# Monte Carlo Sampling Methods Homework 6

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December 17, 2022

## 1 Exercise 71

In this question, we need to use a Metropolized version of the overdamped stochastic Newton scheme to sample from the Rosenbrok density:

$$\pi(x) \propto \exp\left(-\frac{100(x_2 - x_1^2)^2 + (1 - x_1)^2}{20}\right). \tag{1}$$

When $S$ is a $2 \times 2$ matrix with $trace(S) \neq 0$, the matrix:

$$R = \frac{S + \sqrt{det(S)}I}{\sqrt{trace(S) + 2\sqrt{det(S)}}}. \tag{2}$$

is a square root of $S$. The inverse Hessian, $S = (-D^2 \log(\pi(x)))^{-1}$ will not be positive definite for all $x$. When it is not, we need to replace the inverse Hessian with the square of the matrix obtained by taking absolute values before each square root in the formula for $R$. Since we are Metropolizing, we could omit the divergence of the $S$ term in the overdamped proposal step without introducing additional bias, but it may increase the rejection rate. In the lecture note, the stochastic Newton iteration is introduced as:

$$X_h^{(k+1)} = X_h^{(k)} - hH^{-1}\left(X_h^{(k)}\right)\nabla^{\mathrm{T}}V\left(X_h^{(k)}\right) + h\,\mathrm{div}\,H^{-1}\left(X_h^{(k)}\right) + \sqrt{2hH^{-1}\left(X_h^{(k)}\right)}\xi^{(k+1)} \tag{3}$$

Also, we compare the performance of this preconditioned scheme to the Metropolized overdamped scheme with $S = I$. Figure 1 shows the change in the acceptance ratio of these two methods under different time step sizes. Figure 2 and 3 show the respective histogram graph.

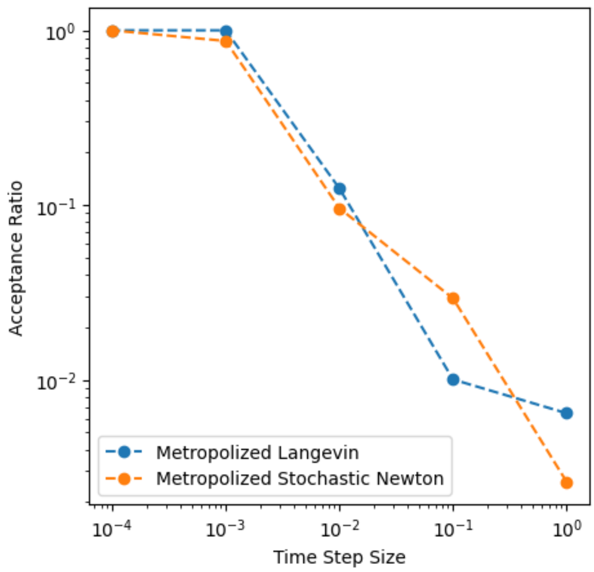


**Figure 1:** Comparison of Metropolized Langevin and Metropolized Stochastic Newton with under different time step size.

