

Chapter 9

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Comments and Proofs

Section 9.2.2.1

Note that in this context, μ is a parameter of the model, not $\frac{1}{1+\exp(-\mathbf{w}^T \mathbf{x})}$.

Section 9.3.1

Because equation 9.77 looks quite different than equation 9.2, I did not find equations 9.81 and 9.82 trivial.

Proof of equation 9.81. We first find $A(\theta)$.

$$\begin{aligned}\int p(y|\theta_i, \sigma^2) &= 1 \\ \int \exp \left[\frac{y\theta_i - A(\theta_i)}{\sigma^2} - c(\theta_i, \sigma^2) \right] dy &= 1 \\ \exp \left[\frac{-A(\theta_i)}{\sigma^2} \right] \int \exp \left[\frac{y\theta_i}{\sigma^2} - c(\theta_i, \sigma^2) \right] dy &= 1 \\ \int \exp \left[\frac{y\theta_i}{\sigma^2} - c(\theta_i, \sigma^2) \right] dy &= \exp \left[\frac{A(\theta_i)}{\sigma^2} \right] \\ \sigma^2 \log \int \exp \left[\frac{y\theta_i}{\sigma^2} - c(\theta_i, \sigma^2) \right] dy &= A(\theta)\end{aligned}$$

Next, we derive in terms of θ :

$$\begin{aligned}\frac{\partial A}{\partial \theta_i} &= \frac{\partial}{\partial \theta_i} \sigma^2 \log \int \exp \left[\frac{y\theta_i}{\sigma^2} - c(\theta_i, \sigma^2) \right] dy \\ &= \sigma^2 \frac{\int \frac{\partial}{\partial \theta} \exp \left[\frac{y\theta_i}{\sigma^2} - c(y_i, \sigma^2) \right] dy}{\int \exp \left[\frac{y\theta_i}{\sigma^2} - c(y_i, \sigma^2) \right] dy} \\ &= \sigma^2 \frac{\int \frac{y}{\sigma^2} \exp \left[\frac{y\theta_i}{\sigma^2} - c(y_i, \sigma^2) \right] dy}{\exp(\frac{A(\theta_i)}{\sigma^2})} \\ &= \int yp(y|\theta_i, \sigma^2) dy = \mathbb{E}[y|\theta_i, \sigma].\end{aligned}$$

Proof of equation 9.8.2. Using the proof of equation 9.8.1, we have

$$\begin{aligned}
\frac{\partial}{\partial \theta_i} \frac{\partial A}{\partial \theta_i} &= \frac{\partial}{\partial \theta_i} \int y \exp \left[\frac{y\theta_i - A(\theta_i)}{\sigma^2} - c(y_i, \sigma^2) \right] dy \\
&= \int y \frac{\partial}{\partial \theta_i} \exp \left[\frac{y\theta_i - A(\theta_i)}{\sigma^2} - c(y_i, \sigma^2) \right] dy \\
&= \int y \left(\frac{y - \frac{\partial A}{\partial \theta_i}}{\sigma^2} \right) \exp \left[\frac{y\theta_i - A(\theta_i)}{\sigma^2} - c(y_i, \sigma^2) \right] dy \\
&= \frac{1}{\sigma^2} \int \left(y^2 - y \frac{\partial A}{\partial \theta_i} \right) p(y|\theta_i, \sigma^2) dy \\
&= \frac{1}{\sigma^2} \left(\int y^2 p(y|\theta_i, \sigma^2) dy - \frac{\partial A}{\partial \theta_i} \int y p(y|\theta_i, \sigma^2) dy \right) \\
&= \frac{1}{\sigma^2} (\mathbb{E}[y^2|\theta_i, \sigma^2] - \mathbb{E}[y|\theta_i, \sigma^2]^2) \\
&= \frac{1}{\sigma^2} \text{var}[y|\theta_i, \sigma^2].
\end{aligned}$$

It follows that

$$\text{var}[y|\theta_i, \sigma^2] = \sigma^2 \frac{\partial^2 A}{\partial \theta_i^2}.$$

Exercises

Exercise 2

$$\begin{aligned}
\mathcal{N}(\mathbf{x}|\mathbf{\Sigma}, \boldsymbol{\theta}) &= \frac{1}{(2\pi)^{\frac{D}{2}} |\mathbf{\Sigma}|^{\frac{1}{2}}} \exp \left(-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu})^T \mathbf{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu}) \right) \\
&= \frac{1}{(2\pi)^{\frac{D}{2}} |\mathbf{\Sigma}|^{\frac{1}{2}}} \exp \left(-\frac{1}{2} (\mathbf{x}^T \mathbf{\Sigma}^{-1} \mathbf{x} - 2\boldsymbol{\mu}^T \mathbf{\Sigma}^{-1} \mathbf{x} + \boldsymbol{\mu}^T \mathbf{\Sigma}^{-1} \boldsymbol{\mu}) \right) \\
&= \frac{1}{(2\pi)^{\frac{D}{2}} |\mathbf{\Sigma}|^{\frac{1}{2}}} \exp \left(-\frac{1}{2} \boldsymbol{\mu}^T \mathbf{\Sigma}^{-1} \boldsymbol{\mu} \right) \exp \left(-\frac{1}{2} \mathbf{x}^T \mathbf{\Sigma}^{-1} \mathbf{x} + \boldsymbol{\mu}^T \mathbf{\Sigma}^{-1} \mathbf{x} \right) \\
&= \frac{1}{A(\mathbf{\Sigma}, \boldsymbol{\mu})} h(\mathbf{x}) \exp (\boldsymbol{\eta}(\mathbf{\Sigma}, \boldsymbol{\mu})^T \boldsymbol{\phi}(\mathbf{x}))
\end{aligned}$$

Where

$$\begin{aligned}
h(\mathbf{x}) &= 1, \\
A(\mathbf{\Sigma}, \boldsymbol{\mu}) &= \frac{1}{(2\pi)^{\frac{D}{2}} |\mathbf{\Sigma}|^{\frac{1}{2}}} \exp \left(-\frac{1}{2} \boldsymbol{\mu}^T \mathbf{\Sigma}^{-1} \boldsymbol{\mu} \right),
\end{aligned}$$

$\boldsymbol{\eta}(\mathbf{\Sigma}, \boldsymbol{\mu})$ is $-\frac{1}{2}$ times a concatenation of the rows (or columns) of $\mathbf{\Sigma}^{-1}$ and $\boldsymbol{\mu}^T \mathbf{\Sigma}$, and $\boldsymbol{\phi}(\mathbf{x})$ is a concatenation of the rows (or columns) of $\mathbf{x}\mathbf{x}^T$ and \mathbf{x} .