CS 480 – Introduction to Artificial Intelligence

Topic: Bayesian Networks
Chapter: 13

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Joint Distribution

- We have n random variables, $V_1, V_2, \ldots, V_n$
- We are interested in the probability of a possible world, where
  - $V_1=v_1, V_2=v_2, \ldots, V_n = v_n$
- $P(V_1, V_2, \ldots, V_n)$ associates a probability for each possible world ≡ the **joint distribution**
- How many independent parameters are needed, if V_i are all binary?

Joint Distribution

- Extremely useful
  - Can answer any type of query
- Extremely inefficient
  - Requires exponential size memory
  - Inference using an exponential-size table requires exponential time
- Chapter 13 $\Rightarrow$ Efficient representation and inference

Chain Rule

- $P(V_1, V_2, \ldots, V_n) =$
  - $P(V_1)P(V_2 | V_1)P(V_3 | V_1,V_2) \ldots P(V_n | V_1, V_2, \ldots, V_{n-1})$
  - $P(V_2)P(V_1 | V_2)P(V_3 | V_2,V_1) \ldots P(V_n | V_2, V_1, \ldots, V_{n-1})$
  - …
- If all V_i are binary, $P(V_1, V_2, \ldots, V_n)$ requires 2^n-1 independent parameters
- $P(V_1)$: How many?
- $P(V_2 | V_1)$: How many?
- $P(V_3 | V_1, V_2)$: How many?
- …
- $P(V_n | V_1, V_2, \ldots, V_{n-1})$: How many?
- How many in total?

Marginal Independence

- Two random variables A and B are **marginally independent** if and only if
  - $P(A, B) = P(A)*P(B)$, equivalently
  - $P(A \mid B) = P(A)$, equivalently
  - $P(B \mid A) = P(B)$

The Joint revisited

- $P(V_1, V_2, \ldots, V_n) =$
  - $P(V_1)P(V_2 | V_1)P(V_3 | V_1,V_2) \ldots P(V_n | V_1, V_2, \ldots, V_{n-1})$
- If $V_i \perp V_j$ for all $i \neq j$
  - $P(V_1, V_2, \ldots, V_n) =$
    - $P(V_1)P(V_2 | V_1)P(V_3 | V_1,V_2) \ldots P(V_n | V_1, V_2, \ldots, V_{n-1})$
    - $P(V_1)P(V_2)P(V_3) \ldots P(V_n)$
    - How many independent parameters now?

Conditional Independence

- Marginal independence is not very common
- Two random variables A and B are conditionally independent given C if and only if
  - $P(A, B|C) = P(A|C)*P(B|C)$, equivalently
  - $P(A|B,C) = P(A|C)$, equivalently
  - $P(B|A,C) = P(B|C)$

WHY INDEPENDENCE?

- The joint distribution for n binary random variables
  - $2^n - 1$ independent entries; exponential
- If the variables were all
  - Marginally independent, then
    - $1 + 1 + \ldots + 1 = n$ independent parameters; polynomial
  - Conditionally independent given one of them, then
    - $1 + 2 + 2 + \ldots + 2 = 1 + 2(n-1) = 2n - 1$ independent parameters; polynomial