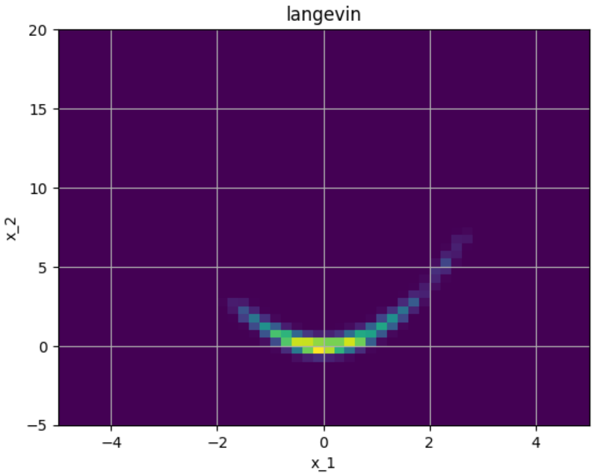


**Figure 2:** Histogram of Langevin sampling scheme by $x_1$ and $x_2$.

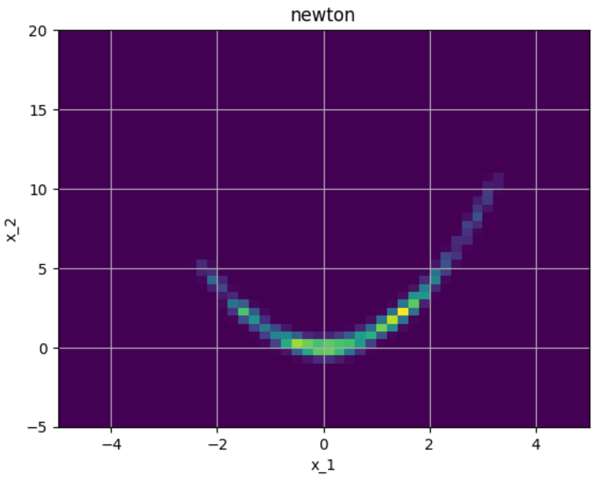


**Figure 3:** Histogram of Stochastic Newton iteration by $x_1$ and $x_2$.

## 2 Exercise 75

Here we would use the Affine Invariant Ensemble Scheme (AIES) to sample from the Rosenbrock density defined in the previous. In Figure 4 and Figure 5, we change $\alpha$ and $L$ and observe the change in acceptance rate. In Figure 6, we also plot the graph by $x_1$ and $x_2$ for AIES and compare it with section 1. Given the same computational power, in the Rosenbrock problem, the efficiency of preconditioned overdamped Langevin is better than the affine-invariant ensemble sampler, where both are much better than the vanilla overdamped Langevin scheme. Still, this problem is not well-conditioned since the Rosenbrock function cannot be affinely transformed into a nice-shaped function. Also, the narrow nonlinear shape of the density makes the acceptance rate in AIES very low.

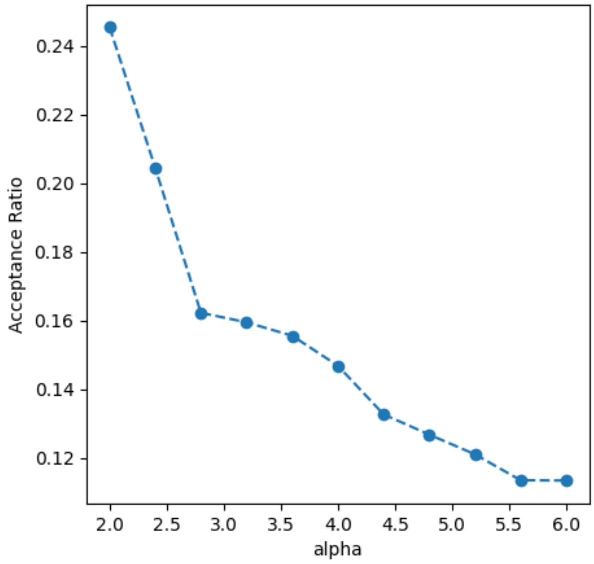


**Figure 4:** Change of acceptance rate in AIES under different $\alpha$.

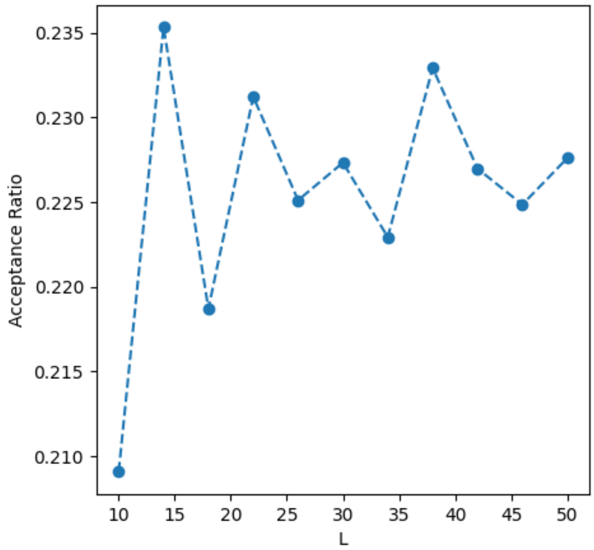


**Figure 5:** Change of acceptance rate in AIES under different $L$.

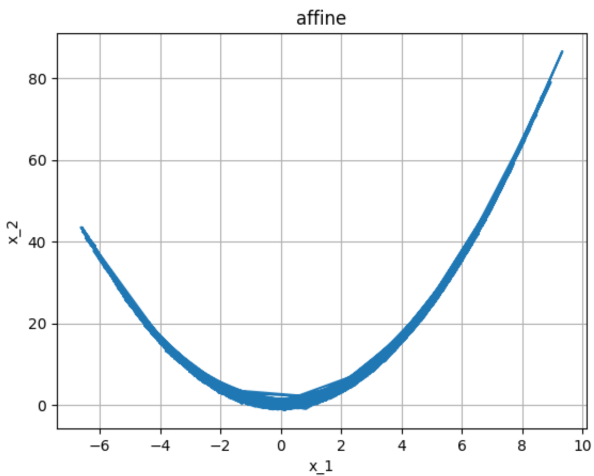


**Figure 6:** Trajectory of AIES by $x_1$ and $x_2$.