Chapter 10

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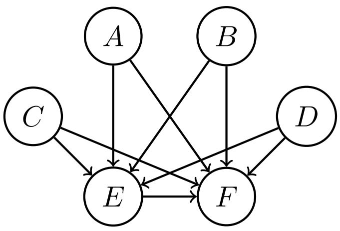
Exercises

Exercise 1

Marginalizing out X gives

$$\begin{aligned}
p(A:F) &= \sum_{x'} p(A:F|X=x') \\
&= p(A)p(B)p(C)p(E)\sum_{x'} p(E,F|C,D,X=x')p(X=x'|A:D) \\
&= p(A)p(B)p(C)p(D)p(E,F|A:D) \\
&= p(A)p(B)p(C)p(D)p(F|A:E)p(E|A:D).
\end{aligned}$$

It follows that the graph G' is



Note that we can reverse $E \to F$.

We know that this is a minimal I-map because removing any edges creates conditional independence assumptions not made by G:

Edge removed in G'	New CI in G'	Counterexample in G
$A \to E$	$A \perp_{G'} E \mid B, C, D$	$A \to X \to E$
$A \to F$	$A \perp_{G'} F \mid B, C, D, E$	$A \to X \to F$
$B \to E$	$B \perp_{G'} E \mid A, C, D$	$B \to X \to E$
$B \to F$	$B \perp_{G'} F \mid A, C, D, E$	$B \to X \to F$
$C \to E$	$C \perp_{G'} E \mid A, B, D$	$C \to X \to E$
$C \to F$	$C \perp_{G'} F \mid A, B, D, E$	$C \to X \to F$
$D \to E$	$D \perp_{G'} E \mid A, B, C$	$D \to X \to E$
$D \to F$	$D \perp_{G'} F \mid A, B, C, E$	$D \to X \to F$
$E \to F$	$E \perp_{G'} F \mid A, B, C, D$	$E \leftarrow X \to F$

The first row of the above table can be read as: removing the $A \to E$ edge in G' creates the conditional independence $A \perp_{G'} E|B, C, D$; however, this CI is not held in G because of the $A \to X \to E$ path.

Note that $A \perp_{G'} E|B, C, D$ rather than $A \perp_{G'} E|B, C, D, F$ because of Berkson's paradox (this also applies for all above CIs conditioned on three nodes).

This exercise highlights the power of hidden nodes: G' has 50% more edges than G, despite having fewer nodes.

Exercise 3

Let $Y_i = \mathrm{ch}(X_i)$, $B_i = \mathrm{pa}(X_i)$, $Z_i = \mathrm{copa}(X_i)$, and $U_i = X_{-i} \setminus (Z_i \cup Z_i \cup B_i)$.

$$
\begin{aligned}
p(X_i|X_{-i}) &= \frac{p(X_i, X_{-i})}{p(X_{-i})} \\
&\propto p(X_i, X_{-i}) \\
&= p(X_i, B_i, Y_i, Z_i, U_i) \\
&= p(Y_i|X_i, Z_i)p(X_i|B_i)p(B_i, Z_i, U_i) \\
&\propto p(Y_i|X_i, Z_i)p(X_i|B_i) \\
&= p(X_i|B_i) \prod_{y \in Y_i} p(y|pa(y)).
\end{aligned}
$$