℓ_1 norm) of the diagonal elements. Minimization of the ℓ_1 norm is a well-known heuristic for the cardinality minimization problem, employed as early as the 1970s by geophysicists attempting to deconvolve seismic activity [21, 74]. Since then, ℓ_1 minimization has been applied to a variety of cardinality minimization problems including image denoising [65], model selection in statistics [76], sparse approximation [20], portfolio optimization with fixed transaction costs [50], design of sparse interconnect wiring in circuits [80], and design of sparse feedback gains in control systems [43].

We can write this as
 $\langle A_1, X \rangle = b_1$
 \vdots
 $\langle A_p, X \rangle = b_p$
as in the constraints
for the primal
version of SDP
(but there, the
 A_i and X
are symmetric)

Recently, results pioneered by Candès and Tao [16] and Donoho [25] have characterized a vast set of instances for which the ℓ_1 heuristic can be a priori guaranteed to yield the optimal solution. These techniques provide the foundations of the recently developed *compressed sensing* or *compressive sampling* frameworks for measurement, coding, and signal estimation. As has been shown by a number of research groups (e.g., [4, 14, 15, 16]), the ℓ_1 heuristic for cardinality minimization provably recovers the sparsest solution whenever the sensing matrix has certain “basis incoherence” properties, and, in particular, when it is randomly chosen according to certain specific ensembles.

The fact that the ℓ_1 heuristic is a special case of the nuclear norm heuristic suggests that these results from the compressed sensing literature might be extended to provide guarantees about the nuclear norm heuristic for the more general rank minimization problem. In this paper, we show that this is indeed the case, and the parallels are surprisingly strong. Following the program laid out in the work of Candès and Tao, our main contribution is the development of a restricted isometry property (RIP), under which the nuclear norm heuristic can be *guaranteed* to produce the minimum-rank solution. Furthermore, as in the case for the ℓ_1 heuristic, we provide several specific examples of matrix ensembles for which the RIP holds with overwhelming probability. Our results considerably extend the compressed sensing machinery in a so far undeveloped direction, by allowing a much more general notion of parsimonious models that rely on low-rank assumptions instead of cardinality restrictions.

To make the parallels as clear as possible, we begin by establishing a dictionary between the matrix rank and nuclear norm minimization problems and the vector sparsity and ℓ_1 norm problems in section 2. In the process of this discussion, we present a review of many useful properties of the matrices and matrix norms necessary for the main results. We then generalize in section 3 the notion of restricted isometry to matrices, and show that when linear mappings are restricted isometries, recovering low-rank solutions of underdetermined systems can be achieved by nuclear norm minimization. In section 4, we present several families of random linear maps that are restricted isometries with overwhelming probability when the dimensions are sufficiently large. In section 5, we briefly discuss three different algorithms designed for solving the nuclear norm minimization problem and their relative strengths and weaknesses: interior point methods, gradient projection methods, and a low-rank factorization technique. In section 6, we present several numerical experiments and demonstrate that in practice nuclear norm minimization recovers the lowest rank solutions of affine sets with even fewer constraints than those guaranteed by our mathematical analysis. Finally, in section 7, we list a number of possible directions for future research.

1.1. When Are Random Constraints Interesting for Rank Minimization? As in the case of compressed sensing, the conditions we derive to guarantee properties of the nuclear norm heuristic are deterministic, but they are at least as difficult to check as solving the rank minimization problem itself. We are only able to guarantee that the nuclear norm heuristic recovers the minimum-rank solution of $\mathcal{A}(X) = b$ when \mathcal{A} is sampled from specific ensembles of random maps. The constraints appearing in many of the applications mentioned above, such as low-order control system design, are typically not random at all and have structured demands according to the specifics of the design problem. Furthermore, in many of these applications, the problem is formulated as minimizing rank subject to some more general convex constraints than the linear equalities we are considering. It thus behooves us to present several

examples where random affine constraints manifest themselves in practical scenarios for which no practical solution procedure is known.

Minimum-Order Linear System Realization. Rank minimization forms the basis of many model reduction and low-order system identification problems for linear time-invariant (LTI) systems. The following example illustrates how random constraints might arise in this context. Consider the problem of finding the minimum-order discrete-time LTI system that is consistent with a set of time-domain observations. In particular, suppose our observations are the system output sampled at a fixed time N after a random Gaussian input signal is applied from $t = 0$ to $t = N$. Suppose we make such measurements for p different randomly generated input signals, $a_i(t)$. Then we observe the outputs $y_i(N) = \sum_{t=0}^N a_i(N-t)h(t)$ for $i = 1, \dots, p$, where $h(t)$ denotes the impulse response. We can write this compactly as $y = Ah$, where $h = [h(0), \dots, h(N)]'$ and $A_{ij} = a_i(N-j)$.

