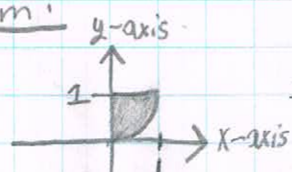
MATH 365 Assignment #3:

① Evaluate the iterate integrals listed below:

a) $\int_0^1\int_{x^2}^1 \frac{\cos\sqrt{y}}{y}\,dy\,dx \Rightarrow$ we must change the order of integration

↳ Diagram:

$x^2 \le y \le 1$
$0 \le x \le 1$ $\Rightarrow$



$\Rightarrow$ $\boxed{\begin{matrix} 0 \le y \le 1 \\ 0 \le x \le \sqrt{y} \end{matrix}}$

[Change integrative order]

thus, we can change the order of integration.

Same

We may also solve this Algorithmically:

convert for x from

$\boxed{x^2} \le y \le 1 \Rightarrow$ Constant

$0 \le x \le \boxed{1}$

$\left[\begin{matrix} x^2 = y \text{ so} \\ x = \sqrt{y} \end{matrix}\right]$

$\Downarrow$ Compute

$\boxed{\begin{matrix} 0^2 \le y \le 1 \; [0 \le y \le 1] \\ 0 \le x \le \sqrt{y} \end{matrix}}$

$$= \int_0^1\int_0^{\sqrt{y}} \frac{\cos(\sqrt{y})}{y}\,dx\,dy = \int_0^1 \left[(x)\left(\frac{\cos\sqrt{y}}{y}\right)\right]_0^{\sqrt{y}} dy$$

$$= \int_0^1 \left[\frac{\sqrt{y}\cos\sqrt{y}}{y} - \cancel{\frac{(0)\cos\sqrt{y}}{y}}\right] dy = \int_0^1 \frac{\cos\sqrt{y}}{\sqrt{y}}\,dy$$

let $u = \sqrt{y}$ then $du = \frac{1}{2\sqrt{y}}dy$
When $y=1, u=1$; when $y=0, u=0$

[integration by Substitution]

[Substitute for u] ↳ $\int_0^1 (2)\left(\frac{\cos(u)}{1}\right) du$ $\left[\begin{matrix}\text{if } du = \frac{1}{2\sqrt{y}}dy \text{ then} \\ 2du = \frac{1}{\sqrt{y}}dy\end{matrix}\right]$

$$= 2\left[\sin(u)\right]_0^1 = 2\sin(1) - \cancel{2\sin(0)} = 2\sin(1)$$

thus:

$$\int_0^1\int_{x^2}^1 \frac{\cos(\sqrt{y})}{y}\,dy\,dx = \boxed{2\sin(1)}$$

b) $\int_R\int e^{y/x}\,dA \Rightarrow \begin{matrix} 1 \le x \le 2 \\ 0 \le y \le x \end{matrix} \Rightarrow \int_1^2\int_0^x e^{y/x}\,dy\,dx$

let $u = y/x$ then $du = \frac{1}{x}dy$ So $x\,du = dy$
When $y=0, u=0$; when $y=x, u=1$

[integration by Substitution]

[Substitute for u] ↳ $\int_1^2\int_0^1 xe^u\,du\,dx = \int_1^2 \left[xe^u\right]_0^1 dx$

$$= \int_1^2 x(e^1 - e^0)dx = \int_1^2 x(e-1)\,dx$$

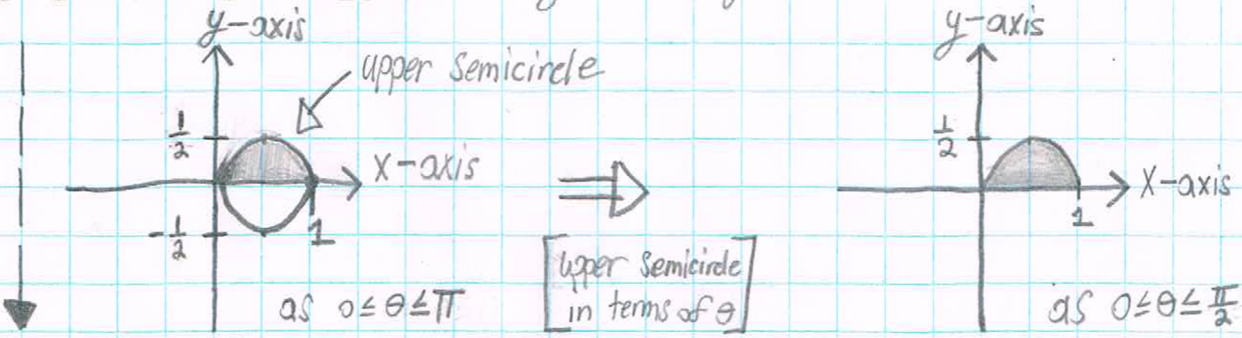
$$\int_1^2 x(e-1)\,dx = \left[\frac{x^2(e-1)}{2}\right]_1^2 = \frac{4(e-1)}{2} - \frac{(e-1)}{2} = \boxed{\frac{3}{2}(e^1-1)}$$

