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MATH 3101 Assignment 7:

① Draw the Subgroup lattice for $\mathbb{Z}_{24}$:

the Positive divisons of 24 are 1, 2, 3, 4, 6, 8, 12, 24

the Subgroups of $\mathbb{Z}_{28}$ are thus:

$\langle 1\cdot[1]\rangle = \langle[1]\rangle$

$\langle 2\cdot[1]\rangle = \langle[2]\rangle$

$\langle 3\cdot[1]\rangle = \langle[3]\rangle$

$\langle 4\cdot[1]\rangle = \langle[4]\rangle$

$\langle 6\cdot[1]\rangle = \langle[6]\rangle$

$\langle 8\cdot[1]\rangle = \langle[8]\rangle$

$\langle 12\cdot[1]\rangle = \langle[12]\rangle$

$\langle 24\cdot[1]\rangle = \langle[24]\rangle$

Since the t^{th} dirisor of 12 1, 2, 3, 4, 6, & 12, the Subgroups of $\langle[2]\rangle$ are:

$|\langle[2]\rangle| = \frac{24}{2} = 12$

$\langle 1\cdot[2]\rangle = \langle[2]\rangle$

$\langle 2\cdot[2]\rangle = \langle[4]\rangle$

$\langle 3\cdot[2]\rangle = \langle[6]\rangle$

$\langle 4\cdot[2]\rangle = \langle[8]\rangle$

$\langle 6\cdot[2]\rangle = \langle[12]\rangle$

$\langle 12\cdot[2]\rangle = \langle[24]\rangle = \langle[0]\rangle$

$|\langle[3]\rangle| = \frac{24}{3} = 8$, Since the t^{th} divisor of 8 is 1, 2, 4, & 8:

$\langle 1\cdot[3]\rangle = \langle[3]\rangle$

$\langle 2\cdot[3]\rangle = \langle[6]\rangle$

$\langle 4\cdot[3]\rangle = \langle[12]\rangle$

$\langle 8\cdot[3]\rangle = \langle[24]\rangle = \langle[0]\rangle$

$|\langle[4]\rangle| = \frac{24}{4} = 6$, Since the t^{th} divisor of 6 is 1, 2, 3, & 6:

$\langle 1\cdot[4]\rangle = \langle[4]\rangle$

$\langle 2\cdot[4]\rangle = \langle[8]\rangle$

$\langle 3\cdot[4]\rangle = \langle[12]\rangle$

$\langle 6\cdot[4]\rangle = \langle[24]\rangle = \langle[0]\rangle$

$|\langle[6]\rangle| = \frac{24}{6} = 4$, Since the t^{th} divisor of 4 is 1, 2, & 4:

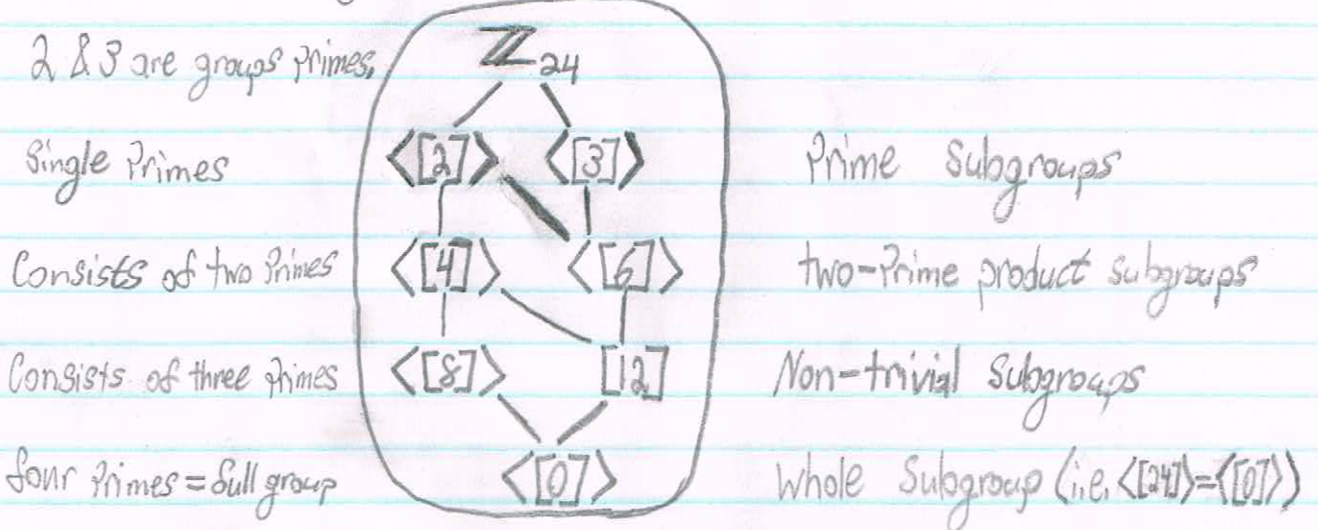
$\langle 1\cdot[6]\rangle = \langle[6]\rangle$

