Ex1 Prove that the quadratic cone is convex

proof: quadratic cone: $C = \{(x,t) \in \mathbb{R}^{n+1} \mid \|x\|_2 \le t\}$

$$= \left\{ \begin{bmatrix} x \\ t \end{bmatrix} \middle| \begin{bmatrix} x \\ t \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ t \end{bmatrix} \le 0,\ t \ge 0 \right\}$$

suppose $C_1, C_2 \in C$ $\quad C_1 = \begin{bmatrix} x_1 \\ t_1 \end{bmatrix} \quad C_2 = \begin{bmatrix} x_2 \\ t_2 \end{bmatrix}$

Want to Show: $\theta C_1 + (1-\theta) C_2 \in C$ for $\theta \in (0,1)$ firstly as $\theta \in (0,1)$, $t_1, t_2 \ge 0$

it's clear $\theta t_1 + (1-\theta) t_1 \ge 0$

consider

$$\begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta t_1 + (1-\theta) t_2 \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \theta x_1 + (1-\theta) x_2 \\ \theta t_1 + (1-\theta) t_2 \end{bmatrix} \quad (*)$$

$$= (\theta x_1 + (1-\theta) x_2)^T (\theta x_1 + (1-\theta) x_2) - (\theta t_1 + (1-\theta) t_2)^2$$

$$= \underbrace{\theta^2 x_1^T x_1 - \theta^2 t_1}_{(\mathrm{I})} + \underbrace{(1-\theta)^2 x_2^T x_2 - (1-\theta)^2 t_2^2}_{(\mathrm{II})} + 2\theta(1-\theta) x_1^T x_2 - 2\theta(1-\theta) t_1 t_2.$$

As $\begin{bmatrix} x_1 \\ t_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ t_2 \end{bmatrix} \in C$ so $(\mathrm{I}) \le 0$ $\quad (\mathrm{II}) \le 0$

Consider Cauchy Shwartz Inequality:

$$|\langle x_1, x_2 \rangle| = |x_1^T x_2| \le \|x_1\|_2 \|x_2\|_2 \iff |x_1^T x_2| \le t_1 t_2$$

So $2\theta(1-\theta)(x_1^T x_2 - t_1 t_2) \le 0 \quad \Rightarrow \quad (*) \le 0$

So $\theta C_1 + (1-\theta) C_2 \in C$ which means C is convex

Ex2 Prove that the intersection of two convex sets is convex

proof: Suppose there are two convex sets C_1 & C_2

choose $x_1, x_2 \in C_1 \cap C_2$ consider $\theta_1 x_1 + (1-\theta) x_2$ $\quad \theta \in (0,1)$

$x_1, x_2 \in C_1 \cap C_2$ means $x_1, x_2 \in C_1$ and $x_1, x_2 \in C_2$

As C_1, C_2 are convex so $\theta x_1 + (1-\theta) x_1 \in C_1$ and $\theta x_1 + (1-\theta) x_2 \in C_2$

so $\theta x_1 + (1-\theta) x_2 \in C_1 \cap C_2$ when $\theta \in (0,1)$

$\Rightarrow C_1 \cap C_2$ is still convex

Ex 3 Prove that the image of a convex set under an affine function is convex, and that the inverse image is also convex

proof: Suppose $f(x) = Ax+b$ is an affine function and C is a convex set.

consider $f(C) = \{ f(x) \mid x \in C \}$

suppose $x_1, x_2 \in C$, $f(x_1), f(x_2) \in f(C)$, as C is convex so

$\theta x_1 + (1-\theta) x_2 \in C$ for $\theta \in (0,1)$ $\Rightarrow f(\theta x_1 + (1-\theta) x_2) \in f(C)$

$$
\begin{aligned}
f(\theta x_1 + (1-\theta) x_2) &= A(\theta x_1 + (1-\theta) x_2) + b \\
&= \theta A x_1 + \theta b + (1-\theta) A x_2 + (1-\theta) b \\
&= \theta f(x_1) + (1-\theta) f(x_2) \in f(C) \quad \text{for } \theta \in (0,1)
\end{aligned}
$$

so $f(C)$ is also convex.