From linear system theory, the order of the minimal realization for such a system is given by the rank of the following Hankel matrix (see, e.g., [39, 69]):

$$\text{hank}(h) := \begin{bmatrix} h(0) & h(1) & \cdots & h(N) \\ h(1) & h(2) & \cdots & h(N+1) \\ \vdots & \vdots & & \vdots \\ h(N) & h(N+1) & \cdots & h(2N) \end{bmatrix}.$$

Therefore, the problem can be expressed as

$$\begin{aligned} & \text{minimize} && \text{rank } \text{hank}(h) \\ & \text{subject to} && Ah = y, \end{aligned}$$

where the optimization variables are $h(0), \dots, h(2N)$ and the matrix A consists of independent, identically distributed (i.i.d.) zero-mean Gaussian entries.

Low-Rank Matrix Completion. In the matrix completion problem, we are given a random subset of entries of a matrix and would like to fill in the missing entries such that the resulting matrix has the lowest possible rank. This problem arises in machine learning scenarios where we are given partially observed examples of a process with a low-rank covariance matrix and would like to estimate the missing data. Such models are ubiquitous in factor analysis, collaborative filtering, and latent semantic analysis [63, 70]. In many of these settings, some prior probability distribution (such as a Bernoulli model or uniform distribution on subsets) is assumed to generate the set of available entries.

Suppose we are presented with a set of triples $(I(i), J(i), S(i))$ for $i = 1, \dots, p$ and wish to find the matrix with $S(i)$ in the entry corresponding to row $I(i)$ and column $J(i)$ for all i . The matrix completion problem seeks to

$$\begin{aligned} & \text{minimize} && \text{rank } Y \\ & \text{subject to} && Y_{I(i), J(i)} = S(i), \quad i = 1, \dots, K, \end{aligned}$$

which is a special case of the affine rank minimization problem.

Low-Dimensional Euclidean Embedding Problems. A problem that arises in a variety of fields is the determination of configurations of points in low-dimensional Euclidean spaces subject to some given distance information. In computational chemistry, these problems arise in inferring the three-dimensional structure of a molecule

(molecular conformation) from information about interatomic distances [78]. In manifold learning, one may be given high-dimensional data with low-dimensional structure that can be recovered by searching for a low-dimensional embedding of the data preserving local distance information [64, 75].

A symmetric matrix $D \in \mathcal{S}^n$ is called a *Euclidean distance matrix* (EDM) if there exist points x_1, \dots, x_n in \mathbb{R}^d such that $D_{ij} = \|x_i - x_j\|^2$. Let $V := I_n - \frac{1}{n}\mathbf{1}\mathbf{1}^T$ be the orthogonal projection matrix onto the hyperplane $\{v \in \mathbb{R}^n : \mathbf{1}^T v = 0\}$. A classical result by Schoenberg states that D is an EDM of n points in \mathbb{R}^d if and only if $D_{ii} = 0$, the matrix VDV is negative semidefinite, and $\text{rank } VDV$ is less than or equal to d [67]. If the matrix D is known exactly, the corresponding configuration of points (up to a unitary transform) is obtained by simply taking a matrix square root of $-\frac{1}{2}VDV$. In many cases, however, only a random sampling collection of the distances are available. The problem of finding a valid EDM consistent with the known pairwise distances and with the smallest embedding dimension can be expressed as the rank optimization problem

$$\begin{aligned} & \text{minimize} && \text{rank } VDV \\ & \text{subject to} && VDV \preceq 0, \\ & && \mathcal{A}(D) = b, \end{aligned}$$

where $\mathcal{A} : \mathcal{S}^n \rightarrow \mathbb{R}^p$ is a random sampling operator as discussed in the matrix completion problem.

This problem involves a linear matrix inequality (LMI) and appears to be more general than the equality constrained rank minimization problem. However, general LMIs can equivalently be expressed as rank constraints on an appropriately defined block matrix. The rank of a block symmetric matrix is equal to the rank of a diagonal block plus the rank of its Schur complement (see, e.g., [46, section 2.2]). Given a function f that maps matrices into $q \times q$ symmetric matrices, the condition that $f(X)$ is positive semidefinite can be equivalently expressed through a rank constraint as

$$f(X) \succeq 0 \iff \text{rank} \left(\begin{bmatrix} I_q & B \\ B' & f(X) \end{bmatrix} \right) \leq q \quad \text{for some } B \in \mathbb{R}^{q \times q}.$$

That is, if there exists a matrix B satisfying the inequality above, then $f(X) = B'B \succeq 0$. Using this equivalent representation allows us to rewrite problem (1.1) with general LMI constraints as an affine rank minimization problem.

Image Compression. A simple and well-known method to compress two-dimensional images can be obtained by using the SVD (e.g., [3]). The basic idea is to associate to the given grayscale image a rectangular matrix M , with the entries M_{ij} corresponding to the gray level of the (i, j) pixel. The best rank- k approximation of M is given by

$$X_k^* := \underset{\text{rank } X \leq k}{\operatorname{argmin}} \|M - X\|,$$

where $\|\cdot\|$ is any unitarily invariant norm. By the classical Eckart–Young–Mirsky theorem [28, 55], the optimal approximant is given by a truncated SVD of M , i.e., if $M = U\Sigma V^T$, then $X_k^* = U\Sigma_k V^T$, where the first k diagonal entries of Σ_k are the largest k singular values and the rest of the entries are zero. If, for a given rank k , the approximation error $\|M - X_k^*\|$ is small enough, then the amount of data needed to encode the information about the image is $k(m + n - k)$ real numbers, which can be much smaller than the mn required to transmit the values of all the entries.