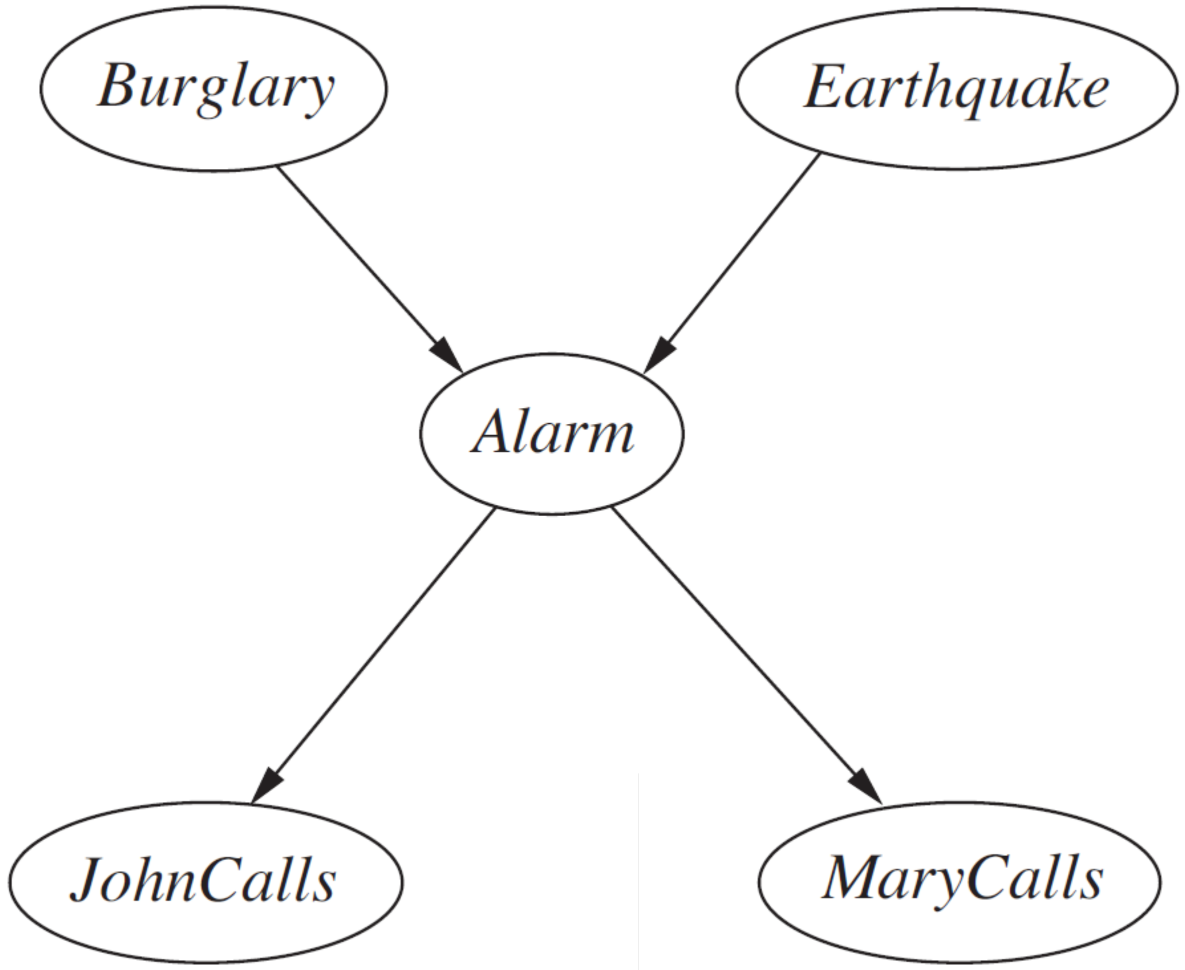
Advantages of more compact representation

- Fewer parameters
  - Makes learning and reasoning easier
- Consider asking an expert the probability of specific entry in a huge probability table

Bayesian Networks

- Random variables = nodes
- Direct relationships = directed edges
- BNs capture independencies
  - More compact than full joint representation
- Graphs provide
  - Graph theory / efficient reasoning
  - Intuition

BURGLARY EXAMPLE



Directed Graphs

- A **graph** consists of **nodes** and **edges**
- **Nodes:** $\mathcal{X}=\{X_1, X_2, \ldots, X_n\}$
- **Undirected Edge:** $X_i - X_j$
- **Directed Edge:** $X_i \rightarrow X_j$
- A graph is **directed** if its *all* edges are directed

