Reasoning about Unforeseen Possibilities During Policy Learning

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Abstract

Methods for learning optimal policies in autonomous agents often assume that the way the domain is *conceptualised*—its possible states and actions and their causal structure—is known in advance and does not change during learning. This is an unrealistic assumption in many scenarios, because new evidence can reveal important information about what is possible, possibilities that the agent was not aware existed prior to learning. We present a model of an agent which both discovers and learns to exploit unforeseen possibilities using two sources of evidence: direct interaction with the world and communication with a domain expert. We use a combination of probabilistic and symbolic reasoning to estimate all components of the decision problem, including its set of random variables and their causal dependencies. Agent simulations show that the agent converges on optimal polices even when it starts out unaware of factors that are critical to behaving optimally.

1. Introduction

Consider the following decision problem from the domain of crop farming (inspired by Kristensen and Rasmussen (2002)): Each harvest season, an agent is responsible for deciding how best to grow barley on the land it owns. At the start of the season, the agent makes some decisions about which grain variety to plant and how much fertiliser to use. Come harvest time, those initial decisions affect the yield and quality of the crops harvested. We can think of this problem as a single-stage (or "one-shot") decision problem, in which the agent chooses one action based on a set of observations, then receives a final reward based on the outcome of its action.

Suppose the agent has experience from several harvests, and believes it has a good idea of the best seeds and fertiliser to use for a given climate. One harvest, something totally unexpected happens: Despite choosing what it thought were the best grains and

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fertiliser, many of the crops have come out deformed and of poor quality. A neighbouring farmer tells the agent the crops have been infected by a fungus that spreads in high temperatures, and that the best way to protect crops in future is to apply fungicide. The agent must now revise its model of the environment and its reward function to account for fungus (a concept it previously was not aware of), extend its available actions to include fungicide application (an action it previously did not realise existed), reason about the probabilistic dependencies these new concepts have to the ones it was already aware of, and reason about how they affect rewards.

This example illustrates at least three challenges, the combination of which typically is not handled by current methods for learning optimal decision policies (we defer a detailed discussion of related work to Section 6):

- In addition to starting unaware of the probabilistic dependency structure of the decision problem, the agent starts unaware of even the true *hypothesis space* of possible problem structures, including the sets of possible actions and environment variables, and their causal relations.
- Domain exploration alone might not be enough to discover these unknown factors. For instance, it is unlikely the agent would discover the concept of fungus or the action of fungicide application by just continuing to plant crops. External expert instruction is necessary to overcome unawareness.
- An expert might interject with contextually relevant advice during learning, not just at the beginning of the problem. Further, that advice may refer to concepts which are not a part of the agent's current model of the domain.

In the face of such strong unawareness, one might be tempted to side-step learning an explicit model of the problem, and instead learn optimal behaviour directly from the data. Deep reinforcement learning (e.g. Mnih et al. (2015)) has proved extremely useful for learning implicit representations of large problems, particularly in domains where the input sensory streams are complex and high-dimensional (e.g computer vision). However, in many such models, the focus of attention for abstraction and representation learning is on perceptual features rather than causal relations (see, e.g., Pearl (2017)) and other decision-centric attributes. For instance, in works such as by Chen et al. (2015), who demonstrate an end-to-end solution to the problem of autonomous driving, the decisions are not elaborated beyond "follow the lane" or "change lanes" although significant perceptual representation learning may need to happen in order to work with the predicate "lane" within the raw sensory streams. For safety critical decisions, or ones involving significant investment (e.g. driving a car, advising on a medical procedure, deciding on crops to grow for the year) it is important ¹ that a system can explain the reasoning behind its decisions so the user can trust its judgement.

We present a learning agent that, in a complementary approach to representation learning in batch mode from large corpora, uses evidence and a reasoning mechanism to incrementally construct an *interpretable* model of the decision problem, based on which optimal decision policies are computed. The main contributions of this paper are:

¹As evidenced by significant recent interest from agencies interested in the application of AI, e.g., the DARPA Explainable AI programme DARPA-BAA-16-53 (2016)

- An agent which, starting unaware of factors on which an optimal policy depends, learns optimal behaviour for single-stage decision problems via direct experience and advice from a domain expert. The learning uses *decision networks* to represent beliefs about the variables and causal structure of the decision problem, providing a compact and interpretable model. Crucially, the agent can revise *all* components of this model, including the set of random variables, their causal dependencies, and the domain of the reward function (Section 4).
- A communication framework via which an expert can offer both solicited and unsolicited advice to the agent during learning: that is, the expert advice is offered piecemeal, in reaction to the agent's latest attempts to solve the task, rather than all of it being conveyed prior to learning. Messages from the expert can include entirely new concepts which the agent was previously unaware of, and provide important qualitative information to complement the quantitative information conveyed by statistical correlations in the domain trials (Section 3.2).
- Experiments across a suite of randomly generated decision problems, which demonstrate that our agent can learn the optimal policy from evidence, even if it were when initially unaware of variables that are critical to its success (Section 5).

The kinds of applications we ultimately have in mind for this work include tasks in which there is a need for flexible and robust responses to a vast array of contingencies. In particular, we are interested in the paradigm of continual (or life-long) learning (Silver, 2011; Thrun and Pratt, 2012), wherein the agent must continually and incrementally add to its knowledge, and so revise the hypothesis space of possible states and actions within which decisions are made. In this context, there is a need for autonomous model management (Liebman et al., 2017), which calls for reasoning about what the hypothesis space is, in addition to policy learning within those hypothesis spaces.

2. The Learning Task

We consider learning in *single-stage* decision problems. In these problems, the agent chooses an action based on a set of initial observations, then immediately receives a final reward based on the outcome of its action. Subsequent repetitions of the same decision problem have mutually independent initial observations, and the immediate reward depends only on the current action and its outcome. Solving the problem of unforeseen possibilities for single-stage scenarios is a necessary first step towards the long-term goal of extending this work to multi-stage, sequential decision problems.

To learn an optimal decision policy, the agent must compute which action will maximise its expected reward, given its observations of the state in which the action is to be performed. Formally, the optimal action \vec{a}^* given observations \vec{e} is the action which maximizes expected utility:

$$\vec{a}^* = \arg\max_{\vec{a} \in v(\mathcal{A})} EU(\vec{a}|\vec{e}) = \arg\max_{\vec{a} \in v(\mathcal{A})} \sum_{\vec{s} \in v(\mathcal{C})} Pr(\vec{s}|\vec{a}, \vec{e}) \mathcal{R}(\vec{s})$$
(1)

Here, $\mathcal{A} = \{A_1, A_2, \dots, A_n\}$ is the set of action variables and $\mathcal{C} = \{C_1, C_2, \dots, C_m\}$ is the set of chance variables (or state variables). An action $\vec{a} \in \mathcal{V}(\mathcal{A})$ is an assignment

to each of the action variables in \mathcal{A} , such that $\vec{a} = (a_1, a_2, \dots, a_n)$ with $a_1 \in A_1, a_2 \in A_2$, etc. (We use v(Q) to denote the Cartesian product of all sets in Q.) Similarly, a *state* $\vec{s} \in \mathcal{C}$ is an assignment to each chance variable in \mathcal{C} . The reward received in state \vec{s} is $\mathcal{R}(\vec{s})$.

In our task, the agent faces the extra difficulty of starting unaware of certain actions and concepts (i.e., chance variables) on which the true optimal policy depends. Consequently, it begins learning with an incomplete hypothesis space. It also has an incorrect model of the domain's causal structure—i.e., there may be missing or incorrect dependencies. Further, the learning agent will start with an incomplete or incorrect reward function. The agent's learning task, then, is to use evidence to converge on an optimal policy, despite beginning with an initial model defined over an incomplete set of possible states and actions and incorrect dependencies among them.

Since we are interested in interpretable solutions, our approach is that the learning agent uses evidence to dynamically construct an *interpretable model* of the decision problem from which an optimal policy can be computed. Formally, we treat this task of learning an optimal policy as one of learning a *decision network* (DN). DNs capture preferences with a numeric reward function and beliefs with a Bayes Net, thereby providing a compact representation of all the components in equation (1).

Definition 1. Decision Network

A Decision Network (DN) is a tuple $\langle \mathcal{C}, \mathcal{A}, \Pi, \theta, \mathcal{R} \rangle$, where \mathcal{C} is a set of chance variables, \mathcal{A} is a set of action variables (the agent controls their values), and \mathcal{R} is a reward function whose domain is $\Pi_R \subseteq \mathcal{C}$ and range is \mathbb{R}^2 $\langle \Pi, \theta \rangle$ is a Bayes Net defined over $\mathcal{C} \cup \mathcal{A}$. That is, Π is a directed acyclic graph (DAG), defining for each $C \in \mathcal{C}$ its parents $\Pi_c \subseteq \mathcal{C} \cup \mathcal{A}$, such that C is conditionally independent of $(\mathcal{C} \cup \mathcal{A}) \setminus \Pi_c$ given Π_c . θ defines for each variable $C \in \mathcal{C}$ its conditional probability distribution $\theta_c = Pr(C|\Pi_c)$.

A policy for a DN is a function π from the observed portion \vec{e} of the current state (i.e., \vec{e} is a subvector of $v(\mathcal{C})$) to an action $v(\mathcal{A})$. As is usual with DNs, any variable X whose value depends on the action performed (in other words, X is a descendant of \mathcal{A}) cannot be observed until after performing an action. More formally, $\mathcal{C} = \mathcal{B} \cup \mathcal{O}$, where the "before" variables \mathcal{B} are non-descendants of \mathcal{A} , and the "outcome" variables \mathcal{O} are descendants of \mathcal{A} (i.e., where Π^* is the closure of Π , $X \in \mathcal{O}$ iff $\exists A \in \mathcal{A}$ such that $A \in \Pi_X^*$). So a policy π for the DN is a function from $v(\mathcal{B}')$ to $v(\mathcal{A})$, where $\mathcal{B}' \subseteq \mathcal{B}$ are the observable variables in \mathcal{B} .

Figure 1a shows a DN representation of the barley example from the introduction. The action variables (rectangles) are $\mathcal{A} = \{Grain, Harrow, Fungicide, Fertiliser, Pesticide\}$, the chance variables (ovals) are $\mathcal{C} = \mathcal{B} \cup \mathcal{O}$ where \mathcal{B} includes Precipitation, Temperature, $Soil\ Type$ etc. and \mathcal{O} includes Precipitation, Prec

Our DN formulation has some similarities to influence diagrams (Howard and Matheson, 2005). In contrast to DNs, influence diagrams allow action nodes to have parents via "information arcs" from chance nodes. While this feature is useful to model multistage decision problems, we do not require it here since all action variables are assigned simultaneously.

² Agents do not have intrinsic preferences over actions, but rather over their outcomes.

Our agent will incrementally learn all components of the DN, including its set of random variables, dependencies and reward function. We denote the true DN by dn_+ , and the agent's current model of the DN at time t as dn_t . A similar convention is used for each component of the DN (so, for instance, \mathcal{A}_t is the set of action variables the agent is aware of at time t). Thus, we compute an update function $dn_t = update(dn_{t-1}, e_t)$, where e_t is the latest body of evidence and the DNs dn_{t-1} and dn_t may differ in any respect. Figure 1b gives an example of an agent's possible starting model dn_0 for the barley example.

Notice that dn_0 is missing factors that influence the optimal policy defined by dn_+ (e.g. it is unaware of the concept of fungus). It is also missing dependencies that are a part of dn_+ (e.g. it does not think the choice of grain has any influence on the amount of crops that will grow). One usually assumes that all variables in a DN are connected to the utility node because a variable that is not has no effects on optimal behaviour. So the agent knows that dn_+ is so connected, but dn_+ 's set of variables, causal structure and reward function are all hidden and must be inferred from evidence.

We make four main assumptions to restrict the scope of the learning task:

- 1. The agent can observe the values of all the variables it is aware of (so the domain of its policy function at time step t is $v(\mathcal{B}_t)$). However, it cannot observe a chance variable's values at times before it was aware of it, even after becoming aware of it—i.e. the agent cannot re-perceive past domain trials upon discovering an unforeseen factor.
- 2. The agent cannot perform an action it is unaware of. Formally, if X ∈ A₊ but X ∉ A_t, then the fully aware expert perceives the learning agent's action as entailing X = 0 (plus other values of other variables in A₊). This differs from the learning agent's own perception of its action: X = 0 is not a part of the agent's representation of what it just did because it is not aware of X! Once the agent becomes aware of X, then knowing that inadvertent actions are not possible, it can infer all its past actions entail X = 0. This contrasts with chance variables, whose values at times when the agent was unaware of them will always be hidden. Clearly, this assumption does not hold across all decision problems (e.g. an agent might inadvertently lean on a button, despite not knowing the button exists). If we wished to lift this assumption, we could simply treat action variable unawareness in the same way as we treat chance variable unawareness (as described in Section 4).
- 3. The set of random variables in the agent's initial DN dn_0 is incomplete rather than wrong: that is, $\mathcal{B}_0 \subseteq \mathcal{B}_+$, $\mathcal{O}_0 \subseteq \mathcal{O}_+$ and $\mathcal{A}_0 \subseteq \mathcal{A}_+$. Further, the initial domain of \mathcal{R} is a subset of its true domain: i.e., $\Pi_R^0 \subseteq \Pi_R^+$. This constraint together with the dialogue strategies in Section 3.2 simplify reasoning: DN updates may add new random variables but never removes them; and may extend the domain of \mathcal{R} but never retracts it. However, the causal structure Π and reward function \mathcal{R} can be revised, not just refined.
- 4. The expert has complete knowledge of the actual decision problem dn_+ but lacks complete knowledge of dn_t —the learning agent's perception of its decision problem at time t. Further, we make the expert *cooperative*—her advice is always sincere, competent and relevant (see Section 3.2 for details).

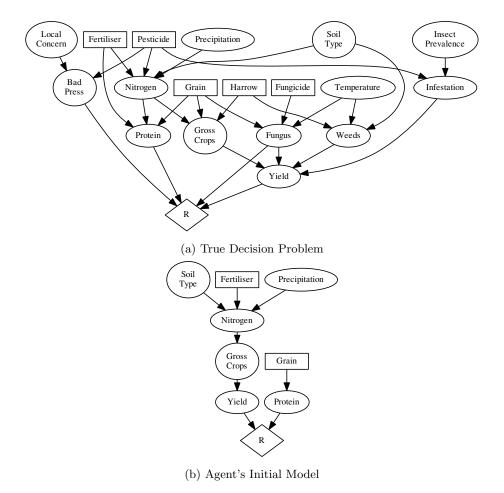


Figure 1: The graphical component of the DN for the "Barley" problem. Action variables are represented by rectangles, chance variables by ovals, and the reward node by a diamond.

Any competent agent attempting this learning task should obey two key principles:

Consistency: At all times t, dn_t should be consistent. That is, Π_t should be a DAG defined over the vocabulary $\mathcal{C}_t \cup \mathcal{A}_t$, θ_t should abide by the basic laws of probability, and \mathcal{R}_t should be a well-defined function that captures an asymmetric and transitive preference relation over \mathcal{C}_t .

Satisfaction: Evidence is informative about what can happen (however rarely), not just informative about likelihood. At all times t, dn_t should satisfy all the possibilities that are entailed by the observed evidence so far.

Consistency is clearly desirable, because anything can be inferred from an inconsistent DN, making any action optimal. Satisfaction is not an issue in traditional approaches to learning optimal policies, because evidence never reveals a possibility that is not

within the hypothesis space of the learning agent's initial DN. But in our task, without Satisfaction, the posterior DN may fail to represent an unforeseen possibility that is monotonically entailed by observed evidence. It would then fail to capture the unforeseen possibility's effects on expected utilities and optimal policy. Our experiments in Section 5 show that when starting out unaware of factors on which optimal policies depend, a baseline agent that does not comply with Satisfaction performs worse than an agent that does. We will say dn_t is valid if it complies with Consistency and Satisfaction; it is invalid otherwise.

