

Phoenix Dynamics Robotics Manual

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Chapter 1

Dynamics

1.1 Kinematics

Take a vector with components $\mathbf{P} = (a_x, b_y, c_z)$. A scale factor w can be added (to the matrix form) to give

$$\mathbf{P} = \begin{bmatrix} P_x \\ P_y \\ P_z \\ w \end{bmatrix} \quad (1.1)$$

where $(a_x, b_y, c_z) = (P_x/w, P_y/w, P_z/w)$. A direction vector can be represented by a scale factor of zero ($w = 0$).

A universe reference frame is represented by $F_{x,y,z}$ and a moving frame is represented by $F_{n,o,a}$ where the letters n, o, and a come from the words normal, orientation and approach. Relative to the gripper, the z -axis is the approach axis by which the gripper approaches an object. The orientation with which the gripper frame approaches the part is the orientation axis. The normal-axis or x -axis is normal to both. A fourth vector which gives the location of a frame relative to a reference frame can be added to the vectors representing the components of the n -, o -, and a -axes to give a homogeneous matrix representation of this relative frame

$$F = \begin{bmatrix} 1 & 0 & 0 & d_x \\ 0 & 1 & 0 & d_y \\ 0 & 0 & 1 & d_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (1.2)$$

Pre-multiplying the frame matrix by the transformation matrix will yield the new location of the frame.

Rotation matrices about the x -, y - and z -axes are given by

$$\begin{aligned} \text{rot}(x, \theta) &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \theta & \sin \theta \\ 0 & \sin \theta & \cos \theta \end{bmatrix} \\ \text{rot}(y, \theta) &= \begin{bmatrix} \cos \theta & 0 & \sin \theta \\ 0 & 1 & 0 \\ -\sin \theta & 0 & \cos \theta \end{bmatrix} \\ \text{rot}(z, \theta) &= \begin{bmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned} \quad (1.3)$$

Denoting the transformation of frame R relative to frame U (universe) as ${}^U T_R$, denoting the p relative to the frame R as ${}^R p = p_{noa}$, and denoting p relative to frame U as ${}^U p = p_{xyz}$ we have

$${}^U p = {}^U T_R \times {}^R p \quad (1.4)$$

1.1.1 Twists and Screws

The special orthogonal group is denoted SO and we may define it as the space of rotation matrices in $\mathbb{R}^{n \times n}$ by

$$SO(n) = \{R \in \mathbb{R}^{n \times n} : RR^T = I, \det R = +1\} \quad (1.5)$$

The space of $n \times n$ skew-symmetric matrices is given by

$$so(n) = \{S \in \mathbb{R}^{n \times n} : S^T = -S\} \quad (1.6)$$

The special Euclidean group, generalized to n dimensions, is given by

$$SE(n) \equiv \mathbb{R}^n \times SO(n) \quad (1.7)$$

In other words, the special Euclidean group is comprised of a configuration pair (p, R) which is the product space of \mathbb{R}^n with $SO(n)$ and is given in 3 dimensions by

$$SE(3) = \{(p, R) : p \in \mathbb{R}^3, R \in SO(3)\} = \mathbb{R}^3 \times SO(3) \quad (1.8)$$

We can define $se(n)$ as being comprised of the configuration pair (v, ω) where v is an element of \mathbb{R}^n and ω is a skew-symmetric matrix from $so(n)$. In three dimensions we have

$$se(3) \equiv \{(v, \hat{\omega}) : v \in \mathbb{R}^3, \hat{\omega} \in so(3)\} \quad (1.9)$$

A *twist* is an element of $se(3)$, $\hat{\xi} \in se(3)$. The twist coordinates of $\hat{\xi}$ are given by $\xi \equiv (v, \omega)$.

A rigid body motion which consists of rotation about an axis in space through an angle of θ radians, followed by a translation along the same axis by and amount d is referred to as a screw motion. A *screw* is composed of an axis l , a pitch h , and a magnitude M .