$\langle 2\cdot[6]\rangle = \langle[12]\rangle$

$\langle 4\cdot[6]\rangle = \langle[24]\rangle = \langle[0]\rangle$

$|\langle[8]\rangle| = \frac{24}{8} = 3$, $|\langle[12]\rangle| = \frac{24}{12} = 2$; Since 3 & 2 are prime numbers numbers, it follows that [8] & [12] are non-trivial Subgroups.

Subgroups lattice for $\mathbb{Z}_{24}$:



② let $\phi : \mathbb{Z}_{14} \to \mathbb{Z}_{14}$ be defined by the rule that for $[x] \in \mathbb{Z}_{14}$, $\phi([x]) = [3][x]$. Show that ϕ is an automorphism of $\mathbb{Z}_{14}$:

We must prove an isomorphism from G onto itself:

1) Define Mapping:

let $\phi : \mathbb{Z}_{14} \to \mathbb{Z}_{14}$ $\therefore$ Defined

2) Show ϕ is one-to-one:

Suppose for some $[x], [y] \in \mathbb{Z}_{14}$, $\phi([x]) = \phi([y])$,

then $[3][x] = [3][y]$

Multiply both sides of the above equation by $[3] = [3]^{-1}$,

we get that $[3]([3][x]) = [3]([3][y])$

$\therefore ([3][3])[y] = ([3][3])[y]$

$[1][x] = [1][y]$

$[x] = [y]$ $\therefore$ one-to-one Proven true

3) Show that ϕ is onto:

let $[y] \in \mathbb{Z}_{14}$, Set $[x] = [3][y]$

Then $\phi([x]) = \phi([3][y]) = [3]([3][y]) = ([3][3])[y] = [1][y] = [y]$

$\therefore$ Onto Proven true

4) Show that ϕ preserves the operation:

let $[x], [y] \in \mathbb{Z}_{14}$. Then $\phi([x]+[y]) = \phi([x+y]) = [3][x+y]$

$= [3]([x]+[y]) = [3][x] + [3][y]$

$= \phi([x]) + \phi([y])$ $\therefore$ Operation Preservation Proven true

$\therefore$ the two groups are isomorphic; Since they are both $\mathbb{Z}_{14}$, we can say that ϕ is an automorphism of $\mathbb{Z}_{14}$ $\therefore$ Proven true

③ In your Solution to Problem #2a from Assignment 6, you showed that $H=\{[0],[4],[8]\}$ is a Subgroup of $\mathbb{Z}_{12}$. let $\psi: \mathbb{Z}_{12} \to H$ be defined by the rule that for $[x]\in\mathbb{Z}_{12}$, $\psi([x])=[4x]$. Show that ψ is a homomorphism, & Compute $\mathrm{Ker}(\psi)$:

A homorphism from G to G' is a mapping $\phi: G \to G'$ Such that:

$$\phi(x \circledast y) = \phi(x) \boxast \phi(y)$$

let $[x],[y] \in \mathbb{Z}_{12}$. Then
$$\begin{aligned}\psi([x]+[y]) &= \psi([x+y]) = [4(x+y)] \\ &= [4][x+y] = [4]([x]+[y]) = [4][x]+[4][y] \\ &= \psi([x])+\psi([y])\end{aligned}$$

$\therefore \psi$ is a homomorphism

$$\begin{aligned}\mathrm{Ker}\,\phi &= \{x\in\mathbb{Z} \mid x=Kn \text{ for Some } K\in\mathbb{Z}\} \\ &= \{x\in\mathbb{Z} \mid x=12K \text{ for Some } K\in\mathbb{Z}\}\end{aligned}$$

