Chapter 5

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Exercise 1

a. Since x and y are independent, we have

$$p(x, y|\theta) = p(y|x, \boldsymbol{\theta})p(x|\boldsymbol{\theta}).$$

The author gives both $p(y|x, \boldsymbol{\theta})$ and $p(x|\boldsymbol{\theta})$. Plugging in the given values, we get

$$\begin{align}
p(x = 0, y = 0|\boldsymbol{\theta}) &= \theta_2(1 - \theta_1) \tag{1}\\
p(x = 0, y = 1|\boldsymbol{\theta}) &= (1 - \theta_2)\theta_1 \tag{2}\\
p(x = 0, y = 1|\boldsymbol{\theta}) &= (1 - \theta_2)(1 - \theta_1) \tag{3}\\
p(x = 1, y = 1|\boldsymbol{\theta}) &= \theta_2\theta_1. \tag{4}
\end{align}$$

b. We find the likelihood by plugging in (2) (3) (4) and (5) to the likelihood function:

$$\begin{split}
p(\mathcal{D}|\hat{\boldsymbol{\theta}}) &= \prod p(x_i, y_i|\boldsymbol{\theta})\\
&= \theta_1^4\theta_2^4(1 - \theta_1)^3(1 - \theta_2)^3.
\end{split} \tag{5}$$

The MLE is given by

$$\begin{aligned}
\hat{\boldsymbol{\theta}} &= \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\, p(\mathcal{D}|\boldsymbol{\theta})\\
&= \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\, \theta_1^4\theta_2^4(1 - \theta_1)^3(1 - \theta_2)^3.
\end{aligned}$$

Since $p(\mathcal{D}|\boldsymbol{\theta})$ is differentiable with respect to θ_1 and θ_2, we can differentiable and set equal to zero to obtain $\hat{\boldsymbol{\theta}}$:

$$\begin{aligned}
\frac{\partial}{\partial\theta_1}p(\mathcal{D}|\boldsymbol{\theta}) &= \frac{\partial}{\partial\theta_1}\theta_1^4\theta_2^4(1 - \theta_1)^3(1 - \theta_2)^3\\
&= 4\theta_1^3\theta_2^4(1 - \theta_1)^3(1 - \theta_2)^3 - 3\theta_1^4\theta_2^4(1 - \theta_1)^2(1 - \theta_2)^3\\
&= 0.
\end{aligned}$$

Solving for θ_1, we get that $\hat{\theta}_1 = \frac{4}{7}$. A similar process yields that $\hat{\theta}_2 = \frac{4}{7}$. This result is not too surprising as θ_1 is how often $x = 1$ in the data and θ_2 is how often our observer was correct, both of which are $\frac{4}{7}$. We do not prove that $\hat{\theta}_1 = \hat{\theta}_2 = \frac{4}{7}$ is a global maximum, given the context, it is not to far-fetched to assume the only critical point is the global maximum. Plugging in the MLE's of $\hat{\theta}_1$ and $\hat{\theta}_2$ into equation (5) gives

$$\begin{aligned} p(\mathcal{D}|\hat{\boldsymbol{\theta}}, M_2) &= \left(\frac{4}{7}\right)^4 \left(\frac{4}{7}\right)^4 \left(1 - \frac{4}{7}\right)^3 \left(1 - \frac{4}{7}\right)^3 \\ &= \frac{4^8 3^6}{7^{14}}. \end{aligned}$$

c. We take a different approach than part **b.** by maximizing the log-likelihood (rather than the likelihood) and by using Lagrange multipliers (rather than differentiating and setting equal to zero). To put it in math notation, we are trying to maximize

$$\begin{aligned} \log p(\mathcal{D}|\boldsymbol{\theta}, M_4) &= \log \theta_{00}^2 \theta_{01} \theta_{10}^2 \theta_{11}^2 \\ &= 2 \log \theta_{00} + \log \theta_{01} + 2 \log \theta_{10} + 2 \log \theta_{11} \end{aligned} \tag{6}$$

over the constraint

$$G(\boldsymbol{\theta}) = \theta_{00} + \theta_{01} + \theta_{10} + \theta_{11} = 1. \tag{7}$$