A generalized force acting on a rigid body consists of a linear component, i.e. a ‘pure force’, and an angular component, i.e. a ‘pure moment’, acting at a point. This generalized force can be represented as a vector in \mathbb{R}^6

$$F = \begin{bmatrix} f \\ \tau \end{bmatrix} \quad (1.10)$$

where f is the linear component in \mathbb{R}^3 and τ is the rotational component in \mathbb{R}^3 . We refer to this force and moment pair as a *wrench*.

1.2 Differential Motion

1.3 Dynamic Analysis

The Lagrangian is given by $L = T - V$ where T and V are the kinetic and potential energy of a system, respectively. If F_i is the summation of all external forces acting on the i th generalized coordinate q_i , the equations of motion are given by

$$F_i = \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{q}_i} \right) - \frac{\partial L}{\partial q_i} \quad (1.11)$$

When this is solved, the resulting equations of manipulator dynamics can be written

$$M(q)\ddot{q} + V(q, \dot{q}) + G(q) = \tau \quad (1.12)$$

or, alternatively,

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + N(q, \dot{q}) = \tau \quad (1.13)$$

where $M(q)$ is the inertia matrix, $V(q, \dot{q})$ is the Coriolis and centripetal acceleration vector, $C(q, \dot{q})$ is the Coriolis matrix, $G(q)$ is the gravity vector, $N(q, \dot{q})$ represents gravity and other non-linear terms and τ is the n -vector of generalized forces which could be, for example, the actuator torques.

To be concrete, we can give an example of this in which we have two coordinates, x and θ , for the i th of n linkages. If F_i is the summation of all

external forces for a linear motion and T_i is the summation of all external torques for a rotational motion, then

$$\begin{aligned} F_i &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{x}_i} \right) - \frac{\partial L}{\partial x_i} \\ T_i &= \frac{d}{dt} \left(\frac{\partial L}{\partial \dot{\theta}_i} \right) - \frac{\partial L}{\partial \theta_i} \end{aligned} \quad (1.14)$$

We can simplify the equations of motion for a 2-DOF system which is given by

$$\begin{bmatrix} \tau_i \\ \tau_j \end{bmatrix} = \begin{bmatrix} M_{ii} & M_{ij} \\ M_{ji} & M_{jj} \end{bmatrix} \begin{bmatrix} \ddot{\theta}_i \\ \ddot{\theta}_j \end{bmatrix} + \begin{bmatrix} C_{iii} & C_{ijj} \\ C_{jii} & C_{jjj} \end{bmatrix} \begin{bmatrix} \dot{\theta}_i \\ \dot{\theta}_j \end{bmatrix} + \begin{bmatrix} C_{iij} & C_{iji} \\ C_{jij} & C_{jji} \end{bmatrix} \begin{bmatrix} \dot{\theta}_i \dot{\theta}_j \\ \dot{\theta}_j \dot{\theta}_i \end{bmatrix} + \begin{bmatrix} G_i \\ G_j \end{bmatrix} \quad (1.15)$$

Where the coefficient M_{ii} is the effective inertia at joint i , such that an acceleration at joint i causes a torque at joint i equal to $M_{ii}\ddot{\theta}_i$, and the coefficient M_{ij} is the coupling inertia between joints i and j such that an acceleration at joint i or j causes a torque at joint j or i equal to $M_{ij}\ddot{\theta}_j$ or $M_{ji}\ddot{\theta}_i$. $C_{ijj}\dot{\theta}_j^2$ terms represent centripetal forces acting at joint i due to a velocity at joint j . All terms with $\dot{\theta}_i\dot{\theta}_j$ represent Coriolis accelerations and, when multiplied by corresponding inertias, represent Coriolis forces. G_i represents gravity forces at joint i .

1.4 Trajectory Planning

Chapter 2

Controls

2.1 State Variable Representation

Powerful tools from matrix algebra can be used to solve sets of first-order differential equations. That is why it is sometimes helpful to transform a system described by n th-order differential equations into a system of first-order differential equations.

