MATH 3855 Assignment Two:

① let b_n be the number of different ways in which he can climb n stairs in this fashion. It is easy to see that $b_1 = 1$ & $b_2 = 2$:

a) Find a recurrence relation for the sequence $\{b_n\}$:

we can simplify this by breaking the ways of climbing the stairs into two steps:

I) The last stride was is one stair:

- The first n−1 stairs can be climbed in any valid set of ways $\therefore b_{n-1}$

II) The last stride was two stairs:

- Since what was shown above, we must see that: b_{n-2} is representative for the second part

Thus: $b_n = b_{n-1} + b_{n-2}$ $(n \geq 3)$

As is given, it becomes clear how $b_1 = 1$ & $b_2 = 2$

This pattern results in recursion over b_n

this gives us the recurrence relation: $b_n = b_{n-1} + b_{n-2}$

b) Solve the above recurrence relation:

This Recurrence can be expanded:

If $n=3$:

$$b_3 = b_{3-1} + b_{3-2} = b_2 + b_1 = 1 + 2 = 3$$

If $n=4$:

$$b_4 = b_{4-1} + b_{4-2} = b_3 + b_2 = 3 + 2 = 5$$

If $n=5$:

$$b_5 = b_{5-1} + b_{5-2} = b_4 + b_3 = 5 + 3 = 8$$

If $n=6$:

$$b_6 = b_{6-1} + b_{6-2} = b_5 + b_4 = 8 + 5 = 13$$

If $n=7$:

$$b_7 = b_{7-1} + b_{7-2} = b_6 + b_5 = 13 + 8 = 21$$

We note the following pattern:

this follows the sequence of numbers in the fibonacci sequence for $n \geq 3$. The fib. sequence goes $\{0,1,1,2,3,5,8,13,21,\ldots\}$; so removing the first two entries (for $n \geq 3$) gives the sequence $\{1,2,3,5,8,13,21,\ldots\}$ matching what the above recurrence gives.

the fib. sequence is of the form:

$$f_{n+1} = f_n + f_{n-1} \quad \text{for all } n \geq 2$$

Due to the above Pattern it is clear that:

$b_n = f_{n+1}$ for all $n \geq 3$ in b_n

the fib. Sequence is a well understood recurrence, but to Simplify it we can Show the following:

Since: $f_{n+1} = f_n + f_{n-1}$

$x^2 = 1, x = -1, b = -1$

When $f_{n+1} - f_n - f_{n-1} = 0$ Since x^2, x, b are by increasing subscript values.

the Auxiliary equation is thus:

$$0 = t^2 - t - 1$$

to Solve this:

$$\frac{-(b) \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2(1)}$$

$$= \frac{1 \pm \sqrt{5}}{2}$$ thus the roots are $p = \frac{1+\sqrt{5}}{2}$ & $q = \frac{1-\sqrt{5}}{2}$.

Hence, $f_n = Ap^n + Bq^n$:

Since $n=1$ is $b_1 = 1$ & $n=2$ is $b_2 = 2$:

(I) $A\left(\frac{1+\sqrt{5}}{2}\right)^1 + B\left(\frac{1-\sqrt{5}}{2}\right)^1 = 1$

(II) $A\left(\frac{1+\sqrt{5}}{2}\right)^2 + B\left(\frac{1-\sqrt{5}}{2}\right)^2 = 2$

We know the Sequence is Shifted from the fib. Sequence:

$$b_n = Ap^{n+1} + Bp^{n+1}$$

Since:

(I) $Ap = 1 - Bq$ We Sub. into (II)

$$\left(1 - B\left(\frac{1-\sqrt{5}}{2}\right)\right)\left(\frac{1+\sqrt{5}}{2}\right) + B\left(\frac{1-\sqrt{5}}{2}\right)^2 = 2$$

$$B\left(\frac{1-2\sqrt{5}+5}{2} - \frac{1-\sqrt{5}}{2}\right) = 2 - \frac{1+\sqrt{5}}{2}$$

$$B = \frac{5-\sqrt{5}}{10} \quad \text{So} \quad A = \frac{1-Bq}{p} = \frac{1 - \left(\frac{5-\sqrt{5}}{10}\right)\left(\frac{1-\sqrt{5}}{2}\right)}{\left(\frac{1+\sqrt{5}}{2}\right)} = \frac{5+\sqrt{5}}{10}$$