A valid DN is necessary, but not sufficient: dn_t should not only be valid, but in addition its probabilistic component $\langle \Pi_t, \theta_t \rangle$ should reflect the relative frequencies in the domain trials. Section 4 will define a DN update procedure with all of these properties.

The set of valid DNs is always unbounded—it is always possible to add a random variable to a valid DN while preserving its validity. So in addition to the above two monotonic constraints on DN update, we adopt two intuitively compelling defeasible principles that make DN update tractable. Indeed, the agent needn't enumerate all valid DNs; instead, DN update uses the defeasible principles to dynamically construct a single DN from evidence:

Minimality: The DN should have minimal complexity. In other words, its random variables discriminate one possible state from another, two states have distinct payoffs, and/or two factors are probabilistically dependent only when evidence justifies this.

Conservativity: The agent should minimise changes to the DN's hypothesis space when observing new evidence.

Minimality is a form of Occam's razor: make the model as simple as possible while accounting for evidence. Conservativity captures the compelling intuition that you preserve as much as possible of what you inferred from past evidence even when you have to revise the DN to restore consistency with current evidence. Minimality and Conservativity underlie existing symbolic models of commonsense reasoning (Poole, 1993; Hobbs et al., 1993; Alchourrón et al., 1985), the acquisition of causal dependencies (Bramley et al., 2015; Buntine, 1991) and preference change (Hansson, 1995; Cadilhac et al., 2015).

Our final desirable feature is to support **active learning**: to give the agent some control over the evidence it learns from next, both from the domain trials (Section 3.1) and the dialogue content (Section 3.2).

3. Evidence

We must use evidence to overcome ignorance about how to conceptualise the domain, not just ignorance about likelihoods and payoffs. We regiment this by associating each piece of evidence e_t with a formula δ_t that expresses (partial) information about the true decision problem dn_+ , where δ_t follows monotonically from e_t , in the sense that it would be *impossible* to observe e_t unless δ_t is true of the decision problem dn_+ that generated e_t . Thus δ_t is a partial description of a DN that must be satisfied by the actual (complete) decision network dn_+ .

We illustrate this with three examples. First, given the assumptions we made about the relationship between the agent's initial DN dn_0 and dn_+ (i.e., that $\mathcal{B}_0 \subseteq \mathcal{B}_+$, $\mathcal{A}_0 \subseteq \mathcal{A}_+$, $\mathcal{O}_0 \subseteq \mathcal{O}_+$ and $\Pi_R^0 \subseteq \Pi_R^+$), dn_0 in Figure 1b yields the partial description δ_0 given in (2):

$$\{Grain, Fertiliser\} \subseteq \mathcal{A}_{+} \land$$

$$\{Soil\ Type, Precipitation\} \subseteq \mathcal{B}_{+} \land$$

$$\{Nitrogen, Gross\ Crops, Yield, Protein\} \subseteq \mathcal{O}_{+} \land$$

$$\{Yield, Protein\} \subseteq \Pi_{R}^{+}$$

$$(2)$$

The second example concerns domain trials: suppose the agent is in a state where it experiences the reward r and observes that the Yield variable has value y and the Protein variable has value p. Equation (3) must be true of dn_+ , where the domain of quantification is dn_{+} 's atomic states:

$$\exists s(s \to y \land p \land \mathcal{R}_{+}(s) = r) \tag{3}$$

Thirdly, suppose the expert advises the agent to apply pesticide. Then, $Pesticide \in \mathcal{A}_+$ must be true.

We capture this relationship between a complete DN and formulae like (2) and (3) by defining a syntax and semantics of a language for partially describing DNs (details are in the Appendix). Each model for interpreting the formulae δ in this language corresponds to a unique complete DN, and $dn \models \delta$ if and only if δ (partially) describes dn. Where δ_t represents the properties that a DN generating observed evidence e_t must have, $dn_+ \models \delta_t$ (because dn_+ generated e_t), although other DNs may satisfy δ_t too. This relationship between dn_+ , e_t and δ_t enables the agent to accumulate an increasingly specific partial description of dn_+ as it observes more and more evidence: where $e_{0:t-1}$ is the sequence of evidence e_1, \ldots, e_{t-1} and $\delta_{0:t-1}$ its associated partial description of dn_+ , observing the latest evidence e_t yields $e_{0:t}$ with an associated partial description $\delta_{0:t}$ that is the conjunction of $\delta_{0:t-1}$ and δ_t . The agent will thus estimate a valid DN from evidence (as defined in Section 2) under the following conditions:

- 1. $\delta_{0:t}$ captures the necessary properties of a DN that generates $e_{0:t}$
- 2. The agent's model obeys this partial description. That is, $dn_t \models \delta_{0:t}$

This section describes how we achieve the first condition; Section 4 defines how we achieve the second, with Section 4.2 also ensuring that the probabilistic component $\langle \Pi_t, \theta_t \rangle$ reflects the relative frequencies in the domain trials.

3.1. Domain Evidence: Sample 0.1

Domain evidence consists of a set of domain trials. In a domain trial τ_i , the agent observes $\vec{b_i} \in v(\mathcal{B}_i)$, performs an action $\vec{a_i} \in v(\mathcal{A}_i)$, and observes its outcome $\vec{o_i} \in$ $v(\mathcal{O}_i)$ and reward r_i . From now on, for notational convenience, we may omit the vector notation, or freely interchange vectors with conjunctions of their values. Each domain trial therefore consists of a tuple:

$$Sample_{0:t} = [\langle b_i, a_i, o_i, r_i \rangle : 0 < i \le t]$$

$$(4)$$

The domain trial $\tau_i = \langle b_i, a_i, o_i, r_i \rangle$ entails the partial description (5) of dn_+ : in words, there is an atomic state $s \in v(\mathcal{C}_+ \cup \mathcal{A}_+)$ in dn_+ that entails the observed values b_i , a_i and o_i and which has the payoff r_i .

$$\exists s((s \to (b_i \land a_i \land o_i)) \land \mathcal{R}_+(s) = r_i)$$
 (5)

Formula (5) follows monotonically from τ_i because the agent's perception of a domain trial is incomplete but never wrong: the agent's random variables have observable values, and they are always a subset of those in dn_+ thanks to the agent's starting point dn_0 and the dialogue strategies (see Section 3.2). Thus, even if the agent subsequently discovers a new random variable X, (5) still follows from τ_i , regardless of X's value at time i (which remains hidden to the agent). But on discovering X, tuples in (4) get extended—the agent will observe X's value in subsequent domain trials. Thus in contrast to standard domain evidence, the size of the tuples in (4) is dynamic. We discuss in Section 4 how the agent copes with these dynamics. The expert also keeps a record $Sample_{0:t}^+$ of the domain trials; these influence her dialogue moves (see Section 3.2). Her representation of each trial is at least as specific as the agent's because she is aware of all the variables—so the size of the tuples in $Sample_{0:t}^+$ is static.

Domain trials can reveal to the agent that its conceptualisation of the domain is deficient. If there are two trials in $Sample_{0:t}$ with the same observed value for Π_R^t (i.e., the current estimated domain of the reward function) but the rewards are different, then this entails that Π_R^t is invalid. If $Sample_{0:t}$ contains two domain trials with the same observed values for every chance variable C_t of which the agent is currently aware, but the rewards are different, then the vocabulary C_t is invalid. Section 4 will define how the agent detects and learns from these circumstances.

The agent's strategy for choosing an action mixes exploitation and exploration in an ϵ -greedy approach: In a proportion $(1 - \epsilon)$ of the trials, the agent chooses what it currently thinks is an optimal action. In the remainder, the agent chooses an action at random.

3.2. Dialogue Evidence: $\mathcal{D}_{0:t}$

The interaction between the agent and the expert consists of the agent asking questions that the expert then answers, and unsolicited advice from the expert. All the dialogue moves are about dn_+ , and the signals are in the formal language for partially describing DNs (see the Appendix), but with the addition of a sense ambiguous term, which we motivate and describe shortly. The agent's and expert's lexica are different, however: the agent's vocabulary lacks the random variables in dn_+ that it is currently unaware of, and so the expert's utterances may feature neologisms. As we said earlier, we bypass learning how to ground neologisms (but see Larsson (2013); Forbes et al. (2015); Yu et al. (2016)) by assuming that once the agent has heard a neologism, it can observe its denotation in all subsequent domain trials.

The sense ambiguous term is w_t^b : its intended denotation is what the expert observes about the state at time t before the agent acts—i.e., $\llbracket w_t^b \rrbracket \in v(\mathcal{B}_+)$. But this denotation is hidden to the agent, whose default interpretation is $\llbracket w_t^b \rrbracket$ projected onto \mathcal{B}_t —i.e., it is restricted by the agent's conceptualisation of the domain at time t. The expert uses w_t^b to advise the agent of a better action than the one it performed at t. We will see in Section 3.2.1 that using w_t^b in the expert's signal minimises its neologisms, which

makes learning more tractable. But hidden messages may create misunderstandings; Section 3.2.2 describes how the agent detects and learns from them.

The agent and expert both keep a dialogue history: $\mathcal{D}_{0:t}$ for the agent and $\mathcal{D}_{0:t}^+$ for the expert. Each utterance in a dialogue history is a tuple $\langle \omega_i, \sigma_i, \mu_i \rangle$, where ω_i is the speaker (i.e., either the expert or the learning agent), σ_i the signal, and μ_i its (default) interpretation. Since w_i^b may be misinterpreted, μ_i and μ_i^+ may differ and so μ_i is not equivalent to $\delta_i - \delta_i$ is the information about dn_+ that follows from the signal σ_i whatever the denotation of w_i^b might be. We will specify δ_i for each signal σ_i in Section 3.2.1.

3.2.1. The expert's dialogue strategy

As noted earlier, the expert's dialogue strategy is Cooperative: the message μ^+ that she intends to convey with her signal σ is satisfied by the actual decision problem dn_+ , so that $dn_+ \models \mu^+$. This makes her sincere, competent and relevant. In many realistic scenarios, this assumption may be untrue (a human teacher, for example, might occasionally make mistakes). We intend to explore relaxations of this assumption in future work.

Further, the expert's dialogue strategy limits the amount of information she is allowed to send in each signal. There are two motivations for limiting the amount of information. The first stems from the definition itself of our learning task; and the second is practical. First, recall from Section 1 that we allow the expert advice to occur piecemeal, with signals being interleaved among the learner's domain trials. This is because a major aim of our learning task is to reflect the kind of teacher-apprentice learning seen between humans, where the teacher only occasionally interjects to say things that relate to the learner's latest attempts to solve the task. There are many tasks where an expert may be incapable of exhaustively expressing everything they know about the problem domain, but rather can only express relevant information in reaction to specific contingencies that they experience.

The second, more practical motivation is to make learning tractable. The number of possible causal structures Π is hyperexponential in the number of random variables (Buntine, 1991). Prior work utilises defeasible principles such as Conservativity (Bramley et al., 2015) and Minimality (Buntine, 1991) to make inferring Π tractable. However, if the expert's signal σ_t features a set \mathcal{N} of variables that the agent was unaware of, then the agent must add each variable in \mathcal{N} to the causal structure Π_t , and moreover by Consistency and Satisfaction, each of these must be connected to Π_t 's utility node. But Π_{t-1} did not feature these variables at all, and so the number of maximally conservative and minimal updates that satisfy this connectedness is hyperexponential in the size of \mathcal{N} . Thus, an expert utterance with many neologisms undermines the efficiency and incrementality of learning.

We avoid this complexity by restricting all expert signals to containing at most one neologism. Specifically, the expert must have conclusive evidence that the agent is aware of all but one of the random variables that feature in her signal σ . The expert knows the agent is aware of a variable X if: (a) X has already been mentioned in the dialogue, either by her or by the agent; or (b) X is an action variable and the agent has performed its positive value x (recall that inadvertent action is not possible):

$$\mathcal{C}_{e} = \{X \in \mathcal{C}_{+} \mid \exists \sigma_{i} \in \mathcal{D}_{0:t}^{+} : X \in \sigma_{i}\}
\mathcal{A}_{e} = \{X \in \mathcal{A}_{+} \mid \exists \sigma_{i} \in \mathcal{D}_{0:t}^{+} : X \in \sigma_{i} \text{ or } \exists \tau_{i} \in Sample_{0:t}^{+} : x \in \tau_{i}\}
10$$
(6)

Here, C_e and A_e respectively are the set of chance and action variables that the expert knows the agent is aware of. The following principle, which we refer to as 1N, applies to all the expert's signals:

At Most One Neologism (1N): Each expert signal σ features at most one variable from $(A_+ \cup C_+) \setminus (A_e \cup C_e)$ (i.e., at most one neologism). Furthermore, if σ features such a variable X, then σ declares its type: i.e., σ includes the conjunct $X \in \mathcal{B}_+$, $X \in \mathcal{O}_+$ or $X \in \mathcal{A}_+$, as appropriate.

The expert uses an ambiguous term w_t^b to comply with Cooperativity and 1N in contexts where, without w_t^b , she would be unable express any advice. For instance, assume that the expert's knowledge of the agent's "before" vocabulary is given by \mathcal{B}_e . The expert would typically be unable to express that, given $b_t \in v(\mathcal{B}_+)$, the agent should have performed an alternative action a' to the action a_t that the agent actually performed—by 1N, she cannot use the vector $b_t^+ \in v(\mathcal{B}_+)$ in her signal if $|\mathcal{B}_+| - |\mathcal{B}_e| > 1$. If instead she were to use the vector b_t^e , where $b_t^e = b_t^+ \upharpoonright \mathcal{B}_e$ (i.e., b_t^+ projected onto \mathcal{B}_e), then the resulting statement (7) may be false (and so violate Cooperativity), because these expected utilities marginalise over all possible values of $\mathcal{B}_+ \setminus \mathcal{B}_e$, rather than using their actual values:

$$EU(a'|b_t^e) > EU(a_t|b_t^e) \tag{7}$$

Alternatively, by replacing b_t^e in the signal (7) with the ambiguous term w_t^b , her intended message becomes hidden but it abides by both Cooperativity and 1N.

The expert's dialogue policy is to answer all the agent's queries as and when they arise, and to occasionally offer unsolicited advice about a better action. She does the latter when two conditions hold:

- (i) The agent has been behaving sufficiently poorly to justify the need for advice
- (ii) The current context is one where she can express a better option while abiding by Cooperativity and 1N.

Condition (i) is defined via two parameters γ and β : in words, the last piece of advice was offered greater than γ time steps ago, and from then until now, the fraction of suboptimal actions taken by the agent is greater than β . This is formalised in equation (8), where t' is the time of the last advice and $a_i^{+,*}$ is the optimal action given dn_+ and b_i^+ . In the experiments in Section 5, we vary γ and β to test how changing the expert's penchant for offering unsolicited advice affects the learning agent's convergence on optimal policies. Condition (ii) for giving advice is satisfied when the observed reward r_t is no higher than the expected payoff from the agent's action a_t^+ (equation (9)), and there is an alternative action a' with a higher expected payoff, which can be expressed while complying with Cooperativity and 1N (equation (10)).

$$|t - t'| > \gamma \wedge \frac{|\{a_i^+ : t' \le i \le t \text{ and } EU(a_i^+|b_i^+) < EU(a_i^{+,*}|b_i^+)\}|}{|t - t'|} > \beta$$
 (8)

$$EU(a_t^+|b_t^+) \ge r_t \tag{9}$$

$$\exists A \exists a'(a' \in v(\mathcal{A}_e \cup \{A\}) \land EU(a'|b_t^+) \ge EU(a_t^+ \upharpoonright (\mathcal{A}_e \cup \{A\})|b_t^+))$$

$$11$$

$$(10)$$

When the context satisfies these conditions, then there are witness constants A and a' that satisfy (10). These constants are used to articulate the advice: the expert utters (11), where a_t is the expression formed by projecting the vector $a_t^+ \in v(\mathcal{A}_+)$ onto $\mathcal{A}_e \cup \{A\}$.

$$EU(a'|w_t^b) > EU(a_t|w_t^b) \tag{11}$$

In our Barley example, the message (11) might be paraphrased as: in the current circumstances, it would have been better to apply pesticide and not use fertiliser than to not apply pesticide and use fertiliser.