2.1.1 Systems Modeled by Linear Differential Equations

Consider a system represented by a n th-order, single-input linear constant coefficient differential equation

$$\sum_{i=0}^n a_i \frac{d^i y}{dt^i} = u \quad (2.1)$$

This equation can be replaced by n first-order differential equations

$$\begin{cases} \frac{dx_k}{dt} = x_{k+1}, & 1 < k < n \\ \frac{dx_n}{dt} = \frac{1}{a_n} \left[\sum_{k=0}^{n-1} a_k x_{k+1} \right] + \frac{1}{a_n} u \end{cases} \quad (2.2)$$

where $x_1 \equiv y$ and $i, k \in \mathbb{W}$. This can be written as a matrix equation

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -\frac{a_0}{a_n} & -\frac{a_1}{a_n} & -\frac{a_2}{a_n} & \cdots & -\frac{a_{n-1}}{a_n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{a_n} \end{bmatrix} u \quad (2.3)$$

or

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + \mathbf{b}u \quad (2.4)$$

A multi-input-multi-output (MIMO) system can be represented by

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \\ \vdots \\ \frac{dx_n}{dt} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1r} \\ b_{21} & b_{22} & \cdots & b_{2r} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nr} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_r \end{bmatrix} \quad (2.5)$$

or

$$\frac{d\mathbf{x}}{dt} = A\mathbf{x} + B\mathbf{u} \quad (2.6)$$

where \mathbf{u} is an r -vector of input functions.

Let Φ be the $n \times n$ *transition matrix* of the differential equation given above which is described by the matrix equation

$$\frac{d\Phi}{dt} = A\Phi \quad (2.7)$$

If $\Phi(0) = I$ (initial condition) then $\Phi(t) = e^{At}$ where

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!} \quad (2.8)$$

The solution to (2.6) on the interval $0 \leq t < \infty$ is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}(0) + \int_0^t e^{A(t-\tau)} B\mathbf{u}(\tau) d\tau \quad (2.9)$$

2.1.2 Systems Modeled by Constant Coefficient Linear Difference Equations

An n -th order (linear constant-coefficient) difference equation is given by

$$\sum_{i=0}^n a_i y(k+i) = \sum_{i=0}^m b_i u(k+i) \quad (2.10)$$

Define a shift operator by the equation

$$Z[y(k)] \equiv y(k+1) \quad (2.11)$$

The n -th order linear constant-coefficient difference equation

$$y(k+n) + \sum_{i=0}^{n-1} a_i y(k+i) = u(k) \quad (2.12)$$

can be written as

$$(Z^n + \sum_{i=0}^{n-1} a_i Z^i)[y(k)] = u(k) \quad (2.13)$$

The characteristic equation of this difference equation is

$$Z^n + \sum_{i=0}^{n-1} a_i Z^i = 0 \quad (2.14)$$

2.2 Signal Flow Graphs

2.3 Stability, Sensitivity and Error

2.3.1 Stability

Definition 2.3.1. A continuous system *stable* if its impulse response $y_\delta(t)$ approaches zero as time approaches infinity. Similarly, a discrete-time system is stable if its Kronecker delta response $y_\delta(k)$ approaches zero as time approaches infinity.

A continuous or discrete-time system can also be defined as stable if every bounded input results in a bounded output.

2.3.2 Sensitivity

Sensitivity can be given for either the transfer or the frequency response function. The sensitivity of a system to its parameters is a measure of how much either of these system functions differ from its nominal when each of its parameters differs from its nominal value. Sensitivity can also be given for systems expressed in the time domain.

For a mathematical model $T(k)$ with k regarded as the only parameter, the sensitivity of $T(k)$ with respect to the parameter k is defined by

$$S_k^{T(k)} \equiv \frac{d \ln T(k)}{d \ln k} = \frac{dT(k)}{dk} \frac{k}{T(k)} \quad (2.15)$$

2.3.3 Error

For the canonical feedback system, the open-loop transfer function is given by

$$GH = \frac{K s^a \prod_{i=1}^{m-a} (s + z_i)}{s^b \prod_{i=1}^{n-b} s + p_i} \quad (2.16)$$

We only consider the case where $b \geq a$ and $l \equiv b - a$.

