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MS FINANCE ET GESTION DES RISQUES

Pricing early-exercise and discrete barrier options by Fourier-cosine series expansions

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1 Introduction

The main objective of this article is to present a methodology of pricing based on the **Fourier-cosine expansion** for the **early-exercise and and discretely-monitored barrier options** which also works for European options [2]. The Fast Fourier Transform (FFT) is an efficient tool both for the pricing of European options and calibration since it can be applied to many models requiring only the **characteristic function**. As seen in the course, this is the case for regular affine processes. Therefore this methodology of *transform methods* can be applied to exponential Lévy models.

In the article, the *transform methods* are generalized to more complex options such as Bermudan, American and barrier options. This paper shows that the **COS method** where the transition density probability is replaced by its Fourier-cosine series expansion can also be applied in order to price the early-exercise and barrier options with an exponential convergence for Lévy models.

In this work, firstly, we will expose the COS methodology applied for the pricing of the Bermudan and barrier options. Then, we will discuss the results and error analysis obtained and try to replicate one part of the paper. Finally, we will focus on the advance brought by the article, the literature around this topic and its links to the course of *Numerical Methods for Finance* followed this semester

2 The COS methodology for pricing Bermudan and barrier options

2.1 The Bermudan options

In this subsection, we will rapidly remind the main characteristics of the Bermudan option.

Let's denote t_0 the initial time and $t_1, ..., t_M = T$ all the possible exercised dates with $\Delta_t = t_m - t_{m-1}$ for $m = 1, ..., M$. We obtain the following formula for the pricing of a Bermudan option with M exercises dates:

$$\begin{cases} c(x, t_{m-1}) = e^{-r\Delta t} \displaystyle\int_{\mathbb{R}} v(y, t_m) f(y|x) \, \mathrm{d}y \\ v(x, t_{m-1}) = max(g(x, t_{m-1}), c(x, t_{m-1})) \end{cases} \tag{1}$$

where :

$$v(x, t_0) = e^{-r\Delta t} \int_{\mathbb{R}} v(y, t_1) f(y|x) \, \mathrm{d}y \tag{2}$$

$v(x, t), c(x, t)$ and $g(x, t)$ represent respectively the **option value, the continuation value and the payoff** at time t. Intuitively, a Bermudan option can be exercised at pre-settled dates before maturity. At the exercise, the holder receives the payoff otherwise between two dates, the option can be seen as European option where a classical risk-neutral approach for the pricing can be used.

2.2 The COS methodology for the pricing of an European option

This methodology relies on the fact that the **characteristic function** and the **Fourier-cosine series coefficients** of $f(y|x)$ are related. Moreover, using the fact that $\lim_{y\to\infty} f(y|x) = 0$, the infinite integration range in the equation 1 can be truncated without loosing too much accuracy. Let's denote TOL a given tolerance and suppose :

$$\int_{\mathbb{R}\setminus[a,b]} f(y|x) \, \mathrm{d}y < TOL \tag{3}$$

The term $c(x, t_{m-1})$ can be approximated by :

$$c_1(x, t_{m-1}) = e^{-r\Delta t} \int_a^b v(y, t_m) f(y|x) \, \mathrm{d}y = \frac{1}{2}(b-a) e^{-r\Delta t} \sum_{k=0}^{+\infty}{}' A_k(x) V_k(t_m) \tag{4}$$

where the density function $f(y|x)$ has been replaced by its Fourier cosine expansion and the coefficients of the Fourier-cosine expansion of the density and the function $v(y, t_m)$ respectively denoted by $A_k(x)$ and $V_k(t_m)$ are defined as follow :

$$A_k(x) := \frac{2}{b-a} \int_a^b f(y|x) cos\left(k\pi \frac{y-a}{b-a}\right) \mathrm{d}y \tag{5}$$

$$V_k(t_m) := \frac{2}{b-a} \int_a^b v(y, t_m) cos\left(k\pi \frac{y-a}{b-a}\right) \mathrm{d}y \tag{6}$$

