Partial Differential Equations

Edited by Stewart Nash

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Preface

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Chapter 1

Difference Methods for Parabolic Equations

1.1 Difference Equations

The partial derivatives of a sufficiently smooth function $u(x,t)$ may be approximated by *finite difference quotients*. Each such approximation consists of a discrete difference operator plus a *truncation error*, which arises from neglecting higher-order terms in the Taylor expansion.

1.1.1 Forward difference approximation of u_t

Using Taylor's theorem in time, we obtain the following approximation for u_t and expression for the truncation error R_t

$$u_t(x,t) = \frac{u(x,t+k) - u(x,t)}{k} + R_t(x,t), \tag{1.1}$$

$$R_t(x,t) = -\frac{k}{2} u_{tt}(x,\bar{t}), \qquad t < \bar{t} < t+k. \qquad \text{(Forward Difference for } u_t\text{)}$$

If u_{tt} is bounded, then

$$R_t(x,t) = \mathcal{O}(k) \qquad (k \to 0)$$

1.1.2 Centered difference approximation of u_x

Expanding in space about x, we obtain

$$u_x(x,t) = \frac{u(x+h,t) - u(x-h,t)}{2h} + R_x(x,t), \tag{1.2}$$
$$R_x(x,t) = -\frac{h^2}{6}u_{xxx}(\bar{x},t), \qquad x-h < \bar{x} < x+h.$$

(Centered Difference for u_x)

If u_{xxx} is bounded, then

$$R_x(x,t) = \mathcal{O}(h^2) \qquad (h \to 0)$$

1.1.3 Centered difference approximation of u_{xx}

The second spatial derivative is approximated by

$$u_{xx}(x,t) = \frac{u(x+h,t) - 2u(x,t) + u(x-h,t)}{h^2} + R_{xx}(x,t), \tag{1.3}$$
$$R_{xx}(x,t) = -\frac{h^2}{12}u_{xxxx}(\bar{x},t), \qquad x-h < \bar{x} < x+h.$$

(Centered Difference for u_{xx})

If u_{xxxx} is bounded, then

$$R_{xx}(x,t) = \mathcal{O}(h^2) \qquad (h \to 0)$$

1.1.4 Centered difference approximation of u_{xt}

A centered approximation to u_{xt} is given by

$$u_{xt}(x,t) = \frac{1}{4hk}\big(u(x+h,t+k) - u(x+h,t-k) \qquad + R_{xt}(x,t)$$
$$- u(x-h,t+k) + u(x-h,t-k)\big)$$

(Centered Difference for u_{xt})

The truncation error has the form

$$R_{xt}(x,t) = -\frac{h^2}{6}u_{xxxt}(\bar{x},\bar{t}) - \frac{k^2}{6}u_{xttt}(\tilde{x},\tilde{t}) \tag{1.4}$$

where

$$x-h < \bar{x},\tilde{x} < x+h, \qquad t-k < \bar{t},\tilde{t} < t+k$$

If u_{xxxt} and u_{xttt} are bounded, then

$$R_{xt}(x,t) = \mathcal{O}(h^2+k^2) \qquad (h,k \to 0).$$

1.1.5 O-notation

Usually it is only the order of magnitude of the truncation error which is of interest. A function $f(h)$ is said to be of the *order* $g(h)$ *as* $h \to 0$, if there exist constants $C > 0$ and $h_0 > 0$ such that

$$|f(h)| \leq C|g(h)|, \qquad 0 < |h| < h_0.$$

In this case, we write

$$f(h) = \mathcal{O}(g(h)) \qquad (h \to 0).$$

If, as $h_1, h_2 \to 0$,$f_1 = \mathcal{O}(g_1)$ and $f_2 = \mathcal{O}(g_2)$, then

$$f_1 + f_2 = \mathcal{O}(g_1 + g_2).$$

Example 1.1. For (9.1), the truncation error is $\mathcal{O}(k)$ $(k \to 0)$, provided u_{tt} is bounded. For (9.2), the truncation error is $\mathcal{O}(h^2)$ $(h \to 0)$, provided u_{xxx} is bounded. For (9.4), the truncation error is $\mathcal{O}(h^2 + k^2)$ $(h \to 0)$, provided u_{xxxt} and u_{xttt} are bounded; we omit as understood the limits $h \to 0$ and $k \to 0$.

