Computational Neuroscience Homework 5

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1 Problem 1

In this question, we would generate correlated N-dimensional input data with zero means and some set covariance. Specifically, we could generate $N \times N$ random matrix $\mathbf{A}$ where each entry is independently drawn from a Gaussian distribution $\sim \mathcal{N}\left(0, \frac{\eta^2}{N}\right)$. Then, we could construct the covariance matrix $\Sigma = \mathbf{A}\mathbf{A}^T$. To sample individual input data points from this distribution, we could sample N-dimensional vectors, $\boldsymbol{\epsilon}$, with entries independently drawn from $\epsilon_i \sim \mathcal{N}(0,1)$. We then generate outcome as $\mathbf{x} = \mathbf{A}\boldsymbol{\epsilon}$. We could generate various $\mathbf{x}$ and combine them as a matrix format. Then, we feed it into a linear neuron and train the feed-forward synapses by Oja's rule with a small learning rate of 0.001. Specifically, consider a simplified model of a neuron y that returns a linear combination of its inputs $\mathbf{x}$ using presynaptic weights $\mathbf{w}$:

$$y(\mathbf{x}) = \sum_{j=1}^{m} x_j w_j, \tag{1}$$

Oja's rule defines the change in presynaptic weights $\mathbf{w}$ given the output response y of a neuron to its inputs $\mathbf{x}$ to be [1]:

$$\Delta \mathbf{w} = \mathbf{w}_{n+1} - \mathbf{w}_n = \eta y_n(\mathbf{x}_n - y_n \mathbf{w}_n). \tag{2}$$

We initialize random weights from Gaussian distribution with zero mean and unit variance. Over learning for at least 1000 steps, the weights end up aligning with the first principal component as the eigenvector with the largest eigenvalue of Σ[1]. Figure 1 shows the randomly generated data 200 points (from $\mathbf{A}$), the first principal component, and the projection of the trained weight vector. Figure 2 shows the results at different stages of training. Figure 3 shows the norm of the weight vector as a function of time to observe the effects of normalization. Figure 4 describes the trajectory of the weight vector and shows its gradual convergence to the first PC vector. Next, we reproduce the calculations in a higher dimensional input space $N = 10$. We use the first two input dimensions for the scatter plot. The results are shown in Figure 5, 6, 7, and 8.

2 Problem 2

In this question, we could implement the nearest-neighbor spike-timing-dependent plasticity (STDP) for a single synapse [2]. Here we describe the presynaptic and postsynaptic input both as Poisson processes with different firing rates, x_0 and x respectively, where x_0 is a fixed value. Thus, the postsynaptic probability density becomes exponential in time, xe^{-xt}. High (low) firing rates x result in predominantly small (large) intervals and hence in potentiation (depression). The expected magnitude of synaptic modification per one presynaptic spike has the form:

$$\begin{aligned} C(x) &= \int_0^{\infty} A_+ e^{-t/\tau_+} x e^{-xt} dt + \int_{-\infty}^{0} A_- e^{t/\tau_-} x e^{xt} dt \\ &= x\left(\frac{A_+}{\tau_+^{-1} + x} + \frac{A_-}{\tau_-^{-1} + x}\right). \end{aligned} \tag{3}$$

We use the nearest-neighbor algorithm to calculate the instantaneous change in weights. Namely, we only consider the (maximum) two postsynaptic events mostly closed to the specific presynaptic event, one before and one after. Specifically:

$$\begin{aligned} \Delta W_1 &= A_+ e^{(t_{\text{pre}} - t_{\text{poss}})/\tau_+} \quad &\text{if} \quad t_{\text{post}} > t_{\text{pre}}, \\ \Delta W_2 &= -A_- e^{-(t_{\text{pre}} - t_{\text{post}})/\tau_-} \quad &\text{if} \quad t_{\text{post}} < t_{\text{pre}}. \end{aligned} \tag{4}$$