The agent's default interpretation of (11) is (12), where $b_t \in v(\mathcal{B}_t)$:

$$\sum_{s \in \mathcal{C}_t \times \mathcal{A}_t} Pr(s|a', b_t) \mathcal{R}_+(s) \ge \sum_{s \in \mathcal{C}_t \times \mathcal{A}_t} Pr(s|a_t, b_t) \mathcal{R}_+(s)$$
(12)

So $\langle expert, (11), (12) \rangle$ is added to $\mathcal{D}_{0:t}$. While the intended message of (11) is true (because (10) is true), (12) may be false, and so no monotonic entailments about probabilities can be drawn from it. For example, in our Barley example, suppose that the agent's current model of the domain is Figure 1b, and the agent has observed soil type n and precipitation c. Then the agent's defeasible interpretation of the expert's advice is that it would have been better to apply pesticide (p) and not use fertiliser $(\neg f)$ (than to not apply pesticide and use fertiliser) in any state where $n \wedge c$ is true (note that relative to the DN shown in Figure 1b, the expert mentioning applying pesticide leads it to discover this entirely new action):

$$\sum_{s \in \mathcal{C}_t \times \mathcal{A}_t} Pr(s|p, \neg f, n, c) \mathcal{R}_+(s) \ge \sum_{s \in \mathcal{C}_t \times \mathcal{A}_t} Pr(s|\neg p, f, n, c) \mathcal{R}_+(s)$$
 (13)

But this (defeasible) interpretation could be false: the expert's (true) intended message may have been that $p \land \neg f$ is better in a much more specific situation: one where not only is $n \land c$ true, but also the local concern is low and the insect prevalence is high (in other words, the probabilities in (13) should have been conditioned on $\neg l$ and i as well). However, (11) and the mutually known dialogue policy monotonically entails (14), whatever the true referent for w_b^t might be.

$$\exists s((s \to (a' \land b_t)) \land \mathcal{R}_+(s) > r_t) \tag{14}$$

So (14) is added to $\delta_{0:t}$, and for dn_t to be valid it must satisfy it; similarly for $A \in \mathcal{A}_+$. In our example, the agent adds (15) to $\delta_{0:t}$:

$$\exists s((s \to (p \land \neg f \land n \land c)) \land \mathcal{R}_{+}(s) > r_{t})$$
(15)

Thus the expert's unsolicited advice can result in the agent discovering an unforeseen action term A (in this case, adding pesticide) and/or prompt a revision to the reward function \mathcal{R}_t , which in turn may reveal to the agent that its conceptualisation of the domain is deficient (just as (5) may do).

³Our learning algorithms in Section ⁴ do not exploit the (defeasible) information about likelihood that is expressed in (12); that is a matter for future work.

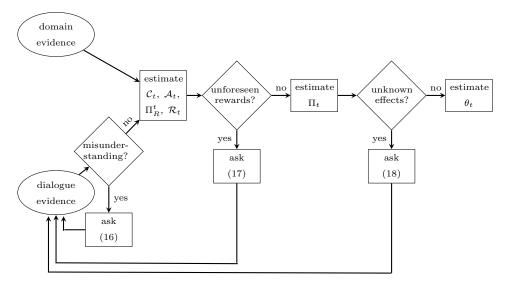


Figure 2: Information flow when updating a DN with the latest evidence. Diamonds are tests and rectangles are processes.

3.2.2. The agent's dialogue strategy

The agent aims to minimise the expert's effort during the learning process, and so asks a question only when DN update fails to discriminate among a large number of DNs. In this paper, we identify three such contexts (as shown in Figure 2): (i) Misunderstandings; (ii) Unforeseen Rewards; and (iii) Unknown Effects. We now describe each of these in turn.

Misunderstandings. The agent checks whether the (default) interpretation of the current signal is consistent with those of prior signals; when they are inconsistent, the agent knows there has been a misunderstanding (because of Cooperativity). For instance, suppose $\mathcal{B}_t = \emptyset$ —so the agent assumes $[w_i^b] = \top$ for all $i \leq t$ —and the expert advised $EU(a|w_{t-n}^b) > EU(a'|w_{t-n}^b)$ and $EU(a'|w_t^b) > EU(a|w_t^b)$. The agent's default interpretations of these signals fails the consistency test. Thus the agent infers that it is unaware of a \mathcal{B} variable, but does not know its name. If the agent were to guess what variable to add, then learning would need to support retracting it on the basis of subsequent evidence, or reason about when it is identical to some subsequent factor the agent becomes aware of. This is a major potential complexity in reasoning and learning. We avoid it by defining a dialogue strategy that ensures that the agent's vocabulary is always a subset of the true vocabulary. Here, that means the agent asks the expert for the name of a \mathcal{B} variable. In words, question (16) expresses: what is different about the state before I acted just now and the state before I acted n time steps ago?⁴

$$?\lambda V(V \in \mathcal{B}_{+} \land value(V, t - n) \neq value(V, t))$$
(16)

⁴Strictly speaking, answers to this question are partial descriptions of the domain trials $Sample_{0:t}^+$ as well as dn_+ . The formal details of this are straightforward but tedious, so we gloss over it here.

This signal uses a standard representation for questions (Groenendijk and Stokhof, 1982): ? is an operator that maps a function to a set of propositions (the true answers), and the function's arguments correspond to the wh-elements of the question (which, what etc). Its semantics, given in the Appendix, follows (Asher and Lascarides, 2003): a proposition is a true answer if and only if it substitutes specific individual(s) for the λ -term(s) to create a true proposition—so "something" is not an answer but "nothing" is an answer, corresponding to the empty set. An answer can thus be non-exhaustive—it needn't include all the referents that satisfy the body of the λ -expression.

The agent's policy for when to ask (16) guarantees that it has a true positive answer. By the 1N rule, the expert's answer includes one variable, chosen at random if there is a choice. E.g., she answers with the signal $X \in \mathcal{B}_+$; so $X \in \mathcal{B}_+$ is added to $\delta_{0:t}$. In our barley example, the answer might be, for instance, "the temperature was high", where the concept "temperature" is a neologism to the learning agent.

The agent now knows that its default interpretations of prior signals that feature w_i^b ($i \prec t$) are incorrect—on learning $X \in \mathcal{B}_+$, it now knows that these terms do not denote \top . But the agent cannot observe the correct interpretation. Testing whether subsequent messages are consistent with prior messages would thus involve reasoning about past hidden values of X, which would be complex and fallible. For the sake of simplicity, we avoid it: if the agent discovers a new \mathcal{B} -variable at time t, then she ceases to test whether subsequent messages μ_{t+m} are consistent with past ones $\mu_{t-m'}$ ($m, m' \geq 0$). In effect, the agent "forgets" past signals and their (default) interpretations, but does not forget their monotonic consequences, which are retained in $\delta_{0:t}$.

Unforeseen rewards. The current reward function's domain, Π_R^t , may be too small to be valid (see Sections 2 and 3.1). If the agent were to guess what variable to add to Π_R^t , then learning would need to support retracting it on the basis of subsequent evidence—this is again a major potential complexity in reasoning and learning. Instead, when the agent infers Π_R^t is too small (we define how the agent infers this in Section 4), the agent seeks monotonic evidence for fixing it by asking the expert (17):

$$?\lambda V(V \in \Pi_R^+ \bigwedge_{X \in \Pi_R^t} V \neq X) \tag{17}$$

For instance, in the barley domain, this question could express something like "Other than having a high yield and high protein crops, what else do I care about?" to which the answer might be "You care about bad publicity". This is a true (non-exhaustive) answer, even though the agent (also) cares about avoiding outbreaks of fungus.

Non-exhaustive answers enable the expert to answer the query while abiding by Cooperativity and the 1N rule. But they also generate a *choice* on which answer to give. The expert's choice is driven by a desire to be informative: she includes in her answer all variables in Π_R^+ that she knows the agent is aware of (i.e., the variables $Y \in \mathcal{C}_e \cap \Pi_R^+$) and one potential neologism (i.e., $X \in (\mathcal{C}_+ \setminus \mathcal{C}_e) \cap \Pi_R^+$) if it exists, with priority given to a variable whose value is different in the latest trial τ_t from a prior trial $\tau_{t'}$ that had the same values as τ_t on all the variables mentioned in the query (17) but a different reward. If her answer includes a potential neologism, then by 1N she declares its type.

For example, let $\Pi_R^{t-1} = \{O_1, O_2\}$ and $\Pi_R^+ = \{O_1, O_2, O_3, O_4, O_5\}$. Suppose that $O_3 \in \mathcal{C}_e$ but $O_4, O_5 \notin \mathcal{C}_e$. Now suppose that the latest trial τ_t entails $o_1, \neg o_2$ and reward

of 1, but a prior trial entails $o_1, \neg o_2$ and a reward of 0.5. Then even if O_3 has different values in these trials, the agent does not infer (defeasibly) that $O_3 \in \Pi_R^+$. Rather, she asks (17): i.e., $?\lambda V(V \in \Pi_R^+ \wedge V \neq O_1 \wedge V \neq O_2)$. The expert's answer includes O_3 because $O_3 \in \mathcal{C}_e \cap \Pi_R^+$. In addition, she can mention either O_4 or O_5 , but not both (because of 1N). Suppose that τ_t entails o_4 and a prior trial with a different reward entails $o_1 \wedge \neg o_2 \wedge \neg o_4$. However, O_5 lacks this property. Then her answer is: $O_3 \in \Pi_R^+ \wedge O_4 \in \Pi_R^+ \wedge O_4 \in \mathcal{O}_+$.

The ambiguous term w_t^b is not a part of the expert's answer, and so the message is observable—indeed, it is the same as the signal. This becomes a conjunct in $\delta_{0:t}$, and so in our example, for dn_t to be valid it must satisfy $O_4 \in \Pi_R^t$ and $O_4 \in \mathcal{O}_t$ (and so $\Pi_R^t \supset \Pi_R^{t-1}$ and if O_4 truly was a neologism to the agent then $\mathcal{O}_t \supset \mathcal{O}_{t-1}$ as well). Section 4 defines how to estimate a valid dn_t from this evidence.

Unknown Effects. There are contexts where the search space of possible causal structures remains very large in spite of the defeasible principles for restricting it, in which case the agent asks a question whose answer will help to restrict the search space:

$$?\lambda V(X \in \Pi_V) \tag{18}$$

In words, what does X affect? For instance, in the barley domain, it might express: What does the temperature affect? (to which an answer might be the risk of weeds). In Section 4 we will define precisely the contexts in which the agent asks this question, including which variable X it asks about. The expert's answer $X \in \Pi_Y^+$ gives priority to a variable Y that she believes the agent is unaware of. This increases the chances that the agent will learn potentially valuable information about the hypothesis space.

The expert's answer $X \in \Pi_Y^+$ has an observable interpretation; $X \in \Pi_Y^+$ is added to $\delta_{0:t}$ and since $dn_t \models \delta_{0:t}$, $X \in \Pi_Y^t$. Thus some dependencies in Π_t are inferred monotonically via observable expert messages. Others are inferred defeasibly via statistical pattern recognition in the domain trials (see Section 4).

4. The Model for Learning

We now define DN update in a way that meets the criteria from Section 2. We must estimate all components of the DN from the latest evidence e_t , in a way that satisfies the partial description $\delta_{0:t}$ that has accumulated so far (this is required to make the DN valid). That is, one must estimate the set of random variables C_t and A_t , the domain $\Pi_R^t \subseteq C_t$ of the reward function as well as the function \mathcal{R}_t itself, the dependencies Π_t among $C_t \cup A_t$, and the conditional probability tables (CPTs) θ_t , given Π_t . These components are estimated from the prior DN dn_{t-1} , the latest evidence e_t and the partial description $\delta_{0:t}$ in the following order, as shown in Figure 2:

- 1. (a) Estimate \mathcal{B}_t , \mathcal{O}_t , \mathcal{A}_t and Π_R^t ;
 - (b) Estimate \mathcal{R}_t , given Π_R^t ;
- 2. (a) Estimate Π_t , given $\mathcal{B}_t \cup \mathcal{O}_t \cup \mathcal{A}_t$ and Π_R^t ;
 - (b) Estimate θ_t , given Π_t .

Step 1 proceeds via constraint solving (see Section 4.1); step 2 via a combination of symbolic and statistical inference (see Section 4.2). Update proceeds in this order because the set of dependencies one must deliberate over depends on the set of random variables it is defined over, and the global constraint on dependencies—namely, that the utility node is a descendant of all nodes in the DN—must be defined in terms of the domain Π_R^t of the reward function. Likewise, the set of CPTs θ_t that one must estimate is defined by the dependency structure Π_t . The search for a valid DN can prompt backtracking, however: e.g., failure to derive a valid dependency structure Π_t may ultimately lead to a re-estimate of the set of random variables $\mathcal{B}_t \cup \mathcal{O}_t \cup \mathcal{A}_t$. We now proceed to describe in detail each of these components of DN update.

4.1. Random Variables and Reward Function

The first step in DN update is to identify dn_t 's random variables—that is, the sets \mathcal{B}_t , \mathcal{O}_t and \mathcal{A}_t —and the reward function \mathcal{R}_t . This is achieved via **constraint solving**, with the constraints provided by $\delta_{0:t}$.

The number of valid vocabularies is always unbounded, because any superset of a valid vocabulary is valid. As motivated in Section 2, we make search tractable via greedy search for a *minimal* valid vocabulary: the agent (defeasibly) infers the vocabulary in (19a-c) (this covers all the variables the agent is aware of thanks to the dialogue strategy from Section 3.2), and also defeasibly infers the minimal domain for \mathcal{R}_t , defined in (19d).

$$\mathcal{B}_t = \{X : X \in \mathcal{B}_+ \text{ is a conjunct in } \delta_{0:t}\}$$
(19a)

$$\mathcal{A}_t = \{X : X \in \mathcal{A}_+ \text{ is a conjunct in } \delta_{0:t}\}$$
(19b)

$$\mathcal{O}_t = \{X : X \in \mathcal{O}_+ \text{ is a conjunct in } \delta_{0:t}\}$$
 (19c)

$$\Pi_R^t = \{X : X \in \Pi_R^+ \text{ is a conjunct in } \delta_{0:t}\}$$
(19d)

The evidence described in Section 3 yields three kinds of formulae in $\delta_{0:t}$ that (partially) describe \mathcal{R}_+ : (5), (14), and $X \in \Pi_R^+$. The agent uses constraint solving to find, or fail to find, a reward function \mathcal{R}_t that satisfies all conjuncts in $\delta_{0:t}$ of the form (5) and (14), plus the constraint (20), which will ensure \mathcal{R}_t is well-defined with respect to its (estimated) domain (19d).

$$\forall s_1 \forall s_2 ((s_1 \upharpoonright \Pi_R^t \leftrightarrow s_2 \upharpoonright \Pi_R^t) \to \mathcal{R}_t(s_1) = \mathcal{R}_t(s_2)) \tag{20}$$

Constraints of the form (5) and (14) are skolemized and fed into an off-the-shelf constraint solver (with \mathcal{R}_+ replaced with \mathcal{R}_t). The possible denotations of these skolem constants are the atomic states defined by the vocabulary (19a–c). If there is a solution, the constraint solver returns for each skolem constant a specific denotation $x \in v(\mathcal{C}_t \times \mathcal{A}_t)$.