By factoring, we note:

$$\boxed{b_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right)}$$

② Given a sequence $\{q_n\}$ with $q_0 = 0$. Assume that $\{q_n\}$ satisfies a non-homogenous recurrence relation as follows:

$$q_{n+1} - 3q_n = 5^n \quad (n \geq 0) \rightarrow 0 = q_{n+1} - 3q_n - 5^n$$

a) Find the generating function $Q(x)$ for $\{q_n\}$:

For the sequence;

If $n=0$:

$$q_{0+1} = 5^0 + 3q_0 \rightarrow q_1 = 5^0 + 3(0) = 1$$

If $n=1$:

$$q_{1+1} = 5^1 + 3q_1 \rightarrow q_2 = 5 + 3(1) = 8$$

If $n=2$:

$$q_{2+1} = 5^2 + 3q_2 \rightarrow q_3 = 25 + 3(8) = 49$$

If $n=3$:

$$q_{3+1} = 5^3 + 3q_3 \rightarrow q_4 = 272$$

Thus, the sequence is $q_n = \{0, 1, 8, 49, 272, \ldots\}$

Using the standard method: [Note: $q_{n+1} = 5^n + 3q_n$]

$$Q(x) = q_0 + q_1x + q_2x^2 + q_3x^3 + \ldots = q_0 + (3q_0 + 5^0)x + (3q_1 + 5^1)x^2 + (3q_2 + 5^2)x^3 + \ldots$$

$$= q_0 + 3x[q_0 + q_1x + q_2x^2 + \ldots] + x[5^0 + 5^1x + 5^2x^2 + \ldots]$$

$$= q_0 + 3xQ(x) + x(1 + 5x + 5^2x^2 + \ldots)$$

this occurs since $q_{n+1} = 5^n + 3q_n$ lets us sub. for q_n's in $Q(x)$'s original equation.

Since we know $q_0 = 0$, we can rearrange:

$$Q(x) - 3xQ(x) = q_0 + x(1 + 5x + 5^2x^2 + \ldots)$$

$$\rightarrow (1-3x)Q(x) = 0 + \frac{x}{1-5x} \rightarrow Q(x) = \frac{x}{(1-5x)(1-3x)}$$

thus, the recurrence relation is:

$$\boxed{Q(x) = \frac{x}{(1-5x)(1-3x)}}$$

b) Show that $q_n = \frac{1}{2}(5^n - 3^n)$: [Start w/ partial fraction decomp.]

$$Q(x) = \frac{x}{(1-5x)(1-3x)} = \frac{1}{\ell}\left(\frac{1}{1-5x} - \frac{1}{1-3x}\right)$$

Since: $[(1-3x) - (1-5x)] = [-3x + 5x] = 2x$, thus if:

$$\frac{1}{\ell}\left(\frac{1}{1-5x} - \frac{1}{1-3x}\right) = \frac{1}{\ell}\left(\frac{2x}{(1-5x)(1-3x)}\right) = \frac{x}{(1-5x)(1-3x)} \text{ then } \ell = 2$$

thus: $Q(x) = \frac{1}{2}\left(\frac{1}{1-5x} - \frac{1}{1-3x}\right)$

By Theorm 25.3:

under the binomial expansions for $(1-5x)^{-1}$ & $(1-3x)^{-1}$ we obtain:

$$Q(X) = \frac{1}{2}\left(\left[1+5x+(5x)^2+\dots\right]-\left[1+3x+(3x)^2+\dots\right]\right)$$