[1]See attached code

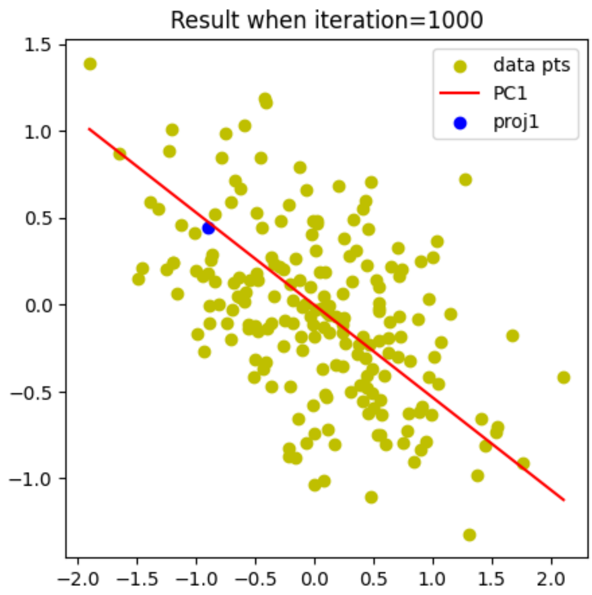

Figure 1: Instance of scattered data points, first PC, and the projection of the weight vector onto the first PC when $N = 2$.

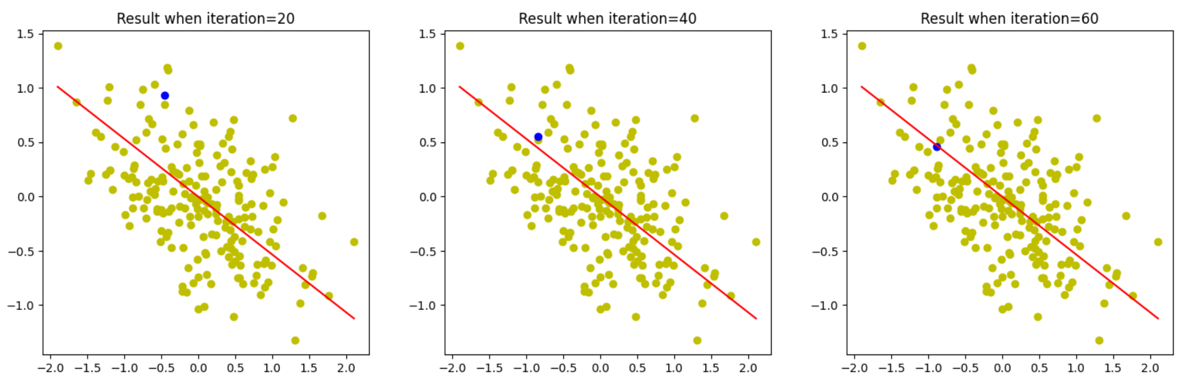

Figure 2: Results at various early stages of iterations $(20, 40, 60)$ before convergence when $N = 2$.

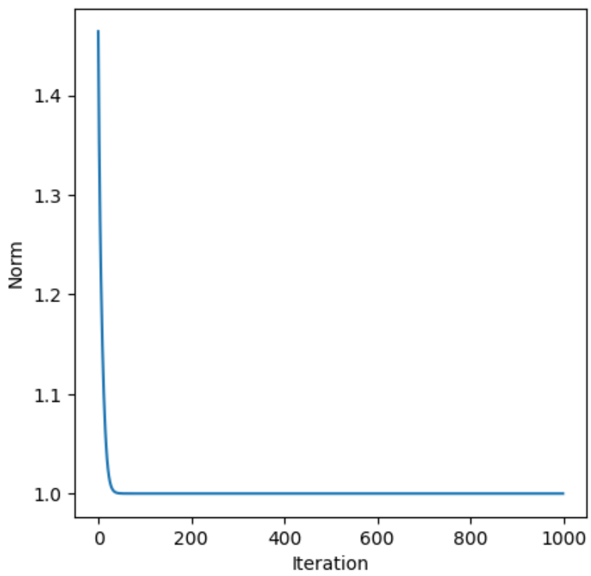

Figure 3: Norm of the weight vector when $N = 2$.