Next. I want to show the inverse image is also convex

$f^{-1}(C) = \{ x \mid f(x) \in C \}$

Want to show $\theta x_1 + (1-\theta) x_2 \in f^{-1}(C)$ if $f(x_1) \in C$, $f(x_2) \in C$ and $\theta \in (0,1)$

As C is convex so $\theta f(x_1) + (1-\theta) f(x_2) \in C$

i.e. $\theta f(x_1) + (1-\theta) f(x_2) \in C$

$\Leftrightarrow \theta (A x_1 + b) + (1-\theta)(A x_2 + b) \in C$

$\Leftrightarrow A(\theta x_1 + (1-\theta) x_2) + b \in C$

$\Leftrightarrow f(\theta x_1 + (1-\theta) x_2) \in C \quad \Leftrightarrow \theta x_1 + (1-\theta) x_2 \in f^{-1}(C)$

so $f^{-1}(C)$ is also convex

Ex 4 **2.1** Let $C \subseteq \mathbf{R}^n$ be a convex set, with $x_1, \ldots, x_k \in C$, and let $\theta_1, \ldots, \theta_k \in \mathbf{R}$ satisfy $\theta_i \geq 0$, $\theta_1 + \cdots + \theta_k = 1$. Show that $\theta_1 x_1 + \cdots + \theta_k x_k \in C$. (The definition of convexity is that this holds for $k = 2$; you must show it for arbitrary k.) *Hint.* Use induction on k.

proof: when $k=2$, if C is a convex set if $\theta_1 + \theta_2 = 1$

$\theta_1 x_1 + \theta_2 x_2 \in C$

Suppose that $\sum^{k} \theta_i x_i \in C$ if $\sum^{k} \theta_i = 1$ and $x_1, \cdots x_k \in C$

Now want to show $\sum_{i=1}^{k+1} \theta_i x_i \in C$ if $\sum_{i=1}^{k+1} \theta_i = 1$

$$y = \theta_1 x_1 + \theta_2 x_2 + \cdots + \theta_k x_k + \theta_{k+1} x_{k+1}$$
$$= (1-\theta_{k+1})\left(\frac{\theta_1}{1-\theta_{k+1}} x_1 + \frac{\theta_2}{1-\theta_{k+1}} x_2 + \cdots + \frac{\theta_k}{1-\theta_{k+1}} x_k\right) + \theta_{k+1} x_{k+1}$$

Recall that $\sum_{i=1}^{k} \frac{\theta_i}{1-\theta_{k+1}}$

$$= \frac{\sum_{i=1}^{k}\theta_i}{1-\theta_{k+1}} = \frac{1-\theta_{k+1}}{1-\theta_{k+1}} = 1$$

and as $\theta_i \in (0,1)$ $i=1,\dots k+1$ $\theta_i < 1-\theta_{k+1}$ and $\theta_i > 0$ $1-\theta_{k+1} > 0$

so $\frac{\theta_i}{1-\theta_{k+1}} \in (0,1)$ $i = 1, \dots k$

By assumption, $\frac{\theta_1}{1-\theta_{k+1}} x_1 + \frac{\theta_2}{1-\theta_{k+1}} x_2 + \cdots + \frac{\theta_k}{1-\theta_{k+1}} x_k \in C$

since C is convex so $\theta_{k+1} y_{k+1} + (1-\theta_{k+1})\left(\frac{\theta_1}{1-\theta_{k+1}} x_1 + \cdots + \frac{\theta_k}{1-\theta_{k+1}} x_k\right) \in C$

QED.