So that:

$$\boxed{q_n = \frac{1}{2}\left(5^n-3^n\right)}$$

3

a) Show that $x(1+x)/(1-x)^3$ is the generating function for the Sequence whose n^{th} term is n^2:

$$\text{Let } P(X) = \frac{X(1+X)}{(1-X)^3} = -\frac{(X)(X+1)}{(X-1)^3}$$

Using Partial fractions:

$$P(X) = \frac{X(1+X)}{(1-X)^3} = \frac{-A}{X-1} + \frac{-B}{(X-1)^2} + \frac{-C}{(X-1)^3}$$

$\hookrightarrow X(X+1) = A(X-1)^2 + B(X-1) + C$

$\hookrightarrow 0 = A - B + C$

$\hookrightarrow 1 = -2A + B$

$\hookrightarrow 1 = A$

$A = 1$ so $\rightarrow$ $B - C = 1$, $B = 3$; $B = 3$ so $\rightarrow C = 2$

$$P(X) = \frac{-1}{(X-1)} + \frac{-3}{(X-1)^2} + \frac{-2}{(X-1)^3} = (X-1)^{-1} + 3(X-1)^{-2} + 2(X-1)^{-3}$$

thus, by Theorm 25.3 & the binomial expansions:

$$P(X) = -(1 + X + X^2 + \dots) - 3(1 + 2x + 3x^2 + \dots) - 2(1 + 3x + 6x^2 + 10x^3 + \dots)$$

as:

$$(1-ax)^{-m} = 1 + max + \dots + \binom{m+n-1}{n} a^n x^n + \dots\dots$$

thus; the series representation for this is:

$$\Rightarrow \sum_{n=0}^{\infty} X^n n^2 \text{ for } P(X)$$

$\therefore$ the n^{th} term has a coefficient of n^2, meaning it is Proven

$$\left[P(x) = -\sum_{K=0}^{\infty} x^n - 3\sum_{K=0}^{\infty} (1+n)x^n - \sum_{K=0}^{\infty} (n+1)(n+2)x^n\right]$$

$\rightarrow = \binom{n}{0}$ $\quad \rightarrow = 3\binom{n}{1}$ $\quad \rightarrow = 2\binom{n}{2}$ for $n \geq 2$

Note that:

$$P(x) \Rightarrow \left[\binom{1+n-1}{n} + 3\binom{2+n-1}{n} + 2\binom{3+n-1}{n}\right](-x^n)$$

$$= -\left[\binom{n}{0} + 3\binom{n}{1} + 2\binom{n}{2}\right]x^n = -\left[(n+1)^2\right]x^n \text{ for } n \geq 2$$

if we change the indexing such that $n \geq 1$ like the rest of the function $\rightarrow (n+1)^2 \rightarrow (n - \cancel{1} + \cancel{1})^2 = n^2$

See: $\binom{n}{0} = 1$, $\binom{n}{1} = n$, $\binom{n}{2} = \frac{(n)(n-1)}{2}$

b) Let $A(x)$ be the generating function for the Sequence $\{a_n\}$ & define $S_n = a_0 + a_1 + \dots + a_n$ $(n \geq 0)$

Show that the generating function for the Sequence $\{a_n\}$ is $S(x) = \frac{A(x)}{1-x}$;

We know that $(1-x)^{-1} = (1+x+x^2+x^3+\dots+x^n)$ & that S_n is defined as a Series of a_i from i between 0 to n.

$\hookrightarrow (1-a_i)^{-1} = S_n$ for i from 0 to n

As $A(x)$ is the generating function for the Sequence $\{a_n\}$ & the Sequence is defined along x, we note:

We can express $(1-a_i)^{-1}$ as $(1-x)^{-1}$ when x is the x used in $A(x)$ for $S(x)$

Note: [we see the Pattern emerge]

for $u(x) = u_0 + u_1 x + u_2 x^2 + \dots + u_n x^n$

$= (u_0) + (u_1 x) + (u_2 x^2) + \dots$

let $u_i = a_i$ Since the Series correspond

$u(x) = S_n$ as each $a_i \rightarrow u_i x^i$

$(1-x) S(x) = A(x)$ So $S(x) = \frac{A(x)}{1-x}$

c) use the results of a & b to find a formula for $\Sigma_{i=0}^{\infty} i^2$;

we note that:

$\Sigma_{i=0}^{\infty} i^2$ has a generating function of $\frac{x(x+1)}{(1-x)^3}$ by Q.a;

$$A(x) = \frac{x(1+x)}{(1-x)^3} \text{ Such that } s_n = i_0 + i_1 x + i_2 x^2 + \dots + i_n^n$$

for the Sequence i defined by $\Sigma_{i=0}^{n} i^2$

thus $I(x) = A(x)/(1-x) = [x(1+x)]/[1-x]^4$

Using Partial fractions:

$$A(x) = \frac{x(1+x)}{(1-x)^4} = \frac{A}{(x-1)^2} + \frac{B}{(x-1)^3} + \frac{C}{(x-1)^4}$$

$\Rightarrow x(1+x) = (x-1)^2 A + (x-1)B + C = A[x^2-2x+1] + B[x-1] + C$

$\Rightarrow 0 = A - B + C$; $B - C = 1$; $C = B - 1 = 3 - 1 = 2$

$\Rightarrow 1 = -2A + B$; $3 = B$

$\Rightarrow 1 = A \Rightarrow A = 1$

$$A(x) = \frac{1}{(x-1)^2} + \frac{3}{(x-1)^3} + \frac{2}{(x-1)^4}$$

$$= (x-1)^{-2} + 3(x-1)^{-3} + 2(x-1)^{-4}$$

Thus, by Theorm 25.3 & Binomial expansions:

$$A(x) = (1 + 2x + 3x^2 + \dots) + 3(1 + 3x + 6x^2 + 10x^3 + \dots) + 2(1 + 4x + 10x^2 + 20x^3), \text{ as: } (1-ax)^{-m} = 1 + max + \dots + \binom{m+n-1}{n} a^n x^n + \dots$$

Note that:

$$A(x) \Rightarrow x^n\left[\binom{n}{1} + 3\binom{n}{2} + 2\binom{n}{3}\right] = x^n\left[n + 3\left(\frac{n(n+1)}{2}\right) + \left(\frac{n(n+1)(n+2)}{6}\right)2\right]$$

$$= x^n\left[n + \frac{3n(n-1)}{2} + \frac{n(n-1)(n-2)}{3}\right]$$

$$= x^n\left[n + \frac{9n^2 - 9n + 2n(n^2 - 3n + 2)}{6}\right]$$

$$= x^n\left[\left(\frac{1}{6}\right)(6n + \cancel{9n^2} - \cancel{9n} + 2n^3 - \cancel{6n^2} + \cancel{4n})\right]$$

$$= x^n\left[\tfrac{1}{6}(n + 3n^2 + 2n^3)\right] = x^n\left[1/6(n)\underbrace{(2n^2 + 3n + 1)}_{\Rightarrow = (2n+1)(n+1)}\right]$$

thus, the Series representation for $A(x)$ is:

$$A(x) = \Sigma_{n=0}^{\infty} \tfrac{1}{6}(n)(2n+1)(n+1)$$ thus a formula for $\Sigma_{i=0}^{\infty} i^2$

is $\Sigma_{i=0}^{\infty} i^2 = \frac{1}{6}(n)(2n+1)(n+1)$

④ Use the generating functions to find the number of Partitions of 16 in which each part is an odd prime. Give explicitly all those Partitions :

firstly, we note :

odd Prime numbers less then 16 : $\{3,5,7,11,13\} = P$

the generating functions for list P is :

$$(1-x^3)^{-1}(1-x^5)^{-1}(1-x^7)^{-1}(1-x^{11})^{-1}(1-x^{13})^{-1}$$

the terms up to 16 are :