thus:

$$\int_R\int e^{y/x}\,dA \; \{1 \le x \le 2, 0 \le y \le x\} = \frac{3}{2}(e-1)$$

c) $\int_R\int r\,dr\,d\theta$ Where R is the region of the upper semicircle $r=\cos\theta$:

Sketch of Circle $r=\cos\theta$: Find region for integration



thus:

the radius is from 0 to $r=\cos\theta$ $\therefore$ $0\le r\le\cos\theta$

the upper semicircle is defined along $0\le\theta\le\frac{\pi}{2}$

So:

$$\int_0^{\frac{\pi}{2}}\int_0^{\cos\theta} r\,dr\,d\theta = \int_0^{\frac{\pi}{2}}\left[\frac{r^2}{2}\right]_0^{\cos\theta} d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} \cos^2(\theta)-\cancel{0^2}\,d\theta$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\cos^2\theta\,d\theta \quad \left[\text{if } \cos 2\theta = 2\cos^2\theta-1 \text{ then } \frac{\cos 2\theta+1}{2}=\cos^2\theta\right]$$

$$=\frac{1}{2}\int_0^{\frac{\pi}{2}}\frac{\cos 2\theta+1}{2}\,d\theta = \frac{1}{4}\int_0^{\frac{\pi}{2}}\cos(2\theta)+1\,d\theta = \frac{1}{4}\left[\frac{\sin(2\theta)}{2}+\theta\right]_0^{\frac{\pi}{2}}$$

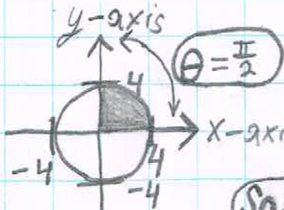
$$=\frac{1}{4}\left[\cancel{\frac{\sin(\pi/2\cdot 2)}{2}}+\frac{\pi}{2}-\left(\cancel{\frac{\sin(\pi/2\cdot 0)}{2}}+\cancel{0}\right)\right]$$

$$=\frac{1}{4}\left(\frac{\pi}{2}\right)=\boxed{\frac{\pi}{8}}$$

thus:

$\int_R\int r\,dr\,d\theta = \boxed{\frac{\pi}{8}}$, When R is the region of the upper semicircle $r=\cos\theta$

d) $\int_R\int \sin(x^2+y^2)\,dA$ Where R is the region in the first quadrant of the circle $x^2+y^2=4$



So: $0 \le \theta \le \frac{\pi}{2}$

if $x^2+y^2=4$ then $r^2\cos^2\theta + r^2\sin^2\theta = 4$

$r^2(\cos^2\theta + \sin^2\theta) = 4$

$r^2 = 4 \rightarrow r = \pm 2$

So the radius of the circle extends from $0 \le r \le 2$.

$$\int_0^{\frac{\pi}{2}}\int_0^2 r\sin(r^2\cancel{(\cos^2\theta+\sin^2\theta)})\,dr\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^2 r\sin(r^2)\,dr\,d\theta$$

let $u = r^2$ so $du = 2r\,dr$ then $\frac{du}{2} = r\,dr$

When $r=2, u=4$; When $r=0, u=0$

[Substitute for u]

$$\int_0^{\frac{\pi}{2}}\int_0^4 \left(\frac{1}{2}\right)\sin(u)\,du\,d\theta = \int_0^{\frac{\pi}{2}} \frac{1}{2}\Big[-\cos(u)\Big]_0^4\,d\theta$$

$$= \int_0^{\frac{\pi}{2}} \frac{1}{2}\Big[-\cos(4) + (+\cos(0))\Big]\,d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} 1-\cos(4)\,d\theta = \frac{1}{2}\Big[\theta - \theta\cos(4)\Big]_0^{\frac{\pi}{2}}$$

$$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{\pi}{2}\cos(4) - \cancel{(0 - 0\cos(4))}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{2}(1-\cos(4))\right] = \boxed{\frac{\pi}{4}(1-\cos(4))}$$

Thus:

$$\int_R\int \sin(x^2+y^2)\,dA = \boxed{\frac{\pi}{4}(1-\cos(4))}$$

When R is the region in the first quadrant of the circle $x^2+y^2=4$.