We first find the critical points by finding $\boldsymbol{\theta}$ such that

$$\nabla G(\boldsymbol{\theta}) = \lambda \nabla \log p(\mathcal{D}|\boldsymbol{\theta}, M_4)$$

for some none-zero λ. Finding the gradients is fairly straightforward, we just differentiate with respect to each θ_{ij}:

$$\begin{aligned} \nabla \log p(\mathcal{D}|\boldsymbol{\theta}, M_4) &= \left(\frac{2}{\theta_{00}}, \frac{2}{\theta_{10}}, \frac{1}{\theta_{01}}, \frac{2}{\theta_{11}} \right) \\ \nabla G(\boldsymbol{\theta}) &= (1, 1, 1, 1) \end{aligned}$$

Now we solve for λ:

$$\lambda \left(\frac{2}{\theta_{00}}, \frac{1}{\theta_{01}}, \frac{2}{\theta_{10}}, \frac{2}{\theta_{11}} \right) = (1, 1, 1, 1) \tag{8}$$

$$\lambda(2, 1, 2, 2) = \theta_{00}, \theta_{01}, \theta_{10}, \theta_{11}. \tag{9}$$

Plugging into our constraint yields

$$\begin{aligned} 2\lambda + \lambda + 2\lambda + 2\lambda &= 1 \\ \lambda &= \frac{1}{7}. \end{aligned} \tag{10}$$

It follows that $\hat{\boldsymbol{\theta}}$ is given by

$$\hat{\theta}_{00} = \frac{2}{7}, \hat{\theta}_{01} = \frac{1}{7}, \hat{\theta}_{10} = \frac{2}{7}, \hat{\theta}_{11} = \frac{2}{7}.$$

This result is not too surprising. Each θ_{ij} is the probability of an event happening. Intuitively, $\hat{\theta_{ij}}$ would be how often the corresponding even happened in the given data.

Now that we have found $\hat{\boldsymbol{\theta}}$, we can find $p(\mathcal{D}|\hat{\boldsymbol{\theta}}, M_4)$:

$$\begin{aligned} p(\mathcal{D}|\hat{\boldsymbol{\theta}}, M_4) &= \theta_{00}^2\theta_{01}\theta_{10}^2\theta_{11}^2 \\ &= \left(\frac{2}{7}\right)^2\left(\frac{1}{7}\right)\left(\frac{2}{7}\right)^2\left(\frac{2}{7}\right)^2 \\ &= \frac{2^6}{7^7}. \end{aligned}$$

d. We use the same methods as above to find $p(x_i, y_i|M_j, D_{-i})$. For the two parameter model, we have:

i	x_i	y_i	$\hat{\theta}_1$	$\hat{\theta}_2$	$p(x_i, y_i|M_2, \hat{\boldsymbol{\theta}}(\mathcal{D}_{-i})$
0	1	1	$\frac{3}{6}$	$\frac{3}{6}$	$\frac{9}{36}$
1	1	0	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{6}{36}$
2	0	0	$\frac{4}{6}$	$\frac{3}{6}$	$\frac{6}{36}$
3	1	0	$\frac{3}{6}$	$\frac{4}{6}$	$\frac{6}{36}$
4	1	1	$\frac{3}{6}$	$\frac{3}{6}$	$\frac{9}{36}$
5	0	0	$\frac{4}{6}$	$\frac{3}{6}$	$\frac{6}{36}$
6	0	1	$\frac{4}{6}$	$\frac{4}{6}$	$\frac{4}{36}$

$$L(M_2) = \log\frac{9\cdot 6\cdot 6\cdot 6\cdot 9\cdot 6\cdot 4}{36^7} \approx -5.271.$$

As for the four parameter model, we have:

i	x_i	y_k	$\hat{\theta}_{00}$	$\hat{\theta}_{01}$	$\hat{\theta}_{10}$	$\hat{\theta}_{11}$	$p(x_i, y_i|M_4, \hat{\boldsymbol{\theta}}(\mathcal{D}_{-i})$
0	1	1	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
1	1	0	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$
2	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$
3	1	0	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$
4	1	1	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
5	0	0	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$
6	0	1	$\frac{2}{6}$	$\frac{0}{6}$	$\frac{2}{6}$	$\frac{2}{6}$	$\frac{1}{6}$

$$L(M_4) = \log\frac{1\cdot 0\cdot 1\cdot 0\cdot 1\cdot 1\cdot 1}{6} = \log\frac{0}{6} = -\infty.$$