Substituting the skolem terms with their denotations projected onto Π_R^t yields equalities $\mathcal{R}_t(y) = r$ (from (5)) and inequalities $\mathcal{R}_t(y) > r$ (from (14)), where $y \in v(\Pi_R^t)$. A complete function \mathcal{R}_t is constructed from this partial function by defaulting to indifference (recall Minimality): for any $y \in v(\Pi_R^t)$ where there is an inequality $\mathcal{R}_t(y) > r$ but no equality $\mathcal{R}_t(y) = r$, we set $\mathcal{R}_t(y) = r + c$ for some constant c (in our experiments, c = 0.1); for any $y \in v(\Pi_R^t)$ for which there are no equalities or inequalities, we set $\mathcal{R}_t(y) = 0$.

The constraint solver may yield no solution: i.e., there is no function \mathcal{R}_t with the currently estimated domain (19d) satisfying all observed evidence about states and their rewards. This is the context "unforeseen rewards" described in Section 3.2 (see also Figure 2): the agent asks (17), deferring DN update until it receives the expert's answer. The expert's answer is guaranteed to provide a new variable to add to Π_R^t . It may also be a neologism—a variable the agent was unaware of—and so after updating $\delta_{0:t}$ with the expert's answer, the agent backtracks to re-compute (19a–d).

4.2. Estimating $\langle \Pi_t, \theta_t \rangle$

Current approaches to incrementally learning Π in a graphical model of belief exploit local inference to make the task tractable. There are essentially two forms of local inference.

The first is a greedy local search over a full structure: remove an edge from the current DAG or add an edge that does not create a cycle, then test whether the result has a higher likelihood given the evidence (e.g., Bramley et al. (2015); Friedman and Goldszmidt (1997)). This is Conservative: Π changes only when evidence justifies it. However, adapting it to our task is problematic. Firstly, such techniques rely heavily on a decent initial DAG to avoid getting stuck in a local maximum; but in our task the agent's initial unawareness of the possibilities makes an initial decent DAG highly unlikely. Secondly, removing an edge can break the global constraint on DNs that all nodes connect to the utility node. We would need to add a third option of doing local search given Π: replace one edge in Π with another edge somewhere else. This additional option expands the search space considerably.

We therefore adopt the alternative form of local inference: assume conditional independence among parent sets. Buntine (1991) assumes that X's parent set is conditionally independent of Y's given evidence $e_{0:t}$ and the total temporal order \succ over the random variables— $X \succ Y$ means that X may be a parent to Y but not vice versa. This independence assumption on its own is not sufficient for making reasoning tractable, however. If a Bayes Net has 21 variables (as the ones we experiment with in Section 5 do), then a variable may have 2^{20} possible parent sets—a search space that is too large to be manageable. So Buntine (1991) prunes X's possible parent sets to those that evidence so far makes reasonably likely (we will define "reasonably likely" in a precise way shortly). There are then two alternative ways of updating:

- Parameter Update: Estimate the posterior probability of a parent set from its prior probability and the *latest* evidence under an assumption that the set of reasonable parent sets *does not change*.
- Structural Update: Review and potentially revise which parent sets are reasonable, given a batch of evidence. (Thus, Structural Update changes the set of possible structures which are considered).

We adapt Buntine's model to our task in two ways. Firstly, Buntine's model assumes that the total ordering on variables is known. In our task the total order is hidden, and marginalising over all possible temporal orders is not tractable—it is exponential on the size of the vocabulary. We therefore make an even stronger initial independence assumption than Buntine when deciding which parent sets are reasonable: Specifically,

that X's parent set is conditionally independent of Y's given evidence $e_{0:t}$ alone. Unfortunately, this allows combinations of parent sets with non-zero probabilities to be cyclic. We therefore have an additional step where we greedily search over the *space of total orderings* (similar to (Friedman and Koller, 2003)), and use *Integer Linear Programming* (Vanderbei, 2015) at each step to find a "most likely" causal structure Π_t which both obeys the currently proposed ordering and that is also a valid DN—in particular, it is a DAG where the utility node is a descendant of all other nodes.

Secondly, as the agent's vocabulary of random variables expands, we need to provide new probability distributions over the larger set of possible parent sets, which in turn will get updated by subsequent evidence. We now describe each of these components of the model in turn.

4.2.1. Parameter Update

Each variable $V \in \mathcal{C}_t$ is associated with a set P_v of reasonable parent sets. Each parent set $\Pi_v \in P_v$ is some combination of variables from $\mathcal{C}_t \cup \mathcal{A}_t$. Parameter Update determines the posterior distribution over P_v given its prior distribution and the latest piece of evidence under the assumption that the possible values of P_v do not change.

Updates from Domain Trials. Suppose that the latest evidence e_t is a domain trial (i.e., $\tau_t \in Sample_{0:t}$). Parameter Update uses Dirichlet distributions to support incremental learning: if $\tau_t \upharpoonright V = i$ and $\tau_t \upharpoonright \Pi_v = j$, then we can calculate the posterior probability of Π_v in a single step using (21):

$$Pr(\Pi_v|e_{0:t}) = Pr(\Pi_v|e_{0:t-1}) \frac{(n_{v=i|j} + \alpha_{v=i|j} - 1)}{(n_{v=i|j} + \alpha_{v=i|j} - 1)}$$
(21)

Here, $n_{v=i|j}$ is the number of trials in $Sample_{0:t}$ where V=i and $\Pi_v=j$, and $\alpha_{v=i|j}$ is a "pseudo-count" which represents the Dirichlet parameter for V=i and $\Pi_v=j$. The sum of all trials where $\Pi_v=j$ is given by $n_{v=\cdot|j}$. Formula (21) follows from the recursive structure of the Γ function in the Dirichlet distribution (the Appendix provides a derivation, which corrects an error in (Buntine, 1991)).

Estimating θ_t , given Π_t , likewise exploits the Dirichlet distribution (Buntine, 1991, p56):

$$E_{\theta|Sample_{0:t},\Pi}(\theta_{v=i|j}) = \frac{\int_{\theta} \theta_{v=i|j} Pr(Sample|\Pi,\theta) Pr(\theta|\Pi)}{\int_{\theta} Pr(Sample|\Pi,\theta) Pr(\theta|\Pi)}$$

$$= \frac{n_{x=i|j} + \alpha_{v=i|j}}{n_{v=.|j} + \alpha_{v=.|j}}$$
(22)

In words, (22) computes the conditional probability tables (CPTs) directly from the counts in the trials and the appropriate Dirichlet parameters, which in turn quantify the extent to which one should trust the counts in the domain trials for estimating likelihoods—the higher the value of the α s relative to the ns, the less the counts influence the probabilities. Note that the α -parameters vary across the values of the variables and their (potential) parent sets. We motivate this shortly, when we describe how to perform DN update when a new random variable needs to be added to it. At the start of the learning process, $\alpha_{v=i|j}=0.5$ for all V, i and j.

Updates from Expert Evidence. Now suppose e_t is an expert utterance σ_t , but not one that introduces a neologism (we discuss DN update over an expanded vocabulary at the end of this section). Then Parameter Update starts by using \land -elimination on δ_t (i.e., the partial description of dn_+ that σ_t entails) to infer a conjunction $\pi_{t,v}$ of all conjuncts in $\delta_{0:t}$ of the form $X \in \Pi_v$. Note that δ_t does not contain conjuncts of the form $X \notin \Pi_v$ (see Section 3.2), although it may entail such formulae (e.g., from conjuncts declaring a variable's type). Formula (23) then computes a posterior distribution over P_v , where η is a normalising factor:

$$Pr(\Pi_v|e_{0:t}) = \begin{cases} \eta Pr(\Pi_v|e_{0:t-1}) & \text{if } \Pi_v \vdash \pi_{t,v} \\ 0 & \text{otherwise} \end{cases}$$
 (23)

Equation (23) is Conservative—it preserves the relative likelihood of parent sets that are consistent with $\pi_{t,v}$. It offers a way to rapidly and incrementally update the probability distribution over P_v with at least some of the information that is revealed by the latest expert evidence.

Enforcing global constraints on Π . Equations (21) and (23) on their own do not comply with Satisfaction nor even with Consistency. If we naively construct the "most likely" global structure by simply picking the highest probability parent set Π_v for each variable V independently, their combination might be cyclic, or violate information about parenthood entailed by $\delta_{0:t}$.

To find a global structure which is valid as well as likely, we combine ILP techniques with a greedy local search over the space of total orderings.

We first compute $Pr(P_v|e_{0:t},\succ)$ from $Pr(P_v|e_{0:t})$, where \succ is a total temporal order over dn_t 's random variables that satisfies the partial order entailed by $\delta_{0:t}$. Next, we use ILP techniques to determine the most likely structure Π^m_{\succ} which obeys the current ordering \succ , and is a DAG with all nodes connected to the utility node. We stop searching when all valid total orders \succ' formed via a local change to \succ yield a less likely structure (i.e., $Pr(\Pi^m_{\succ}|e_{0:t},\succ') \leq Pr(\Pi^m_{\succ}|e_{0:t},\succ)$), and return Π^m_{\succ} as Π_t . We now describe these steps in detail.

The partial description $\delta_{0:t}$ imposes a partial order on the random variables, thanks to the expert's declarations of the form $X \in \Pi_Y$ (this corresponds to condition (i) in Fact 1), its entailments about each variable's type (i.e., \mathcal{A}_t , \mathcal{B}_t or \mathcal{O}_t), which are in effect constraints on the relative position of variables within the DAG (conditions (ii)–(iv) in Fact 1); and information about whether a variable is an immediate parent to the utility node (condition (v)):

Fact 1. Total orders that satisfy $\delta_{0:t}$

A total order \succ of random variables satisfies the partial order entailed by $\delta_{0:t}$ iff it satisfies the following 5 conditions:

- (i) If $\delta_{0:t} \models X \in \Pi_y$, then $X \succ Y$;
- (ii) If $\delta_{0:t} \models A \in \mathcal{A}_+$ then $X \not\succ A$ for any X, and $A \not\succ Y$ for any Y where $\delta_{0:t} \models Y \in \mathcal{B}_t$:
- (iii) If $\delta_{0:t} \models O \in \mathcal{O}_+$ then there is a variable X where $\delta_{0:t} \models X \in \mathcal{A}_+$ and $X \succ O$;
- (iv) If $\delta_{0:t} \models B \in \mathcal{B}_+ \land O \in \mathcal{O}_+$ then $B \succ O$; and

(v) for any X, Y, if $\delta_{0:t} \models X \in \Pi_R^+$ and $\delta_{0:t} \not\models Y \in \Pi_R^+$, then $Y \succ X$.

Note that thanks to dn_0 and the dialogue strategies, where Φ is \mathcal{B}_+ , \mathcal{O}_+ , \mathcal{A}_+ , Π_R^+ or Π_y , one can test whether $\delta_{0:t} \models X \in \Phi$ via \land -elimination.

The agent starts by choosing (at random) a total order ≻ that satisfies Fact 1. The agent then uses (24) to estimate the probabilities over direct parenthood relations (pa(X,Y) means X is a parent to Y) given the evidence and \succ :

$$Pr(pa(X,Y)|e_{0:t},\succ) = \begin{cases} \sum_{\Pi_y \in P_y: X \in \Pi_y} Pr(\Pi_y|e_{0:t}) & \text{if } X \succ Y \\ 0 & \text{if } Y \succ X \end{cases}$$
 (24)

Thanks to Fact 1 this makes any combination of non-zero probability parenthood relations a DAG with no \mathcal{B} -variable being a descendant to an \mathcal{A} variable, and all \mathcal{A} variables have no parents. But it does not guarantee that all nodes are connected to the utility node. We use an ILP step to impose this global constraint. The result is a valid structure Π (i.e., it satisfies $\delta_{0:t}$) that evidence so far also deems to be likely (we give an outline proof of its validity on page 25).

The **ILP Step** is formally defined as follows:

Decision variables: Where $X, Y \in \mathcal{C}_t \cup \mathcal{A}_t$, pa(X, Y) is a Boolean variable with value 1 if X is a parent of Y (i.e., $X \in \Pi_y$), and 0 otherwise

Objective Function: We want to find the most likely combination of valid parenthood relations: i.e., we want to solve (25):

$$\max \sum_{X,Y \in \mathcal{C}_t \cup \mathcal{A}_t} Pr(pa(X,Y)) * pa(X,Y) + (1 - Pr(pa(X,Y))) * (1 - pa(X,Y))$$
 (25)

Constraints: C1 ensures three things: (i) variables cannot have a parent which is not present in at least one of the reasonable parent sets (see (24)); (ii) parent sets obey all expert declarations of the form $Z \in \Pi_{z'}$ (by equation (23)); and (iii) parents obey the currently proposed temporal ordering \succ (by equation (24)) thereby guaranteeing that the final graph is acyclic with no \mathcal{B} variable being a descendant of \mathcal{A} and all \mathcal{A} variables are orphans. Constraints C2 and C3 ensure that the final graph satisfies the necessary global conditions for being a DN, namely that every variable is in some way connected to the utility node, and that every outcome variable has at least one action ancestor. C4 ensures that the decision variables take on binary values (0 or 1).

C1: $\forall X, Y.(Pr(pa(X,Y)) = 0 \implies pa(X,Y) = 0)$ Disallow impossible parents $\forall X \notin \Pi_R^t \cdot (\sum_y pa(X,Y) \ge 1)$ $\forall O \in \mathcal{O} \cdot (\sum_{y \in \mathcal{O} \cup \mathcal{A}} pa(Y,O) \ge 1)$ $\forall X, Y.pa(X,Y) \in \{0,1\}$ C2:One child minimum C3:No orphaned outcomes Binary Restriction

Linear Relaxation of the ILP Step. Finding the solution to a Linear Program where all decision variables must be integers is an NP-hard problem. So we approximate the solution via *Linear Relaxation*: we replace C4 with (26), so that the values of decision variables can be real numbers:

$$\forall X, Y.pa(X,Y) \in [0,1]$$

$$20 \tag{26}$$

We then round the resulting values to integers. This reduces the time needed to compute a solution, but one must be careful in the rounding procedure: simply rounding each $pa^*(X,Y)$ to its nearest integer may create an inconsistent DN. Rounding in two phases avoids this. In the first phase, we produce a set of rounded decision variables pa'(X,Y) that satisfy C1 and C2:

$$pa'(X,Y) = \begin{cases} 1 & \text{if } pa^*(X,Y) \geq 0.5 \lor (X \notin \Pi_R^t \land \forall Z \neq Y.(pa^*(X,Y) \geq pa^*(X,Z))) \\ 0 & \text{otherwise} \end{cases}$$

In the second phase, we ensure that constraint C3 is also satisfied, producing the final integer values for all the variables pa(X,Y):

$$\begin{aligned} pa(X,Y) &= 1 \text{ iff } pa'(X,Y) = 1 & \lor \\ (Y &\in \mathcal{O} \land \\ & \forall Z((Z \neq X \land Z \in \mathcal{O} \cup \mathcal{A}) \rightarrow \\ (pa'(Z,Y) &= 0 \land Pr(pa(X,Y)) > Pr(pa(Z,Y))))) \end{aligned}$$

Because the random variable P_v does not contain *all* possible parent sets to V (so as to keep learning tractable), it is possible that there is in fact no solution to the ILP step for any valid temporal order \succ . This is a context in which the agent performs a Structural Update, which in turn changes the variables that satisfy the antecedent to constraint C1.

4.2.2. Structural Update

To make learning tractable, P_v includes only reasonable parent sets to V—a strict subset of those that are possible. But as evidence changes, so does what is reasonably likely. Therefore, the agent occasionally performs a *Structural Update*: the full batch of evidence $e_{0:t}$ is used to review and potentially revise the possible values of P_v and its probability distribution.