$\hookrightarrow 12 \equiv [4x] \pmod{12}$

$\hookrightarrow x=0,3,6,9$

$0\cdot4\equiv0 \bmod 12$; $4\cdot3\equiv0 \bmod 12$

$6\cdot4\equiv0 \bmod 12$; $9\cdot4\equiv0 \bmod 12$

$\mathrm{Ker}\,\phi=\{[0],[3],[6],[9]\}$

④ let f be the element of S_6 defined as follows: $f(1)=3, f(2)=2, f(3)=4, f(4)=1, f(5)=6,$ & $f(6)=5$. let g be the element of S_6 defined as follows: $g(1)=2, g(2)=3, g(3)=1, g(4)=6, g(5)=5,$ & $g(6)=4$. Compute $f\circ g$ & $g\circ f$:

f has the following matrix representation:

$$\begin{bmatrix}1&2&3&4&5&6\\3&2&4&1&6&5\end{bmatrix}$$

g has the following matrix representation:

$$\begin{bmatrix}1&2&3&4&5&6\\2&3&1&6&5&4\end{bmatrix}$$

Calculate $g\circ f$:

$$g\circ f \begin{array}{c} f \\ g \end{array}\left.\begin{array}{cccccc}1&2&3&4&5&6\\3&2&4&1&6&5\\3&1&6&5&4&2\end{array}\right] \Rightarrow g\circ f=\begin{bmatrix}1&2&3&4&5&6\\3&1&6&5&4&2\end{bmatrix}$$

Calculate $f\circ g$:

$$f\circ g \begin{array}{c} g \\ f \end{array}\left.\begin{array}{cccccc}1&2&3&4&5&6\\2&3&1&6&5&4\\2&4&1&6&5&3\end{array}\right] \Rightarrow f\circ g=\begin{bmatrix}1&2&3&4&5&6\\2&4&1&6&5&3\end{bmatrix}$$

(5) Determine whether the following statements are true or false. Justify your responses:

a) Any two groups of order 4 are isomorphic to one another:

let $G_1 = \mathbb{Z}_4$ & $G_2 = V_4$

↳ this is the Klein four group (See pg. 190 Q10)

$G_2 = V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, has 3 elements of order 2

$G_1 = \mathbb{Z}_4$, has one element of order 2

thus, we cannot satisfy the requirment of bijection since we cannot map between 3 & 1 elements w/ a one-to-one correspondence.

$\therefore$ False

b) Every automorphism is a homomorphism:

An automorphism is an isomorphism from a group G to G itself

↳ So all automorphisms are isomorphisms

A homomorphism from G to G' is a mapping $\phi: G \to G'$ such that:

$$\phi(x \circledast y) = \phi(x) \boxtimes \phi(y) \text{ for all } x,y \in G$$

A isomorphism from G to G' is a mapping $\phi: G \to G'$ such that:

I) ϕ is a one-to-one correspondence (bijection) from G to G'

II) $\phi(x \circledast y) = \phi(x) \boxtimes \phi(y)$ for all $x,y \in G$

thus, by property two of isomorphisms, we note that all isomorphisms are homomorphisms. Since all automorphisms are isomorphisms, we can clearly see all Automorphisms are isomorphisms, & all isomorphisms are homomorphisms.

↳ thus, we infer that all Automorphisms are homomorphisms

$\therefore$ True

c) If $\phi: G \to G'$ is a homorphism, then $Ker(\phi)$ is a subgroup of G:

By Theorm 3.32:

- $\phi(e) = e'$, thus the identity is always in the Kernel

  ↳ thus, the Kernel's non-empty

- Assume $x, y \in Kernel\,\phi$, So $\phi(x) = \phi(y) = e'$, then

$$\phi(xy) = \phi(x) \cdot \phi(y)$$
$$= e' \cdot e' = e'$$

thus, $ab \in Kernel\,\phi$ So the Kernel's closed under products

- Assume $\phi(x) = e$, then $\phi(x^{-1}) = \phi(x)^{-1} = e'$

  thus, the Kernel's closed for inverses

Therefore, the Kernel's a subgroup.

$\therefore$ True

d) $|S_6| = 720$

$|S_6|$ asks to find the order of the group

By lagranges Theorm:

$$|G| = |H| \cdot |G:H|$$

Since the Symmetric group is Symmetric, we see:

$720/6 = 120$
$120/5 = 24$
$24/4 = 6$
$6/3 = 2$
$2/2 = 1$
$1/1 = 1$

this makes sence since each Subgroup H of G divides it by the above theorm, & Symmetric groups have Symmetry.

Thus! $|S_n| = n!$

$|S_6| = 6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

$\boxed{|S_6| = 720}$

$\therefore$ True