Consider a given image, whose associated matrix M has low rank or can be well-approximated by a low-rank matrix. As proposed by Wakin et al. [81], a single-pixel camera would ideally produce measurements that are random linear combinations of all the pixels of the given image. Under this situation, the image reconstruction problem boils down exactly to affine rank minimization, where the constraints are given by the random linear functionals.

It should be remarked that the simple SVD image compression scheme described has certain deficiencies that more sophisticated techniques do not share (in particular, the lack of invariance of the description length under rotations). Nevertheless, due to its simplicity and relatively good practical performance, this method is particularly popular in introductory treatments and numerical linear algebra textbooks.

2. From Compressed Sensing to Rank Minimization. As discussed above, when the matrix variable is constrained to be diagonal, the affine rank minimization problem (1.2) reduces to the cardinality minimization problem of finding the element in the affine space with the smallest number of nonzero components. In this section we will establish a dictionary between the concepts of rank and cardinality minimization. The main elements of this correspondence are outlined in Table 2.1. With these elements in place, the existing proofs of sparsity recovery provide a template for the more general case of low-rank recovery.

In establishing our dictionary, we will provide a review of useful facts regarding matrix norms and their characterization as convex optimization problems. We will show how computing both the operator norm and the nuclear norm of a matrix can be cast as semidefinite programming (SDP) problems. We also establish suitable optimality conditions for the minimization of the nuclear norm subject to affine equality constraints, the main convex optimization problem studied in this article. Our discussion of matrix norms and their connections to SDP and convex optimization will mostly follow the discussion in [9, 37, 79], where extensive lists of references are provided.

Matrix vs. Vector Norms. The three vector norms that play significant roles in the compressed sensing framework are the ℓ_1 , ℓ_2 , and ℓ_∞ norms, denoted by $\|x\|_1$, $\|x\|$, and $\|x\|_\infty$, respectively. These norms have natural generalizations to matrices, inheriting many appealing properties from the vector case. In particular, there is a parallel duality structure.

For a rectangular matrix $X \in \mathbb{R}^{m \times n}$, $\sigma_i(X)$ denotes the i th largest singular value of X and is equal to the square root of the i th largest eigenvalue of XX' . The rank of X will usually be denoted by r and is equal to the number of nonzero singular values. For matrices X and Y of the same dimensions, we define the inner product in $\mathbb{R}^{m \times n}$ as $\langle X, Y \rangle := \text{Tr}(X'Y) = \sum_{i=1}^m \sum_{j=1}^n X_{ij}Y_{ij}$. The norm associated with this inner product is called the Frobenius (or Hilbert–Schmidt) norm $\|\cdot\|_F$. The Frobenius

*X' denotes
 X -transposed
(X^T).*

Table 2.1 A dictionary relating the concepts of cardinality and rank minimization.

Parsimony concept	cardinality	rank
Hilbert space norm	Euclidean	Frobenius
Sparsity inducing norm	ℓ_1	nuclear
Dual norm	ℓ_∞	operator
Norm additivity	disjoint support	orthogonal row/column spaces
Convex optimization	linear programming	semidefinite programming

norm is also equal to the Euclidean, or ℓ_2 , norm of the vector of singular values, i.e.,

$$\|X\|_F := \sqrt{\langle X, X \rangle} = \sqrt{\text{Tr}(X'X)} = \left(\sum_{i=1}^m \sum_{j=1}^n X_{ij}^2 \right)^{\frac{1}{2}} = \left(\sum_{i=1}^r \sigma_i^2 \right)^{\frac{1}{2}}.$$

The operator norm (or induced 2-norm) of a matrix is equal to its largest singular value (i.e., the ℓ_∞ norm of the singular values):

$$\|X\| := \sigma_1(X).$$

The nuclear norm of a matrix is equal to the sum of its singular values, i.e.,

$$\|X\|_* := \sum_{i=1}^r \sigma_i(X),$$

and is alternatively known by several other names including the Schatten 1-norm, the Ky Fan r -norm, and the trace class norm. Since the singular values are all positive, the nuclear norm is also equal to the ℓ_1 norm of the vector of singular values. These three norms are related by the following inequalities which hold for any matrix X of rank at most r :

$$(2.1) \quad \|X\| \leq \|X\|_F \leq \|X\|_* \leq \sqrt{r}\|X\|_F \leq r\|X\|.$$

usually written
 $\|X\|_2$

ff; HW

Dual Norms. For any given norm $\|\cdot\|$ in an inner product space, there exists a dual norm $\|\cdot\|_d$ defined as

$$(2.2) \quad \|X\|_d := \sup\{\langle X, Y \rangle : Y \in \mathbb{R}^{m \times n}, \|Y\| \leq 1\}.$$

Furthermore, the norm dual to the norm $\|\cdot\|_d$ is again the original norm $\|\cdot\|$.