Figure 3 depicts the structure of the algorithm that corresponds roughly to the nodes "estimate Π_t " and "estimate θ_t " in Figure 2. In particular, it depicts the four contexts in which Structural Update is performed in our experiments in Section 5:

- 1. There have been 100 domain trials since last structural update
- 2. The agent has just discovered an unforeseen factor
- 3. The latest dialogue evidence σ_t yields a 0 probability to all parent sets in P_v via equation (23)
- 4. The algorithm for enforcing global constraints on Π fails to produce any valid structure

The **input** to Structural Update consists of a (chance) variable V, the batch of evidence $e_{0:t}$, the prior distribution $Pr_{t-n}(\Pi_v)$ over all possible parent sets to V (where t-n is the time of the previous structural update), and the current partial description $\delta_{0:t}$ of the DN. Its **output** is a (perhaps new) set of values for P_v (i.e., those parent sets to V that are currently deemed "reasonable"), the posterior probability $Pr_t(\Pi_v)$ for each possible parent set Π_v (whether it is in P_v or not) and the posterior probability distribution $Pr_t(P_v)$ (which is computed by normalising $Pr_t(\Pi_v)$ for $\Pi_v \in P_v$). Informally,

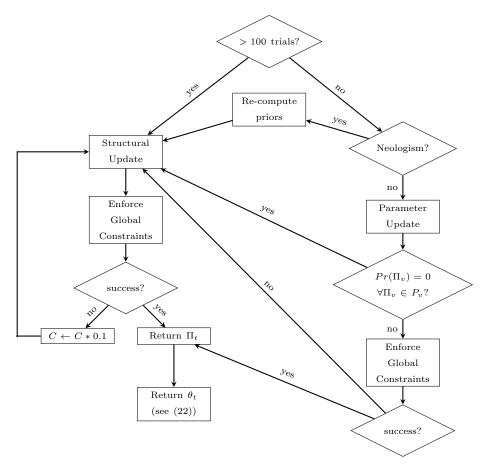


Figure 3: Algorithm for Estimating $\langle \Pi_t, \theta_t \rangle$.

the Structural Update **algorithm** dynamically constructs a parent lattice L_v^t , with each node corresponding to a parent set to V and arcs corresponding to the superset relation. When expanding the lattice by adding a new node, we estimate its posterior probability given the evidence $e_{0:t}$. We stop expanding the lattice when all its leaves have a posterior probability that is below a certain threshold. Thus Minimality is implicit: Structural Update assumes that supersets of sufficiently improbable parent sets will not improve their probability.

More formally, the root(s) of the parent lattice L_v^t are the minimal valid parent sets: i.e., they contain X if $\delta_{0:t} \models X \in \Pi_v$, and if $\delta_{0:t} \models V \in \mathcal{O}_t$ then L_v^t 's root(s) each contain at least one variable from $\mathcal{A}_t \cup \mathcal{O}_t$. For each root Π_v , one computes its posterior probability, given the evidence $e_{0:t}$. If a node was alive in the previous Structural Update, then this posterior probability is already calculated via the sequence of Parameter Updates that were performed between the previous and current structural updates. It is therefore immediately available. If the node was marked as asleep in the previous lattice, however, then the node's posterior probability will not have been updated with the intervening domain trials—it is computed now by performing a sequence of Parameter Updates (see

equation (21)).

The highest of these posterior probabilities is taken to be the Best-Posterior. One then marks the nodes as alive if its posterior probability is $\geq C*Best-Posterior$ for some constant C (in the experiments in Section 5, C=0.001), and asleep otherwise. You then create a new node in the lattice by adding a variable to an alive leaf node. You compute its posterior probability, and if it exceeds the current Best-Posterior, then you update Best-Posterior and re-classify which nodes in the lattice are alive and which are asleep accordingly. You stop when all the leaves in the current lattice are asleep.

The set P_v is defined to be all the alive parent sets in L_v^t and their ancestors in L_v^t . Having (re-)estimated P_v for each variable V, you test whether each variable $X \notin \Pi_R^t$ is in at least one reasonable parent set: if it is not then there is clearly no valid Π that one can generate from them (since X will not connect to the utility node). This test failing is the context "Unknown Effects" in the agent's dialogue strategy, which we described in Section 3.2.2: i.e., instead of expanding the search space by reducing the asleep threshold C, the agent seeks further evidence by asking the expert (18): what is an effect of X? The evidence $e_{0:t}$ and partial description $\delta_{0:t}$ get updated with the expert's answer, and DN update begins again—backtracking to estimating the random variables via (19a–c) may be necessary because the expert's answer may be a neologism. The experiments in Section 5 show that this querying strategy keeps a tight reign on the size of the search space, while still ensuring that many of the dependencies in Π_t are inferred defeasibly from the domain trials, rather than monotonically from explicit expert declarations about parenthood.

The posterior probabilities that are computed during the lattice expansion are assumed to be the posterior probabilities $Pr_t(\Pi_v|L_v^t)$ —the probability that Π_v is Π_v^+ , given that $\Pi_v^+ \in L_v^t$ (if $\Pi_v \notin L_v^t$, then $Pr_t(\Pi_v|L_v^t) = 0$). The posterior probabilities on every possible parent structure is then defined by (27) to (29), where $Pr(L_v^t)$ is the probability that the true parent set Π_v^+ is in L_v^t , and p is the probability mass we wish to assign to the unexplored space:

$$Pr_t(\Pi_v) = Pr_t(L_v^t)Pr_t(\Pi_v|L_v^t) + Pr_t(\neg L_v^t)Pr_t(\Pi_v|\neg L_v^t)$$
(27)

$$Pr_t(\Pi_v|\neg L_v^t) = \begin{cases} 0 & \text{if } \Pi_v \in L_v^t \\ \frac{1}{2^{|\mathcal{C}_t \cup \mathcal{A}_t \setminus V|} - |L_v^t|} & \text{if } \Pi_v \notin L_v^t \end{cases}$$
 (28)

$$Pr_t(\neg L_v) = p Pr_t(L_v) = 1 - Pr_t(\neg L_v)$$
(29)

Thus Structural Update returns for each chance variable V the values P_v , $Pr_t(P_v)$ and $Pr_t(\Pi_v)$ for all possible parent sets Π_v . But the algorithm does not require the agent

⁵This contrasts with Buntine's Structural Update, in which P_v is the alive parent sets only. We include asleep subsets of alive parent sets because our version of Parameter Update abstracts over all possible total temporal orders of the variables. To illustrate the issue, suppose that there is a very strong statistical correlation in the domain trials between X and Y—so strong that the only alive parent set for P_x is $\{Y\}$, and the only alive parent set for P_y is $\{X\}$. If $P_x = \{\{Y\}\}$ and $P_y = \{\{X\}\}$, then these generate no consistent structure. But if P_v includes all subsets of alive parent sets, then we avoid this problem: \emptyset will be a possible value of P_x and of P_y , even though Parameter Update deems \emptyset to be an unlikely parent set for X and for Y.

to enumerate or reason about parent sets within the unexplored space. These become relevant only if in the next Structural Update, the agent expands the parent lattice to include a parent set that was absent from the prior lattice.

The process for estimating Π_t concludes, as always, by applying the algorithm for enforcing global constraints that we described earlier (see Figure 3). If this succeeds, it returns Π_t , and from this one computes θ_t via (22). If not, then the agent reduces the asleep threshold (by multiplying C by 0.1) and attempts Structural Update again.

4.2.3. Initial Distribution

We implicitly encode the defeasible principle of Minimality in our initial distribution over parent sets by making larger parent sets less likely than smaller ones. The agent starts with a small probability ρ that Y is a parent to X (provided this is consistent with the variables' types as stipulated in δ_0). The probability of a parent set Π_x is then the product of each individual parent's presence or absence:

$$Pr_0(pa(Y,X)) \propto \begin{cases} 0 & \text{if } \delta_0 \models \neg pa(Y,X) \\ \rho & \text{otherwise} \end{cases}$$
 (30)

$$Pr_0(pa(Y,X)) \propto \begin{cases} 0 & \text{if } \delta_0 \models \neg pa(Y,X) \\ \rho & \text{otherwise} \end{cases}$$

$$Pr(\Pi_x) = \prod_{Y \in \Pi_X} Pr(pa(Y,X)) \prod_{Z \notin \Pi_X} (1 - Pr(pa(Z,X)))$$
(31)

For each $X \in \mathcal{C}_0$, the agent uses (30), (31) and Structural Update to dynamically construct the set P_x^0 of reasonable parent sets of X and its initial probability distribution.

4.2.4. Updating with Evidence containing a Neologism

Suppose the latest evidence e_t is an expert utterance σ_t that features a neologism Z. Then by (19a-c), Z is a random variable in dn_t but it was not a part of dn_{t-1} . This means that the prior distributions over the possible parent sets no longer cover all possible parent sets! These must be updated to take account of the extra possibilities afforded by the addition of Z, and to ensure that the definitions we have given so far for calculating posterior distributions from evidence (e.g., equation (21)) remain welldefined. In addition, the agent cannot observe Z's past values in $Sample_{0:t}$ —i.e., it cannot observe $n_{z=i|j}$, nor $n_{v=i|j}$ when $Z \in \Pi_v$. However, we need these counts for Parameter Update (see (21)) and estimating θ_t (see (22)). The Dirichlet α -parameters in (21) and (22) that involve Z are also not defined. We now describe how the agent revises all these parameters, so as to ensure that (21) and (22) remain well defined and continue to support probabilistic reasoning over the newly expanded hypothesis space.

We start by defining how the agent computes the probability distribution over the expanded set of possible parent sets. The agent starts this process by performing a Structural Update on the batch of evidence $e_{0:t-1}$ that preceded the latest evidence e_t with the neologism Z. This ensures that the (small) probability mass p that is assigned to the set of "unreasonable" parent sets takes into account all evidence to date. This yields a revised (prior) probability distribution $Pr(\Pi_x|e_{0:t-1})$ for every possible parent set Π_x of $X \in \mathcal{C}_t$ that does not include the new variable Z. Equation (32) then assigns probabilities to all possible parent sets to $X \neq Z$, including parent sets containing Z:

$$Pr(\Pi_{x}|e_{0:t}) \propto \begin{cases} 0 & \text{if } \delta_{0:t} \models \neg \Pi_{x} \\ (1-\rho)Pr(\Pi_{x}|e_{0:t-1}) & \text{if } Z \notin \Pi_{x} \\ \rho Pr(\Pi'_{x}|e_{0:t-1}) & \text{if } \Pi_{x} = \Pi'_{x} \cup \{Z\} \end{cases}$$
(32)

This update is Conservative, because it preserves the relative likelihood among the parent sets that do not include Z—in particular, unreasonable parent sets remain unreasonable. It is also Minimal because it re-assigns only a small proportion ρ of the probability mass of a parent set that does not include Z to the parent set formed by adding Z to it—in particular, adding Z to an unreasonable parent set is unreasonable. As usual, the agent needn't actively enumerate each possible parent set nor compute its probability; rather (32) is used to dynamically construct the lattice L_x^t , from which the agent identifies the reasonable parent sets P_x^t for X and P_x^t 's probability distribution, where the possibilities now include the additional variable Z.

If Z is a chance variable, then equation (33) in combination with (31) defines $Pr(\Pi_z|e_{0:t})$ (if Z is an action variable, then $\Pi_Z = \emptyset$):

$$Pr(pa(Y,Z)|e_{0:t}) = \begin{cases} 1 & \text{if } \delta_{0:t} \models pa(Y,Z) \\ 0 & \text{if } \delta_{0:t} \models \neg pa(Y,Z) \\ \rho & \text{otherwise} \end{cases}$$
(33)

This is Minimal because any variable being a parent to Z is assigned a low default probability ρ ; so the dynamic lattice construction L_z^t for identifying P_z^t and its probability distribution will restrict search considerably. With P_v and its distribution for all $V \in \mathcal{C}_t$ in place, one applies the algorithm for enforcing global constraints to yield Π_t .

We now return to the issue of the counts $n_{v=i|j}$, and the Dirichlet α -parameters. Our model avoids the computational complexity of marginalising over Z's past values by throwing away the past domain trials $Sample_{0:t-1}$ and their counts $n_{v=i|j}$, but retaining the relative likelihoods they gave rise to by packing these into updated values for the Dirichlet α -parameters, as shown in (34) (where K is a constant):

Where
$$Z \in dn_t$$
 but $Z \notin dn_{t-1}$
 $n_{v=i|j} = 0$ for all v , Π_v

$$\alpha_{v=i|j} = \begin{cases} K * Pr_{t-1}(V = i, \Pi_v = j) & \text{if } V \neq Z \text{ and } Z \notin \Pi_v \\ K * 0.5 & \text{if } V = Z \text{ or } Z \in \Pi_v \end{cases}$$
(34)

With the α -parameters and the counts $n_{v=i|j}$ now defined over the expanded set of atomic states, the agent can compute θ_t , given Π_t and $e_{0:t}$, using (22). Furthermore, it ensures that the equations (21) and (22) remain well-defined, should the DN get updated by subsequent observed evidence.

Equation (34) makes the agent Conservative because when it observes subsequent evidence, the revised α -parameters ensure that the likelihoods inferred from the past domain trials bias the estimated likelihoods of Π and of θ that are based on that subsequent evidence (via equations (21) and (22) respectively). Indeed, the larger K is, the more Conservative you are: i.e., the more the probability distribution over dependencies and CPTs before your set of random variables changed influences reasoning about dependencies and CPTs after the set of random variables changes.

4.2.5. DN Update is Valid

We have now defined how to use evidence to update each of the DN's components. It meets the desiderata from Section 2. In particular:

Fact 2. DN update complies with Consistency and Satisfaction: i.e., dn_t is a consistent DN, and $dn_t \models \delta_{0:t}$.

Outline proof. $\delta_{0:t}$ contains three kinds of conjuncts: (i) $X \in \mathcal{B}_+, X \in \mathcal{A}_+, X \in \mathcal{O}_+,$ $X \in \Pi_R^+$; (ii) formulae of the form (5) and (14); and (iii) $X \in \Pi_y$. Equations (19) together with the algorithm for enforcing global constraints guarantees that Π_t satisfies all the conjuncts of type (i), including their consequences on each variable's relative position in the causal structure Π^+ . This is because Π_t satisfies a total order \succ (see equation (24) and C1 in the ILP step), where by Fact $1 \succ$ satisfies the partial order imposed by $\delta_{0:t}$ —e.g., where $\delta_{0:t} \models X \in \mathcal{B}_+$, $\Pi_x^t \subset \mathcal{B}_t$. \mathcal{R}_t is a consistent preference function because: (a) equation (20) makes \mathcal{R}_t well-defined on its domain Π_R^t ; and (b) since its range is \mathbb{R} it defines an asymmetric and transitive relation. Further, since all conjuncts in $\delta_{0:t}$ of form (ii) are constraints on \mathcal{R}_+ , it follows immediately that the constraint solver returns a reward function \mathcal{R}_t that satisfies these. Equation (23) plus C1 in the ILP step guarantees that Π_t entails conjuncts of type (iii). Furthermore, the algorithm for enforcing global constraints guarantees Π_t is a DAG with all nodes connected to the utility node: it is a DAG because Π_t satisfies a consistent total order \succ ; and all nodes are connected to the utility node because of constraints C2 and C3 in the ILP Step. Finally, equation (22) guarantees that θ_t is a consistent probability distribution. Thus dn_t is consistent, and $dn_t \models \delta_{0:t}$.

5. Experiments

Our experiments show that our agent can learn the optimal policy for a given task, even when initially unaware of factors that are critical to its success. Further, we show the usefulness of the defeasible principles our agent adheres to (such as Minimality and Conservativity) by comparing our agent with several variations which abandon those principles.