Ex5 **2.2** Show that a set is convex if and only if its intersection with any line is convex. Show that a set is affine if and only if its intersection with any line is affine.

proof: Part 1: "$\Rightarrow$" if a set is convex then its intersection with any line is convex

Recall that a line is convex as any $x_1, x_2 \in l : x_0 + t\vec{r}$ $\theta x_1 + (1-\theta) x_2 \in l$

when $\theta \in (0,1)$ so the intersection of two convex sets is also convex

"$\Leftarrow$" Suppose that the intersection of a set C and any line $l_{x_0, r}$ is convex

i.e. $C \cap l_{x_0, \vec{r}}$ is convex and $l_{x_0, \vec{r}} = x_0 + t\vec{r}$ $t \in \mathbb{R}$.

now choose $x_1, x_2 \in C$ $\theta x_1 + (1-\theta) x_2 = x_2 + \theta(x_1 - x_2) \in l_{x_2, \vec{r}}$ where $\theta \in (0,1)$

$\vec{r} = x_1 - x_2$ which means we can find a line $l_{x_2, \vec{r}} = x_2 + t\vec{r}$ contains x_1, x_2

so $x_1, x_2 \in C \cap l_{x_2, \vec{r}}$ as $C \cap l_{x_2, \vec{r}}$ is convex so $\theta x_1 + (1-\theta) x_2 \in C \cap l_{x_2, \vec{r}}$

$\Rightarrow \theta x_1 + (1-\theta) x_2 \in C$ $\theta \in (0,1)$ so C is convex

Part (2): We use the same idea, firstly prove the intersection of two affine sets is affine. Assume C_1 and C_2 are affine. so choose $x_1, x_2 \in C_1 \cap C_2$

$\theta x_1 + (1-\theta) x_2 \in C_1$, $\theta \in (0,1)$ as C_1 is affine and $x_1, x_2 \in C_1$.

also $\theta x_1 + (1-\theta) x_2 \in C_2$ $\theta \in (0,1)$ as C_2 is affine

So $\theta x_1 + (1-\theta) x_2 \in C_1 \cap C_2$ i.e. $C_1 \cap C_2$ is affine

it's clear any line l is affine so for any affine set C, $C \cap l$ is also affine.

"$\Leftarrow$": suppose for a set C, $C \cap l$ is affine for any line l

choose $x_1, x_2 \in C$ $\quad \theta x_1 + (1-\theta) x_2 = x_2 + \theta(x_1 - x_2)$ $\quad \theta \in \mathbb{R}$.

so consider the line $l: x_2 + t(x_1 - x_2)$ $\quad t \in \mathbb{R}$ $\quad x_1, x_2 \in l$

so $x_1, x_2 \in C \cap l$ $\quad \Rightarrow \theta x_1 + (1-\theta) x_2 \in C \cap l$ as $C \cap l$ is affine

So $\theta x_1 + (1-\theta) x_2 \in C$ for $\theta \in \mathbb{R}$ so C is affine

Ex 6 *Solution set of a quadratic inequality.* Let $C \subseteq \mathbf{R}^n$ be the solution set of a quadratic inequality,

$$C = \{x \in \mathbf{R}^n \mid x^T A x + b^T x + c \leq 0\},$$

with $A \in \mathbf{S}^n$, $b \in \mathbf{R}^n$, and $c \in \mathbf{R}$.

(a) Show that C is convex if $A \succeq 0$.

(b) Show that the intersection of C and the hyperplane defined by $g^T x + h = 0$ (where $g \neq 0$) is convex if $A + \lambda g g^T \succeq 0$ for some $\lambda \in \mathbf{R}$.

Are the converses of these statements true?

proof: (a) Use the conclusion in Ex.5. I want to prove for any line l $C \cap l$ is convex

Consider the line $l = \{x \mid x = x_0 + t\vec{r} \quad t \in \mathbb{R}\}$

so $l \cap C = \{t \in \mathbb{R} \mid (x_0 + t\vec{r})^T A (x_0 + t\vec{r}) + b^T (x_0 + t\vec{r}) + c \leq 0\}$

$$= \{t \in \mathbb{R} \mid \vec{r}^T A \vec{r} t^2 + (2 x_0^T A \vec{r} + b^T \vec{r}) t + (x_0^T A x_0 + b^T x_0 + c) \leq 0\} \quad (*)$$

This is a quadratic function. consider any $f(x) = ax^2 + bx + c \leq 0$ $\quad (a > 0)$

as $f''(x) = 2a > 0$ so $f(x)$ is always convex when $a > 0$

so the epigraph $\{x \mid f(x) \le 0\}$ is always a convex set

when $a=0$ $f(x)$ degenerates to a line, which is affine, so also convex

so the epigraph $\{x \mid f(x) \le 0\}$ when $a=0$ is also convex.

go back to equation $(*)$ as $A \succeq 0$ so $r^T A r \ge 0$ for any $\vec{r}$

which means the parameter of second-order term is nonnegative

so $L \cap C$ is convex for any line L

use the conclusion in Ex.5. $\Rightarrow$ C is convex

Converse of the statement is false: counter example consider $x \in \mathbb{R}$. $\{x \mid -x^2 \le 0\} = \mathbb{R}$.

