Lecture 2 outline

The Buckingham Pi Theorem

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The Buckingham Π theorem derives from the observation that our choices of measurement units (inches vs. yards; seconds vs. years; ounces vs. kilograms) are arbitrary. Therefore all physical equations must be dimensionally homogeneous. If a physical equation is written as $A + B + C + \ldots = 0$, and any measurement unit is changed by a factor λ (e.g. $\lambda = 36$ if we replace yards by inches), then each term A, B, C, etc., will scale as some power of λ. Dimensional homogeneity means that every term must scale with the *same* power of λ. That power of λ can then be factored out and divided out of the equation, leaving the equation invariant under changes in the arbitrary units, as required.

Since the equations must be dimensionally homogeneous, we can always put them in the form $f(v_i) = 0$, where the v_i $(i = 1 \ldots n)$ are the n variables relevant to the physical situtation being considered, and f is a function homogeneous of degree 0 to all changes of units (i.e. f is a dimensionless function). For example, the previous equation can be rewritten as $f = 1 + (B/A) + (C/A) \ldots = 0$; each term is now dimensionless.

This simple observation has powerful consequences. Applying Euler's theorem for homogeneous functions to $f(v_i)$, and then integrating the resulting linear partial differential equations using the method of characteristics, one can prove the Buckingham† Pi Theorem:

The solution to any physical problem involving n variables $v_1 \ldots v_n$ depending on r independent dimensional units has in general the form

$$0 = f(\Pi_1, \Pi_2, \ldots, \Pi_{n-r}) \,, \tag{1}$$

where the Π_k are $(n - r)$ dimensionless variables of the form

$$\Pi_k = v_1^{\mu_{k1}} v_2^{\mu_{k2}} \cdots v_n^{\mu_{kn}} \,,$$

and the μ_{kj} are rational numbers, chosen so as to make the Π_k dimensionless (homogenenous of degree 0 under scaling of each of the r dimensional units in the variables). Since there are r linear equations determining the n μ_{ki} for each k, in the general case that the r equations are linearly independent of each other, the matrix equation will have $n - r$ independent

† Edgar Buckingham 1914 *Physical Review*, **4**, 345 "On Physically Similar Systems; Illustrations of the Use of Dimensional Equations." Also Buckingham 1921 *Phil. Mag.*, **42**, 696 "Notes on the Method of Dimensions." Buckingham (1867-1940), educated at Harvard and Leipzig, wrote a standard textbook *An Outline of the Theory of Thermodynamics* (1900), and worked at the National Bureau of Standards (1905-1937) on soil physics, gas properties, acoustics, fluid mechanics and blackbody radiation. He was an expert naval consultant on screw and propeller systems, which seems to have inspired his interest in using dimensional analysis to design experiments to determine functional dependences in model systems which could be exactly scaled to real ones (big ships). From the sarcastic remarks in his 1914 paper, Buckingham appears to have been goaded into publishing his famous theorem as a rebuttal to a rather silly paper by Caltech's distinguished chemist and relativist R.C. Tolman.

solutions (unique up to a scale factor in the μ's for each k -i.e. up to powers of the Π_k, since a function $f(\Pi)$ is also a function $g(\Pi^2)$ or a function $h(\Pi^{42/11})$). If only $R < r$ of the equations determining the μ's are linearly independent, then the physical problem will depend on a larger number, $n - R$ of dimensionless Π's.

Notice that if we vary Π_1 keeping the other Π's fixed in equation (1), the function must have a root (or a sequence of roots labeled by some integer dimensionless index a), which determines Π_1 as a function of the other Π's (and perhaps the index a, if more than one physically relevant solution exists):

$$\Pi_1 = g(\Pi_2, \ldots, \Pi_{n-r}, a)\,. \tag{2}$$

The Buckingham Pi Theorem can be a powerful tool in deriving physical equations to within dimensionless factors (usually negligible in order of magnitude estimates!). This is especially true when it is coupled with physical common sense or dimly remembered or empirical facts, which can be used to determine groupings of variables in ways independent of dimensional arguments.

Example: Estimate the electromagnetic power P radiated by a charge e (in esu, which has dimensions $[M]^{1/2}[L]^{3/2}[T]^{-1}$; we use Gaussian, aka God's, units for electromagnetism in this course), which oscillates over a distance a with angular frequency ω.

Since the problem involves electromagnetic radiation, we suspect that the speed of light c might also be an important variable, but we can't think of anything else important. So we infer that the variables are $n = 5$: P, e, a, ω, c. They involve $r = 3$ units $[M]$, $[L]$ and $[T]$.

The Pi Theorem thus tells us that there are $5 - 3 = 2$ dimensionless variables, which by experimentation (usually quickest) or formal solution of the linear system of equations for the μ's, we identify as

$$\Pi_1 = P/(\omega e^2/a), \quad \Pi_2 = \omega a/c\,. \tag{3}$$

Notice that this choice of $\Pi's$ is not unique; we could equally well have chosen $\Pi_1' = \Pi_2^3$, and $\Pi_2' = \Pi_1\Pi_2$ or any other independent products of powers of our Π's. Using form (2) of the Pi Theorem, we find

$$P = \frac{\omega e^2}{a} g(\omega a/c)\,. \tag{4}$$

A dim memory that when $\omega a/c \ll 1$ the radiation is called 'electric dipole' radiation tells us that the variables e and a must appear just in the combination ea, which can only happen if $g(\Pi_2) = \Pi_2^3$ times a constant, so we finally get

$$P = \Pi(ea)^2\omega^4/c^3 \qquad (5)\,,$$

where Π is some dimensionless number (which happens to be 1/3). Thus, up to a factor of 1/3, we have recovered Larmor's formula for electric dipole radiation, without having to learn about Liénard-Wiechart potentials and the fields of accelerated charges.

If instead we had two equal charges e oscillating in antiphase to $\pm a$, then the dipole moment is zero. We dimly recall that there is still electric quadrupole radiation. In that case, e and a must appear just as ea^2, which requires $g(\Pi_2) = \Pi_2^5\Pi$, so

$$P = \Pi(ea^2)^2\omega^6/c^5\,. \tag{6}$$

The exact result in this case gives $\Pi = 1/40$. Dimensionless factors cannot always be neglected even in order of magnitude! Choosing variables can be an art: if we had used frequency ν instead of angular frequency ω as a variable, the dimensionless factor in (5) with ν replacing ω would have been $(2\pi)^4/3 = 520$.