In the case of vector norms in \mathbb{R}^n , it is well known that the dual norm of the ℓ_p norm (with $1 < p < \infty$) is the ℓ_q norm, where $\frac{1}{p} + \frac{1}{q} = 1$. This fact is essentially equivalent to Hölder's inequality. Similarly, the dual norm of the ℓ_∞ norm of a vector is the ℓ_1 norm. These facts also extend to the matrix norms we have defined. For instance, the dual norm of the Frobenius norm is the Frobenius norm. This can be verified by simple calculus (or Cauchy-Schwarz), since

$$\sup\{\text{Tr}(X'Y) : Y \in \mathbb{R}^{m \times n}, \text{Tr}(Y'Y) \leq 1\}$$

is equal to $\|X\|_F$, with this bound being achieved if Y is equal to $X/\|X\|_F$. Similarly, as shown below, the dual norm of the operator norm is the nuclear norm. The proof of this fact will also allow us to present variational characterizations of each of these norms as semidefinite programs.

PROPOSITION 2.1. *The dual norm of the operator norm $\|\cdot\|$ in $\mathbb{R}^{m \times n}$ is the nuclear norm $\|\cdot\|_*$.*

Proof. First consider an $m \times n$ matrix Z . The fact that Z has operator norm less than or equal to t can be expressed as a linear matrix inequality,

$$(2.3) \quad \|Z\| \leq t \iff t^2 I_m - ZZ' \succeq 0 \iff \begin{bmatrix} tI_m & Z \\ Z' & tI_n \end{bmatrix} \succeq 0,$$

proof follows from
 question in HW
 relating eigenvalues
 & singular values

where the last implication follows from a Schur complement argument. As a consequence, we can give a semidefinite optimization characterization of the operator norm, namely,

$$(2.4) \quad \|Z\| = \inf \left\{ t : \begin{bmatrix} tI_m & Z \\ Z' & tI_n \end{bmatrix} \succeq 0 \right\}.$$

Now let $X = U\Sigma V'$ be a singular value decomposition of an $m \times n$ matrix X , where U is an $m \times r$ matrix, V is an $n \times r$ matrix, Σ is an $r \times r$ diagonal matrix, and r is the rank of X . Let $Y := UV'$. Then $\|Y\| = 1$ and $\text{Tr}(XY') = \sum_{i=1}^r \sigma_i(X) = \|X\|_*$, and hence the dual norm is greater than or equal to the nuclear norm.

To provide an upper bound on the dual norm, we appeal to semidefinite programming duality. From the characterization in (2.3), the optimization problem

$$\begin{aligned} & \text{maximize} && \text{Tr}(X'Y) \\ & \text{subject to} && \|Y\| \leq 1 \end{aligned}$$

is equivalent to the semidefinite program

$$(2.5) \quad \begin{aligned} & \text{maximize} && \text{Tr}(X'Y) \\ & \text{subject to} && \begin{bmatrix} I_m & Y \\ Y' & I_n \end{bmatrix} \succeq 0. \end{aligned}$$

The dual of this semidefinite program (after an inconsequential rescaling) is given by

$$(2.6) \quad \begin{aligned} & \text{minimize} && \frac{1}{2}(\text{Tr}(W_1) + \text{Tr}(W_2)) \\ & \text{subject to} && \begin{bmatrix} W_1 & X \\ X' & W_2 \end{bmatrix} \succeq 0. \end{aligned}$$

Set $W_1 := U\Sigma U'$ and $W_2 := V\Sigma V'$. Then the triple (W_1, W_2, X) is feasible for (2.6) since

$$\begin{bmatrix} W_1 & X \\ X' & W_2 \end{bmatrix} = \begin{bmatrix} U & \\ V & \end{bmatrix} \Sigma \begin{bmatrix} U & \\ V & \end{bmatrix}' \succeq 0.$$

Furthermore, we have $\text{Tr}(W_1) = \text{Tr}(W_2) = \text{Tr}(\Sigma)$, and thus the objective function satisfies $(\text{Tr}(W_1) + \text{Tr}(W_2))/2 = \text{Tr} \Sigma = \|X\|_*$. Since any feasible solution of (2.6) provides an upper bound for (2.5), we have that the dual norm is less than or equal to the nuclear norm of X , thus proving the proposition. \square

Notice that the argument given in the proof above further shows that the nuclear norm $\|X\|_*$ can be computed using either the semidefinite program (2.5) or its dual (2.6), since there is no duality gap between them. Alternatively, this could have also been proven using a Slater-type interior point condition since both (2.5) and (2.6) admit strictly feasible solutions. Interested readers can find an in-depth discussion of Slater conditions for semidefinite programming in Chapter 4 of [83].