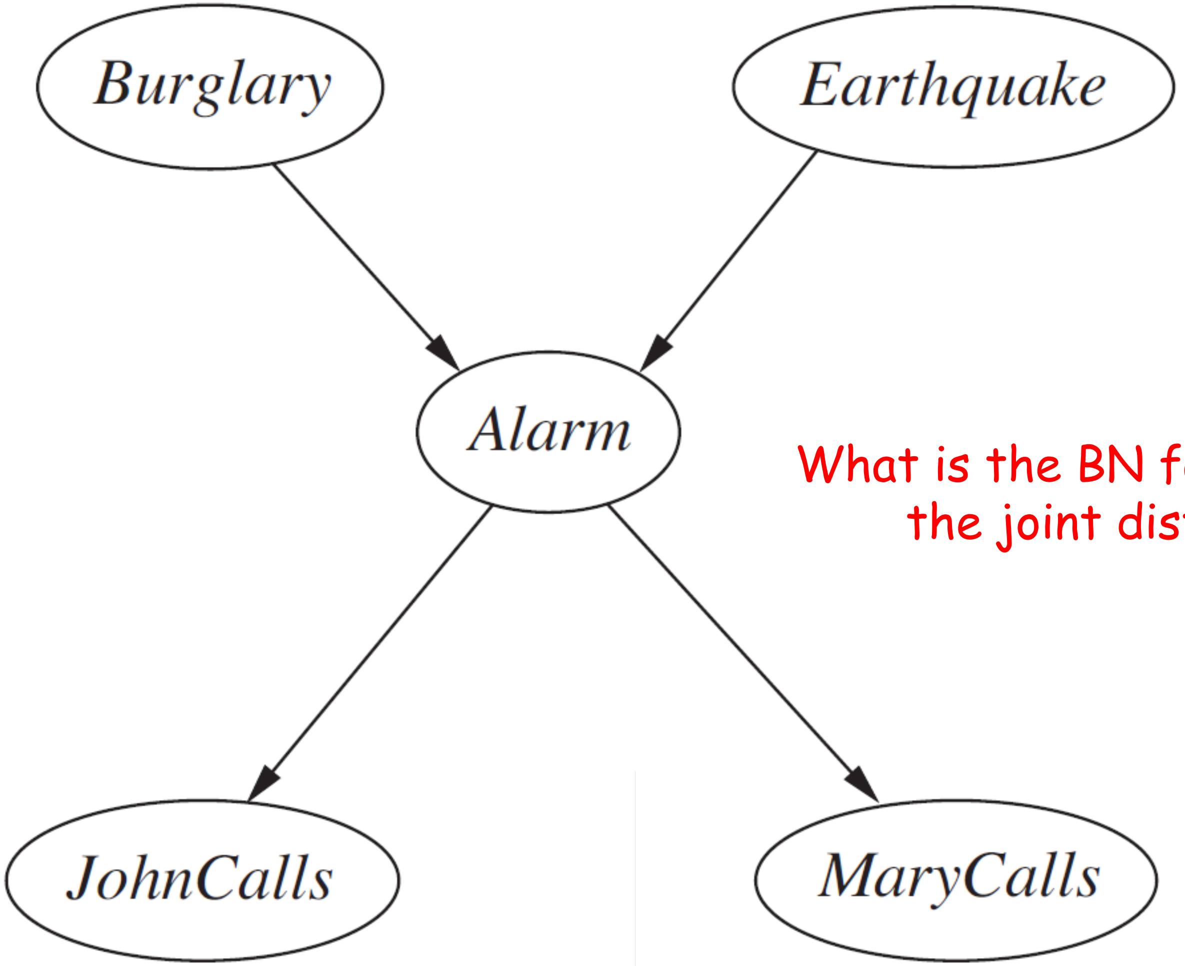
RELATIONSHIPS

- $X_i \rightarrow X_j$
  - X_i is the **parent**
  - X_j is the **child**
- X_i is an **ancestor** of X_j if there is a directed path from X_i to X_j
- X_i is a **descendant** of X_j if there is a directed path from X_j to X_i
- **Nondescendants**(X_i) $\equiv \mathcal{X} \setminus$ Descendants(X_i)

Bayesian network Factorization

$$P(X_1, \ldots, X_n) = \prod_i P(X_i \mid Pa(X_i))$$

BURGLARY EXAMPLE



What is the BN factorization of the joint distribution?