A canonical system whose open-loop transfer function can be written in the form

$$GH = \frac{K \prod_{i=1}^{m-a} (s + z_i)}{s^l \prod_{i=1}^{n-a-l} s + p_i} \equiv \frac{K B_1(s)}{s^l B_2(s)} \quad (2.17)$$

where $l \geq 0$ and $-z_i$ and $-p_i$ are the nonzero finite zeros and poles of GH , respectively, is called a type l system.

Three criteria of the effectiveness (of feedback) in a stable type l unity feedback system are

- position (step) error constant
- velocity (ramp) error constant
- acceleration (parabolic) error constant

2.3.4 Specifications

We define an open-loop frequency response function $GH(\omega)$. For continuous systems $GH(\omega) \equiv GH(j\omega)$ and for discrete-time systems $GH(\omega) \equiv GH(e^{j\omega T})$. There are seven frequency-domain specifications which we will cover:

- Phase crossover frequency, ω_π
- Gain margin
- Gain crossover frequency, ω_1
- Phase margin, ϕ_{PM}
- Delay time, T_d
- Cutoff frequency, ω_c or f_c
- Bandwidth, BW
- Cutoff rate

- Resonance peak, M_p
- Resonant frequency, ω_p

When using time-domain specifications, we define them in terms of responses to either unit step, ramp or parabolic inputs. We look at both steady state and transient responses. Steady state performance specifications include K_p , K_v and K_a . The transient response performance specifications which we will cover are as follows:

- Overshoot
- Delay time, T_d
- Rise time, T_r
- Settling time, T_s
- Dominant time constant, τ

2.4 Nyquist Analysis and Design

2.4.1 Mapping

Let us consider a complex variable $s = \sigma + j\omega$. We will denote a complex transfer function of s as $P(s)$. Let us also consider a complex variable $z = \mu + j\nu$ and denote a discrete-time (system) complex transfer function of z as $P(z)$. For the first variable and transfer function we create two graphs: (1) the s -plane which has $j\omega$ on the ordinate and σ on the abscissa, (2) the $P(s)$ -plane which has $\text{Im } P$ on the ordinate and $\text{Re } P$ on the abscissa. The function P maps points of the s -plane into the $P(s)$ -plane. Similarly, $P(z)$ is a mapping or transformation from the z -plane to the $P(z)$ -plane. For Nyquist stability plots, the locus of points in the s -plane which are chosen to map is called the Nyquist path. A polar plot is constructed in the $P(s)$ -plane by taking $s = 0 + j\omega$.

2.4.2 Examples

Example 2.4.1. Consider a system with the open-loop transfer function

$$GH_1(s) = \frac{K}{s(s + p_1)(s + p_2)} \quad K_1, p_1, p_2 > 0 \quad (2.18)$$

We create the Nyquist (polar) plot using the following code

```

import numpy
import matplotlib.pyplot as plt
import control

K1 = 1
p1 = 0.5
p2 = 1

numerator = K1
denominator = numpy.poly([0, -p1, -p2])
GH = control.TransferFunction(numerator, denominator)

plt.figure()
control.nyquist(GH, omega_limits=(0.01, 100), omega_num=1000)
plt.show()

```

Example 2.4.2. The general transfer function of a continuous system lead compensator is

$$P_{\text{Lead}}(s) = \frac{s + a}{s + b} \quad b > a \quad (2.19)$$

This compensator has a zero at $s = -a$ and a pole at $s = -b$. The general transfer function of a continuous system lag compensator is

$$P_{\text{Lag}}(s) = \frac{a(s + b)}{b(s + a)} \quad b > a \quad (2.20)$$

However, in this case the zero is at $s = -b$ and the pole is at $s = -a$. The gain factor a/b is included because of the way it is usually mechanized. The general transfer function of a continuous system lag-lead compensator is

$$P_{\text{LL}}(s) = \frac{(s + a_1)(s + b_2)}{(s + b_1)(s + a_2)} \quad b_1 > a_1, b_2 > a_2 \quad (2.21)$$

This compensator has two zeros and two poles. For mechanization considerations, the restriction $a_1 b_2 = b_1 a_2$ is usually imposed.

