MATH 3801 Assignment Four:

① Consider the linear Programming Program:

$$(LP)\quad \begin{aligned} &\min\ c^T x \\ &\text{s.t. } Ax = b \\ &\quad\ x \geqslant 0 \end{aligned}$$

Where $A = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 1 \end{bmatrix}$, $b = \begin{bmatrix} 6 \\ 10 \\ 4 \end{bmatrix}$, $C = \begin{bmatrix} -1 \\ -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, & $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ is a tuple of Variables.

you are given that $B = \{1,2,3\}$ determines the basic feasible Solution

$$x^* = \begin{bmatrix} 1 \\ 5 \\ 3 \\ 0 \\ 0 \end{bmatrix}:$$

a) Show that B is an optimal basis for (LP):

the basic Solution determined by B is:

$x^* = \begin{bmatrix} 1 \\ 5 \\ 3 \\ 0 \\ 0 \end{bmatrix}$, Which is a feasible Solution to the Program.

thus, B is a feasible basis, Since the following $Ax^* = b$ is valid:

$$1 + 5 + 3(0) = 6$$
$$2(1) + 5 + 3 = 10$$
$$-(1) + 5 + 3(0) = 4$$

Now, Solving for y^* in $y^{*T} A_B = C_B^T$ gives:

$$\begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 9 & 1 & 0 \\ 2 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & -2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} y_1 + 2y_2 - y_3 = -1 \\ y_1 + y_2 + y_3 = -2 \\ 0y_1 + y_2 + 0y_3 = 0 \end{bmatrix} \Longrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 1 & 1 & 1 & -2 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -1 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 0 & 0 & -3 \\ 1 & 0 & 1 & -2 \\ 0 & 1 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & -2 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1/2 \end{array}\right] \Rightarrow y_1 = -3/2,\ y_2 = 0,\ y_3 = -1/2$$

Soln

$$y^* = \begin{bmatrix} -3/2 \\ 0 \\ -1/2 \end{bmatrix}$$

Clearly: (Since $N = \{4,5\}$)

$$y^{*T} A_N = [-3/2 \ \ 0 \ \ -1/2] \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} = [-\tfrac{3}{2} \ \ -\tfrac{1}{2}] \leq \begin{bmatrix} 0 \\ 0 \end{bmatrix} = [0 \ \ 0] = c_N$$

Hence, y^* is duel feasible. Thus, B is a duel feasible basis.

Hence, B is an optimal basis

b) Suppose that the value of b_1 is changed from 6 to θ. For what values of θ does B remain an optimal basis for (LP):

let $b' = \begin{bmatrix} \theta \\ 10 \\ 4 \end{bmatrix}$; by definition, the basic solution x^* to $Ax = b'$ determined by B satisfies $Ax^* = b'$, $x_j^* = 0$ for all $j \notin B$; or equivalently, $A_B x_B^* = b'$, $x_j = 0$ for all $j \notin B$. So we need to solve:

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} \theta \\ 10 \\ 4 \end{bmatrix} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & 0 & \theta \\ 2 & 1 & 1 & 10 \\ -1 & 1 & 0 & 4 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 0 & 2 & 0 & \theta+4 \\ 0 & 3 & 1 & 18 \\ -1 & 1 & 0 & 4 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & \frac{\theta}{2}+2 \\ -1 & 1 & 0 & 4 \\ 0 & 3 & 1 & 18 \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 0 & 1 & 0 & \frac{\theta}{2}+2 \\ -1 & 0 & 0 & -\frac{\theta}{2}+2 \\ 0 & 0 & 1 & -\frac{3\theta}{2}+12 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{\theta}{2}-2 \\ 0 & 1 & 0 & \frac{\theta}{2}+2 \\ 0 & 0 & 1 & -\frac{3\theta}{2}+12 \end{array}\right]$$

we get $x_1^* = \frac{\theta}{2} - 2$, $x_2^* = \frac{\theta}{2} + 2$, $x_3^* = -\frac{3\theta}{2} + 12$

So B determines an optimal basic feasible Solution if & only if we have:

$$x_1^* \geq 0$$
$$x_2^* \geq 0$$
$$x_3^* \geq 0$$

In terms of θ, the System is:

$$\frac{\theta}{2} - 2 \geq 0$$
$$\frac{\theta}{2} + 2 \geq 0$$
$$-\frac{3\theta}{2} + 12 \geq 0$$

Note:

$$\longrightarrow \frac{\theta}{2} \geq 2$$
$$\frac{\theta}{2} \geq -2$$
$$12 \geq \frac{3\theta}{2}$$

Which Simplifies to:

$$\theta \geq 4$$
$$\theta \geq -4$$
$$\theta \leq 8$$

thus, the range of values for which B remains an optimal basis is:

$-4 \leq 4 \leq \theta \leq 8$ which Simplifies to $4 \leq \theta \leq 8$

c) Give the perturbed problem with respect to B:

$$\begin{aligned} \min\ & -x_1 - 2x_2 \\ \text{s.t. } & x_1 + x_2 \quad + x_4 = 6 \\ & 2x_1 + x_2 + x_3 = 10 \\ & -x_1 + x_2 \quad + x_5 = 4 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

Convert:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b \\ & x \geq 0 \end{aligned}$$

to:

$$\begin{aligned} \min\ & c^T x \\ \text{s.t. } & Ax = b' \\ & x \geq 0 \end{aligned}$$