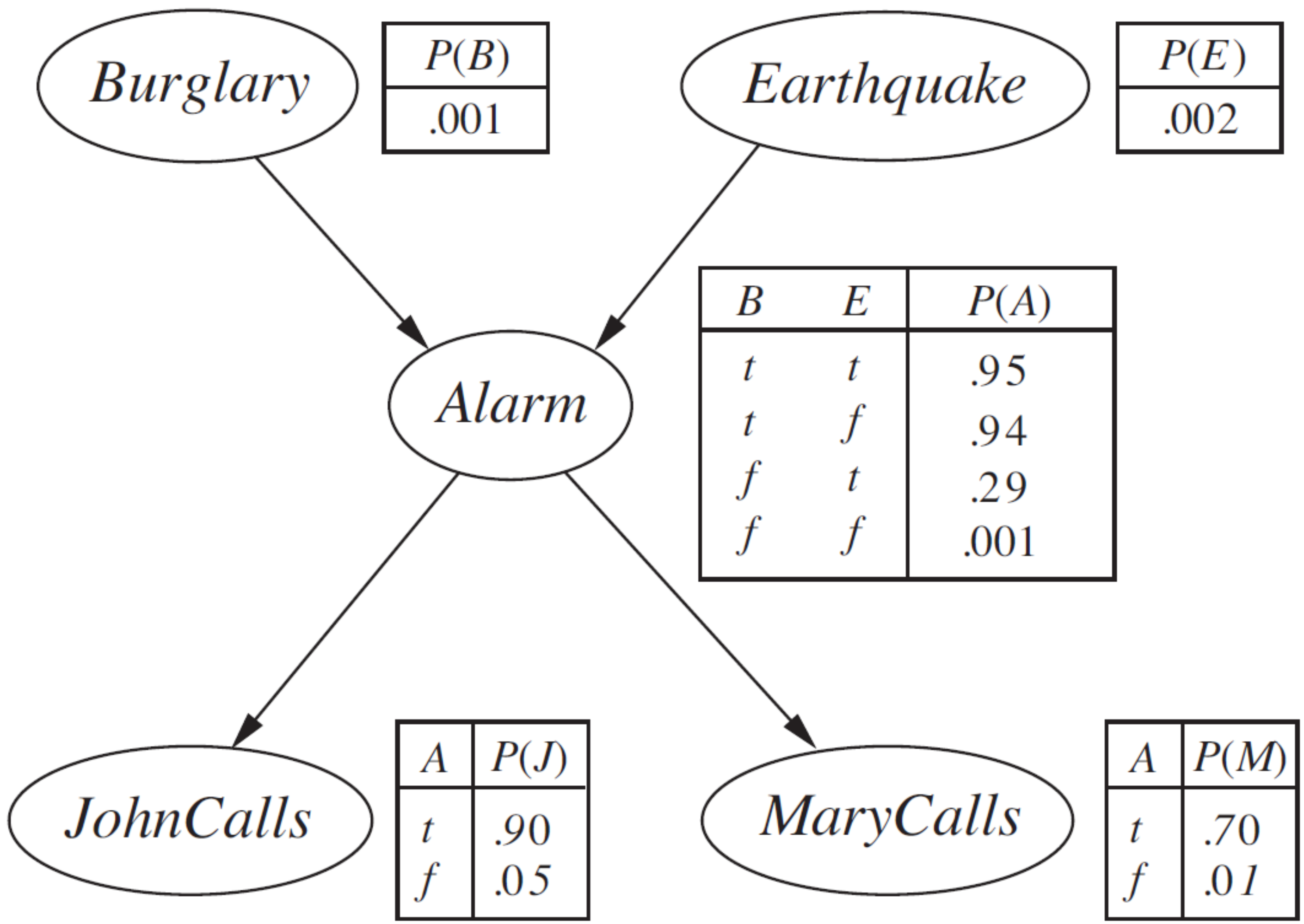
Independencies - Parents

- X is independent of its non-descendants given its parents
  - $X \perp \text{Non-descendants}(X) \mid \text{Parents}(X)$
- What are the independencies in the burglary example?

PARAMETERIZATION

Given the indecencies encoded in a BN, what are the parameters needed to capture the joint representation efficiently?