Since $L(M_2) \approx -5.271 > L(M_4) = -\infty$, CV will pick M_2.
e. We can use part **b.** and **c.** to answer this question. For model M_2,

$$BIC(M_2, \mathcal{D}) = \log \frac{4^8 3^7}{7^{14}} - \frac{2}{2} \log 7 \approx -4.520.$$

For model M_4,

$$BIC(M_4, \mathcal{D}) = \log \frac{2^6}{7^7} - \frac{3}{2} \log 7 \approx -5.377.$$

Recall that M_4, despite having four parameters, only has 3 free parameters because all the parameters must sum to 1.

Once again, M_2 beats out M_4 since $BIC(M_2, \mathcal{D}) \approx -4.520 > BIC(M_4, \mathcal{D}) \approx -5.377$.

Exercise 9

In this question we prove that the posterior median minimizes the posterior ℓ_1 loss. Minimizing this loss is particularly useful when we do not want outliers in out data to skew our predictions.

First, we find the expected ℓ_1 loss in terms of $\boldsymbol{\theta}$:

$$\begin{aligned}
\rho(a, y) &= \mathbb{E}_y[L(a, y)] \\
&= \int_{-\infty}^{\infty} |a - y| p(y|\mathbf{x}) dy \\
&= \int_{-\infty}^{a} (a - y) p(y|\mathbf{x}) dy - \int_{a}^{\infty} (a - y) p(y|\mathbf{x}) dy \\
&= a \cdot P(a \leq y|\mathbf{x}) - \int_{-\infty}^{a} y \cdot p(y|\mathbf{x}) dy - a \cdot P(y > a|\mathbf{x}) + \int_{a}^{\infty} y \cdot p(y|\mathbf{x}) dy
\end{aligned}$$

where a is our prediction for the unknown value y.

Next, we find the critical points by differentiating with respect to a (not y) and setting equal to zero:

$$\begin{aligned}
\frac{\partial \rho}{\partial a} &= P(a \leq y|\mathbf{x}) + 2a \cdot p(a|\mathbf{x}) - P(a > y|\mathbf{x}) - 2a \cdot p(a|\mathbf{x}) \\
&= P(a \leq y|\mathbf{x}) - P(a > y|\mathbf{x}) \\
&= 0.
\end{aligned}$$

Since we also know that $P(a \leq y) + P(a > y) = 1$ it must be the case that

$$P(a \leq l|y\mathbf{x}) = P(a > y|\mathbf{x}) = \frac{1}{2}.$$

Exercise 10

I am fairly sure the question is wrong and should read along the lines of:

> If $L_{FN} = cL_{FP}$ show that we should pick $\hat{y} = 1$ iff $\tau < p(y = 1|\mathbf{x})$, where $\tau = \frac{1}{1+c}$.

($\implies$) The loss matrix is

	$y = 1$	$y = 0$
$\hat{y} = 1$	0	L_{FP}
$\hat{y} = 0$	cL_{FP}	0

We want to predict $\hat{y} = 1$ when

$$\begin{aligned} \mathbb{E}[Loss|\hat{y} = 1] &< \mathbb{E}[Loss|\hat{y} = 0] \\ L_{TP} \cdot p(y = 1|\mathbf{x}) + L_{FP} \cdot p(y = 0|\mathbf{x}) &< L_{TN} \cdot p(y = 0|\mathbf{x}) + L_{FN} \cdot p(y = 1|\mathbf{x}) \\ L_{FP} \cdot p(y = 0|\mathbf{x}) &< cL_{FP} \cdot p(y = 1|\mathbf{x}). \end{aligned}$$

Assuming $L_{FP} \neq 0$, we can solve for $p(y = 1|\mathbf{x})$:

$$\begin{aligned} 1 - p(y = 1|\mathbf{x}) &< c \cdot p(y = 1|\mathbf{x}) \\ 1 &< (c + 1)p(y = 1|\mathbf{x}) \\ \frac{1}{c + 1} &< p(y = 1|\mathbf{x}) \\ \tau &< p(y = 1|\mathbf{x}). \end{aligned}$$

($\impliedby$) We can apply the same logic as above in reverse order.