Computational Neuroscience Homework 2

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This short writeup serves for the supplementary material for the Homework 2 of NEURL-GA 3042 Computational Neuroscience course taught by Professor Xiao-jing Wang in Fall 2022.

1 Part 1

We have implemented the leaky integrate-and-fire (LIF) model based on the RC circuit equation in Python:

$$C\frac{dV_m}{dt} = -G_L(V_m - V_L) + I \tag{1}$$

For the numerical method to solve the ODE numerically, two typical and widely-applied methods are forward Euler method and Runge-Kutta 2nd order (RK2) method. Specifically, for a general ODE system:

$$\frac{dx}{dt} = f(x) \tag{2}$$

The forward Euler method would calculate as:

$$x_{n+1} = x_n + f(x_n) \cdot dt \tag{3}$$

and the RK2 method would calculate as:

$$x_{n+1} = x_n + f(x_n + \frac{1}{2}f(x_n) \cdot dt)dt \tag{4}$$

The RK2 method has a higher order of accuracy, but we are required to use first-order forward Euler for this assignment. Firstly, we set the magnitude of DC as different values, where we inject the current at $10ms$ and end it at $150ms$ with grid step size in t as $0.1ms$. The result is presented in Figure 1. When $I < I_c$, the membrane potential would gradually increase but not spike. When $I \geq I_c$, the action potential is triggered and the spiking frequency is larger for large I. For simplicity, we set the spike potential as a constant value $V_{\text{spike}} = 20mV$. When the injected current $I(t) = 0$ for some t, we would reset the membrane potential back to V_L.

Next, we fix $I = 0.55nA$ and change the timestep for numerical integration to different values. The result is presented in Figure 2. Consistent with our intuition, smaller stepsize indicates a more accurate calculation and a more computationally expensive task. When Δt is relatively large $(1ms)$, the spike is not discharged and reset instantaneously with an observable delayed interval, which is not physiologically accurate. The larger timestep is, the more delay will happen, which will lead to a smaller firing rate than real. When Δt is even larger than the refractory period τ_{ref} $(5ms)$, the potential would not be reset to V_{reset} and consistently stay at V_{spike}, which results in an incorrect description of the dynamic. For convenience, we would choose $dt = 0.1ms$ for further calculation, which is both less time-costly and qualitatively accurate.

2 Part 2

To calculate the critical value I_c of I above which the model fires spikes repetitively, we could approach it theoretically or experimentally (numerically). Theoretically, we should have the steady state of voltage $V_{\text{SS}} = V_{\text{th}}$, or $I_c = G_L(V_{\text{th}} - V_L) = 0.025\mu S \cdot (-50mV - (-70mV)) = 0.50nA$. Experimentally, since the model is deterministic when the initial condition and I are formulated, we could repetitively set I as a different value and find the smallest I that has $V(t)$ cross the threshold V_{th} (at some t_{spike}). Though this approach is decided by how finer our picked grid is for I, we would deduce the same result if ΔI is small enough (say $10^{-3}nA$).