BURGLARY EXAMPLE



$P(B)$
.001

$P(E)$
.002

B	E	$P(A)$
t	t	.95
t	f	.94
f	t	.29
f	f	.001

A	$P(J)$
t	.90
f	.05

A	$P(M)$
t	.70
f	.01

Ind Assumption $\Longleftrightarrow$ Factorization

THEOREMS

- **Theorem 1:** If a probability distribution P holds the independencies encoded in G, then P factorizes according to G
- **Theorem 2:** If P factorizes according to G, then it holds the independencies encoded in G
- Let's see a constructive proof for Theorem 1; we'll not prove Theorem 2

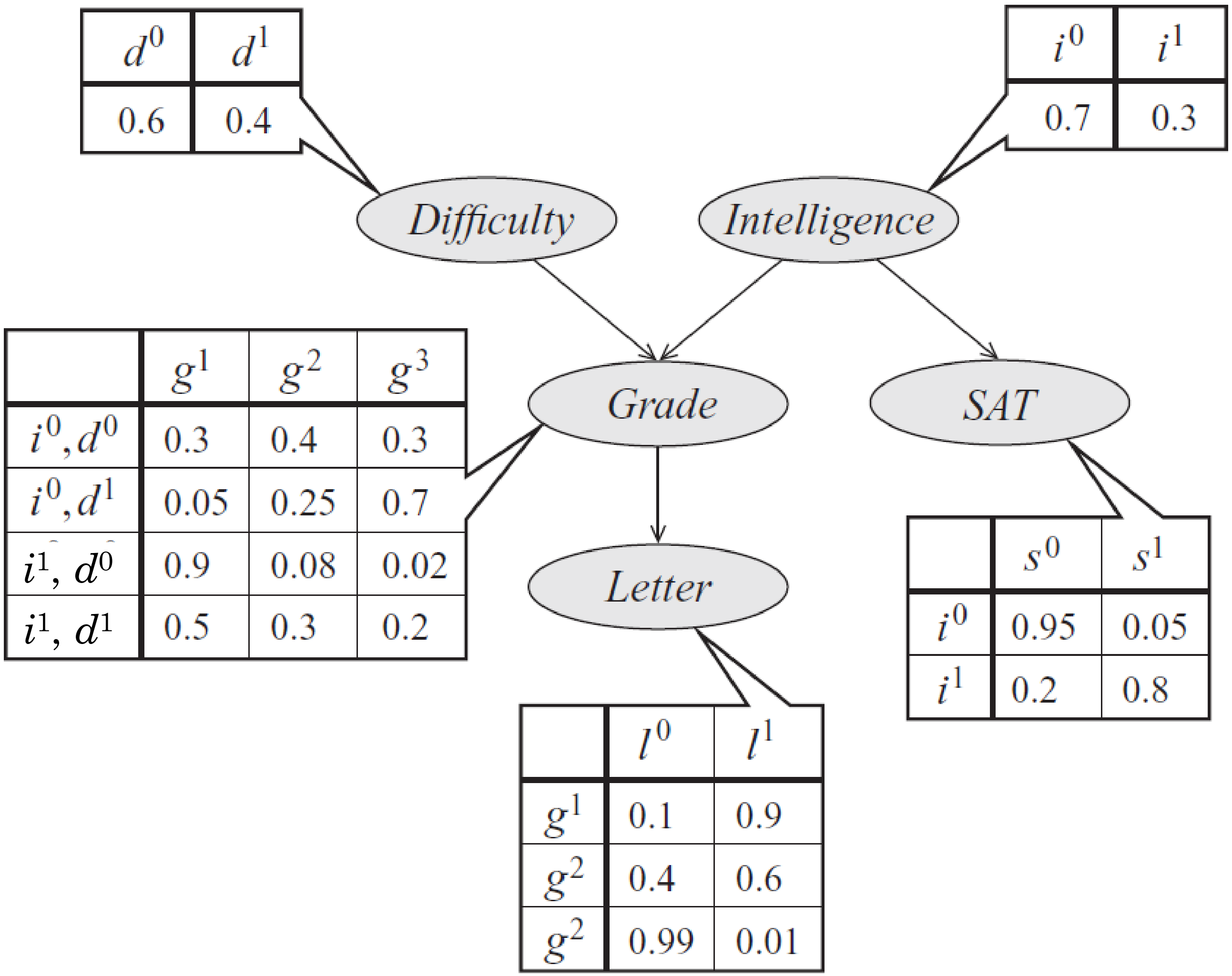
FROM INDEPENDENCE TO FACTORIZATION

- Linear chain example
  - $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E$
- Burglary example