③ $f_1(x): 1+x^3+x^6+x^9+x^{12}+x^{15}$

⑤ $f_2(x): 1+x^5+x^{10}+x^{15}$

⑦ $f_3(x): 1+x^7+x^{14}$

⑪ $f_4(x): 1+x^{11}$

⑬ $f_5(x): 1+x^{13}$

So, we have :

$$f_1(x)f_2(x)f_3(x)f_4(x)f_5(x) = (1+x^3+x^6+x^9+x^{12}+x^{15})(1+x^5+x^{10}+x^{15})(1+x^7+x^{14})(1+x^{11})(1+x^{13})$$

$$= f_1(x)f_2(x)f_3(x)(1+x^{11}+x^{13}+\cancel{x^{24}})$$

$$= f_1(x)f_2(x)(1+x^7+x^{14}+x^{11}+\cancel{x^{18}}+\cancel{x^{25}}+x^{13}+\cancel{x^{20}}+\cancel{x^{27}}+\cancel{x^{24}}+\cancel{x^{31}}+\cancel{x^{38}})$$

$$= f_1(x)(1+x^7+x^{14}+x^{11}+x^{13}+x^5+x^{12}+\cancel{x^{19}}+x^{16}+\cancel{x^{19}}+x^{10}+\cancel{x^{17}}+\cancel{\cdots}+x^{15}+\cancel{\cdots})$$

$$= (1+x^3+x^6+x^9+x^{12}+x^{15})(1+x^5+x^7+x^{10}+x^{11}+x^{12}+x^{13}+x^{14}+x^{15}+x^{16}+\cancel{\cdots})$$

We Can now Count the terms Which Contribute values to the Coefficient of x^{16} :

I. $13+3=16$ III. $7+3+3+3=16$

II. $11+5=16$ IV. $5+5+3+3=16$

thus, by I to IV, we See there are 4 Partitions for the required type.

⑤ Use the generating function method to find the number of partitions of 20 with four parts which are 1, 2, 4, & 10:

the parts are $\{1,2,4,10\}$, so the generating function for these partitions is:

$$1:\ (1-x)^{-1}$$
$$2:\ (1-x^2)^{-1}$$
$$4:\ (1-x^4)^{-1}$$
$$10:\ (1-x^{10})^{-1}$$

thus, the solution is the coefficient of x^{20} in:

$$(1-x)^{-1}(1-x^2)^{-1}(1-x^4)^{-1}(1-x^{10})^{-1}$$

We now look for the coefficient of x^{20}, so only the terms up to x^{20} are needed:

$$(1+x+x^2+x^3+\dots+x^{19}+x^{20})(1+x^2+x^4+x^6+\dots+x^{18}+x^{20})(1+x^4+x^8+x^{12}+x^{16}+x^{20})$$
$$\times(1+x^{10}+x^{20})$$
$$=(1+x+x^2+\dots+x^{20})(1+x^2+x^4+\dots+x^{20})(1+x^4+x^8+x^{12}+x^{16}+x^{20}+x^{10}+x^{14}+x^{18}$$
$$+x^{20}+\dots)$$
$$=(1+x+x^2+\dots+x^{20})(1+x^2+x^4+\dots+x^{20})(1+x^4+x^8+x^{10}+x^{12}+x^{14}+x^{16}+x^{18}+2x^{20})$$
$$=(1+x+x^2+\dots+x^{20})(1+x^4+x^8+x^{10}+x^{12}+x^{14}+x^{16}+x^{18}+2x^{20}+x^2+x^6+x^{10}+x^{18}+x^{14}$$
$$+x^{16}+x^{18}+x^{20}+x^4+x^8+x^{12}+x^{14}+x^{16}+x^{18}+x^{20}+x^6+x^{10}+x^{14}+x^{16}+x^{18}+x^{20}$$
$$+x^8+x^{12}+x^{16}+x^{18}+x^{20}+x^{10}+x^{14}+x^{18}+x^{20}+x^{12}+x^{16}+x^{20}+x^{14}+x^{18}+x^{16}+x^{20}$$
$$+x^{18}+x^{20}+\dots)$$
$$=(1+x+x^2+\dots+x^{20})(1+x^2+2x^4+2x^6+3x^8+4x^{10}+5x^{12}+6x^{14}+7x^{16}+8x^{18}+10x^{20})$$
$$=10x^{20}+8x^{20}+7x^{20}+6x^{20}+5x^{20}+4x^{20}+3x^{20}+2x^{20}+2x^{20}+x^{20}+x^{20}+\dots$$
$$=49x^{20}+\dots \Rightarrow \text{the coefficient of } x^{20} \text{ is } 49$$