We evaluate our agent against two different scenarios. In the first scenario, we have the agent learn in 10 randomly generated decision problems. By varying the true decision problem the agent must learn, these experiments evaluate whether the model's ability to converge on optimal behaviour is robust to the type of domain it has to learn about. In the second scenario, we have the agent learn the hand-crafted "Barley" example from the introduction (see Figure 1). This experiment allows us to show how the model performs on a concrete example of our novel task which is explicitly designed to require the exploitation of unforeseen possibilities to be successful.

Our primary metric is **policy error** (PE). Equation (35) defines PE_t at time t as the weighted average of the expected difference in reward between the true optimal policy π_+^* and the agent's perceived optimal policy π_t^* (observations b are projected onto the agent's current conceptualisation of the domain to ensure π_t^* is defined):

$$PE_{t} = \sum_{b \in v(\mathcal{B}_{+})} Pr_{+}(b) \left[EU_{+}(\pi_{+}^{*}(b)|b) - EU_{+}(\pi_{t}^{*}(b \upharpoonright \mathcal{B}_{t})|b) \right]$$
(35)

We also measure the agent's cumulative reward over the learning period:

$$R_t = \sum_{\tau_i \in Sample_{0:t}} r_i \tag{36}$$

Additionally, we measure the **runtime** the agent takes to reason with the 3000 training samples. We do not measure the agent's performance in terms of, for instance, the

minimum edit distance between dn_t and dn_+ , because the mapping from an optimal policy to its DN is one to many—for us, constructing a DN is merely the means for achieving optimal behaviour, and structural differences between dn_+ and dn_t may be benign in this respect.

All experiments are run over 3000 training examples. However, we assume that each time the expert sends a message, the agent misses out on an opportunity to interact with the domain. This means that the more the expert intervenes, the fewer domain trials the agent gets rewards from.

5.1. Random DNs: Setup

In these experiments, we tested each agent variation against 10 different decision problems. Each true decision network dn_+ consists of 21 Boolean variables: 7 action variables; 7 "before" variables; and 7 "outcome" variables (i.e., $|\mathcal{A}_+| = |\mathcal{B}_+| = |\mathcal{O}_+| = 7$). This results in non-trivial decision problems with over 2 million atomic states. We generate both the structure and parameters for each dn_+ randomly using an adapted version of the Bayesian network generation algorithm in (Ide et al., 2004).⁶ We also create a reward domain for each dn_+ by choosing 5 chance variables at random, then generate a reward function which yields rewards in the range 0–50.

We then assign an agent an initial conceptualisation dn_0 of the decision problem; we test how unawareness affects learning by varying dn_0 . To evaluate an agent's performance in learning, we run 100 simulations, each consisting of the agent observing and learning from 3000 pieces of evidence (a mix of domain trials and expert messages). Running 100 simulations allows us to smooth over the fact that learning may be affected by the non-deterministic outcomes of the actions it performs. Our performance metrics then take the average over the 100 simulations.

We evaluate and compare agents that vary along the following dimensions:

Initial Awareness: We compare an agent which starts at dn_0 with full knowledge of the (true) decision problem's set of random variables against one which starts with just one action and outcome variable:

	Vocab Size at dn_0
Min-Vocab	2
Full-Vocab	21

Expert Tolerance: We vary the expert's tolerance to the agent's suboptimal behaviour when she deliberates over whether to offer unsolicited advice: this is achieved by varying β and γ in (8):

	β	γ
Low-Tolerance	0.001	1
Default-Tolerance	0.9	50

⁶Our adaptation adds additional rules to ensure the algorithm generates only valid DNs. For example, the structure must conform to the rules of each variable's type.

Minimality: An agent can be minimal, slightly minimal, or maximal. They vary on the following three dimensions: (i) adopting a greedy search for a minimal domain of \mathcal{R} (see equation (19d)) vs. defining the domain of \mathcal{R}_t to be all of \mathcal{C}_t (i.e., the maximum possible domain, creating many more arrows in the graphical component of the DN); (ii) the prior probability ρ that is assigned to the parenthood relations (see equations (30) and (32)) can make parenthood (defeasibly) unlikely ($\rho = 0.1$) or equally likely as unlikely ($\rho = 0.5$); and (c) the asleep threshold C that prunes the search space to reasonable parent sets can result in pruning (C = 0.001) or not (C = 0).

	Minimal Π_R^t ?	ho	C
Minimal	yes	0.1	0.001
Slightly-minimal	yes	0.5	0.001
Maximal	no	0.5	0

Conservativity: An agent can be *conservative* or *non-conservative*. They vary on how they calculate the probability distribution over the expanded set of possible options when a new variable gets added to the DN.

All agents set $n_{v=i|j}$ to 0 when it discovers a new variable. But the conservative and non-conservative agents vary on the how they re-set the probability distribution over dependencies and the Dirichlet α -parameters:

	Estimating $Pr(\Pi_v^t)$ and $Pr(\theta_t)$ when adding a new variable
Conservative	Use (32), (33), and (34) with $K = 20$
Non-Con	Use (30) and $\alpha_{n=i j} = \alpha_0 = 0.5$

Domain Strategy: The agent's domain-level strategy is to execute an optimal action vs. some other action in a ratio of $(1 - \epsilon)$: ϵ , and initially $\epsilon = 0.3$ (see Section 3.1). We vary how this strategy changes during learning as follows: (i) ϵ does not change (Static); or (ii) ϵ gets progressively reduced (Decay).

	Initial ϵ	Decay factor (per time step)
Static Decay		1.0 0.999

Unless stated otherwise, the agents we evaluate are *Min-Vocab*, *Minimal*, *Conservative* and *Static*, and the expert has *Default Tolerance*.

The **baseline** agent for our task adopts a standard ϵ -greedy strategy (with ϵ set to 0.3) to learn an optimal policy from dn_0 : it ignores observed evidence that makes dn_0 invalid. So whatever the observed evidence, the baseline agent's random variables $C_0 \cup A_0$ and dependencies Π_0 do not change. All our other agents learn valid DNs (see Fact 2). We compare the performance of the following nine agents on our task: the Baseline agent, Default (i.e., min-vocab, minimal, conservative, static, default tolerance), Low-Tolerance

	PE_t	p-value			PE_t	p-value
Default	0.34	-	•	Default	0.97	
Baseline	1.85	0.00		Baseline	3.05	0.00
Low-Tol	0.51	0.01		Low-Tol	1.10	0.20
Slightly-Min	0.49	0.01		Slightly Min	1.08	0.24
Decay	0.47	0.03		Decay	0.86	0.20
Full-Vocab	0.08	0.00		Full-Vocab	0.03	0.00
Non-Con	0.78	0.00		Non-Con	1.44	0.00
Maximal	1.21	0.00		Maximal	1.30	0.00
(a) dn_+^{best}		•	(b)	dn_{+}^{worst}		

Table 1: Policy error (PE) for agents at t=3000. Average of 100 runs reported.

(expert has low tolerance); Slightly-Min (agent is slightly minimal), Maximal (agent is maximal), Non-Con (agent is non-conservative), Decay (agent explores progressively less during learning), and Full-Vocab (the agent starts out aware of dn_+ 's vocabulary, but not its dependencies or reward function).

In some of the 100 simulations, an agent that lacks the computational advantages of Minimality—a principle we argued would enhance tractability—crashes before it is observed and learned from 3000 pieces of evidence.⁷ In that case, the agent's DN just before the crash occurred is used to compute the performance metric for that simulation.

5.2. Random DNs: Results

Across a variety of randomly generated DNs, each with different dependencies, CPTs, and reward domains, our agent converges to a near-optimal policy with a relatively small amount of data (3000 pieces of evidence). Tables 1 and 2 show the performance metrics for each agent in terms of final policy error (PE) and accumulated reward, respectively. Figure 4 also shows the rewards gained from the domain trials during learning (averaged across blocks of 150 domain trials and 100 simulations). We report numbers for the DN dn_+^{best} where the agent was most successful at achieving a low PE, and for dn_+^{worst} where the agent was least successful, so as to give a measure of the lower and upper bounds of the agent's performance. (Full details on dn_+^{best} and dn_+^{worst} are in the Appendix.) The p-values compare each agent variation to the Default agent, and measure the probability of seeing a result of this magnitude (or greater) given the null hypothesis that the true average of both learning agents is identical.

From these results we see that the Default agent significantly outperforms the Baseline agent, both in terms of final PE and total accumulated reward. Its PE is also significantly lower than the Maximal and Non-Conservative agents.

We now discuss the various factors that affect learning.

⁷We define a crash as an "Out of Memory" error on a Java Virtual Machine with a 4GB Memory Allocation Pool, or a simulation taking longer than its maximum allocated time of 12 hours.

	Reward	p-value
Default	70043	-
Baseline	69541	0.01
Low-Tol	48726	0.00
Slightly-Min	62145	0.00
Decay	70940	0.00
Full-Vocab	72701	0.00
Non-Con	70079	0.85
Maximal	36760	0.00
(;	a) dn_{+}^{best}	

Table 2: Accumulated reward for agents at t=3000. Average of 100 runs reported

Initial awareness. As expected, starting with complete knowledge of the vocabulary is an advantage. Tables 1 and 2 show that the Full-Vocab agent achieves the lowest PE and highest cumulative rewards across all simulations. This is unsurprising: the agent never has to "throw away" counts from the trials, and it deliberates over all actions from the start. However, the Full-Vocab agent takes significantly longer to run than the Default agent, often anywhere from 3 to 8 times as long (see Table 3). There are two reasons for this. First, inference over a DN with more variables is usually more expensivethe Full-Vocab agent infers a 21 variable DN from the start, while the Default agent initially updates a DN with a much smaller vocabulary. Second, starting with a large initial vocabulary and a fairly weak prior makes it more difficult for Structural Update to prune the search space of parent sets. There is often an initial explosion in the number of parent sets that the Full-Vocab agent considers reasonable, because it has not yet gathered enough information from the domain trials to discriminate unreasonable parent sets from reasonable ones. In contrast, when the agent starts out aware of fewer variables, each new variable is incrementally introduced into a context where the agent has observed enough evidence to construct a more aggressive prior via equation (32).

		Time (s)
$\overline{\mathbf{D}}$	efault	efault 1266
Ва	seline	aseline 15
Low	-Tol	7-Tol 2162
Sligh	tly-Min	ntly-Min 4979
Decay		1198
Full-Vo	cab	cab 9465
Non-Cor	n	n 1134
	Default Baseline Low-Tol Slightly-Min Decay Full-Vocab Non-Con	Baseline 15 Low-Tol 2162 Slightly-Min 4979 Decay 1198 Full-Vocab 9465

Table 3: Run time in seconds for agents at t=3000. Average over 100 runs reported. ("Maximal" agent omitted, as none of the agent's runs managed to finish)

Expert tolerance. In comparison to the Default expert, the Low-Tolerance expert results in a marginally worse PE, and significantly worse total accumulated reward. This is partly because each time the agent receives a message from the expert, it misses both an opportunity to receive a reward by taking an action in the domain, and also an opportunity to improve its estimate of Π and θ by observing an additional domain trial. In our experiments, the Low Tolerance expert can end up sending as many as 15 times more messages than the Default. One advantage of the Low-Tolerance expert, however, is that the agent makes early improvements to its policy. This is illustrated by Figure 5, which shows the PEs over time.

Minimality. The results show that at least some level of minimality is necessary. In the vast majority of simulations (70%–100%, depending on dn_+), the Maximal agent runs out of either time or memory, and so fails to collect rewards or improve its policy via the 3000 pieces of evidence. By comparison, the Default agent does not crash in any of the simulations.

Without a mechanism to prune the set of parent sets or the domain of the reward function, the agent must consider an exponential number of options. For more than a small number of variables, this quickly becomes intractable. The differences are less dramatic for the Slightly-Minimal agent: fewer simulations crash, and those that do tend to crash come later in the simulation. However, its PE is still significantly worse than the Default agent's PE. Excluding crashed agents from the results yields no significant difference in PE, suggesting that even with infinite time and resources, there is not necessarily any harm to adopting Minimality. The largest difference is in the run times: the Slightly Minimal Agent takes around 4 times as long to run as the Default agent. Thus, Minimality makes the problem tractable without harming performance.

Conservativity. The Non-Conservative agent has a significantly worse PE than the Default agent. There appear to be two benefits to preserving information about which parent sets are unreasonable when a neologism is introduced. First, it directs the agent towards more accurate representations of the dependencies within the DN, and therefore hopefully allows it to make better decisions with equivalent samples. Second, having a more aggressive prior leads the (Conservative) Default agent to learn more vocabulary—on average 19 versus 17—as it is more likely to encounter a context where it asks the expert question (18) ("What is affected by X?"). This also explains why the Non-Conservative agent is marginally faster than the Default agent—it reasons with a smaller vocabulary.

Domain Strategy. The Decay agent gathered a significantly larger total reward than the Default agent, without a significant difference in their PEs. In other words, gradually reducing the extent to which the agent explores the space as it gathers more evidence helps it to converge faster on a decent policy. But we haven't demonstrated here whether the Decay agent would maintain this advantage as one increases the number of variables of dn_+ : it may be a disadvantage to gradually explore less if the agent is more radically unaware at the start of the learning process than we used in our experiments here. Additionally, while it was not within the scope of these experiments to explore the space of exploration versus exploitation strategies (we arbitrarily chose an initial split of 0.7:0.3), it is possible that superior initial parameters (and decay rates) exist for learning the optimal policy.

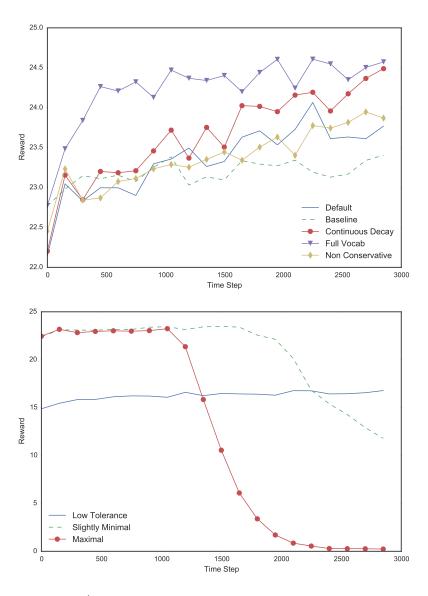


Figure 4: Rewards on dn_+^{best} . Results averaged across 100 simulations, and blocks of 150 time steps. Note the scales in the two graphs differ, to help depict the performance of all agent types—the dips in the second figure are due to the agents crashing before t=3000.

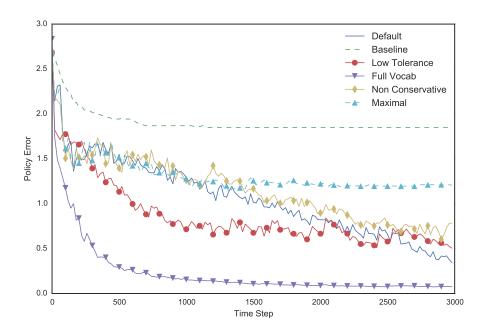


Figure 5: Policy error at each time step on dn_{\perp}^{best} (averaged across 100 simulations).

5.3. Barley DN: Setup

In these experiments, we test our **default** agent's performance on the hand-crafted barley example (see Figure 1) and compare it against the **baseline** agent (which ignores evidence that makes dn_0 invalid). The Barley DN is intended to reflect a somewhat realistic domain for our novel task, and has been explicitly designed so that learning the optimal policy requires knowledge of factors that the agent is initially unaware of.

As in the previous experiment, we run 100 simulations over 3000 pieces of evidence then average the results. The structure of the agent's initial decision network dn_0 and the true decision network dn_+ are shown in Figure 1. The CPTs for each node are in the Appendix.