Now, we **truncate** the infinite serie in the equation 4 which gives the following approximation :

$$c_2(x, t_{m-1}) = \frac{1}{2}(b-a) e^{-r\Delta t} \sum_{k=0}^{N-1}{}' A_k(x) V_k(t_m) \tag{7}$$

Then, approximating the coefficients $A_k(x)$ by $F_k(x)$ defined as follow (where the finite integration between $[a, b]$ has been approximated by the whole integration on $\mathbb{R}$) with respect to the argument developed in the equation 3:

$$F_k(x) := \frac{2}{b-a} Re\left(\phi\left(\frac{k\pi}{b-a}; x\right) e^{-ik\pi\frac{a}{b-a}}\right) \tag{8}$$

where the function ϕ is defined as follow :

$$\phi(\omega; x) := \int_{\mathbb{R}} f(y|x) e^{i\omega y} \, \mathrm{d}y \tag{9}$$

Replacing the coefficients $A_k(x)$ by $F_k(x)$ in the equation 7, we obtain the **COS formula for a European options for different underlying processes** where $\hat{c}(x, t_{m-1})$ represents the approximation of the continuation value $c(x, t_{m-1})$.

$$\hat{c}(x, t_{m-1}) = e^{-r\Delta t} \sum_{k=0}^{N-1}{}' Re\left(\phi\left(\frac{k\pi}{b-a}; x\right) e^{-ik\pi\frac{a}{b-a}}\right) V_k(t_m) \tag{10}$$

As mentioned in the introduction, we are interested in the Lévy processes where the equation 10 can be simplified leading to :

$$\hat{c}(x, t_{m-1}) = e^{-r\Delta t} \sum_{k=0}^{N-1}{}' Re\left(\phi_{levy}\left(\frac{k\pi}{b-a}; x\right) e^{-ik\pi\frac{a}{b-a}}\right) V_k(t_m) \tag{11}$$

using the notation $\phi_{levy}(\omega) := \phi_{levy}(\omega; 0)$. Therefore, assuming that $V_k(t_1)$ is known, it is possible to approximate the equation 2 by :

$$\hat{v}(x, t_0) = e^{-r\Delta t} \sum_{k=0}^{N-1}{}' Re\left(\phi_{levy}\left(\frac{k\pi}{b-a}; x\right) e^{-ik\pi\frac{a}{b-a}}\right) V_k(t_1) \tag{12}$$

2.3 Pricing Bermudan option with the COS method

Now, after all this serie of computations, we are interested in computing the cosine coefficients of the option value at time t_1. To do this, we will use a trick of splitting the integral in the equation 6 in two parts. Denoting x_m^* the *early exercise point* at time t_m in which the **continuation value equals the payoff** (i.e $c(x_m^*, t_m) = g(x_m^*, t_m)$), we split the integral in two parts : one on $[a, x_m^*]$ and the other on $[x_m^*, b]$.

$$V_k(t_m) = \begin{cases} C_k(a, x_m^*, t_m) + G_k(x_m^*, b) & for \quad a \quad call \\ G_k(a, x_m^*) + C_k(x_m^*, b, t_m) & for \quad a \quad put \end{cases} \tag{13}$$

$$V_k(t_M) = \begin{cases} G_k(0, b) & for \quad a \quad call \\ G_k(a, 0) & for \quad a \quad put \end{cases} \tag{14}$$

where :

$$G_k(x_1, x_2) := \frac{2}{b-a} \int_a^b g(x, t_m) cos\left(k\pi \frac{x-a}{b-a}\right) \mathrm{d}x \tag{15}$$

$$C_k(x_1, x_2, t_m) := \frac{2}{b-a} \int_a^b c(x, t_m) cos\left(k\pi \frac{x-a}{b-a}\right) \mathrm{d}x \tag{16}$$

At time t_M, $V_j(t_M)$ is given by the equation 15. At time t_{M-1}, from the equation 11, we obtain the approximation for the continuation value at time t_{M-1} $\hat{c}(x, t_{M-1})$ which is inserted into the equation 16. Therefore, we obtain that for time t_m replacing $V_j(t_{m+1})$ in the equation 16 by its numerical approximation $\hat{V}_j(t_{m+1})$:

$$\hat{C}_k(x_1, x_2, t_m) := e^{-r\Delta t} Re\left(\sum_{j=0}^{N-1} \phi_{levy}\left(\frac{j\pi}{b-a}\right) \hat{V}_j(t_{m+1}).\mathcal{M}_{k.j}(x_1, x_2)\right) \tag{17}$$