It's clear $\mathbb{R}$ is convex but $A=-1 \le 0$ so $A \succeq 0$ doesn't hold but $\{x \mid -x^2 \le 0\}$ is convex

(b) consider the explicit form of $C \cap H$ where $H = \{x \mid g^T x + h = 0\}$

choose $x_0 \in H$. let $x = x_0 + t\vec{r}$ where $t \in \mathbb{R}$ $\vec{r}$ is any direction

if x in H $g^T(x_0 + t\vec{r}) + h = 0$

$$\Leftrightarrow g^T x_0 + h + t g^T \vec{r} = 0 \qquad g^T x_0 + h = 0 \quad \text{as} \quad x_0 \in H$$

$$\Leftrightarrow t g^T \vec{r} = 0$$

so I have two cases (i) $g^T \vec{r} \ne 0 \Rightarrow t = 0$

so $C \cap H = \{x_0\}$ this is the trivial case

$\theta x_0 + (1-\theta) x_0 = x_0 \in C \cap H$.

(ii) $g^T \vec{r} = 0$ the t could be any real value.

so $C \cap H = \{x_0 + t\vec{r} \mid \vec{r}^T A \vec{r}\, t^2 + (2 x_0^T A \vec{r} + b^T \vec{r}) t + (x_0^T A x_0 + b^T x_0 + c) \le 0\}$

by the conclusion in (a) we know that if $\vec{r}^T A \vec{r} \ge 0$ then $C \cap H$ is convex

as we can find $\lambda \in \mathbb{R}$ such that $A + \lambda g g^T \succeq 0$

so $\vec{r}^T (A + \lambda g g^T) r = \vec{r}^T A r + \lambda \underbrace{r^T g g^T r}_{\text{as } g^T r = 0} = r^T A r \ge 0$

so $C \cap H$ is convex.

However, the converse is false, as we can get from the above proof

$r^T A r \geq 0$ only when $g^T r = 0$ not for all r. If $g^T r \neq 0$, we can not get any coclusion of A as $H \cap C$ is a single set. So the converse of the statement is false.

Ex7 **2.16** Show that if S_1 and S_2 are convex sets in $\mathbf{R}^{m+n}$, then so is their partial sum

$$S = \{(x, y_1 + y_2) \mid x \in \mathbf{R}^m,\ y_1,\ y_2 \in \mathbf{R}^n, (x, y_1) \in S_1,\ (x, y_2) \in S_2\}.$$

proof: choose (x, y_1+y_2), $(u, v_1+v_2) \in S$

which means (x, y_1) $(u, v_1) \in S_1$

(x, y_2) $(u, v_2) \in S_2$

as S_1, S_2 are convex so $\theta(x, y_1) + (1-\theta)(u, v_1)$

$= (\theta x_1 + (1-\theta)u,\ \theta y_1 + (1-\theta)v_1) \in S_1$

Similarly $\theta(x, y_2) + (1-\theta)(u, v_2)$

$= (\theta x + (1-\theta)u,\ \theta y_2 + (1-\theta)v_2) \in S_2$

so $\theta(x, y_1+y_2) + (1-\theta)(u, v_1+v_2)$

$= (\theta x + (1-\theta)u,\ \theta(y_1+y_2) + (1-\theta)(v_1+v_2))$

$= (\theta x + (1-\theta)u,\ \theta y_1 + (1-\theta)v_1 + \theta y_2 + (1-\theta)v_2) \in S$