5.4. Barley DN: Results

Figure 7 shows the policy error and average reward over 3000 pieces of evidence for the Barley DN. As expected, the baseline quickly stops improving as it reaches the bounds of what is achievable given its limited level of awareness. On the other hand, the default agent continues to improve its policy as it discovers unforeseen concepts and actions, and eventually converges upon a near-optimal policy.

In addition to learning a near-optimal policy, the agent also learns a fairly accurate and efficient representation of the underlying decision network for the task. Figure 6 shows an example of the agent's learned graph at t=3000. Notice that, in this run, the agent did not manage to discover two pieces of the true vocabulary—namely Weeds and Infestation. Both of these are outcome variables which are not in the reward domain. In future work, we aim to explore methods to allow the agent discover these variables

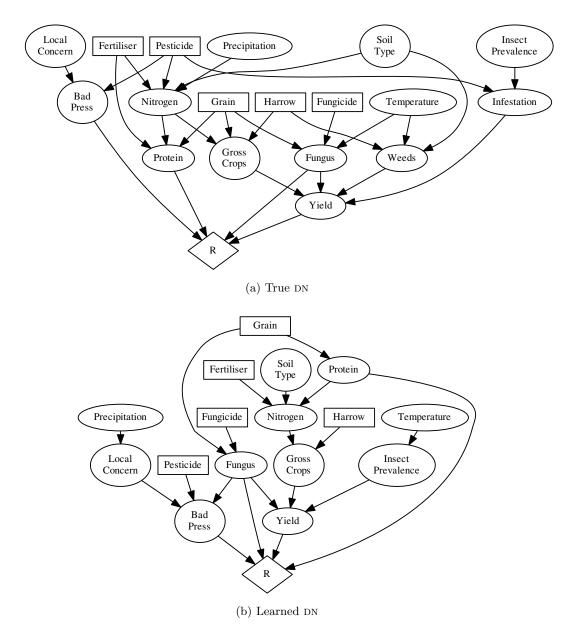


Figure 6: Difference in structure between the true DN and one learned by default agent at t=3000 for a (randomly chosen) simulation

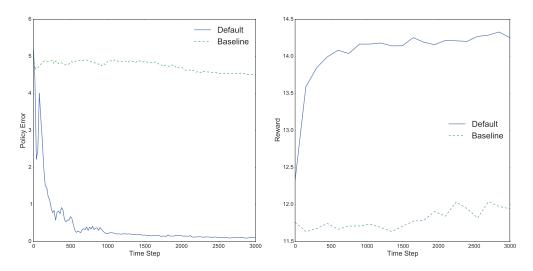


Figure 7: Reward and policy error for agents on the Barley task

which provide a more efficient factorization of the problem if discovered, but which are not necessarily vital for learning the optimal policy.

Despite missing these two variables, the agent attempts to make reasonable compensations for their absence. For example, in the absence of the *Infestation* variable, the agent assumes that *Insect-Prevalence* is directly connected to *Yield*. The few other incorrect edges are either due to reversals of causality (for instance, the model assumes *Protein* influences *Nitrogen* instead of vice versa) or by the model's bias towards simplicity in the absence of sufficient domain trials (for example, *Precipitation* has a relatively weak influence on *Nitrogen* compared to its many other parents, so the agent instead spuriously connects *Precipitation* to *Local-Concern*—a variable with no parents).

6. Related Work

The literature on learning optimal policies includes methods for dealing with numerous types of change, such as non-stationary rewards (Besbes et al., 2014), or noise in sensing and actuation (da Silva et al., 2006). Such methods have also been applied in situations involving hidden variables (e.g., with Partially Observed Markov Decision Processes (Leonetti et al., 2011)), or in cases where the learner must reason about the underlying probabilistic structure of the problem (e.g, with Factored Markov Decision Processes (Degris et al., 2006)). However, these methods require all possible states and actions to be known in advance of learning. That is, the agent starts out with complete knowledge of $\mathcal C$ and $\mathcal A$. In contrast, our agent learns these components from evidence.

There are a few notable exceptions in the reinforcement learning (RL) literature, which attempt to relax the standard assumption that all possible states and actions are known in advance of learning. Rong (2016) defines Markov Decision Processes with Unawareness (MDPUs), and also provides algorithms for learning optimal policies, even when the agent starts out unaware of actions and states that influence what is optimal. This work allows an agent to discover an unforeseen atomic state or unforeseen action via a random

exploration function that is a part of the MDPU. Our work contrasts with this in several ways. First, we support discovering unforeseen possibilities from evidence that stems from communication with an expert, not just via exploration. Secondly, and in light of this quite different kind of evidence, the agent is not simply discovering at most one new atomic state from the latest piece of evidence, but rather discovering an unforeseen concept, which instantly extends the set of atomic states considerably: e.g., an unforeseen concept corresponding to a Boolean random variable doubles the set of atomic states. Accordingly, and in contrast to MDPUs, the set of atomic states in our task is not defined by a single random variable whose set of possible values change as the agent becomes more aware. Rather, the domain states are conceptualised via a set of random variables in a dependency structure, and the agent must learn that dependency structure so that computing joint probability distributions in the domain remains tractable. Learning the concepts and their dependencies is challenging but ultimately worth it, because the probabilistic conditional independence they encapsulate makes learning and reasoning (whether exact or approximate) more tractable. This enables us to conduct experiments in Section 5 where the (true) decision problem consists of 2 million distinct atomic states, compared with the few thousand distinct atomic states in the discretised models that are used in Rong's experiments.

Selective perception methods, such as McCallum's (McCallum and Ballard, 1996) Utree algorithm, were designed for problems in which the true state space may technically be available, but is infeasibly large to reason with. The U-tree method initially treats all states as indistinguishable, then recursively splits states in two as the agent discovers significant differences in their expected reward. The agent thus learns an increasingly complex representation of the state space. The U-tree algorithm handles sequential planning while our model (currently) supports only single-stage decision problems; however, there are three main learning tasks that our model supports that the U-tree algorithm does not. First, our model accommodates situations where the agent is made aware of an entirely new possible action, while the U-Tree algorithm assumes that the set of actions remains fixed. Secondly, our model learns from (qualitative) evidence provided by a domain expert, as well as evidence from trial and error. And thirdly, our model learns unforeseen concepts and causal relations among them, while the U-tree algorithm does not reason about which dimensions define the state space, nor their causal dependencies. This third difference is related to the second: we ultimately want our model to enable an agent to learn about the domain and how to behave in it from a human expert, and it is quite natural for a human to justify why one state is different from another on the basis of the concepts on which optimal behaviour depends by saying, for instance, "sandy soil causes abundant barley yields". Our model supports learning from this type of evidence.

Deep reinforcement learning (DRL, Mnih et al. (2015)) also aims to learn a suitable abstraction of the data to enhance convergence towards optimal policies. DRL combines deep learning (i.e., a convolutional neural network or CNN (Bengio et al., 2013; Hinton et al., 2006)) with RL, and has proven extremely useful (e.g., Chouard (2016)). However, the major successes in DRL have not so far considered changing state-action spaces in the sense of the examples we have discussed. Instead, the focus has been on using a large number (typically, in the millions or tens of millions) of domain trials to re-describe a large state-action space in terms of abstracted versions that render policy learning more tractable. We believe that agents can learn more efficiently if they exploit evidence from both domain trials and messages conveyed by an expert. Unfortunately, expert

evidence about causal relations ("X causes Y") or preference ("If C is true, then doing X is better than doing Y") are inherently symbolic. Adjusting weights in a sub-symbolic representation like a CNN so that it satisfies certain qualitative properties is currently an unsolved problem. Another limitation is that the implicit models generated by DRL lack explainability—they are unable to elaborate on why a given action was recommended over another. Our approach complements methods like DRL by dynamically building an interpretable model of the decision problem. Such interpretable models make it easier for the agent to explain the reasons behind a given decision, and also allow the agent to evaluate its interpretation of expert messages against its current conceptualisation of the decision problem via the standard logical technique of model checking.

The agent must learn how to revise its model of the decision problem to incorporate newly discovered actions and concepts. Hence, part of our task is to learn the domain's causal structure. Several approaches exist for jointly learning causal dependencies and probability distributions (e.g., Bramley et al. (2015); Buntine (1991); Friedman (1998)) and exploiting causal structure to speed-up inference (Albrecht and Ramamoorthy, 2016). We extend this work to meet our objectives. First, models for learning dependencies all assume that the vocabulary of random variables does not change during learning, but in our task it changes when the agent becomes aware of an unforeseen action or concept. Secondly, optimal action depends on payoffs as well as beliefs, and so we must integrate learning dependencies with learning potential payoffs. Finally, Bramley et al. (2015) and Friedman (1998) use only domain trials as evidence, and although Buntine (1991) uses expert evidence, the messages are only about causal relations and they are all declared prior to learning. In contrast, we want to *interleave* dialogue and learning: this enables the expert to offer advice in a timely and contextually relevant manner; consequently, she can explain particular outcomes and correct mistakes.

The DNs used to model the decision problems in this paper resemble a simplified version of *influence diagrams* (Howard and Matheson, 2005). Influence diagrams allow one to express multi-stage decision problems by defining *information arcs* between chance nodes and decision nodes, to help an agent assess whether it is worth observing a given trial or not (known as a *value-of-information* calculation). There has been some work on inferring the probabilistic structure and utilities of other agents in influence diagrams (Bielza et al., 2010; Nielsen and Jensen, 2004; Suryadi and Gmytrasiewicz, 1999). However, as explained in Section 2, the added complexity of the influence diagram definition was unnecessary to tackle the single-stage, fully observable tasks addressed in this paper. Moreover, in our work we assume that the agent knows its own utility function.

There are several areas of research in which expert evidence is used to improve the performance of a learning agent. In *Transfer learning*— where knowledge of the optimal policy for a *source task* is used to improve performance on a related *target task*—the transfer of knowledge is sometimes captured via an expert "giving advice" to a learner (Torrey et al., 2006). Typically, both the source and target task must belong to the same domain. If they do not, an explicit mapping is usually provided from the states and actions in the source task to the states and actions in the target task. In contrast, our agent incrementally learns a single task, but is occasionally made aware of new concepts, which it must learn to accommodate into its existing knowledge. Another difference is that in transfer learning, the expert advice is all declared before the agent begins learning. In our model, the agent and expert engage in a dialogue throughout learning.

Knox and Stone (2009) use human expert evidence to inform RL, but they confine

this evidence to updating the likely outcomes of actions and their rewards; they do not support cases where expert evidence reveals information that the agent was not aware was possible. On the other hand, researchers have developed models for learning optimal policies via a combination of domain actions and natural language instructions, where the instructions may include neologisms, whose semantics the agent must learn (e.g., Liang (2005)). But this work assumes that the neologism denotes an already known concept within the agent's abstract planning language. On encountering the neologism "block", for example, the agent's task is to learn that it maps to the concept block that is already an explicit part of the agent's domain model. In contrast, we support learning optimal policies when the neoglogism denotes a concept that the agent is currently unaware of; e.g., to support learning when block is not a part of the agent's conceptualisation of the domain, and yet this (unforeseen) concept is critical to optimal behaviour.

Forbes et al. (2015) use embodied natural language instructions to support teaching a robot a new skill (a task known as learning by demonstration). Their model learns how to map a natural language neologism to what might be a novel combination of sensory values—in this sense, the meaning of the neologism may be an unforeseen concept—and this novel concept then informs the task of learning new motor controls. In effect, this work links a rapidly growing body of research on learning how to ground natural language neologisms in the embodied environment (known as the symbol grounding problem; (Siskind, 1996; Dobnik et al., 2012; Yu et al., 2016; Hristov et al., 2017) inter alia) with learning a new skill. But it does not support integrating the newly acquired skill into an existing hypothesis space consisting of other actions, consequences and rewards; so they do not support learning optimal policies when not only this new skill but other actions (and related concepts) are needed as well.

Our aim in this paper is to supply a complementary set of learning algorithms to this prior work. Like Forbes et al. (2015), our agent learns from both trial and error and from instruction. But we focus on a complementary task: instead of focussing, as they do, on learning how to execute a new skill, we focus on learning when it is optimal to execute it. Given that we wish to support larger planning tasks, we also broaden the goals to discovering and learning to exploit arbitrary unforeseen concepts, not just the learning of spatial concepts that Forbes et al. (2015) are limited to (see also Dobnik et al. (2012)). On the other hand, natural language ambiguity makes extracting the hidden message from natural language utterances a highly complex process (Grice, 1975; Bos et al., 2004; Zettlemoyer and Collins, 2007; Reddy et al., 2016). We bypass this complexity by using a formal language as the medium of conversation. This formal language can be broadly construed as the kind of language one uses to represent the output of natural language semantic parsing (Artzi and Zettlemoyer, 2013). Replacing the expert that we deploy in our experiments with an expert human would require linking the model we present in this paper with existing work on grounded natural language acquisition and understanding. This task is beyond the scope of the current paper, and forms a major focus of our future work.

Lakkaraju et al. (2017) tackle learning "unknown unknowns" via interaction with an oracle. This work addresses a specific type of unawareness: an agent assigns an incorrect label to a trial with high confidence. Crucially however, they assume that the correct label for this trial must be a label that the agent is already currently aware of. In other words, they exclude the option that the hypothesis space itself is incomplete, which is the type of unawareness that we are interested in.

There is a growing body of work on modelling unaware agents (e.g., Feinberg (2004); Halpern and Rêgo (2013); Board et al. (2011); Heifetz et al. (2013)). These theories predict when the agent's unawareness causes it to deviate from what is actually optimal. But with the exception of Rong (2016), this work does not address how an agent learns from evidence: i.e., how to use evidence to become aware of an unforeseen factor and to estimate how this gets incorporated into its updated conceptualisation of the decision problem. We fill this gap. This prior work interprets awareness with respect to models that include every possible option. A fully aware agent (e.g., the domain expert in our experiments) or an analyst can model an unaware agent this way, but it does not characterise the unaware agent's own subjective perspective (Li, 2008). In contrast, our learning task requires the unaware agent to use evidence to change its set of possible options: it dynamically constructs this set and its causal structure from evidence.

Finally, the barley example from Section 1 shows that becoming aware of an unforeseen possibility can prompt revisions, rather than refinements, to the reward function. But the models that support preference revision on discovering an unforeseen possibility assume a qualitative preference model (Hansson, 1995; Cadilhac et al., 2015). Following Cadilhac et al. (2015), we use evidence to dynamically construct an ever more specific partial description of preferences, and defeasible reasoning yields a complete preference model from the (partial) description—the agent defaults to indifference. But unlike Cadilhac et al. (2015), the learning agent uses evidence to estimate numeric payoffs rather than qualitative preference relations.

7. Conclusions and Future Work

This paper presents a method for discovering and learning to exploit unforeseen possibilities while attempting to converge on optimal behaviour in single-stage decision problems. The method supports learning by trial and error as well as learning by instruction via messages from a domain expert. We model the problem as one of dynamically building an *interpretable conceptualisation* of the domain: we argued that building an interpretable model offers a straightforward way to exploit both the *qualitative* evidence and the *quantitative* evidence. Specifically, our model is the first to support learning *all* components of a decision network, including its set of random variables.

Our algorithms build upon the common sense notions of Minimality and Conservativity. Minimality is encoded in the following ways: greedy search for a minimal vocabulary and domain for the reward function; defaulting to indifference; defeasibly low prior probabilities of dependencies; and the assumption that any superset of a sufficiently improbable parent set is at least as improbable. Conservativity is encoded by retaining the relative likelihoods among dependencies and CPTs learned so far when a new variable gets added to the DN.

Our experiments show that these principles significantly enhance the performance of the agent in its learning task, as measured by policy error and cumulative reward, when learning a DN of 21 random variables from 3000 pieces of evidence. The model was also shown to be robust to the topology of the true DN: in other words, it learns effective policies, whatever the true conceptualisation of the domain might be. An agent that starts with a minimal subset of the true hypothesis space is also competitive with one that starts with full knowledge of the set of random variables that define the atomic states.