The authors show by a serie of calculus that it is possible to determine an analytical formula for $G_k(x_1, x_2)$. Therefore, we obtain the Fourier cosine coefficients of the option value by replacing C_k by $\hat{C}_k$ in the equations 13 which finally leads to the following equations (vector form, here $\hat{C}$ and G are vectors of size N):

$$\hat{V}_k(t_m) = \begin{cases} \hat{C}_k(a, x_m^*, t_m) + G_k(x_m^*, b) & for \quad a \quad call \\ G_k(a, x_m^*) + \hat{C}_k(x_m^*, b, t_m) & for \quad a \quad put \end{cases} \tag{18}$$

Thanks to all these computations, we are able to build an algorithm in order to price Bermudan option with the COS method. One of the main drawbacks is that the computation of the equations 18 relies on a **matrix product** which leads to a complexity in $\mathcal{O}(N^2)$. To fix this issue, the authors develop a **FFT-based algorithm** for computing the matrix product of the equations 18 using the fact that matrix $\mathcal{M}$ is the sum of a Hankel and Toeplitz matrix which gives a complexity of $\mathcal{O}(N \log_2 N)$ instead of $\mathcal{O}(N^2)$.

Algorithm 1 Pricing of a Bermudan call option with the COS method

for $k = 0, ..., N-1$ **do**
  $V_k(t_m) = G_k(0, b)$
end for
for $m = M-1, ..., 1$ **do**
  `Use Newton's method to determine the early-exercise point` x_m^*
  `Compute` $\hat{V}_k(t_m)$ `with the FFT algorithm`
end for
Insert $V_k(t_1)$ **into the equation 12 to obtain** $\hat{v}(x, t_0)$

2.4 Pricing Discrete Barrier Options using the COS method

In this subsection, first we will remind some characteristics of this kind of option and then see how can we apply the COS method.
We call **Discrete-monitored out barrier options**, options that cease to exist if the asset price hits a barrier level denoted H. If $H > S_0$, these options are called "up-and-out" options otherwise they are called "down-and-out" options. Therefore, we obtain for the payoff the following expression :

$$v(x,T) = 1_{S_{t_i}<H}((\alpha(S_T - K))^+ - Rb) + Rb \tag{19}$$

where $\alpha = 1$ for a call and $\alpha = -1$ for a put and Rb is the rebate. Adopting the same notation as the subsection 2.1, the price of an up-and-out option satisfies the following equation :

$$\begin{cases} c(x,t_{m-1}) = e^{-r\Delta t}\displaystyle\int_{\mathbb{R}} v(y,t_m)f(y|x)\,\mathrm{d}y \\ v(x,t_{m-1}) = \begin{cases} e^{-r(T-t_{m-1})}Rb, & x \geq h \\ c(x,t_{m-1}) & x < h \end{cases} \end{cases} \tag{20}$$

We immediately remark that here the **barrier point is known in advance**. Therefore it is not necessary to use a root-searching algorithm conversely to the Bermudan case where it was required in order to define the early-exercise point x_m^*. The pricing of barrier option is therefore easier than the Bermudan's one. Only two steps are required :

- Recover the Fourier cosine series $V_k(t_1)$.
- Compute the COS formula (the equation 12) for European option.

3 Error analysis of the COS method

In the previous section, we deeply focus on the technical part of the COS methodology and see that this provides an algorithm which is very useful for our pricing objectives. In this section, we will focus on the efficiency and the accuracy of the method analyzing the rate of convergence and the stability of the COS method.