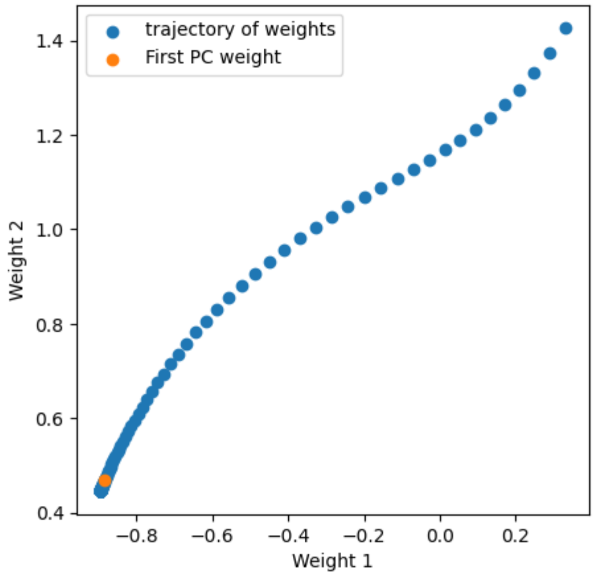

Figure 4: Trajectory of weight vector and the weight for the first PC when $N = 2$.

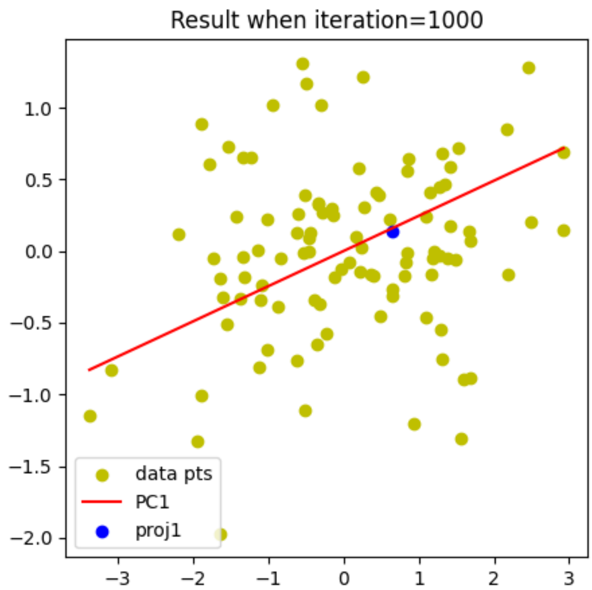

Figure 5: Instance of scattered data points, first PC, and the projection of the weight vector onto the first PC when $N = 10$.

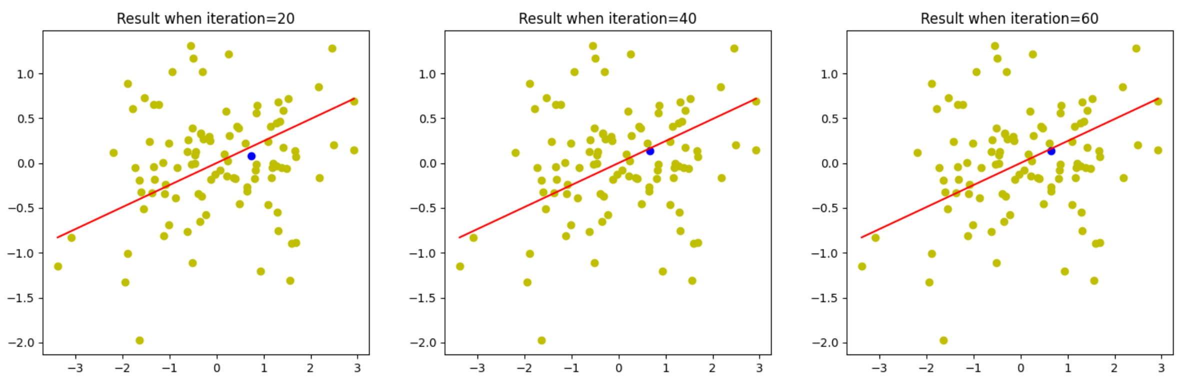

Figure 6: Results at various early stages of iterations $(20, 40, 60)$ before convergence when $N = 10$.