∴ We see there are 49 partitions for the required type

⑥ Use the generating function method to find the number of ways to write $11 = (S_1) + (S_2) + (S_3) + (S_4) + (S_5)$ Such that each S_i is either 0 or a Sum of 1's for at most 5 times:

— the S_i's are values of either 0, 1, 2, 3, 4, or 5:

$$S_i = \{0,1,2,3,4,5\} \text{ for } i\text{'s from 1 to 5}$$

— We know there are 5 S_i variables:

$$(f(x))^5$$

— We can conclude their's a generating function of:

$$(1 + x + x^2 + x^3 + x^4 + x^5)^5$$

- Where each x represents the Sums from 1 to 5 & the Coefficient of 1 represents the Case nothing's given
- the Power of 5 over the function represents the fact there are 5 S_i's to distribe the value of 11 along
- Since $f(x) = 1+x+x^2+x^3+x^4+x^5$, we can rewrite to the form:

$$f(x) = \frac{1-x^6}{1-x}$$

- Since we are Summing upto 11, we must look for the coefficient of x^{11} in the function
- We use the geometric Series as:

$$a + ar + ar^2 + \ldots + ar^n = a \times \frac{r^n - 1}{r-1}, \text{ which Simplifies } f(x)$$

- We derive the following Generating Relation:

$$A(x) = ([1-x^6]/[1-x])^5$$

— Simplifing for x^{11} we note

$$= \left(\frac{1-x^6}{1-x}\right)^5 = (1-x^6)^5 \left(\frac{1}{1-x}\right)^5$$

(Binomial expansion by modulo x^{11})

$$= [x^{11}]\left(\sum_{i=0}^{5} \binom{5}{i}(-1)^i (x^{6i})\right)\left(\sum_{i=0}^{\infty} \binom{-5}{i}(-x)^i\right)$$

(Ignore all x^i if $i > 11$)

$$= \left[\binom{5}{0}[x^{11}] - \binom{5}{1}[x^5]\right] \underbrace{\sum_{i=0}^{\infty} \binom{n+4}{4} x^n}$$

chooses n based off x^i in left equation

$$= \binom{5}{0}\binom{11+4}{4} - \binom{5}{1}\binom{5+4}{4} = (1)(1365) - 5(126) = \boxed{735}$$

[We can also find that $x^{11} = 735$ if we expand $(1+x+x^2+x^3+x^4+x^5)^5$ into $x^{25} + 5x^{24} + \ldots + 735x^{11} + \ldots + 5x + 1$]

∴ there are a total of 735 ways to write 11 based off the S_i Summation Constraint

⑦ Show that the number of Partitions of a positive integer n where **all** Parts appear less than 4 times equals the number of Partitions of n where no part is divisible by 4:

I) the number of Partitions of a positive integer n where all Parts appear less than 4 times:

this is requesting the following:

$$x_1 + 2x_2 + 3x_3 + \dots + (n-1)x_{n-1} + nx_n \;,\; \text{When } 0 \le x_i \le 3 \text{ for all } i$$

to find the generating function for this is:

$$\prod_{i=1}^{\infty}(1+x^i+\dots+x^{Ki}) \Rightarrow \prod_{i=1}^{\infty}(1+x^i+x^{2i}+x^{3i})$$ as each part appears at most four times,

this **equals**:

$$\prod_{i=1}^{\infty}(1+x^i+x^{2i}+x^{3i}) = (1+x+x^2+x^3)(1+x^2+x^4+x^6)(1+x^3+x^6+x^9)\dots$$
$$\dots(1+x^{\ell}+x^{2\ell}+x^{3\ell})\dots$$

where ℓ is a value in the range of the Partition

II) the number of Partitions of n where no part is divisible by 4:

Since the partitions for the numbers $P(n)$ of Partitions of n can be written as the infinite product:

$$P(x) = \prod_{i=1}^{\infty}(1-x^i)^{-1}$$

Since no Part's divisible by 4:

$$P(x) = \prod_{i=1,4\nmid i}^{\infty}(1-x^i)^{-1}$$

Since I & II, we can use generating function relationships for the Proof:

let $P(x)$ & $D(x)$ be the generating functions:

I) $D(x) = \prod_{i=1}^{\infty}(1+x^i+x^{2i}+x^{3i}) = (1+x+x^2+x^3)\dots\dots = \prod_{i=1}^{\infty}\left(\frac{1-x^{(3+1)i}}{1-x^i}\right)$

II) $P(x) = \prod_{i=1,4\nmid i}^{\infty}(1-x^i)^{-1} = (1-x)^{-1}(1-x^2)^{-1}(1-x^3)^{-1}\dots\dots$

Since: $\prod_{i=1}^{\infty}(1+x^i+\dots+x^{Ki}) = \prod_{i=1}^{\infty}\left(\frac{1-x^{(K+1)i}}{1-x^i}\right)$ & $K=4-1$ as each part appears at most four times

thus, we note the following Pattern Emerges:

$$D(x)\prod_{i=1}^{\infty}(1-x^i) = \prod_{i=1}^{\infty}\left(\frac{1-x^{4i}}{\cancel{1-x^i}}\right)\cancel{\prod(1-x^i)} = \prod_{i=1}^{\infty}(1-x^{4i})$$

$$= \prod_{i=1,4\mid i}^{\infty}(1-x^i) = \prod_{i=1,4\nmid i}^{\infty}(1-x^i)^{-1}\prod_{i=1}(1-x^i)$$

$$= P(X) \prod_{i=1}^{\infty}(1-X^i)$$

Since: $D(x)\cancel{\prod_{i=1}^{\infty}(1-x^i)} = P(X)\cancel{\prod_{i=1}^{\infty}(1-x^i)}$ we can Conclude $D(x)=P(x)$

$\therefore$ It is Proven

(8) In Mathematics, a Composition of an integer n is a way of writting n as the Sum of a Sequence of (Strictly) positive integers. Two sequences that differ in the order of their terms define different Compositions of their Sum, while they are considered to define the Same partition of that number:

a) Show that the number of compositions of n into exactly K Parts is $\binom{n-1}{K-1}$:

We can view this as there being a partition of n elements with $(K-1)$ breaks between them.

For example, to partition 7 elements into 3 Parts we can:

I) $X_1, X_2, X_3, X_4, X_5, X_6, X_7$

$\Downarrow$ Partition into 3 Parts

II) $X_1, X_2 \mid X_3, X_4 \mid X_5, X_6, X_7$

$n = 7$ elements

$K = 3$ Part-Partition

- as can be seen in I, there are 6 Commas (,) between each X_i for i's from 1 to 7. Thus, there are $n-1$ Spaces.
- as can be seen in II, we choose exactly 2 Commas (marked as '|') to partition. Thus, there are $K-1$ Spaces.
- Note that since the Partitions cannot be empty, there are no Partition marks ('|') at the beginnings, next-to-eachother, or at the end.
- Thus, there are $(n-1)$ Spaces for $(K-1)$ Splits
  - $\Rightarrow$ therefore, there are $(n-1)$ places to be Chosen for $(K-1)$ objects. This is a simple $(n-1)$ Choose $(K-1)$, where we choose $K-1$ Spots from the pool of $n-1$ total Spots.