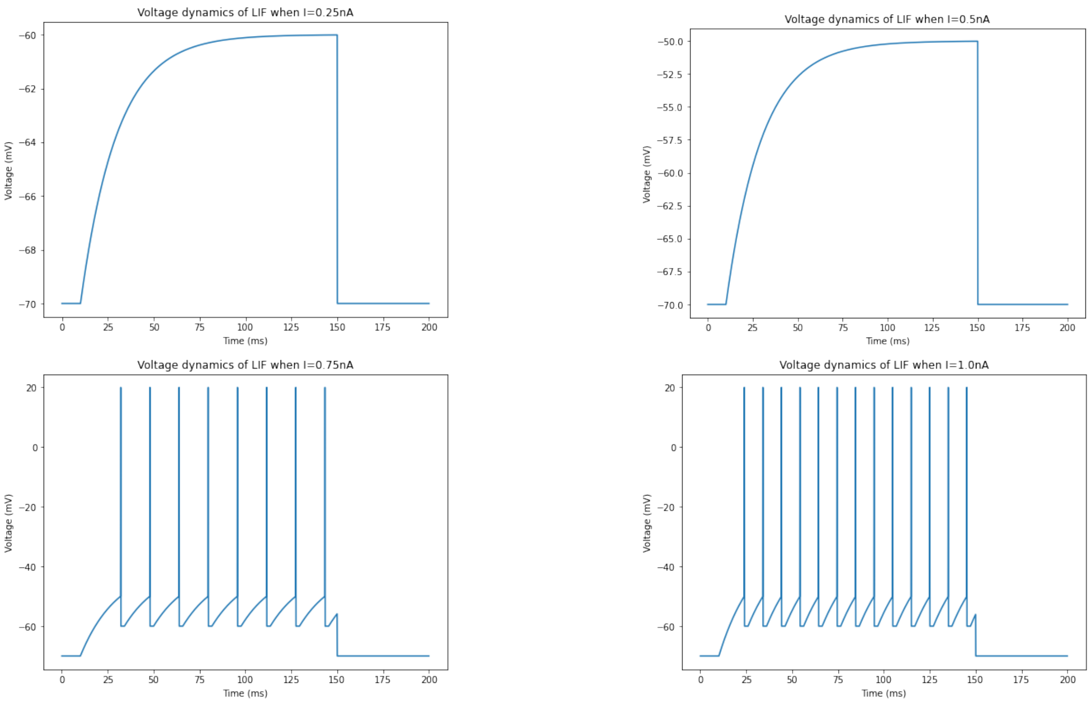

Figure 1: Relationship of injected current I and membrane potential V when $I = 0.25nA, 0.50nA, 0.75nA$ and $1.0nA$ respectively.

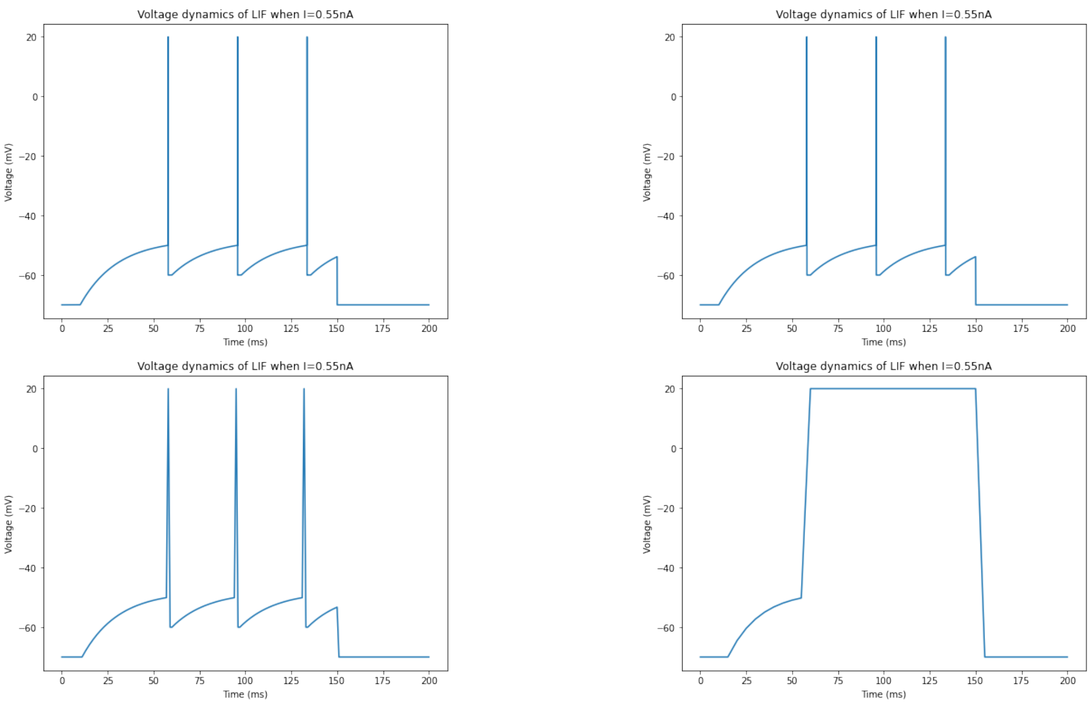

Figure 2: Result when stepsize $\Delta t = 0.01ms, 0.1ms, 1ms$ and $5ms$ respectively.

Then, we count the number of spikes during the period of current injection and calculate the firing frequency f in the number of spikes per second (Hertz). Also, using Equation (2.4) in Xiao-jing Wang's *Theoretical Neuroscience of Cognition*, we have an analytical expression of the firing rate r as a function of the current I:

$$1/r = \begin{cases} \tau_{\text{ref}} + \dfrac{C}{G_L} \ln \left[1 + \dfrac{G_L \Delta V}{(I - I_c)}\right] & \text{if } I > I_c \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

where $\Delta V = V_{\text{th}} - V_{\text{reset}}$ is the voltage difference between the firing threshold and reset. For the constants, we have $V_{\text{th}} = -50mV$, $V_{\text{reset}} = -60mV$, $\tau_{\text{ref}} = 2ms$, $C = 0.5nF$, $G_L = 0.025\mu S$, $V_L = -70mV$. Thus, we obtain the 2 versions of the frequency-current relationship ($f - I$ curve), namely the neuronal input-output transfer function. We run the experiment for $2000ms$, and the result is presented in Figure 3. Noticeably, there exists a strict threshold of current for firing and the firing rate can go to zero with current going to I_c (infinite latency), which is a typical characteristic of type I membrane.