Convex Envelopes of Rank and Cardinality Functions. Let \mathcal{C} be a given convex set. The *convex envelope* of a (possibly nonconvex) function $f : \mathcal{C} \rightarrow \mathbb{R}$ is defined as the largest convex function g such that $g(x) \leq f(x)$ for all $x \in \mathcal{C}$ (see, e.g., [44]). This means that, among all convex functions that lower bound f , g is the best pointwise approximation to f . In particular, if g can be conveniently described, it can serve as an approximation to f that can be minimized efficiently.

directly from $\|X\|_* = \sum_{i=1}^r \sigma_i(X) \leq r \cdot 1 \quad (\#)$

By the chain of inequalities in (2.1), we have that $\text{rank } X \geq \|X\|_*/\|X\|$ for all X . For all matrices with $\|X\| \leq 1$, we must have that $\text{rank } X \geq \|X\|_*$, so the nuclear norm is a convex lower bound of the rank function on the unit ball in the operator norm. In fact, it can be shown that this is the tightest convex lower bound.

THEOREM 2.2 (see [37]). *The convex envelope of $\text{rank } X$ on the set $\{X \in \mathbb{R}^{m \times n} : \|X\| \leq 1\}$ is the nuclear norm $\|X\|_*$.*

The proof of this statement is given in [37]. It relies on von Neumann's trace inequality [45] and a basic result from convex analysis that establishes that, under certain technical conditions, the biconjugate of a function is its convex envelope [44].

Theorem 2.2 provides the following interpretation of the nuclear norm heuristic for the affine rank minimization problem. Suppose X_0 is the minimum-rank solution of $\mathcal{A}(X) = b$ and $M = \|X_0\|$. The convex envelope of the rank on the set $\mathcal{C} = \{X \in \mathbb{R}^{m \times n} : \|X\| \leq M\}$ is $\|X\|_*/M$. Let X_* be the minimum nuclear norm solution of $\mathcal{A}(X) = b$. Then we have

$$\|X_*\|_*/M \leq \text{rank } X_0 \leq \text{rank } X_*,$$

providing an upper and lower bound on the optimal rank when the norm of the optimal solution is known. Furthermore, this is the tightest lower bound among all convex lower bounds of the rank function on the set \mathcal{C} .

For vectors, we have a similar inequality. Let $\text{card}(x)$ denote the cardinality function which counts the number of nonzero entries in the vector x ; then we have $\text{card}(x) \geq \|x\|_1/\|x\|_\infty$. Not surprisingly, the ℓ_1 norm is also the convex envelope of the cardinality function over the set $\{x \in \mathbb{R}^n : \|x\|_\infty \leq 1\}$. This result can be either proven directly or can be seen as a special case of the above theorem.

Additivity of Rank and Nuclear Norm. A function f mapping a linear space into \mathbb{R} is called subadditive if $f(x+y) \leq f(x)+f(y)$. It is additive if $f(x+y) = f(x)+f(y)$. In the case of vectors, both the cardinality function and the ℓ_1 norm are subadditive. That is, it always holds that the number of nonzeros in $x+y$ is less than or equal to the number of nonzeros in x plus the number of nonzeros of y ; furthermore (by the triangle inequality), $\|x+y\|_1 \leq \|x\|_1 + \|y\|_1$. In particular, the cardinality function is additive exactly when the vectors x and y have disjoint support. In this case, the ℓ_1 norm is also additive, in the sense that $\|x+y\|_1 = \|x\|_1 + \|y\|_1$.

For matrices, the rank function is subadditive. For the rank to be additive, it is necessary and sufficient that the row and column spaces of the two matrices intersect only at the origin, since in this case they operate in essentially disjoint spaces (see, e.g., [53]). As we will show below, a related condition that ensures that the nuclear norm is additive is that the matrices A and B have row and column spaces that are orthogonal. In fact, a compact sufficient condition for the additivity of the nuclear norm will be that $AB' = 0$ and $A'B = 0$. This is a stronger requirement than the aforementioned condition for rank additivity, as orthogonal subspaces only intersect at the origin. The disparity arises because the nuclear norm of a linear map depends on the choice of inner products on the spaces \mathbb{R}^m and \mathbb{R}^n on which the map acts, whereas its rank is independent of such a choice.

LEMMA 2.3. *Let A and B be matrices of the same dimensions. If $AB' = 0$ and $A'B = 0$, then $\|A+B\|_* = \|A\|_* + \|B\|_*$.*

Proof. Consider SVDs of A and B :

$$A = U_A \Sigma_A V_A', \quad B = U_B \Sigma_B V_B'.$$

Since U_A and U_B are left-invertible, the condition $AB' = 0$ implies that $V_A' V_B = 0$.

means: $\exists T \text{ s.t. } TU_A = I$

namely: $T = V_A'$ as V_A has orthonormal cols.

$$\text{so } \sum_A^{-1} V_A' A = V_A'.$$

$$\text{and } \sum_B^{-1} V_B' B = V_B'.$$

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$$\text{so } V_A' V_B = \sum_A^{-1} V_A' A B' V_B \sum_B^{-1} = 0.$$

because:
(1) $\|X_*\|_* \leq \|X_0\|_*$
as X_* minimizes
the nuclear norm
and
(2) $\|X_0\|_* \leq \text{rank}(X_0)$
from ^M(#) above.