as $(\theta x + (1-\theta)u,\ \theta y_1 + (1-\theta)v_1) \in S_1$

$(\theta x + (1-\theta)u,\ \theta y_2 + (1-\theta)v_2) \in S_2$

so S is convex

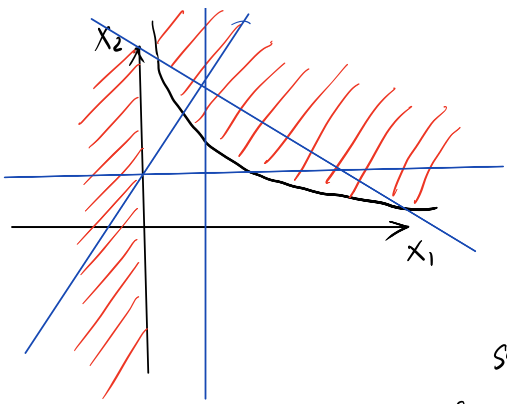
Ex 8 **2.23** Give an example of two closed convex sets that are disjoint but cannot be strictly separated.

consider two set $S_1 = \{(x_1, x_2) \mid x_1 \leq 0\}$ $S_2 = \{(x_1, x_2) \mid x_1 x_2 \geq 1, x_1 > 0, x_2 > 0\}$

Firstly show S_1 and S_2 are convex choose $x, y \in S_1$ $x = (x_1, x_2)$ $y = (y_1, y_2)$

$\theta x + (1-\theta)y = (\theta x_1 + (1-\theta)y_1,\ \theta x_2 + (1-\theta)y_2)$ as $\theta \in (0,1)$ $x_1, y_1 \leq 0$

so $\theta x_1 \leq 0$ $(1-\theta)y_1 \leq 0$ so $\theta x_1 + (1-\theta)y_1 \leq 0$ $\Rightarrow \theta x + (1-\theta)y \in S_1$



S_2 is also convex

consider $f(x) = \frac{1}{x}$ $(x>0)$

$f''(x) = \frac{2}{x^3} > 0$ when $x>0$

so $f(x) = \frac{1}{x}$ is a convex function

so for $y>0$ $\{x,y \mid \frac{1}{x} \le y\}$ is the epigraph of $f(x)$

so $\{x,y \mid \frac{1}{x} \le y\}$ is convex because $f(x)$ is convex

which means $S_2 = \{(x_1,x_2) \mid x_1x_2 \ge 1, x_1>0, x_2>0\}$ is convex

It's clear, $S_1 \cap S_2 = \phi$ as $x_1, x_2 > 0$ in S_2 and $x_1 \le 0$ in S_1

It's obvious S_1 and S_2 are closed as it's easy to show S_1^c, S_2^c are open.

Now for any line $l: x_2 = ax_1 + b$, if l is not vertical, we can alway find enough big y_1 such that for (x_1, y_1) $ax_1 + b < y_1$

and enough small y_2. $ax_2 + b > y_2$ and (x_1, y_1) $(x_1, y_2) \in S_1$

so l cannot be separating hyperplane

for vertical line $l: x = a$ if $a=0$ this not the strict separating hyperplane. if $a>0$ $x=a$ and $f(x) = \frac{1}{x}$ always has intersection

if $a<0$ $x = a \in S_1$

So we never find the strict separating hyperplane

Ex 9

(b) Let $C = \{x \in \mathbf{R}^n \mid \|x\|_\infty \le 1\}$, the ℓ_∞-norm unit ball in $\mathbf{R}^n$, and let $\hat{x}$ be a point in the boundary of C. Identify the supporting hyperplanes of C at $\hat{x}$ explicitly.

Solution: I want to find a, such that $a^Tx \le a^T\hat{x}$

Suppose $\hat{x}$ is on the boundary, so there is at least one element of $\hat{x}$ is 1. or -1 suppose $\hat{x}$ only have one element with absolute value 1.

