### 3.6.5 Estimating change in density

Frequently, we wish to draw inference on change in density over time, or difference in density between habitats. We consider here simple comparisons between two density estimates. An example involving estimation of trend over time is given in Section 8.4.

Consider two density estimates, $\hat{D}_1$ and $\hat{D}_2$. Suppose first that they are independently estimated. We then estimate the difference in density by $\hat{D}_1 - \hat{D}_2$ with variance

$$\widehat{\text{var}}\left(\hat{D}_1 - \hat{D}_2\right) = \widehat{\text{var}}\left(\hat{D}_1\right) + \widehat{\text{var}}\left(\hat{D}_2\right) \qquad (3.99)$$

Distance provides approximate degrees of freedom $\text{df}_1$ for $\hat{D}_1$ and $\text{df}_2$ for $\hat{D}_2$, based on Satterthwaite's approximation. We can use these to obtain an approximate $t$ -statistic:

$$T = \frac{\left(\hat{D}_1 - \hat{D}_2\right) - (D_1 - D_2)}{\sqrt{\widehat{\text{var}}\left(\hat{D}_1 - \hat{D}_2\right)}} \sim t_{\text{df}} \qquad (3.100)$$

where

$$\text{df} \simeq \frac{\left\{\widehat{var}\left(\hat{D}_1\right) + \widehat{var}\left(\hat{D}_2\right)\right\}^2}{\left\{\widehat{var}\left(\hat{D}_1\right)\right\}^2/\text{df}_1 + \left\{\widehat{var}\left(\hat{D}_2\right)\right\}^2/\text{df}_2} \qquad (3.101)$$

Provided df are around 30 or more, the simpler $z$ -statistic provides a good approximation:

$$Z = \frac{\left(\hat{D}_1 - \hat{D}_2\right) - (D_1 - D_2)}{\sqrt{\widehat{\text{var}}\left(\hat{D}_1 - \hat{D}_2\right)}} \sim N(0,1) \qquad (3.102)$$

For either statistic, we can test the null hypothesis $H_0 : D_1 = D_2$ by substituting $D_1 - D_2 = 0$ in eqn (3.100) or $(3.102)$, and looking at the resulting value in $t$-tables or $z$-tables. Approximate $100(1-2\alpha)\%$ confidence limits for $(D_1 - D_2)$ are given by

$$\left(\hat{D}_1 - \hat{D}_2\right) \pm t_{\text{df}}(\alpha)\sqrt{\widehat{\text{var}}\left(\hat{D}_1 - \hat{D}_2\right)} \qquad (3.103)$$

for df $< 30$, or

$$\left(\hat{D}_1 - \hat{D}_2\right) \pm z(\alpha)\sqrt{\widehat{\text{var}}\left(\hat{D}_1 - \hat{D}_2\right)} \qquad (3.104)$$

otherwise. Often, a single detection function is fitted to pooled data, so that

$$\hat{D}_1 = \frac{n_1\hat{f}(0)\hat{E}_1(s)}{2L_1} \quad \text{and} \quad \hat{D}_2 = \frac{n_2\hat{f}(0)\hat{E}_2(s)}{2L_2} \qquad (3.105)$$

(where the terms $\hat{E}_i(s)$ are omitted if the objects are not in clusters). Because $\hat{f}(0)$ appears in both equations, we can no longer assume that $\hat{D}_1$ and $\hat{D}_2$ are independent. Instead, we can write

$$\hat{D}_i = \hat{M}_i \hat{f}(0), \quad i = 1, 2 \tag{3.106}$$

where

$$\hat{M}_i = \frac{n_i \hat{E}_i(s)}{2L_i} \tag{3.107}$$

As the $M_i$ are independently estimated, and are assumed to be independent of $\hat{f}(0)$, we can now find the variance of $\hat{D}_1 - \hat{D}_2$ using the delta method:

$$\hat{D}_1 - \hat{D}_2 = \left(\hat{M}_1 - \hat{M}_2\right) \hat{f}(0) \tag{3.108}$$

so that

$$\begin{aligned} \widehat{\text{var}}\left(\hat{D}_1 - \hat{D}_2\right) &= \left(\hat{D}_1 - \hat{D}_2\right)^2 \left[ \frac{\widehat{\text{var}}\left(\hat{M}_1 - \hat{M}_2\right)}{\left(\hat{M}_1 - \hat{M}_2\right)^2} + \frac{\widehat{\text{var}}\{\hat{f}(0)\}}{\{\hat{f}(0)\}^2} \right] \\ &= \{\hat{f}(0)\}^2 \widehat{\text{var}}\left(\hat{M}_1 - \hat{M}_2\right) + \left(\hat{M}_1 - \hat{M}_2\right)^2 \widehat{var}\{\hat{f}(0)\} \end{aligned} \tag{3.109}$$

where

$$\widehat{\text{var}}\left(\hat{M}_1 - \hat{M}_2\right) = \widehat{\text{var}}\left(\hat{M}_1\right) + \widehat{\text{var}}\left(\hat{M}_2\right) \tag{3.110}$$

and

$$\widehat{\text{var}}\left(\hat{M}_i\right) = \hat{M}_i^2 \left[ \frac{\widehat{\text{var}}(n_i)}{n_i^2} + \frac{\widehat{\text{var}}\left\{\hat{E}_i(s)\right\}}{\left\{\hat{E}_i(s)\right\}^2} \right], \quad i = 1, 2 \tag{3.111}$$

Note that the second form of eqn (3.109) still applies when $\left(\hat{M}_1 - \hat{M}_2\right) = 0$, whereas the first form leads to a ratio of zero over zero. Inference can now proceed as before, either with additional applications of Satterthwaite's approximation in conjunction with eqn (3.100), if an approximate $t$-statistic is required, or more usually, by straightforward application of eqn (3.102)