1.1.6 Grid

A *grid* or *mesh* in the xt-plane is a set of points $(x_n, t_m) = (x_0 + nh, t_0 + mk)$, where n and m are integers and (x_0, t_0) is a reference point. The (x_n, t_m) are called *grid points*, *mesh points*, or *nodes*. The positive numbers h and k are respectively the x and t *grid spacings* or *grid sizes*. If h and k are constants, the grid is called *uniform*; if $h = k$ are constant, the grid is said to be *square*. The compact subscript notation

$$u_{nm} \equiv u(x_n, t_m)$$

is convenient and widely used.

Example 1.2. The difference formulas (9.1) and (9.3) may be written

$$\begin{aligned} u_t(x_n, t_m) &= \frac{u_{n,m+1} - u_{nm}}{k} + \mathcal{O}(k) \\ u_{xx}(x_n, t_m) &= \frac{u_{n+1,m} - 2u_{nm} + u_{n,m+1}}{h^2} + \mathcal{O}(h^2) \\ &\equiv \frac{\delta_x^2 u_{nm}}{h^2} + \mathcal{O}(h^2) \end{aligned}$$

where the *difference operator* δ_x^2 is the analog of the differential operator $\delta^2/\delta x^2$. We say that (9.1) is *two-level* (in t) because it involves only two m-values, these being consecutive.

Let a region Ω in the xt-plane be covered by a grid, (x_n, t_m). If all the derivatives in the PDE

$$L[u] = f \qquad (x,t)\text{in}\Omega \tag{1.5}$$

are replaced by difference quotients, the result is the *finite-difference equation*

$$D[U_{nm}] = f_{nm} \qquad (x_n, t_m)\text{in}\Omega \tag{1.6}$$

The continuous problem (9.5) was *differenced* or *discretized* to produce the discrete problem (9.6), whose solution, U_{nm}, approximates $u(x,t)$ at the grid points.

1.2 Consistency and Convergence

If discretization is to provide a useful approximation, the solution to (9.5) should very nearly satisfy (9.6), when h and k are taken sufficiently small. The amount by which the solution to $L[u] = f$ fails to satisfy the difference equation is called the *local truncation error*; it may be expressed as

$$R_{nm} \equiv D[u_{nm}] - f_{nm}$$

The difference equation (9.6) is said to be *consistent with* the PDE (9.5) if

$$\lim_{h,k\to 0} R_{nm} = 0 \tag{1.7}$$

With the exception of the DuFort-Frankel method (Problem 9.10), all difference methods to be treated are consistent with their corresponding PDEs.
In addition to consistency, we want the accuracy of the approximation to improve as $h, k \to 0$. If U_{nm} is the exact solution to (9.6) and u_{nm} is the solution of (9.5) evaluated at (x_n, t_m), the *discretization error* is defined as $U_{nm} - u_{nm}$. The difference method (9.6) is said to be *convergent* if

$$\lim_{h,k\to 0} |U_{nm} - u_{nm}| = 0 \qquad (x_n, t_m)\,\text{in}\,\Omega \tag{1.8}$$

It is possible for a difference method to be consistent but not convergent.

1.3 Stability

Let U_{nm} satisfy (9.6), with initial values U_{n0} and possibly boundary values prescribed. Let V_{nm} be the solution to a perturbed difference system which

differs only in the initial values, and write $V_{n0} \equiv U_{n0} + E_{n0}$. Then, assuming exact arithmetic, the initial perturbation, or "error," E_{n0}, can be shown to propagate, with increasing m, according to the homogeneous difference equation

$$D[E_{nm}] = 0$$

subject to homogeneous boundary conditions.

1.4 Parabolic Equations

The one-dimensional diffusion equation

$$u_t = \kappa^2 u_{xx} \tag{1.9}$$

is used as a guide in developing finite-difference methods for parabolic PDEs in general. For the grid $(x_n, t_m) = (nh, mk)$, we shall state three commonly used difference equations for (9.10). All three are two-level equations whereby the solution, known at level m, is advanced to level $m + 1$.