3.1 Decomposition of the error for European options

The authors define the error as :

$$\epsilon(x;N,[a,b]) := c(x) - \hat{c}(x;N,[a,b]) \tag{21}$$

Reminding that the COS formula for an European option was derived in 3 ways following the results of the subsection 2.2, we decompose the error into three terms : the **integration range truncation** ϵ_1, the **series truncation error** ϵ_2 and the **approximation's error** ϵ_3 of approximate $A_k(x)$ by $F_k(x)$.

$$\epsilon_1(x;N,[a,b]) := c(x) - \hat{c}_1(x;N,[a,b]) = \int_{\mathbb{R}} v(y)f(y|x)\,\mathrm{d}y \sim \mathcal{O}\left(\int_{\mathbb{R}\setminus[a-x,b-x]} f(z)\,\mathrm{d}z\right) \tag{22}$$

$$\epsilon_1(x;N,[a,b]) := c_1(x;[a,b]) - c_2(x;N,[a,b]) = \frac{1}{2}(b-a)e^{-r\Delta t}\sum_{k=N}^{\infty} A_k(x).V_k \tag{23}$$

$$\epsilon_3(x;N,[a,b]) := c_2(x;N,[a,b]) - \hat{c}(x;N,[a,b]) = e^{-r\Delta t}\sum_{k=0}^{N-1} Re\left(\int_{\mathbb{R}\setminus[a,b]} e^{ik\pi} f(y|x)\,\mathrm{d}y\right) V_k \tag{24}$$

The complexity in the equation 22 is obtained using the property for Lévy processes ($f(y|x) = f(y-x)$) and a change of variable. The larger the interval $[a, b]$, the smaller will ϵ_1 be. Therefore, the **truncation range has to be well chosen**. We can show that ϵ_2 **converges exponentially** for probability density functions $\mathcal{C}^\infty([a, b])$. ϵ_3 **can also be bounded independently of** x (see the details in [2]). After gathering all the bounds which are independent of x, applying the triangular inequality, we obtain that for smooth density functions, ϵ **converges exponentially**.

3.2 Error in the Fourier coefficients

Now, we focus on the error for the Bermudan option values which is determined by the error in the Fourier coefficients and its propagation in the backward recursion :

$$\epsilon(k, t_m) := V_k(t_m) - \hat{V}_k(t_m) \tag{25}$$

The authors show that for a sufficient large interval $[a, b]$ and a smooth density function in $\mathcal{C}^\infty([a, b])$, the error $\epsilon(k, t_m)$ converges exponentially in N.

As mentioned before, the choice of the truncation range is important since a very small one will lead to a significant integration-range truncation error whereas a very large needs a large value of N in order to achieve a certain level of accuracy. The authors provide the following formula to define the range of integration :

$$[a, b] := \left[(c_1 + x_0) - L\sqrt{c_2 + \sqrt{c_4}}, (c_1 + x_0) + L\sqrt{c_2 + \sqrt{c_4}}\right] \tag{26}$$

where : $x_0 := ln(\frac{S_0}{K})$ and L depends on the tolerance TOL defined in the equation 3 and $c_1, ..., c_4$ the cumulants of the characteristic function of the underlying process.
The authors determine the setting parameter L using the relation between TOL and L. Indeed, taking $N = 2^{14}$, the series truncation can be put aside and we can only focus on the integration range error. This error decreases with exponentially with L and graphically $L = 8$ is a good choice to obtain a small error.

3.3 The results

In the last section of the article, the authors price the two kinds of options highlighting their convergence. They present the results for the *Black & Scholes (BS)*, *CGMY* and *Normal Inverse Gaussian (NIG)*. They introduce the following ratio in order to analyze the exponential error convergence :

$$\frac{\ln(|err(2^{d+1})|)}{\ln(|err(2^d)|)} \tag{27}$$

with $err(2^d)$ denoting the error between reference solution and approximated obtained with $N = 2^d$. The pricing of a Bermudan put options highlights the exponential convergence of the *COS* method. For both the *BS* and *CGMY*, an accurate solution is obtained in less of 20 ms with an exponential convergence using the *COS* method whereas a second-order convergence for a *CONV* method.

For barrier options, the ratio obtained in the equation 27 is around to 2 in less than 5 ms. It is in adequation with the fact that *COS* method is more efficient for discrete barrier options than Bermudan ones since the barrier levels is known in advance.

