Chapter 9

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Comments and Proofs

Section 9.2.2.1

Note that in this context, μ is a parameter of the model, not $\frac{1}{1+\exp(-\mathbf{w}^T\mathbf{x})}$.

Section 9.3.1

Because equation 9.77 looks quite different than equation 9.2, I did not find equations 9.81 and 9.82 trivial.

Proof of equation 9.81. We first find $A(\theta)$.

$$\begin{aligned}
\int p(y|\theta_i, \sigma^2) &= 1 \\
\int \exp\left[\frac{y\theta_i - A(\theta_i)}{\sigma^2} - c(\theta_i, \sigma^2)\right] dy &= 1 \\
\exp\left[\frac{-A(\theta_i)}{\sigma^2}\right] \int \exp\left[\frac{y\theta_i}{\sigma^2} - c(\theta_i, \sigma^2)\right] dy &= 1 \\
\int \exp\left[\frac{y\theta_i}{\sigma^2} - c(\theta_i, \sigma^2)\right] dy &= \exp\left[\frac{A(\theta_i)}{\sigma^2}\right] \\
\sigma^2 \log \int \exp\left[\frac{y\theta_i}{\sigma^2} - c(\theta_i, \sigma^2)\right] dy &= A(\theta)
\end{aligned}$$

Next, we derive in terms of θ:

$$\begin{aligned}
\frac{\partial A}{\partial \theta_i} &= \frac{\partial}{\partial \theta_i} \sigma^2 \log \int \exp\left[\frac{y\theta_i}{\sigma^2} - c(\theta_i, \sigma^2)\right] dy \\
&= \sigma^2 \frac{\int \frac{\partial}{\partial \theta} \exp\left[\frac{y\theta_i}{\sigma^2} - c(y_i, \sigma^2)\right] dy}{\int \exp\left[\frac{y\theta_i}{\sigma^2} - c(y_i, \sigma^2)\right] dy} \\
&= \sigma^2 \frac{\int \frac{y}{\sigma^2} \exp\left[\frac{y\theta_i}{\sigma^2} - c(y_i, \sigma^2)\right] dy}{\exp(\frac{A(\theta_i)}{\sigma^2})} \\
&= \int y p(y|\theta_i, \sigma^2) dy = \mathbb{E}[y|\theta_i, \sigma].
\end{aligned}$$

Proof of equation 9.8.2. Using the proof of equation 9.8.1, we have

$$\begin{aligned}
\frac{\partial}{\partial \theta_i}\frac{\partial A}{\partial \theta_i} &= \frac{\partial}{\partial \theta_i}\int y \exp\left[\frac{y\theta_i - A(\theta_i)}{\sigma^2} - c(y_i, \sigma^2)\right] dy \\
&= \int y \frac{\partial}{\partial \theta_i} \exp\left[\frac{y\theta_i - A(\theta_i)}{\sigma^2} - c(y_i, \sigma^2)\right] dy \\
&= \int y \left(\frac{y - \frac{\partial A}{\partial \theta_i}}{\sigma^2}\right) \exp\left[\frac{y\theta_i - A(\theta_i)}{\sigma^2} - c(y_i, \sigma^2)\right] dy \\
&= \frac{1}{\sigma^2}\int \left(y^2 - y\frac{\partial A}{\partial \theta_i}\right) p(y|\theta_i, \sigma^2) dy \\
&= \frac{1}{\sigma}\left(\int y^2 p(y|\theta_i, \sigma^2) dy - \frac{\partial A}{\partial \theta_i}\int y p(y|\theta_i, \sigma^2) dy\right) \\
&= \frac{1}{\sigma^2}\left(\mathbb{E}[y^2|\theta_i, \sigma^2] - \mathbb{E}[y|\theta_i, \sigma^2]^2\right) \\
&= \frac{1}{\sigma^2}\operatorname{var}[y|\theta_i, \sigma^2].
\end{aligned}$$

It follows that

$$\operatorname{var}[y|\theta_i, \sigma^2] = \sigma^2 \frac{\partial^2 A}{\partial \theta_i^2}.$$

Exercises

Exercise 2

$$\begin{aligned}
\mathcal{N}(\mathbf{x}|\boldsymbol{\Sigma}, \boldsymbol{\theta}) &= \frac{1}{(2\pi)^{\frac{D}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1}(\mathbf{x} - \boldsymbol{\mu})\right) \\
&= \frac{1}{(2\pi)^{\frac{D}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}(\mathbf{x}^T \boldsymbol{\Sigma}^{-1}\mathbf{x} - 2\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1}\mathbf{x} + \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})\right) \\
&= \frac{1}{(2\pi)^{\frac{D}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right) \exp\left(-\frac{1}{2}\mathbf{x}^T \boldsymbol{\Sigma}^{-1}\mathbf{x} + \boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1}\mathbf{x}\right) \\
&= \frac{1}{A(\boldsymbol{\Sigma}, \boldsymbol{\mu})} h(\mathbf{x}) \exp\left(\boldsymbol{\eta}(\boldsymbol{\Sigma}, \boldsymbol{\mu})^T \boldsymbol{\phi}(\mathbf{x})\right)
\end{aligned}$$

Where

$$\begin{aligned}
h(\mathbf{x}) &= 1, \\
A(\boldsymbol{\Sigma}, \boldsymbol{\mu}) &= \frac{1}{(2\pi)^{\frac{D}{2}}|\boldsymbol{\Sigma}|^{\frac{1}{2}}} \exp\left(-\frac{1}{2}\boldsymbol{\mu}^T \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\right),
\end{aligned}$$

$\boldsymbol{\eta}(\boldsymbol{\Sigma}, \boldsymbol{\mu})$ is $-\frac{1}{2}$ times a concatenation of the rows (or columns) of $\boldsymbol{\Sigma}^{-1}$ and $\boldsymbol{\mu}^T \boldsymbol{\Sigma}$, and $\boldsymbol{\phi}(\mathbf{x})$ is a concatenation of the rows (or columns) of $\mathbf{x}\mathbf{x}^T$ and $\mathbf{x}$.