 X_* is a matrix
 $\|X_*\|_*$ is its
nuclear norm.

Similarly, $A'B = 0$ implies that $U_A' U_B = 0$. Thus, the following is a valid singular value decomposition of A for $A + B$:

$$A + B = [U_A \quad U_B] \begin{bmatrix} \Sigma_A & \\ & \Sigma_B \end{bmatrix} [V_A \quad V_B]'.$$

This shows that the singular values of $A + B$ are equal to the union (with repetition) of the singular values of A and B . Hence, $\|A + B\|_* = \|A\|_* + \|B\|_*$ as desired. \square

COROLLARY 2.4. *Let A and B be matrices of the same dimensions. If the row and column spaces of A and B are orthogonal, then $\|A + B\|_* = \|A\|_* + \|B\|_*$.*

Proof. It suffices to show that if the row and column spaces of A and B are orthogonal, then $AB' = 0$ and $A'B = 0$. But this is immediate: if the columns of A are orthogonal to the columns of B , we have $A'B = 0$. Similarly, orthogonal row spaces imply that $AB' = 0$ as well. \square

Nuclear Norm Minimization. Let us turn now to the study of equality-constrained norm minimization problems where we are searching for a matrix $X \in \mathbb{R}^{m \times n}$ of minimum nuclear norm belonging to a given affine subspace. In our applications, the subspace is usually described by linear equations of the form $\mathcal{A}(X) = b$, where $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ is a linear mapping. This problem admits the primal-dual convex formulation

$$(2.7) \quad \begin{array}{ll} \text{minimize} & \|X\|_* \\ \text{subject to} & \mathcal{A}(X) = b, \end{array} \quad \begin{array}{ll} \text{maximize} & b'z \\ \text{subject to} & \|\mathcal{A}^*(z)\| \leq 1, \end{array}$$

where $\mathcal{A}^* : \mathbb{R}^p \rightarrow \mathbb{R}^{m \times n}$ is the adjoint of \mathcal{A} . By a primal-dual pair, we mean that each optimization problem is the Lagrangian dual of the other, and hence that the minimum of the first optimization problem is equal to the maximum of the second. This notion of duality generalizes the well-known case of linear programming, and is in fact applicable to all convex optimization problems; see, e.g., [8, 9].

The formulation (2.7) is valid for any norm minimization problem, by replacing the norms appearing above by any dual pair of norms. In particular, if we replace the nuclear norm with the ℓ_1 norm and the operator norm with the ℓ_∞ norm, we obtain a primal-dual pair of optimization problems that can be reformulated in terms of linear programming.

Using the SDP characterizations of the nuclear and operator norms given in (2.5)–(2.6) above allows us to rewrite (2.7) as the following primal-dual pair of semidefinite programs:

$$(2.8) \quad \begin{array}{ll} \text{minimize} & \frac{1}{2}(\text{Tr}(W_1) + \text{Tr}(W_2)) \\ \text{subject to} & \begin{bmatrix} W_1 & X \\ X' & W_2 \end{bmatrix} \succeq 0 \\ & \mathcal{A}(X) = b, \end{array} \quad \begin{array}{ll} \text{maximize} & b'z \\ \text{subject to} & \begin{bmatrix} I_m & \mathcal{A}^*(z) \\ \mathcal{A}^*(z)' & I_n \end{bmatrix} \succeq 0. \end{array} \quad \text{HW.}$$

Optimality Conditions. In order to describe the optimality conditions for the norm minimization problem (2.7), we must first characterize the set of all subgradients (i.e., the subdifferential) of the nuclear norm. Recall that for a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, the subdifferential of f at $x \in \mathbb{R}^n$ is the compact convex set

$$\partial f(x) := \{d \in \mathbb{R}^n : f(y) \geq f(x) + \langle d, y - x \rangle \quad \forall y \in \mathbb{R}^n\}.$$

Let X be an $m \times n$ matrix with rank r and let $X = U\Sigma V'$ be an SVD where $U \in \mathbb{R}^{m \times r}$, $V \in \mathbb{R}^{n \times r}$, and Σ is an $r \times r$ diagonal matrix. The subdifferential of the nuclear norm at X is then given by (e.g., [47, 82])

$$(2.9) \quad \partial\|X\|_* = \{UV' + W : W \text{ and } X \text{ have orthogonal row/column spaces and } \|W\| \leq 1\}.$$

For comparison, recall the case of the ℓ_1 norm, where

$$(2.10) \quad \partial\|x\|_1 = \{\text{sign}(x) + w : w \text{ and } x \text{ have disjoint support and } \|w\|_\infty \leq 1\}.$$

The similarity between (2.9) and (2.10) is particularly transparent if we recall the *polar decomposition* of a matrix into a product of orthogonal and positive semidefinite matrices (see, e.g., [46]). The “angular” component of the matrix X is given exactly by UV' . Thus, these subgradients always have the form of an “angle” (or sign), plus possibly a contraction in an orthogonal direction if the norm is not differentiable at the current point.