1.4.1 Explicit (Forward-Difference) Method

$$\frac{U_{n,m+1} - U_{nm}}{k} = \kappa^2 \frac{U_{n+1,m} - 2U_{nm} + U_{n-1,m}}{h^2} \tag{1.10}$$

or

$$U_{n,m+1} = (1 + r\delta_x^2) U_{nm} \qquad (r \equiv \kappa^2 k/h^2) \tag{1.11}$$

1.4.2 Implicit (Backward-Difference) Method

$$\frac{U_{n,m+1} - U_{nm}}{k} = \kappa^2 \frac{U_{n+1,m+1} - 2U_{n,m+1} + U_{n-1,m+1}}{h^2} \tag{1.12}$$

or

$$(1 - r\delta_x^2) U_{n,m+1} = U_{n,m} \qquad (r \equiv \kappa^2 k/h^2) \tag{1.13}$$

1.4.3 Implicit (Crank-Nicolson) Method

$$\frac{U_{n,m+1} - U_{nm}}{k} = \frac{\kappa^2}{2} \frac{\delta_x^2 U_{nm} + \delta_x^2 U_{n,m+1}}{h^2} \tag{1.14}$$

or

$$(1 - r\delta_x^2) U_{n,m+1} = (1 + r\delta_x^2) U_{nm} \qquad (r \equiv \kappa^2 k/h^2) \tag{1.15}$$

1.4.4 Error and stability

Theorem 1.1. The forward-difference method (9.11) has local truncation error $\mathcal{O}(k + h^2)$; it is (conditionally) stable if and only if $r \leq 1/2$.

Theorem 1.2. The backward-difference method (9.12) has local truncation error $\mathcal{O}(k + h^2)$; it is stable.

Theorem 1.3. The Crank-Nicolson method (9.13) has local truncation error $\mathcal{O}(k^2 + h^2)$; it is stable.

1.4.5 Two-dimensions

For the two-dimensional diffusion equation,

$$u_t = \kappa^2(u_{xx} + u_{yy}) \tag{1.16}$$

let $(x_m, y_n, t_l) = (mh, nh, lk)$ and $U_{mnl} \approx u_{mnl} = u(x_m, y_n, t_l)$. The above methods have as their counterparts:

Chapter 2

Difference Methods for Hyperbolic Equations

2.1 One-Dimensional Wave Equation

Methods similar to those given in Section 9.4 may be used to approximate smooth solutions to

$$u_{tt} = c^2 u_{xx} \tag{2.1}$$

Let $(x_n, t_j) = (nh, jk)$ $(n, j = 0, 1, 2, \dots)$ and write $s \equiv k/h$; we have as representatives of the two sorts of methods:

Chapter 3

Variational Formulation of Boundary Value Problems

3.1 Introduction

In certain cases, the solution of a boundary value problem for a PDE is also a solution of an associated calculus of variations problem. A typical problem in the calculus of variations is to find, for a function u belonging a prescribed set $\mathscr{A}$, the extreme values of the integral expression

$$J[u(\mathbf{x})] = \int_\Omega F(\mathbf{x}, u(\mathbf{x}), \nabla u(\mathbf{x}))\, d\Omega \tag{3.1}$$

where F denotes a given function. Hence we shall begin by describing some of the structure of the domain $\mathscr{A}$ of J.

3.2 The Function Space $L^2(\Omega)$

In Chapter 6 we considered the space $L^2(a,b)$ of functions $f(x)$ that are defined and square integrable on (a,b) in $\mathbb{R}^1$. More generally, let let Ω denote a bounded region in $\mathbb{R}^n$ and consider the set $L^2(\Omega)$ of all real-valued functions $u(\mathbf{x})$ defined on Ω which satisfy

$$\int_\Omega u^2(\mathbf{x})\, d\Omega < \infty \tag{3.2}$$

Like $L^2(a,b)$, $L^2(\Omega)$ is a vector space over the real numbers, and the expected definition

$$\langle u, v\rangle \equiv \int_\Omega u(\mathbf{x})v(\mathbf{x})\, d\Omega \tag{3.3}$$

makes it an inner product space.
A subset of $L^2(\Omega)$ is said to be a *subspace* of $L^2(\Omega)$ if the subset is closed under the operation of forming linear combinations.