There are several novel formal components to our model. For instance, we have combined symbolic constraint solving with Dirichlet distribution based estimates to guide learning, and the novel ILP step ensures that the estimated causal structure of the domain is valid—i.e., all nodes are connected to the utility node, and the dependencies satisfy all observed evidence to date. This component on its own presents an important extension to existing work on learning dependencies, since it supports learning a DN as opposed to just a Bayesian network.

These are just the first steps towards building an agent that can learn to conceptualise and exploit its domain via its own action and via messages from teachers or experts. We would like to extend this model to handle decision problems where some of the critical factors for solving the task have hidden values even after the agent becomes aware of them; this would involve replacing our Parameter Update algorithm with a modified version of Structural EM (Friedman, 1998). We would also like to extend it to handle dynamic environments: i.e., to learn (factored) MDPs, where the DNs learned here correspond to just one time slice. This would not only extend the scope of this work from single-stage decision making to sequential planning; it would also be essential if we were to replace defining the agent's and expert's dialogue strategy with learning an optimal dialogue strategy. We would also like to enrich dialogue in three ways. First, we would like the learner to utilise defeasible messages about likelihood. Second, we would like to relax the assumption that the expert's intended message is always true. And finally, we would like to support dialogues in natural language—the latter two enhancements are a necessary step for an agent to discover and exploit unforeseen possible options via interaction with a human teacher.

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Appendix

The model for learning dn_+ made use of a language \mathcal{L} for partially describing DNs. Its syntax and semantics is defined as follows:

Definition 2. The syntax of the language \mathcal{L}

- Terms of various sorts: X, Y... are random variable (RV) constants; Π_Y , $\mathcal{V}, \mathcal{A}, \mathcal{O}, \mathcal{B}$ are Sets of Random Variables (SRV) constants (denoting sets of random variables in the model); where X is an RV constant, x is a proposition term (we also say that $\neg x$ is a propositional term); s is an atomic state (AS) term (denoting an atomic state in the model). The language also includes RV variables and AS variables.
- If p is a propositional term or an AS term, then it is a well-formed formula (WFF) in \mathcal{L} .
- If p is a conjunction of positive and negative propositional terms and \mathcal{X} is an SRV constant, then $p \upharpoonright \mathcal{X}$ is a well-formed formula (WFF) in \mathcal{L} .
- If X is an RV term and Y is an SRV term, then $X \in \mathcal{Y}$ is a WFF in \mathcal{L} .
- If s is an AS term and n is a number, then $\mathcal{R}(s) = n$ is a WFF in \mathcal{L} .
- Where ϕ, ψ are WFFs, so are $\neg \phi, \phi \land \psi, \exists s\phi, \forall s\phi$ (where s is an AS variable variable) and $?\lambda V\phi$ (where V is an RV variable).

Each model dn for interpreting \mathcal{L} corresponds to a (unique) complete DN (see Definition 1). Definition 3 then evaluates the formulae of \mathcal{L} as partial descriptions of dn.

Definition 3. The semantics of \mathcal{L}

Let $dn = \langle \mathcal{C}_{dn}, \mathcal{A}_{dn}, \Pi_{dn}, \theta_{dn}, \mathcal{R}_{dn} \rangle$ be a DN and g a variable assignment function.

- For an RV constant X, $[\![X]\!]^{\langle dn,g\rangle} = X$; similarly for SRV constants.⁸ This ensures that $[\![A]\!]^{dn,g}$, $[\![B]\!]^{dn,g}$ and $[\![\mathcal{O}]\!]^{dn,g}$ denote sets of variables in the appropriate position in the causal structure of dn.
- For an AS variable s, $[s]^{\langle dn,g\rangle} = g(s)$ where $g(s) \in 2^{\mathcal{C}_{dn} \times \mathcal{A}_{dn}}$. For an RV variable V, $[V]^{\langle dn,g\rangle} = g(V) \in \mathcal{C}_{dn} \cup \mathcal{A}_{dn}$.
- For an RV term a and an SRV term b, $[a \in b]^{\langle dn,g \rangle} = 1$ iff $[a]^{\langle dn,g \rangle} \in [b]^{\langle dn,g \rangle}$.
- Where p is a propositional term, $\llbracket p \rrbracket^{\langle dn, g \rangle} = p$.
- For an AS term s and number n, $[\mathcal{R}(s) = n]^{\langle dn, g \rangle} = 1$ iff $\mathcal{R}_{dn}([s]) = n$.

⁸If $X \notin \mathcal{C}_{dn} \cup \mathcal{A}_{dn}$, then $\llbracket X \rrbracket^{\langle dn,g \rangle}$ is undefined and Definition 3 ensures that any formula ϕ featuring X is such that $dn \not\models \phi$; similarly for propositional terms p featuring a value of a variable that is not a part of dn.

- Where p is a conjunction of positive and negative propositional terms and \mathcal{X} is a set constant, $\llbracket p \upharpoonright \mathcal{X} \rrbracket^{\langle dn,g \rangle} = \llbracket p \rrbracket^{\langle dn,g \rangle} \upharpoonright \llbracket \mathcal{X} \rrbracket^{\langle dn,g \rangle}$ (i.e., the projection of the denotation of p onto the set of variables denoted by \mathcal{X}).
- For formulae ϕ , ψ : $\llbracket \phi \wedge \psi \rrbracket^{\langle dn,g \rangle} = 1$ iff $\llbracket \phi \rrbracket^{\langle dn,g \rangle} = 1$ and $\llbracket \phi \rrbracket^{\langle dn,g \rangle} = 1$; $\llbracket \neg \phi \rrbracket^{\langle dn,g \rangle} = 1$ iff $\llbracket \phi \rrbracket^{\langle dn,g \rangle} = 0$; $\llbracket \exists s \phi \rrbracket^{\langle dn,g \rangle} = 1$ iff there is a variable assignment function g' = g[s/p] such that $\llbracket \phi \rrbracket^{\langle dn,g' \rangle} = 1$.
- Where V is a RV variable and ϕ is a formula: $[\![?\lambda V\phi]\!]^{\langle dn,g\rangle} = \{\phi[V/X]: X \text{ is an RV constant and } [\![\phi[V/X]]\!]^{\langle dn,g\rangle} = 1\}.$

These interpretations yield a satisfaction relation in the usual way: $dn \models \phi$ iff there is a function g such that $\llbracket \phi \rrbracket^{\langle dn,g \rangle} = 1$.

Derivation of Parameter Update Given a Domain Trial

We now show that (21)—the formula for incrementally updating the probability of a parent set given the latest domain trial—follows from the recursive Γ function in the Dirichlet distribution.

$$Pr(\Pi_v|e_{0:t}) = Pr(\Pi_v|e_{0:t-1}) \frac{(n_{v=i|j} + \alpha_{v=i|j} - 1)}{(n_{v=i|j} + \alpha_{v-i|j} - 1)}$$
(21)

Exploiting the Dirichlet distribution to compute the posterior probability yields the following, where m_v is the number of possible values for variable V (so in our case, $m_v = 2$):

$$Pr(\Pi_v|e_{0:t}) \propto Pr(\Pi_v) \prod_{j \in values(\Pi_v)} \frac{Beta_{m_v}(n_{v=1|j} + \alpha_{v=1|j}, \dots n_{v=m_v|j} + \alpha_{v=m_v|j})}{Beta_{m_v}(\alpha_{v=1|j}, \dots, \alpha_{v=m_v|j})}$$

If e_t makes $V=i, \Pi_v=j$, then the only term in the product that differs between updating with $e_{0:t-1}$ vs. $e_{0:t}$ is when $\Pi_v=j$. For $e_{0:t-1}$, this term is:

$$\frac{Beta_{m_v}(n_{v=1|j} + \alpha_{v=1|j}, \dots, n_{v=i|j} - 1 + \alpha_{v=i|j}, \dots, n_{v=m_v|j} + \alpha_{v=m_v|j})}{Beta_{m_v}(\alpha_{v=1|j}, \dots, \alpha_{v=m_v|j})}$$

So

$$\begin{split} Pr(\Pi_{v}|e_{0:t}) &\propto & Pr(\Pi_{v}|e_{0:t-1}) \frac{Beta_{m_{v}}(n_{v=1|j} + \alpha_{v=1|j}, \dots n_{v=m_{v}|j} + \alpha_{v=m_{v}|j})}{Beta_{m_{v}}(n_{v=1|j} + \alpha_{v=1|j}, \dots, n_{i|j} - 1 + \alpha_{v=i|j}, \dots, n_{m_{v}|j} + \alpha_{v=m_{v}|j})} \\ &= Pr(\Pi_{v}|e_{0:t-1}) \frac{\prod_{k \in values(V)} \Gamma(n_{v=k|j} + \alpha_{v=k|j})}{\Gamma(n_{v=k|j} + \alpha_{v=k|j})} \frac{\Gamma(n_{v=i|j} - 1 + \alpha_{v=i|j})}{\Gamma(n_{v=i|j} - 1 + \alpha_{v=i|j})} \frac{\Gamma(n_{v=k|j} + \alpha_{v=k|j})}{\Gamma(n_{v=i|j} - 1 + \alpha_{v=i|j})} \\ &= & Pr(\Pi_{v}|e_{0:t-1}) \frac{\Gamma(n_{n=i|j} + \alpha_{v=i|j})}{\Gamma(n_{v=i|j} + m_{v} \alpha_{v=i|j})} \frac{\Gamma(n_{v=i|j} - 1 + \alpha_{v=i|j})}{\Gamma(n_{v=i|j} - 1 + \alpha_{v=i|j})} \end{split}$$

By the definition of Γ (i.e., $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$), we have:

$$\begin{split} Pr(\Pi_v|e_{0:t}) = & Pr(\Pi_v|e_{0:t-1}) \frac{(n_{v=i|j}-1+\alpha_{v=i|j})\Gamma(n_{v=i|j}-1+\alpha_{v=i|j})}{(n_{v=.|j}-1+\alpha_{v=.|j})\Gamma(n_{v=.|j}-1+\alpha_{v=.|j})} \frac{\Gamma(n_{v=.|j}-1+\alpha_{v=.|j})}{\Gamma(n_{v=i|j}-1+\alpha_{v=i|j})} \\ = & Pr(\Pi_v|e_{0:t-1}) \frac{(n_{v=i|j}-1+\alpha_{v=i|j})}{(n_{v=.|j}-1+\alpha_{v=.|j})} \end{split}$$

Decision Networks Used in Experiments

Tables 4, 5, and 6 specify the true conditional probability tables (CPTs) and reward functions for the three Barley DNs used in the experiments of section 5 $(dn_+^{best}, dn_+^{worst},$ and $dn_+^{barley})$. For each CPT, each row lists a variable's name, its immediate parents, and the probability of that variable having the value 1 given each possible assignment to the variable's parents (assignments are enumerated according to the order specified in the second column). The reward table lists the reward domain and the reward received given each possible assignment to the domain.

Table 4: dn_{+}^{best}

X	Π_X	$P(X=1 \mid \Pi_X)$
В1	В5	0.596, 0.774
B2	Ø	0.653
В3	В7	0.457, 0.457
B4	Ø	0.354
B5	B3, B4	0.639, 0.224, 0.35, 0.273
B6	Ø	0.738
B7	Ø	0.313
O1	O2, O6, A3	0.286, 0.478, 0.401, 0.956, 0.53, 0.084, 0.766, 0.923
O2	A1, A7	0.31, 0.554, 0.213, 0.197
O3	A2, A5, B2	0.74, 0.945, 0.92, 0.721, 0.371, 0.963, 0.129, 0.029
O4	B2, O5, B1	0.937, 0.93, 0.484, 0.255, 0.637, 0.191, 0.136, 0.149
O_5	O7	0.36, 0.43
O6	A6, B5	0.677, 0.209, 0.696, 0.521
07	A4, B6	0.821, 0.379, 0.211, 0.383

Π_R	$\mathcal{R}(\Pi_R)$
O1, O4, O3, O5, B7	43.55, 27.42, 35.48, 8.06, 20.97, 11.29, 30.65, 50.0, 46.77, 6.45, 12.9, 32.26, 41.94, 0.0, 33.87, 37.1, 25.81, 4.84, 22.58, 29.03, 14.52, 9.68, 45.16, 24.19, 17.74, 19.35, 48.39, 16.13, 40.32, 3.23, 38.71, 1.61

Table 5: dn_{+}^{worst}

\overline{X}	$\mid \Pi_X \mid$	$P(X=1 \mid \Pi_X)$
B1	Ø	0.49
B2	B3, B7	0.779, 0.547, 0.727, 0.197
B3	Ø	0.198
B4	Ø	0.36
B5	Ø	0.883
B6	Ø	0.237
B7	B1	0.43, 0.992
O1	O7, A1, A3	0.088, 0.467, 0.13, 0.548, 0.7, 0.372, 0.498, 0.047
O2	B5, O7	0.461, 0.599, 0.806, 0.37
O3	A6	0.881, 0.125
O4	O3, A5, A4	0.111, 0.033, 0.315, 0.322, 0.034, 0.579, 0.94, 0.644
O_5	O2, B4, O4	0.816, 0.979, 0.577, 0.467, 0.459, 0.751, 0.191, 0.541
O6	O4, A2	0.818, 0.583, 0.188, 0.957
Ο7	O6, A7, B1	0.314, 0.418, 0.48, 0.822, 0.957, 0.889, 0.697, 0.061

Π_R	$\mathcal{R}(\Pi_R)$
O1, B6, B7, O5, B2	43.55, 6.45, 22.58, 46.77, 37.1, 48.39, 1.61, 8.06, 16.13, 29.03, 50.0, 9.68, 11.29, 14.52, 19.35, 40.32, 4.84, 12.9, 38.71, 20.97, 45.16, 3.23, 32.26, 17.74, 33.87, 25.81, 24.19, 35.48, 27.42, 30.65, 41.94, 0.0

Table 6: dn_+^{barley}

X	$\mid \Pi_X$	$P(X=1 \mid \Pi_X)$
Soil Type	Ø	0.5
Temperature	Ø	0.5
Precipitation	Ø	0.5
Insect-Prevalence	Ø	0.5
Local-Concern	Ø	0.5
Nitrogen	Soil Type, Precipitation, Pesticide, Fertiliser	0.4, 0.6, 0.5, 0.7, 0.3, 0.5, 0.4, 0.6, 0.65, 0.85, 0.75, 0.95, 0.55, 0.75, 0.65, 0.85
Gross Crops	Harrow, Nitrogen, Grain	0.5, 0.4, 0.8, 0.7, 0.6, 0.5, 0.9, 0.8
Fungus	Temperature, Fungicide, Grain	0.2, 0.5, 0.02, 0.04, 0.3, 0.6, 0.03, 0.06
Weeds	Temperature, Harrow, Soil Type	0.2, 0.1, 0.02, 0.01, 0.3, 0.15, 0.03, 0.015
Infestation	Insect-Prevalence, Pesticide	0.1, 0.5, 0.01, 0.05
Yield	Gross Crops, Fungus, Weeds, Infestation	0.2, 0.95, 0.1, 0.5, 0.1, 0.7, 0.05, 0.3, 0.05, 0.65, 0.01, 0.2, 0.01, 0.45, 0.005, 0.1
Protein	Nitrogen, Fertiliser, Grain	0.5, 0.9, 0.4, 0.8, 0.4, 0.8, 0.3, 0.7
Bad Press	Local-Concern, Pesticide	0.0, 0.0, 0.01, 0.5

Π_R	$\mid \mathcal{R}(\Pi_R)$
Yield, Protein, Fungus, Bad Press	10, 15, 15, 20, 0, 5, 5, 10, -10, -5, -5, 0, -20, -15, -15, -10