We can now write concise optimality conditions for the optimization problem (2.7). A matrix X is an optimal solution for (2.7) if there exists a vector $z \in \mathbb{R}^p$ such that

$$(2.11) \quad \mathcal{A}(X) = b, \quad \mathcal{A}^*(z) \in \partial\|X\|_*.$$

The first condition in (2.11) requires feasibility of the linear equations, and the second one guarantees that there is no feasible direction of improvement. Indeed, since $\mathcal{A}^*(z)$ is in the subdifferential at X , for any Y in the primal feasible set of (2.7) we have

$$\|Y\|_* \geq \|X\|_* + \langle \mathcal{A}^*(z), Y - X \rangle = \|X\|_* + \langle z, \mathcal{A}(Y - X) \rangle = \|X\|_*,$$

where the last step follows from the feasibility of X and Y . As we can see, the optimality conditions (2.11) for the nuclear norm minimization problem exactly parallel those of the ℓ_1 optimization case.

These optimality conditions can be used to check and certify whether a given candidate X is indeed a minimum nuclear norm solution. For this, it is sufficient (and necessary) to find a vector $z \in \mathbb{R}^p$ in the subdifferential of the norm, i.e., such that the left- and right-singular spaces of $\mathcal{A}^*(z)$ are aligned with those of X , and that behaves like a contraction on the orthogonal complement.

3. Restricted Isometry and Recovery of Low-Rank Matrices. Let us now turn to the central problem analyzed in this paper. Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear map and let X_0 be a matrix of rank r . Set $b := \mathcal{A}(X_0)$, and define

$$(3.1) \quad \boxed{X^* := \arg \min_X \|X\|_* \quad \text{subject to} \quad \mathcal{A}(X) = b.}$$

That is, X^* is the element of the affine space defined by \mathcal{A} and b with smallest nuclear norm. In this section, we will characterize specific cases when we can a priori guarantee that $X^* = X_0$. The key conditions will be determined by the values of a sequence of parameters δ_r that quantify the behavior of the linear map \mathcal{A} when restricted to the set of matrices of rank r . The following definition is the natural generalization of the restricted isometry property (RIP) from vectors to matrices.

DEFINITION 3.1. Let $\mathcal{A} : \mathbb{R}^{m \times n} \rightarrow \mathbb{R}^p$ be a linear map. Without loss of generality, assume $m \leq n$. For every integer r with $1 \leq r \leq m$, define the r -restricted isometry constant to be the smallest number $\delta_r(\mathcal{A})$ such that

$$(3.2) \quad (1 - \delta_r(\mathcal{A}))\|X\|_F \leq \|\mathcal{A}(X)\| \leq (1 + \delta_r(\mathcal{A}))\|X\|_F$$

holds for all matrices X of rank at most r .

*X₀ no longer a minimal rank solution
X* now instead of X_{*}.*

Extreme examples: (1) $\mathcal{A}(X) = [x_{11} \dots x_{1n} x_{m1} \dots x_{mn}]^T = \text{vec}(X)$. Then \mathcal{A} is essentially the identity and $\|\mathcal{A}(X)\| = \|X\|_F$ so $\delta_r = 0$.

(2) $\mathcal{A}(X) = x_{11}$. This tells us very little about X . Even if we assume $r=1$, for $n=2$ there is no $\delta_r(\mathcal{A}) < 1$.

(3) $n=2, r=1, A(X) = [x_{11} \ x_{12} \ x_{21}]'$. Since $r=1$ this also tells us what x_{22} is, but it is not hard to see that there is no $\delta_r(A) < 1$.

(4) $n=2, r=1, A(X) = [x_{11}, x_{12}+x_{21}, x_{22}]'$. Since $r=1$ this tells us that $x_{11}x_{22} = x_{12}x_{21}$ which together with the easy inequality $|ab| \leq (a^2 + b^2)/2$ allows us to derive

$$\|A(X)\|_2^2 \geq \frac{1}{2} \|X\|_F^2$$

and

$$\|A(X)\|_2^2 \leq \frac{3}{2} \|X\|_F^2$$

so we can take

$$\delta_r = 1 - \frac{1}{\sqrt{2}}.$$

An example shows this is the minimal value. (Thanks to Sarah Stewart for this example.)

THEOREM 3.2. Suppose that $\delta_{2r} < 1$ for some integer $r \geq 1$. Then X_0 is the only matrix of rank at most r satisfying $A(X) = b$.

Proof. Assume, on the contrary, that there exists a rank- r matrix X satisfying $A(X) = b$ and $X \neq X_0$. Then $Z := X_0 - X$ is a nonzero matrix of rank at most $2r$, and $A(Z) = 0$. But then we would have $0 = \|A(Z)\| \geq (1 - \delta_{2r})\|Z\|_F > 0$, which is a contradiction. \square

The proof of the preceding theorem is identical to the argument given by Candès and Tao and is an immediate consequence of our definition of the constant δ_r . No adjustment is necessary in the transition from sparse vectors to low-rank matrices. The key property used is the subadditivity of the rank.