Example 3.1. (a) For k a nonnegative integer, the subset $C^k(\bar{\Omega})$ of all u in $L^2(\Omega)$ which, together with all derivatives of order k or less, are continuous on $\bar{\Omega}$ is a subspace of $L^2(\Omega)$. (b) Let $u_1, \ldots, u_N$ denote N elements of $L^2(\Omega)$. The subset $\mathscr{M}$ of all linear combinations of $u_1, \ldots, u_N$ is a subspace of $L^2(\Omega)$—the subspace *spanned* by the u_i. (c) For m a positive integer, the subset $H^m(\Omega)$ of all u in $L^2(\Omega)$ whose derivative of order m or less are also in $L^2(\Omega)$ is a subspace of $L^2(\Omega)$.

A subspace $\mathscr{M}$ of $L^2(\Omega)$ is *dense in* $L^2(\Omega)$ if for any $\epsilon > 0$ and any f in $L^2(\Omega)$ there exists a v in $\mathscr{M}$ such that

$$\|v - f\|^2 = \int_\Omega (v - f)^2 \, d\Omega < \epsilon \tag{3.4}$$

i.e., if any f in $L^2(\Omega)$ can be approximated with arbitrary precision in the least-squares sense by a function from $\mathscr{M}$.

Theorem 3.1. For each positive integer m, the following subspaces are dense in $L^2(\Omega)$: $C^m(\bar{\Omega})$, $H^m(\Omega)$, and the set of all u in $C^m(\bar{\Omega})$ such that $u = 0$ on S, the boundary of Ω.

Theorem 3.2. If $\mathscr{M}$ is a dense subspace in $L^2(\Omega)$ and if an element u of $L^2(\Omega)$ satisfies $\langle u, v \rangle = 0$ for all v in $\mathscr{M}$, then $u = 0$.

Chapter 4

Variational Approximation Methods

This chapter presents some techniques for constructing approximate solutions to boundary value problems. These techniques are based on the ideas of Chapter 12 and differ markedly from the finite-difference methods of Chapters 9, 10, and 11.

4.1 The Rayleigh-Ritz Procedure

This approximation procedure is limited to boundary value problems admitting the variational formulation "Find $u*$ in $\mathscr{A}$ such that functional $J[u]$ is minimized over $\mathscr{A}$ by $u*$." It was seen in Chapter 12 that such boundary value problems arise in connection with self-adjoint elliptic PDEs.
We suppose $\mathscr{A}$ to be some subset of $L^2(\Omega)$, where Ω denotes a bounded set in $\mathbb{R}^n$ with smooth boundary S consisting of complementary portion S_1 and S_2. Specifically, we take $\mathscr{A} = \{u \text{ in } H^1(\Omega) : u = g \text{ on } S_1\}$ for a given g in $C(\bar{\Omega}$; the associated subspace of comparison functions is taken as $\mathscr{M} = \{v \text{ in } H^1(\Omega) : v = 0 \text{ on } S_1\}$.
Let ϕ_0 denote an arbitrary function from $\mathscr{A}$ (e.g., $\phi_0 = g$) and let $\phi_1, \ldots, \phi_N$ denote N linearly independent functions $\mathscr{M}$. Then,

$$u_N(\mathbf{x}) \equiv \phi_0(\mathbf{x}) + \sum_{j=1}^{N} c_j \phi_j(\mathbf{x}) \tag{4.1}$$

belongs to $\mathscr{A}$ for *all* choices of the constants $c_1, \ldots, c_N$; we denote by $\mathscr{A}_N$ the subset of $\mathscr{A}$ consisting of all such functions u_N. Let u_N^* denote the function

in $\mathscr{A}_N$ that minimizes J over $\mathscr{A}_N$. It can be shown that u_N^* represents the best approximation, in the least-squares sense, from $\mathscr{A}_N$ to the exact solution u^*. This function u_N^* is called the *Rayleigh-Ritz approximation* to the solution of the boundary value problem.
The minimization of J over $\mathscr{A}_N$ is tantamount to the minimization over all $\mathbf{c}$ in $\mathbb{R}^N$ of the ordinary function

$$H(c_1, \ldots, c_N) \equiv J\left[\phi_0 + \sum_{j=1}^{N} c_j \phi_j\right] \tag{4.2}$$

The minimizing constants $c_1^*, c_2^*, \ldots, c_N^*$ must satisfy

$$\frac{\partial H}{\partial c_m}(c_1, \ldots, c_N) = 0 \quad (m = 1, \ldots, N) \tag{4.3}$$

which is a system of N equations in N unknowns.
The Rayleigh-Ritz procedure may also be applied to eigenvalue problems of the sort treated in Section 12.4. In any eigenvalue problem the boundary conditions are all homogeneous. Hence, we take $\phi_0 \equiv 0$ in (13.1) and minimize the Rayleigh quotient J over the subspace $\mathscr{A}_N = \mathscr{M}_N$.