Example 2.4.3. The general transfer function of a digital lead compensator is

$$P_{\text{Lead}}(z) = \frac{K_{\text{Lead}}(z - z_c)}{z - p_c} \quad z_c > p_c \quad (2.22)$$

This compensator has a zero at $z = z_c$ and a pole at $z = p_c$. Its steady state gain is

$$P_{\text{Lead}}(1) = \frac{K_{\text{Lead}}(1 - z_c)}{1 - p_c} \quad (2.23)$$

The gain factor K_{Lead} is included in the transfer function to adjust its gain at a given ω to a desired value. The general transfer function of a digital lag compensator is

$$P_{\text{Lag}}(z) = \frac{(1 - p_c)(z - z_c)}{(1 - z_c)(z - p_c)} \quad z_c < p_c \quad (2.24)$$

This compensator has a zero at $z = z_c$ and a pole at $z = p_c$. The gain factor $(1 - p_c)/(1 - z_c)$ is included so that the low frequency or steady state gain $P_{\text{Lag}}(1) = 1$, analogous to the continuous-time lag compensator.

Digital lag and lead compensators can be designed directly from s -domain specifications by using the transform between the s -domain and the z -domain defined by $z = e^{sT}$.

Example 2.4.4. A proportional (P) controller has an output u proportional to its input e , that is, $u = K_p e$, where K_p is the proportionality constant. A derivative (D) controller has an output proportional to the derivative of its input e , that is, $u = K_D de/dt$, where K_D is a proportionality constant. An integral (I) controller has an output u proportional to the integral of its input e , that is, $u = K_I \int e(t) dt$, where K_I is a proportionality constant. PD, PI, DI, and PID controllers are combinations of proportional (P), derivative (D), and integral (I) controllers. For example, the output u of a PD controller has the form

$$u_{PD} = K_P e + K_D \frac{de}{dt} \quad (2.25)$$

and the output of a PID controller has the form

$$u_{PID} = K_P e + K_D \frac{de}{dt} + K_I \int e(t) dt \quad (2.26)$$

The transfer function of this PID controller is

$$P_{\text{PID}}(s) \equiv \frac{U_{\text{PID}}(s)}{E(s)} = K_P + K_D s + \frac{K_I}{s} = \frac{K_D s^2 + K_P s + K_I}{s} \quad (2.27)$$

This controller has two zeros and one pole. It is similar to the lag-lead compensator of the previous example except that the smallest pole is at the origin (an integrator) and it does not have the second pole. It is typically mechanized in an analog or digital computer.

2.5 Root Locus Analysis and Design

2.6 Bode Analysis and Design

2.7 Miscellaneous Topics

2.7.1 Non-linear Control Systems

2.7.2 Controllability and Observability

2.7.3 State Feedback

2.7.4 Random Inputs

2.7.5 Optimal Control Systems

2.7.6 Adaptive Control Systems

Chapter 3

Image Processing

3.1 Image Registration

Brown divides image registration into four classes of problems: (1) multimodal registration, (2) template matching, (3) viewpoint registration and (4) temporal registration. Multimodal registration is registration of the same scene acquired from different sensors. Template registration is finding a match for a reference pattern in an image. Viewpoint registration is registration of images taken from different viewpoints. Finally, temporal registration is registration of the same scene taken at different times or under different conditions.

Given two images $I_1(x, y)$ and $I_2(x, y)$, image registration is the mapping between the two images which can be expressed as

$$I_2(x, y) = g(I_1(f(x, y))) \tag{3.1}$$

$$= g(I_1(f_x(x, y), f_y(x, y))) \tag{3.2}$$