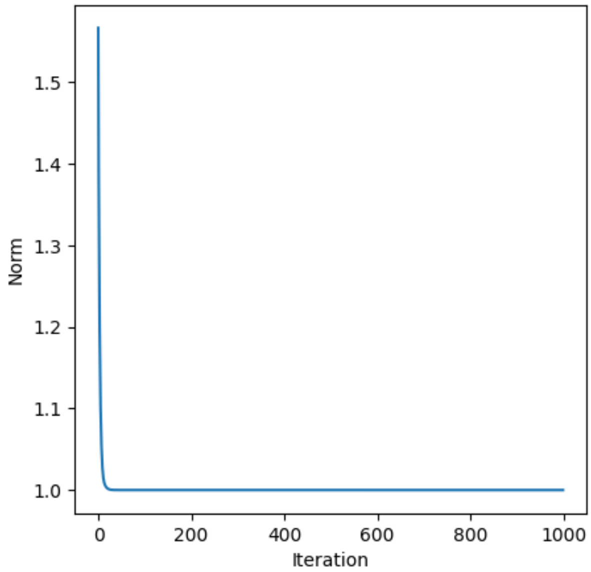

Figure 7: Norm of the weight vector when $N = 10$.

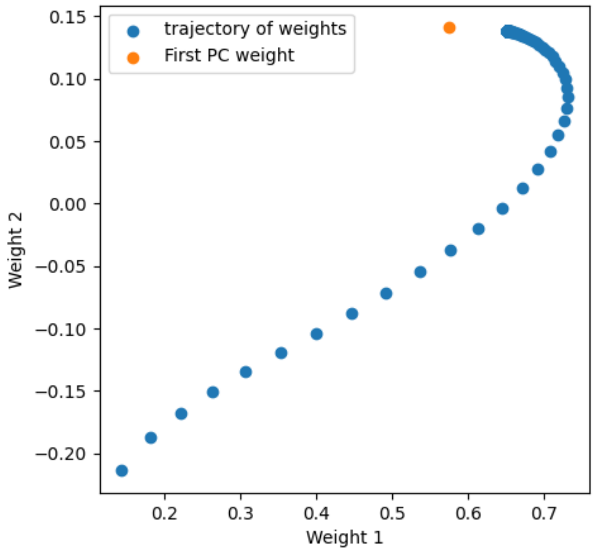

Figure 8: Trajectory of weight vector and the weight for the first PC when $N = 10$.

We define the (total) changes in the synaptic weight as:

$$\Delta W = \Delta W_1 + \Delta W_2. \tag{5}$$

If there is no postsynaptic event after (before) a certain presynaptic event, then $\Delta W_1(\Delta W_2) = 0$. Here we fix $x_0 = 15$Hz and $x \in [0, 25]$Hz. We simulate 10s with a timestep of 0.01ms. We calculate the average change for different presynaptic events, resample the Poisson process of presynaptic draws for 1000 independent trails, then further calculate the mean value for each given x. Figure 9 describes the consistency between the theoretical curve in Equation (3) and the results from experimental simulations.

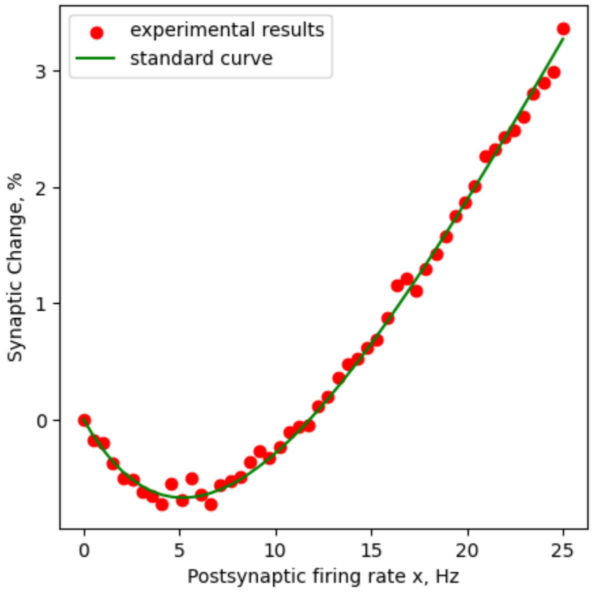

Figure 9: Relationship between postsynaptic firing rate and synaptic change in percentage.

References

[1] Erkki Oja. "Simplified neuron model as a principal component analyzer". In: *Journal of Mathematical Biology* 15 (1982), pp. 267–273.

[2] Eugene Izhikevich and Niraj Desai. "Relating stdp to bcm". In: *Neural computation* 15 (Aug. 2003), pp. 1511–23. DOI: 10.1162/089976603321891783.