3.4 Critical analysis of the article

In this subsection, we will give a critical point of view of the article highlighting the pros and cons.
One of the main advantage of the COS method is its **flexibility** since it can be used as far as the characteristic function is known. Moreover, if the series of coefficients of the option values at first early-exercise is known, the COS formula for European option can be used for the pricing of the Bermudan. Another advantage is the exponential convergence in N for smooth density functions and impressive computational speed. However, an implementation in a **compiled language** as C instead of Matlab to increase the speed of calculations.
In order to set the parameter L of the truncation range in the equation(26), the authors uses a numerical experiments which is not very rigorous. It can more be seen as a guidance than an accurate results. As mentioned in the last section, the choice of the truncation range was improved by a recent paper.
Another drawback is that the procedure to determine the early-exercise point x_m^* is not well detailed. The function $\phi_{Lévy}$ is not explicitly mentioned in the article which makes the task more complex in order to apply a Newton's method using the equation 11.

4 Our implementation

In order to replicate the results of this paper, we read the first paper of the same authors [2]. In this paper, the authors introduce the **Fourier-cosine expansion to price European options**. First, we replicate this framework to price European options, and we add some part to replicate our actual paper which price Bermudan options. In the last part of this section, we will make some comments about these methods and their implementation.

4.1 European option prices

As in the first paper, we price European options using *log moneyness*. For this implementation, we are using the characteristic function of the Geometric Brownian Motion, and the Heston model one. The method is very simple as it is European options. First, we compute the cosine series coefficients with equation 14, then we plug the coefficients into the equation 10 to obtain the option premium.
We used the following parameters for the *GMB* model, risk free rate at 0.0%, maturity one month, $\sigma = 15\%$, and the underlying price at \$100. For the Heston model, we took risk free rate at 0.0%, maturity one month, $\kappa = 1.5$, $\theta = 12^2\%$, $\eta = 0.57$, $\sigma = 12^2\%$, and the underlying price at \$100. For both models, we used a grid size of 2 000 strikes. We obtain Figures 1. and 2.

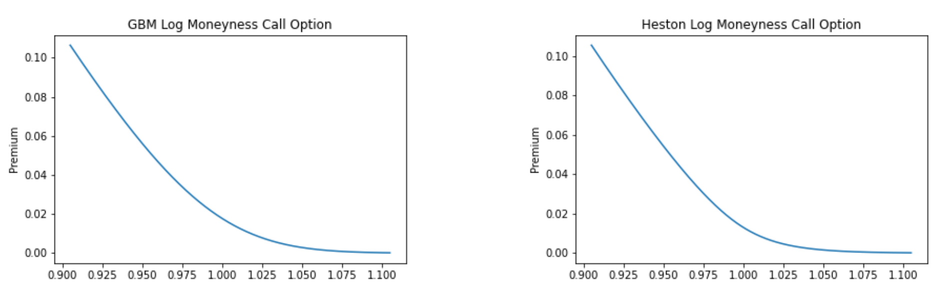

Figure 1: European Call Options

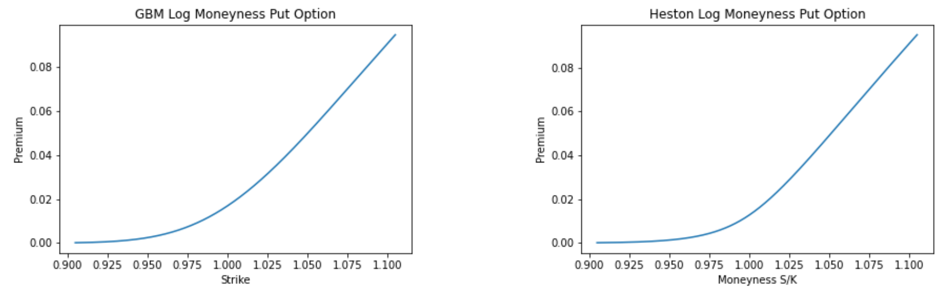

Figure 2: European Put Options

4.2 Bermudan option prices

For this implementation, we are following the **Algorithm 1** that we showed on section 2.3. We are using the same parameters as the European options but, as it is a Bermuda options, we are considering one stopping time before maturity. As we didn't achieve to find the optimal early-exercised point x_m^*, we fixed different point and we obtain the following results :