Next we state a simple condition which guarantees $X^* = X_0$. The proof follows the approach in [14], but a few details need to be adjusted when switching from vectors to matrices.

THEOREM 3.3. Suppose that $r \geq 1$ is such that $\delta_{5r} < 1/10$. Then $X^* = X_0$.

We will need the following technical lemma that shows, for any two matrices A and B , that we can decompose B as the sum of two matrices B_1 and B_2 such that rank B_1 is not too large and B_2 satisfies the conditions of Lemma 2.3. This will be the key decomposition for proving Theorem 3.3.

LEMMA 3.4. Let A and B be matrices of the same dimensions. Then there exist matrices B_1 and B_2 such that

- (1) $B = B_1 + B_2$,
- (2) $\text{rank } B_1 \leq 2 \text{ rank } A$,
- (3) $AB'_2 = 0$ and $A'B_2 = 0$,
- (4) $\langle B_1, B_2 \rangle = 0$.

¹In [16], the authors define the RIP with squared norms. We note here that the analysis is identical modulo some algebraic rescaling of constants. We choose to drop the squares as it greatly simplifies the analysis in section 4.

STOP HERE.

Proof. Consider a full singular value decomposition of A ,

$$A = U \begin{bmatrix} \Sigma & 0 \\ 0 & 0 \end{bmatrix} V^*,$$

and let $\hat{B} := U' BV$. Partition \hat{B} as

$$\hat{B} = \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & \hat{B}_{22} \end{bmatrix}.$$

Defining now

$$B_1 := U \begin{bmatrix} \hat{B}_{11} & \hat{B}_{12} \\ \hat{B}_{21} & 0 \end{bmatrix} V^*, \quad B_2 := U \begin{bmatrix} 0 & 0 \\ 0 & \hat{B}_{22} \end{bmatrix} V^*,$$

it can be easily verified that B_1 and B_2 satisfy the conditions (1)–(4). \square

We now proceed to a proof of Theorem 3.3.

Proof of Theorem 3.3. By optimality of X^* , we have $\|X_0\|_* \geq \|X^*\|_*$. Let $R := X^* - X_0$. Applying Lemma 3.4 to the matrices X_0 and R , there exist matrices R_0 and R_c such that $R = R_0 + R_c$, $\text{rank } R_0 \leq 2 \text{ rank } X_0$, and $X_0 R'_c = 0$ and $X'_0 R_c = 0$. Then

$$(3.3) \quad \|X_0\|_* \geq \|X_0 + R\|_* \geq \|X_0 + R_c\|_* - \|R_0\|_* = \|X_0\|_* + \|R_c\|_* - \|R_0\|_*,$$

where the middle assertion follows from the triangle inequality and the last one from Lemma 2.3. Rearranging terms, we can conclude that

$$(3.4) \quad \|R_0\|_* \geq \|R_c\|_*.$$

Next we partition R_c into a sum of matrices R_1, R_2, \dots , each of rank at most $3r$. Let $R_c = U \text{diag}(\sigma)V'$ be the SVD of R_c . For each $i \geq 1$, define the index set $I_i = \{3r(i-1) + 1, \dots, 3ri\}$, and let $R_i := U_{I_i} \text{diag}(\sigma_{I_i})V'_{I_i}$ (notice that $\langle R_i, R_j \rangle = 0$ if $i \neq j$). By construction, we have

$$(3.5) \quad \sigma_k \leq \frac{1}{3r} \sum_{j \in I_i} \sigma_j \quad \forall k \in I_{i+1},$$

which implies $\|R_{i+1}\|_F^2 \leq \frac{1}{3r} \|R_i\|_*^2$. We can then compute the following bound:

$$(3.6) \quad \sum_{j \geq 2} \|R_j\|_F \leq \frac{1}{\sqrt{3r}} \sum_{j \geq 1} \|R_j\|_* = \frac{1}{\sqrt{3r}} \|R_c\|_* \leq \frac{1}{\sqrt{3r}} \|R_0\|_* \leq \frac{\sqrt{2r}}{\sqrt{3r}} \|R_0\|_F,$$

where the last inequality follows from (2.1) and the fact that $\text{rank } R_0 \leq 2r$. Finally, note that the rank of $R_0 + R_1$ is at most $5r$, so we may put all this together as

$$(3.7) \quad \begin{aligned} \|\mathcal{A}(R)\| &\geq \|\mathcal{A}(R_0 + R_1)\| - \sum_{j \geq 2} \|\mathcal{A}(R_j)\| \\ &\geq (1 - \delta_{5r}) \|R_0 + R_1\|_F - (1 + \delta_{3r}) \sum_{j \geq 2} \|R_j\|_F \\ &\geq \left((1 - \delta_{5r}) - \sqrt{\frac{2}{3}} (1 + \delta_{3r}) \right) \|R_0\|_F \\ &\geq \left((1 - \delta_{5r}) - \frac{9}{11} (1 + \delta_{3r}) \right) \|R_0\|_F. \end{aligned}$$