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MA 240

Homework 4

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I pledge my honor that I have abided by the Stevens Honor System.

1a. Theorem. $F_1^2 + F_2^2 + \ldots + F_n^2 = F_n F_{n+1}$ for all $n \in \mathbb{N}$.

Proof. Let $P(n)$ be true if the claim holds for n. By induction:

Base case: $n = 1$
$1^2 = 1 * 1$, claim holds.

Inductive hypothesis: Assume $P(k)$ is true for some $k \in \mathbb{N}$:

$$F_1^2 + \ldots + F_k^2 = F_k F_{k+1}.$$

Inductive step: Add F_{k+1}^2 to the equation:

$$\begin{aligned} F_1^2 + \ldots + F_k^2 + F_{k+1}^2 &= F_k F_{k+1} + F_{k+1}^2 \\ &= F_{k+1}(F_k + F_{k+1}) \\ &= F_{k+1} F_{k+2} && \text{(Apply } F_n = F_{n-1} + F_{n-2}\text{)} \end{aligned}$$

Thus, $P(k) \implies P(k+1)$ for $k \geq 2$ and $P(1)$ is true, and the claim holds. ■

1b. Theorem. If S_n is the set of all nonempty binary strings of length n with no consecutive 1's, then $|S_n| = F_{n+2}$.

Proof. By strong induction:

Base case: $n = 1$
$S_n = \{0, 1\}, |S_n| = F_3 = 2$, the claim holds.

Inductive hypothesis: Assume $|S_k| = F_{k+2}$ for all $k \leq n \,\big|\, k \in \mathbb{N}$.

Inductive step: Let $z(S_i)$ be the number of strings in S_i that end in 0. Consider how the set S_{k+1} is constructed for each string $r \in S_k$: if r ends in "0", two strings are added to the set representing $r \oplus$ "0" and $r \oplus$ "1". Otherwise, only one string is added of the form $r \oplus$ "0". Since only the strings ending in "0" produce a new element, the length can be described as

$$|S_{k+1}| = |S_k| + z(S_k).$$

Since for each string, $r \oplus$ "0" is added regardless of its last bit, it is also true that $z(S_k) = |S_{k-1}|$. Then, we have

$$\begin{aligned} |S_{k+1}| &= |S_k| + |S_{k-1}| \\ |S_{k+1}| &= F_{k+2} + F_{k+1} && \text{(Inductive Hypothesis)} \\ |S_{k+1}| &= F_{k+3}. \blacksquare && \text{(Apply } F_n = F_{n-1} + F_{n-2}\text{)} \end{aligned}$$

2. Theorem. A plane divided into regions by n straight lines can always be colored such that no two colors are adjacent.

Proof. By induction:

Base case: $n = 1$
One line divides the plane into two regions, which can be colored red and blue.

Inductive Hypothesis: Suppose we have some 2-colored plane divided by k lines.

Inductive Step: Draw the $k + 1^{\text{th}}$ line. Observe that for any region divided by the line, two new same-colored regions adjacent about the line emerge. Pick one half of the plane which the new line bisects. Then, invert the coloring on that half of the plane. Since all the regions adjacent to the line are of the same color, inverting one half resolves the adjacent same-colorings while preserving the 2-coloring property of that half of the plane. Thus, we have produced a 2-coloring of a plane divided by $k + 1$ lines. ■