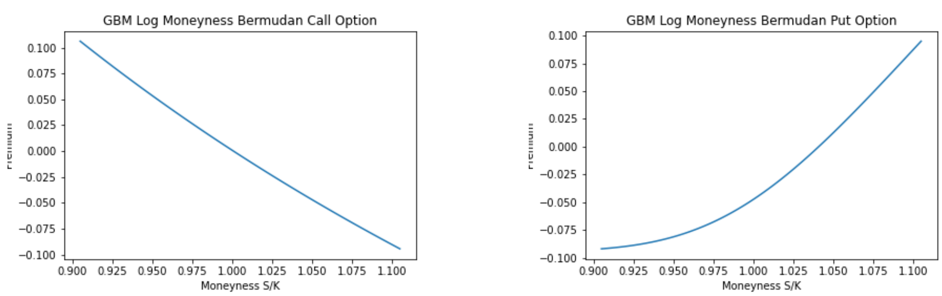

Figure 3: Bermudan Options, $x_m^* = x_1$

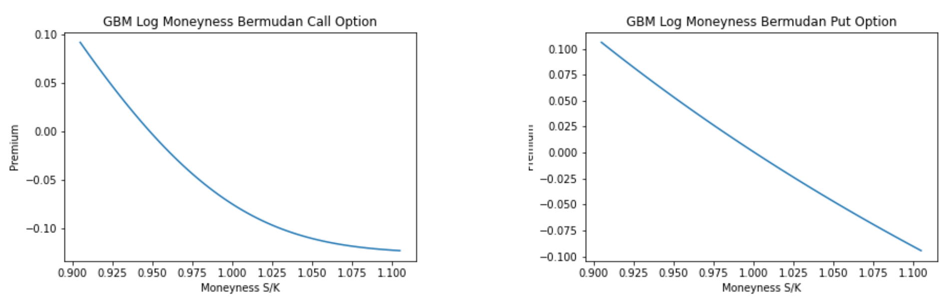

Figure 4: Bermudan Options, $x_m^* = \frac{(x_2 - x_1)}{2}$

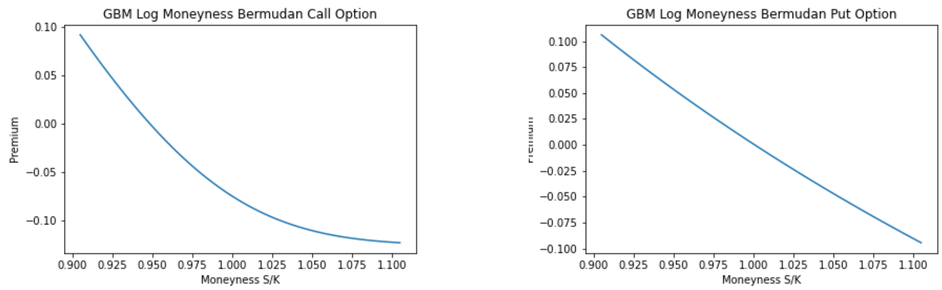

Figure 5: Bermudan Options, $x_m^* = x_2$

4.3 Comments on the implementation

We can make few comments on the implementation.

- **European Options**: The implementation was quite easy compared to the one for the Bermudan options. In fact, everything is well explained in the first paper, and all the formulas are given. The results are good if we compare with the *Black* & *Scholes* closed-form formula. On the computational time side, even with python, the pricing is very fast, and convenient for a large range of strikes.