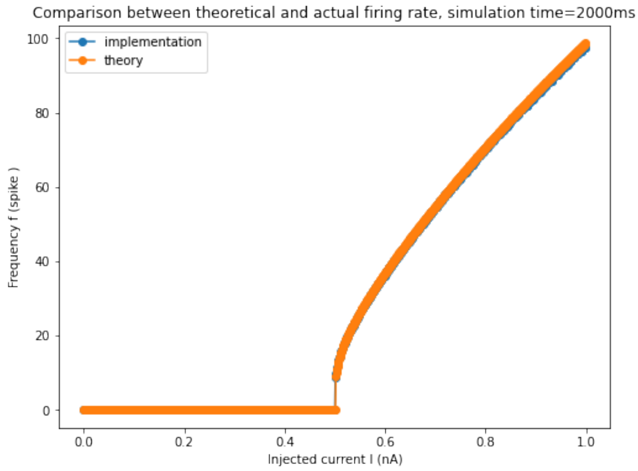

Figure 3: The relationship of firing rate and input current, where the simulation result somehow perfectly recovers the analytic result.

3 Part 3

Then, we add a noise term into the input current:

$$C\frac{dV_m}{dt} = -G_L(V_m - V_L) + I + \sigma\omega(t) \tag{6}$$

where I is the mean current, σ is the noise level, and $\omega(t)$ is the white noise. We would compare two examples where I is just below and above the deterministic threshold I_c by running the experiments for $200ms$. The result is presented in Figure 4. When the σ level increases, more interference is added, and the process becomes more stochastic compared to the control group. When I is close to I_c, the spiking pattern would be very unpredictable for each independent trail, especially for large σ, since the $\sigma\omega(t)$ term would easily lead V_m to cross the threshold.

Additionally, we calculate the firing rate f as the inverse of the mean interspike intervals. To make the smooth effect clearer, we set the noise level $\sigma = 1.0nA\sqrt{ms}$, then superimpose it with the $f - I$ curve without noise. To average out the noise and obtain reliable estimates, we simulate $20s$. The result is presented in Figure 5. We notice that there is no more strict threshold value for input current to lead to repetitive firing, meaning that even a small external current can cause slow repetitive firing. Also, the simulated firing rate rises monotonously with the external current, which is similar to the deterministic condition. Plus, when the injected current is large, the simulated firing rate approaches the analytical value; thus, it suggests that the dynamics of the neuron are mainly driven by the external current instead of the noise.

Furthermore, with noise added, the interspike interval (ISI) is a random variable. We would calculate its coefficient of variation (CV), as the ratio of the standard variation and the mean. We calculate the CVs of ISIs for different external currents and plot them versus the mean firing rate. The result is presented in Figure 6. For the low-frequency situation, the CV is large, which indicates a large variation in the firing time of the neuron. It is reasonable since it corresponds to the situation when the external current is small. Thus, to spike, the neuron needs a large noise (fluctuation) to cross the threshold, which corresponds to the fluctuation-driven setting. However, with a larger firing rate, the CV decreases quickly as the external current is large enough to dominate the dynamics of the neuron, which corresponds to the mean-driven setting.