Burglary Example

- The joint representation
  - Equation
- Contrast number of parameters for
  - Probability table for joint
  - Bayesian network

Student Example



d^0	d^1
0.6	0.4

i^0	i^1
0.7	0.3

	g^1	g^2	g^3
i^0, d^0	0.3	0.4	0.3
i^0, d^1	0.05	0.25	0.7
i^1, d^0	0.9	0.08	0.02
i^1, d^1	0.5	0.3	0.2

	s^0	s^1
i^0	0.95	0.05
i^1	0.2	0.8

	l^0	l^1
g^1	0.1	0.9
g^2	0.4	0.6
g^2	0.99	0.01

Student Example

- The joint representation
  - Equation
- Contrast number of parameters for
  - Probability table for joint
  - Bayesian network

So far

- We've discussed the representation
- Now, it's time for inference

Reasoning Patterns

- **Causal reasoning**
  - From causes to effects
    - E.g., Burglary to Alarm to MaryCalls
    - E.g., Intelligence to Grade to Letter
- **Evidential reasoning**
  - From effects to the causes
    - E.g., JohnCalls to Alarm to Earthquake
    - E.g, Letter to Grade to Difficulty
- **Explaining away/inter-causal reasoning**
  - Causes of a common effect interact
    - E.g., Earthquake, Burglary, and Alarm (and Alarm's descendants)
    - E.g., Difficulty, Intelligence, and Grade (and Grade's descendants)

Inference in Bayesian Networks

- There are several methods, some are exact and some are approximate
- We will study only one in this class
- *Variable Elimination*

Variable Elimination

- Let
  - **V** be the set of all variables, **Q** be the set of query variables, **E** be the set of evidence variables
  - P(**Q** | **E**) be the query

1. Write down the joint dist. using the Bayesian network structure
2. Set the variables in **E** to their respective values
3. Sum over all variables in **V\\(Q∪E)**
   a) Pick an order for variables in **V\\(Q∪E)**
   b) For each variable V_i in **V\\(Q∪E),** create a new factor by
      - Multiplying all the factors that contains V_i, and
      - Summing over possible values of V_i
4. Normalize the last remaining factor (this step is unnecessary if **E** is empty)

Examples

- Given the following BNs, compute the requested probabilities efficiently (without computing the full joint)
  - $A \rightarrow B \rightarrow C$;
    - P(A) = <0.6, 0.4>,
    - P(B | A=t) = <0.8, 0.2>, P(B | A=f) = <0.1, 0.9>
    - P(C | B=t) = <0.7, 0.3>, P(C | B=f) = <0.4, 0.6>
  - Compute P(A), P(B), P(C), P(C | A=t), P(A | C=t)

IRRELEVANT

- Let
  - **V** be the set of all variables, **Q** be the set of query variables, **E** be the set of evidence variables
  - P(**Q** | **E**) be the query
- $Y \in \boldsymbol{V} \setminus \{\boldsymbol{Q} \cup \boldsymbol{E}\}$ is irrelevant iff
  - $Y \notin Ancestors\ of\{\boldsymbol{Q} \cup \boldsymbol{E}\}$
    - or
  - $Y \perp \boldsymbol{Q} \mid \boldsymbol{E}$
- Examples

Why Variable Elimination?

- We could compute P(D) by
  - Computing the full joint table, and then
  - Summing over the remaining variables
- Variable elimination, with a *good* ordering, can
  - Save memory, and
  - Save time

Applications of Bayesian Networks

- Too many to list
- Here is a book about it: http://www.wiley.com/WileyCDA/WileyTitle/productCd-0470060301.html
- Chapters include:
  - Medical diagnosis
  - Complex genetic models
  - Crime risk factors analysis
  - Inference problems in forensic science
  - Classifiers for modeling of mineral potential
  - Reliability analysis of systems
  - Credit-rating of companies
  - Classification of Chilean wines
  - Complex industrial process operation
  - Probability of default for large corporates
  - Risk management in robotics