Homework #1 (NEURL-GA 3042, Fall 2022)

Due date: Sunday September 18

A white noise $w(t)$ is a Gaussian process without correlation in time. Assuming that the mean is zero and the variance is 1, the probability density for w is give by $p(w) = 1/\sqrt{2\pi}\exp(-w^2/2)$. Denote by $< \cdot >$ an average over the probability distribution, we have $< w >= 0, < w(t)w(t') >= \delta(t - t')$, a Delta-function which is 0 if $t \neq t'$ and ∞ if $t = t'$.

A continuous-time random walk is the integral of a white noise. In 1D space it is described by a stochastic differential equation

$$\frac{dx}{dt} = \sigma w(t) \tag{1}$$

where σ is the noise level.

(1a) With the initial condition $x(t = 0) = x_0$, show analytically that at time t $< x(t) >= x_0; < (x - x_0)^2 >= \sigma^2 t$.

(1b) Let $\sigma = 1$. With the initial condition $x_0 = 0$, simulate $x(t)$ with $dt = 0.5$ ms, and $T = 1$ sec. Run the simulation in 1000 trials, each using a different random number generator seed. Compute the trial average and variance as a function of time. Plot and compare the variance as a function of time for the numerical simulation result and the analytical expression from (1a).

Save the histograms of x values over the 1000 trials at $t = 0.2, 0.5, 0.75, 1$ s.

Note: Using the Euler method, the iteration is given by $x_{(n+1)} = x_n + \sigma * \sqrt{dt} * w_n$, where w_n is from a Gaussian distribution, independently sampled at each timestep.

(2) The time-dependent probability $p(x, t)$ of the 1D diffusion (Eq. (1)) is given by the following Fokker-Planck equation

$$\frac{\partial p(x,t)}{\partial t} = D\frac{\partial^2 p(x,t)}{\partial x^2} \tag{2}$$

where $D = \sigma^2/2$.

(1a) With the initial condition $p(x, t = 0) = \delta(x - x_0)$, verify that

$$p(x,t) = \frac{1}{\sqrt{(2\pi t)}\sigma} \exp\left[-\frac{(x-x_0)^2}{2\sigma^2 t}\right] \tag{3}$$

is the solution, i.e. it satisfies Eq. (2) with the correct initial condition.

(2b) Plot $p(x)$ at several time points, overlapping with the normalized histograms of numerically simulated x from (1) at $t = 0.2, 0.5, 0.75, 1$ s. Do they agree? Describe your observations about how the distribution evolves in time.