- **Bermudan Options**: The implementation was a bit more challenging. As we said in the previous section, there were information missing to replicate the paper. Especially around the x_m^* point, and on the characteristic function of Lévy processes denoted $\phi_{Lévy}$. So, we tried our best to replicate as much as we could the results of the paper. We fixed a point instead of finding x_m^* in order to still achieve to find a result, and we used the ϕ_{GBM} instead of $\phi_{Lévy}$. With all those approximations, we don't get accurate results (e.g some of our premium are negative because of the choice of x_m^*) but still retrieve the same shape as for the European options for a good choice of x_m^*. The computational time is much longer for Bermudan options because we need to compute the M_c and M_s matrix multiple times, and we don't have the true early-exercised point x_m^*. Moreover, the computational time could be decreased with the implementation of a *FFT* method. However, as our actual implementation wasn't working perfectly we didn't try to implement this method.

In order to improve our implementation, one could consider finding the optimal early-exercise point. This would give better result for the Bermudan options. Finally, we could consider implement the *FFT* method to improve the computational time, and try to implement the discrete barrier option as in the last section of the paper.

5 Link with the course and the literature

5.1 Link with the course

Obviously, this article is deeply related to the fifth lecture of the course on Fourier methods where it was introduced for the pricing of European options a Fourier inversion based on the Black & Scholes control variates [5]. Through the COS method, we were able to obtain a **relation between the characteristic function with the series of coefficients of the Fourier-cosine expansion of the density function**. The article is deeply related to the third domain of numerical methods in finance which is the **numerical integration techniques** which have in common that they rely on a transformation to the Fourier domain and are mainly use for calibration purposes. As seen during the lecture, the Carr-Madan method is one of the best known examples of this class. In the Fourier domain it is possible to solve various derivative contracts, as long as the characteristic function is available. By means of the Fast Fourier Transform

(FFT), integration can be performed with a computational complexity of $O(Nlog_2N)$, with N the number of integration points.

5.2 Overview of the literature

In the literature, numerical methods can be divided into three subfields : **partial (integro) differential equation (PIDE), Monte Carlo simulations and numerical integration** which corresponds to the one of the article. In order to compare the results of the COS method, the authors use the CONV method which is another transform method based on FFT algorithm widely used for pricing Bermudan and American options. However, it is not the highest efficient method for solving Fourier transformed integrand which are highly oscillatory implying to use a very fine grid in order to obtain a good level of accuracy which is an important drawback compared to the COS method which requires a smaller value of N. For barrier options, another method based on **Hilbert transform** [4] has also be developed. Its error convergence is exponential for models with a characteristic functions with a rapid decay and a computational complexity of $O((M-1)Nlog_2N)$ for a barrier option with M monitoring dates. One of the main drawback of this method is that it is not applicable for Bermudan options.

As we have seen along this work, the COS method is very useful and recent research papers use it and try to optimize it. For instance, in the article of Junike and Pankraskin [3] published in January 2022, the authors derive a new formula to obtain the truncation range and prove that the range is large enough to ensure convergence of the COS method within a predefined error tolerance. Another example of the use of COS method is for the valuation of **electricity storage contracts** [1] where the authors estimate the coefficients V_k in the equation 12 which depends on the the energy level e in storage.

One of the main future perspective will be the use of transform methods to calibrate to these exotic products and price **very fast huge portfolios at the end of a trading day**.

6 Conclusion

To conclude this lecture, we have presented the so-called COS method where the transition density probability is replaced by its Fourier-cosine series expansion series and its applications for option pricing. Given some approximation, we have seen that this method helps to improve substantially the execution times when pricing a range of strike while staying a closed-form formula. Furthermore, this method is very flexible as it can be used whenever we know the characteristic function of the model. As an example, the authors gave some results with different models such as *Geometric Browian Motions*, *Heston model*, *CGMY model*. Thus, this method would be very useful to evaluate rapidly a portfolio of options. Besides these great results, if one wants to apply this methods he would have to choose properly its truncation range.

In a second part, we tried to replicate the results of our paper by implementing the COS method to price European and Bermudan options using Python. We had no problem to implement the method for European options using the first paper of the same authors [2]. However, we faced some troubles trying to replicate the Bermudan options results. In fact, we did not achieve to find the optimal early-exercised point x_m^* at each time which, as we can seen in section 4.2, gave wrong results.

Finally, regarding the new literature on the subject, we have seen that this methods has been improve to new subject such as electricity storage contracts.

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