$|\hat{x}_i| = 1$ and $-1 < \hat{x}_j < 1$ for all $j \ne i$, case (i) $\hat{x}_i = 1$ for $a_i > 0$ and

$a_j = 0 \quad j \neq i \qquad a^T \hat{x} = a_i \geq a^T x = a_i x_i$ for all $x \in C$

as if $x \in C$ $\|x\|_\infty \leq 1$ $x_i \leq 1$ so $a^T = (0, 0, \cdots \underset{\substack{\uparrow \\ i\text{-th element},\ a_i > 0}}{a_i}, 0, \cdots 0)$ $a^T x$ is

the supporting hyperplane

Case (ii) $\hat{x}_i = -1$ when $a_i < 0$ and $a_j = 0$ for $j \neq i$

$a^T x = -a_i \geq a^T x = a_i x_i$ for all $x \in C$ as $-1 \leq x_i$ so the supporting hyperplane

is $a^T x$ $\quad a^T = (0, \cdots \underset{\substack{\downarrow \\ i\text{-th element}\ a_i < 0}}{a_i}, 0 \cdots 0)$

Next for the general case. if $\hat{x}$ has j elements with absolute value 1

$\hat{x} = (x_{i_1}, x_2, \cdots, x_{i_2}, \cdots x_{i_j})$ where $|x_{i_k}| = 1$ when $i_k \in \{i_1, \cdots i_j\}$

$-1 < x_{i_k} < 1$ when $i_k \in \{1, \cdots n\} \setminus \{i_1, \cdots i_j\}$

we only need to assign the element of a^T as

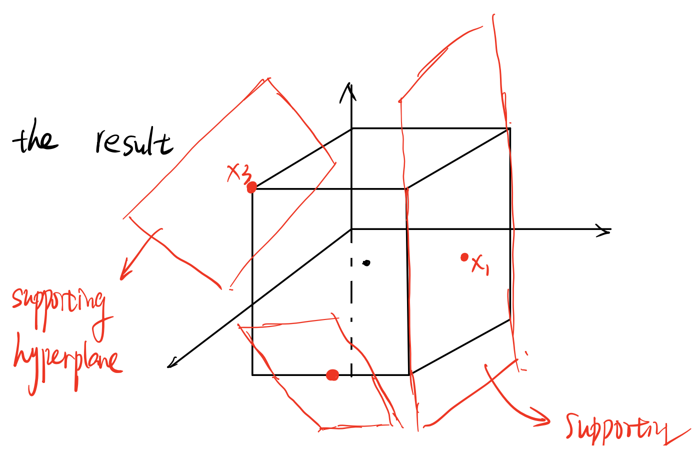
$$\begin{cases} a_i > 0 & \text{if } \hat{x}_i = 1 \\ a_i < 0 & \text{if } \hat{x}_i = -1 \\ a_i = 0 & \text{if } -1 < \hat{x}_i < 1 \end{cases}$$

so $a^T \hat{x} = \sum_{k=1}^{j} |a_i| \geq a^T x = \sum_{k=1}^{j} a_{i_k} x_{i_k}$ when $x \in C$ as if $\|x\|_\infty \leq 1$ $-1 \leq x_i \leq 1$ for every $i = 1, \cdots n$.

So the supporting hyperplane $a^T x$ is

$$\begin{cases} a_i > 0 & \text{if } \hat{x}_i = 1 \\ a_i < 0 & \text{if } \hat{x}_i = -1 \\ a_i = 0 & \text{if } -1 < \hat{x}_i < 1 \end{cases}$$

I plot a figure to illustrate the result



$\parallel \quad 0$
hyperplane.

Ex 10 Verify that as stated on BV p.39, the hyperplane cone is the inverse image of the second order cone under the given affine transformation

solution : C_1 : second order cone : $\{x \mid x^T P x \le (c^T x)^2,\ c^T x \ge 0\}$ $\quad P \in S_+^n \quad c \in \mathbb{R}^n$

C_2 : second order cone : $\{(z,t) \mid z^T z \le t^2,\ t \ge 0\}$

affine function : $f(x) = (P^{1/2} x,\ c^T x)$

The inverse image of the second order cone C_2.

$\{x \mid f(x) \in C_2\}$

$\Leftrightarrow \{x \mid (P^{1/2} x, c^T x) \in C_2\}$ $\quad \Leftrightarrow \{x \mid (P^{1/2} x)^T (P^{\frac{1}{2}} x) \le (c^T x)^2,\ c^T x \ge 0\}$

$\Leftrightarrow \{x \mid x^T P x \le (c^T x)^2,\ c^T x \ge 0\} = C_2$