Where $b' = b + A_B e$

where $B = \{1,2,3\}$ is a basis determining the basic feasible Solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \\ 0 \\ 0 \end{bmatrix}. \text{ Let } e = \begin{bmatrix} \varepsilon \\ \varepsilon^2 \\ \vdots \\ \varepsilon^m \end{bmatrix} \text{ where } \varepsilon > 0$$

Here:

$$A_B = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ -1 & 1 & 0 \end{bmatrix}. \text{ The Perturbed Problem is then:}$$

thus:

$$\begin{aligned} \min\ & -x_1 - 2x_2 \\ \text{s.t. } & x_1 + x_2 \quad + x_4 = 6 + \varepsilon + \varepsilon^2 \\ & 2x_1 + x_2 + x_3 = 10 + 2\varepsilon + \varepsilon^2 + \varepsilon^3 \\ & -x_1 + x_2 \quad + x_5 = 4 - \varepsilon + \varepsilon^2 \\ & x_1, x_2, x_3, x_4, x_5 \geq 0 \end{aligned}$$

② let (P) denote the linear Programming problem:

$$\min\ x_1+3x_2+Cx_3$$
$$\text{s.t. } 2x_1+x_2+5x_3=6$$
$$x_1+2x_2+5x_3=7$$
$$x_1,x_2,x_3 \geq 0$$

Where C is a real number. Find all values of C so that every feasible Solution to (P) is an optimal Solution. Justify your answer:

I) let B denote the basis $\{1,2\}$

Show that B is an optimal basis:

$$\left[\begin{array}{cc|c} 2 & 1 & 6 \\ 1 & 2 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & -3 & -8 \\ 1 & 2 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 7 \\ 0 & 1 & 8/3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 5/3 \\ 0 & 1 & 8/3 \end{array}\right]$$

thus, the basic Solution determined by B is $x^*=\begin{bmatrix} 5/3 \\ 8/3 \\ 0 \end{bmatrix}$

which is a feasable Solution to P; Hence B's a feasible basis.

Now, Solving for y^* in $y^{*T}A_B=C_B^T$ gives:

$$\begin{bmatrix} y_1 & y_2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}=\begin{bmatrix} 1 & 3 \end{bmatrix} \rightarrow \left.\begin{array}{l} 2y_1+y_2=1 \\ y_1+2y_2=3 \end{array}\right\} \Rightarrow \left[\begin{array}{cc|c} 2 & 1 & 1 \\ 1 & 2 & 3 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|c} 0 & -3 & -5 \\ 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0 & 1 & 5/3 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1/3 \\ 0 & 1 & 5/3 \end{array}\right] \Rightarrow y_1=-\frac{1}{3},\ y_2=\frac{5}{3}$$

So,

$$y^*=\begin{bmatrix} -1/3 \\ 5/3 \end{bmatrix}$$

Since: $(N=\{3\})$

$$y^{*T}A_N=\begin{bmatrix} -\frac{1}{3} & \frac{5}{3} \end{bmatrix}\begin{bmatrix} 5 \\ 5 \end{bmatrix}=\frac{20}{3} \leq C = C_3$$

Since C_3 is not in B:

B remains optimal if & only if: $C \leq \frac{20}{3}$

II) let $\mathcal{B}$ determine basis $\{1,3\}$:

$$\left[\begin{array}{cc|c} 2 & 5 & 6 \\ 1 & 5 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & -5 & -8 \\ 1 & 5 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & 8/5 \end{array}\right]$$

thus x^* is $\begin{bmatrix} -1 \\ 8/5 \end{bmatrix}$ which is NOT feasible $(x \geq 0, -1 \ngeq 0)$

↳ Hence $\mathcal{B}$ is NOT feasible

III) let $\mathcal{B}$ determine basis $\{2,3\}$:

$$\left[\begin{array}{cc|c} 1 & 5 & 6 \\ 2 & 5 & 7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 5 & 6 \\ 0 & -5 & -5 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

thus x^* is $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ which is a feasible solution to P, Hence B's a feasible basis

Now, Solving for y^* in $y^{*T} A_B = c_B^T$ gives:

$$\begin{bmatrix} y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 2 & c \end{bmatrix}$$

$$\left.\begin{aligned} y_2 + 2y_3 &= 2 \\ 5y_2 + 5y_3 &= c \end{aligned}\right\} \Rightarrow \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 5 & 5 & c \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 2 & 2 \\ 1 & 1 & c/5 \end{array}\right] \sim \left[\begin{array}{cc|c} 0 & 1 & 2-c/5 \\ 1 & 1 & c/5 \end{array}\right] \sim$$

$$\left[\begin{array}{cc|c} 1 & 0 & 2c/5-2 \\ 0 & 1 & 2-c/5 \end{array}\right] \Rightarrow y_1 = \frac{2c}{5} - 2,\ y_2 = 2 - \frac{c}{5}$$

So,

$$y^* = \begin{bmatrix} 2c/5 - 2 \\ 2 - c/5 \end{bmatrix},$$ So since $N = \{1\}$:

$$y^{*T} A_N = \begin{bmatrix} \frac{2c}{5} - 2 & 2 - \frac{c}{5} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{4c}{5} - 4 + 2 - \frac{c}{5} = \frac{3c}{5} - 2$$

↳ $\frac{3c}{5} - 2 \leq c = c_3$ thus $-2 \leq 2c/5$

$2c \geq -10$

$c \geq -5$

thus, Since in:

I) we find $C \leq \frac{20}{3}$

II) we find $C \geq -5$

We Conclude:

$\boxed{-5 \leq C \leq \frac{20}{3}}$ is the range of values of C Such that every feasible Sol$^{\underline{n}}$ to (P) is an optimal Solution.