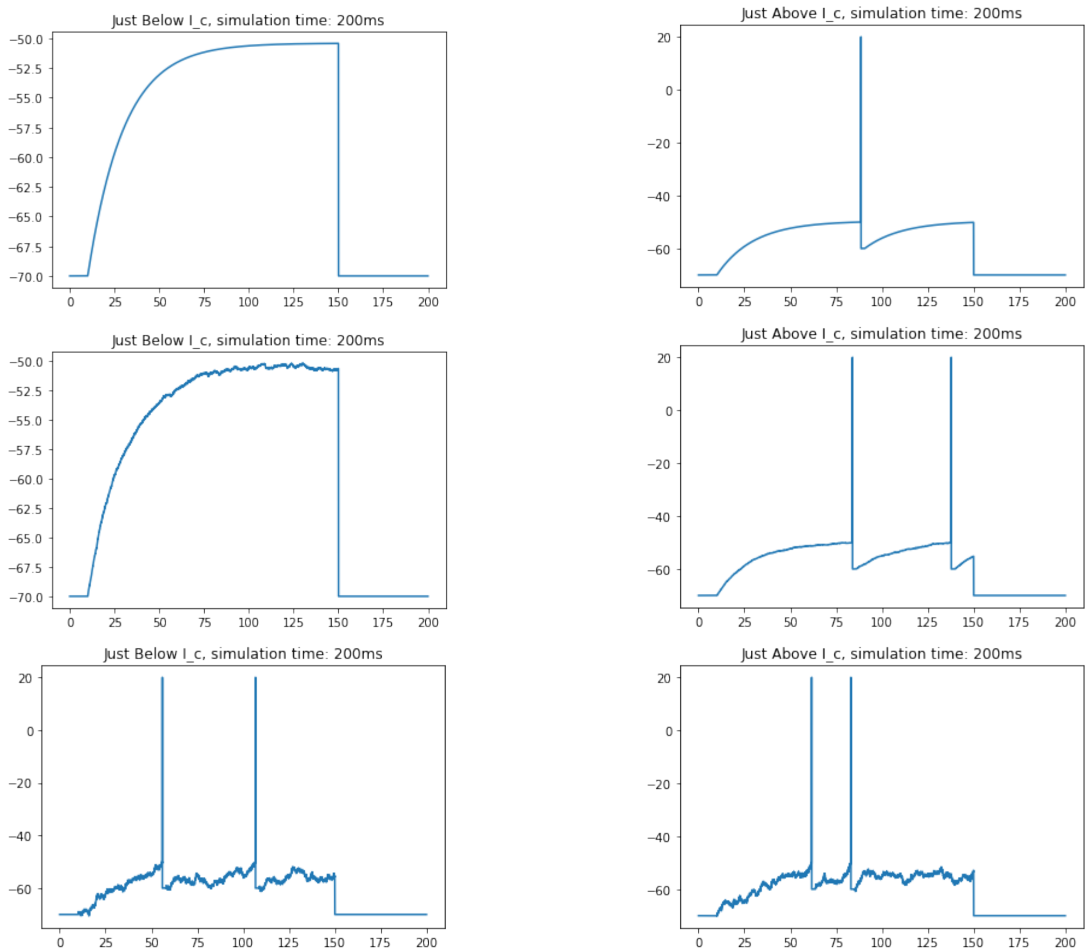

Figure 4: Comparison of $V_m(t)$ when I is just below (Column 1) and just above (Column 2) the deterministic current threshold I_c meanwhile noise is diminished (Row 1), $\sigma = 0.3nA\sqrt{ms}$ (Row 2) and $= 1.0nA\sqrt{ms}$ (Row 3).

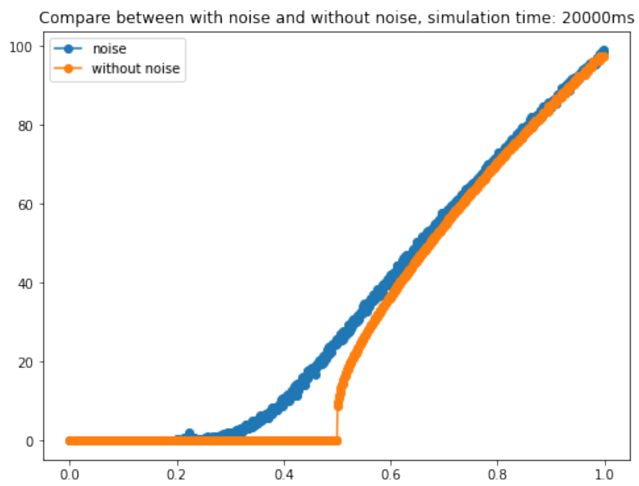

Figure 5: The relationship of firing rate and input current with white noise added.

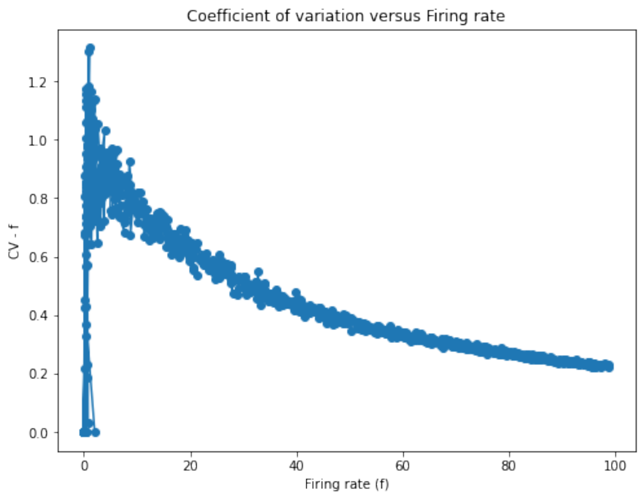

Figure 6: The relationship of CV of ISI v.s. the mean firing rate.