4.2 The Galerkin Procedure

This approximation method is employed when the boundary value problem admits only a weak formulation; e.g., in the case of a linear elliptic PDE containing odd-order derivates. In the event that conditions (12.24) hold in the weak formulation, so that a variational formulation also exists, it can be shown (see Problem 13.4(b)) that the Galerkin and Rayleigh-Ritz procedure coincide.
Consider, then, a boundary value problem with the weak formulation "Find u^* in $\mathscr{A}$ such that $K[u^*, v] = F[v]$ for all v in $\mathscr{M}$." Here we suppose that Ω, S, $\mathscr{A}$, and $\mathscr{M}$ are as described in Section 13.1. Let $phi_1, \ldots, \phi_N$ denote N linearly independent *trial functions* in $\mathscr{M}$ and let ϕ_0 denote an arbitrary function in $\mathscr{A}$. As in the Rayleigh-Ritz procedure, we seek an approximation u_N^* to the weak solution u^* of the form (13.1). In addition, let $\psi_1, \ldots, \psi_N$ denote N linearly independent *weight functions* in $\mathscr{M}$; these may or may not be the same as the trail functions. The Galerkin approximate solution is required to satisfy

$$K[u_N^*, \psi_j] = F[\psi_j] \quad (j = 1, \ldots, N) \tag{4.4}$$

This is a set of N equations in the N unknowns $c_1^*, \ldots, c_N^*$.

Chapter 5

The Finite Element Method: An Introduction

The success of the approximation methods presented in Chapter 13 is largely dependent on the selection of an effective collection of trial functions ϕ_j and/or weight functions ψ_j. If these functions are chosen from certain families of piecewise polynomials, called *finite element spaces*, the following advantages are realized:

(i) It is possible to deal in a systematic fashion with regions Ω having curved boundaries of rather arbitrary shape.

(ii) One can systematically estimate the accuracy of the approximate solution in terms of the adjustable parameters associated with the finite element family.

(iii) The coefficient matrix and data vector for the system of algebraic equations defining the approximate solution can be efficiently generated by computer.

5.1 Finite Element Spaces in One Dimension

Suppose that the interval $[0, 1]$ is subdivided into N subintervals each of length $h = 1/N$. Let $x_j = jh$ $(j = 0, 1, \ldots, N)$ denote the nodes in the interval $[0, 1]$. Then the finite element space denoted by $S^h[k, r]$ shall consist all functions $\phi(x)$ defined on $[0, 1]$ such that (i) on each subinterval $[x_j, x_{j+1}]$, $\phi(x)$ is a polynomial of degree at most k; (ii) $\phi(x)$ has r continuous derivatives on $[0, 1]$, which is to say, ϕ belongs to $C'[0, 1]$.

If $r = 0$, ϕ is continuous but not necessarily differentiable at nodes. If ϕ is to be allowed to be discontinuous at nodes, we set $r = -1$. Evidently, $S^h[k, r]$ is a finite-dimensional vector space (a subspace of $L^2(0, 1)$) and so may be characterized by giving a *basis*; i.e., a linearly independent set of elements $\{\phi_j\}$ that spans the space.
Modifications for the case of nonuniform grids are easily developed.

Example 5.1. A basis for $S^h[0, -1]$, the *piecewise constants*, is given by

$$\phi_j(x) = \begin{cases} 1 & x_{j-1} \leq x \leq x_j \\ 0 & \text{otherwise} \end{cases} \tag{5.1}$$

for $j = 1, 2, \ldots, N$ $(Nh = 1)$. See Fig. 14-1. The functions in $S^h[0, -1]$ are in $L^2(0, 1)$ but are not continuous.