$\therefore$ $(n-1)$ Choose $(K-1)$ is Simplified as $\binom{n-1}{K-1}$

It is Proven

b) let $P_d(n,K)$ be the number of Partitions of n into K distinct Parts, Show that:

$$P_d(n,K) \leq \frac{1}{K!}\binom{n-1}{K-1} \leq P_K(n)$$

Firstly, since $P_d(n,K)$ is a division of n into K distinct Parts, we Know $P_d(n,K)$ is a Proper Subset of $P_K(n)$ because:

$$P_K(n) = \sum_{K=1}^{n} P(n,K) \text{ for all } n \geq 1$$

↳ So for Some K (K must be less then or equal to n Since you Cant make more Splits then elements) we Know that $P_d(n,K) \in P_K(n)$ Since $P_d(n,K)$ is in $\sum_{K=1}^{n} P(n,K)$

thus, we Know that $P_d(n,K) \leq P_K(n)$

to Partition n into K Parts we note the following:

$P_n = \sum_{K=1}^{n} P_K(n)$, theres a Correspondence between the Partitions of n into K Parts, & the Partitions of n−K into K parts together with the Partitions of n−1 into K−1 parts, so: $P_K(n) = P_K(n-K) + P_{K+1}(n-1)$

It is Clear that two Cases exist:

I) $P_K(n) \rightarrow$ Partitions of $n-1$ into $K-1$ Parts Since the Smallest Parts Size's 1 (Since were using Integer Partitions),

II) $P_K(n) \rightarrow$ otherwise Partitions of $n-K$ into K Parts with a 1−1 Correspondence to the above,

thus, it is Clear that $P_K(n) = \sum_K^n \frac{n!}{(n-K)!}$ is the case, Since repitition is allowed, Also, as $P_K(n)$ has two valid Cases [$\binom{n-1}{K-1}$ & $\binom{n-K}{K}$, which Can represent repeats, we Note]:

$$\frac{1}{K!}\binom{n-1}{K-1} \leq \frac{n!}{(n-K)!} = P_K(n)$$

Ferthermore, as $P_d(n,K)$ has no repeats, it, unlike above is:

this is a Stirling Number ~~of the~~ Second Kind:

$$\left\{ {n \atop K} \right\} = \frac{1}{K!}\sum_{i=0}^{K}(-K)^i\binom{K}{i}(K-i)^n$$

as $K < n$ we see its Clear that $\left\{ {n \atop K} \right\}$ is less then $\frac{1}{K!}\binom{n-1}{K-1}$:

$$\left\{ {n \atop K} \right\} = P_d(n,K) \leq \frac{1}{K!}\binom{n-1}{K-1}$$

c) let $m = n + \binom{k}{2}$. Show that:

$$P_k(n) \leq \frac{1}{k!}\binom{m-1}{k-1}$$

We note:

$$\left[\begin{array}{l} n = m - \binom{k}{2} \\ P_k\left(m - \binom{k}{2}\right) = P_k\left(m - \binom{k}{2} - k\right) + P_{k+1}\left(m - \binom{k}{2} - 1\right) \end{array}\right.$$

this follows I & II from b.

In m we see that:

$n + \binom{k}{2}$ results in the choice of two for k, this has the effect to allow repitition in the numerator since:

$$\binom{2}{2} = 1, \binom{3}{2} = 3, \binom{4}{2} = 6, \binom{5}{2} = 10, \ldots$$

In each case, we get the total number of possible permutations for the elements being removed (i.e. $3C2 = 3$ means we can choose 3 ways).

Since $P_k(n)$ counts the total permutations of k for n, & $\binom{m-1}{k-1}$ chooses from permutations since m includes them w/ its $\binom{k}{2}$ term allowing for non-distinct partitions, we see that $P_k(n) \leq \frac{1}{k!}\binom{m-1}{k-1}$ for much the same reason $\left\{ {n \atop k} \right\} \leq \frac{1}{k!}\binom{n-1}{k-1}$. Only now the term $\binom{m-1}{k-1}$ represents permutations, not combinations.

By extension, it's clearly the case that:

$$P_n(n) \leq \frac{1}{k!}\binom{m-1}{k-1}$$