② let $x=4u\cos(v)$, $y=4u\sin(v)$, & $z=u$ for $0\le u\le 1$ & $0\le v\le 2\pi$

a) Eliminate the Parameters to obtain an equation in rectangular Coordinates, & describe the Surface:

$$\begin{aligned} x^2+y^2 &= (4u\cos v)^2+(4u\sin v)^2 \\ &= 16u^2\cos^2(v)+16u^2\sin^2(v) \\ &= 16u^2(\cos^2(v)+\sin^2(v)) \quad [\text{Pythagorean Identity: } \cos^2(x)+\sin^2(x)=1] \\ &= 16u^2(1) \\ &= 16u^2 \quad [\text{Substitute } z=u] \\ &= 16(z)^2 \\ &= 16z^2 \end{aligned}$$

therefore:

this Simplifies to the following equation in Rectangular Coordinates:

$$\boxed{x^2+y^2=16z^2}$$

$16z^2=x^2+y^2$:

- This is of the form: $z^2-\frac{x^2}{a^2}-\frac{y^2}{b^2}$ as $16z^2=x^2+y^2$ is $z^2-\frac{x^2}{16}-\frac{y^2}{16}=0$
  - ↳ ∴ this is an Elliptic Cone
- The Centre is at $(0,0,0)$
- The Cone opens up & down along both the Positive & negative z-axis
- The Cone is Centered along the z-axis

b) Find the equation of the tangent plane to the Parametric Surface at the point $u=1$ & $v=0$:

the equation to the tangent Plane to a Surface at a Point is:

$$\nabla f(P)=f_x(x,y,z)(x-a)+f_y(x,y,z)(y-b)+f_z(x,y,z)(z-c)=0$$

Where $P=(x,y,z)$

$P=(x,y,z)$ when $u=1$ & $v=0$ So:

$$\left.\begin{aligned} x&=4(u)\cos(v)=4(1)\cos(0)=4 \\ y&=4(u)\sin(v)=4(1)\sin(0)=4(0)=0 \\ z&=u=1 \end{aligned}\right\} P(x,y,z)=P(4,0,1) \text{ when } u=1,\ v=0$$

So if $P=(4,0,1)$, $\nabla f(P)$ is:

From Q2 Part A we know the function simplifies to $f(x,y,z)=x^2+y^2-16z^2$

thus, $\nabla f(P)=2x(4,0,1)(x-4)+2y(4,0,1)(y-0)+(-32z)(4,0,1)(z-1)=0$

$$0=8(x-4)+\cancel{0(y)}-32(z-1)$$

$$\frac{0}{8}=x-4-4(z-1)$$

$$0=x-4z-\cancel{4}+\cancel{4} \Rightarrow \text{So } \boxed{4z=x \text{ OR } x-4z=0}$$

We can also determine this Parametrically:

$P=(4,0,1)$ & $r(u,v)=\langle 4u\cos v, 4u\sin v, u\rangle$

So:

$$r_u(u,v)=\langle 4\cos v, 4\sin v, 1\rangle$$

$$r_v(u,v)=\langle -4u\sin v, 4u\cos v, 0\rangle$$

to find the normal to the plane we use:

$$n = \frac{r_u \times r_v}{\|r_u \times r_v\|}$$

So: $r_u \times r_v$

$$\rightarrow \begin{vmatrix} i & j & k \\ 4\cos v & 4\sin v & 1 \\ -4u\sin v & 4u\cos v & 0 \end{vmatrix} \begin{matrix} i & j \\ 4\cos v & 4\sin v \\ -4u\sin v & 4u\cos v \end{matrix}$$

$$= -4u\sin(v)j + 16u\cancel{\cos^2(v)}k - 4u\cos(v)i + 16u\,\cancel{\sin^2(v)}k$$

$$= \langle -4u\cos v, -4u\sin v, 16u \rangle$$

So: $\|r_u \times r_v\|$

$$\rightarrow \sqrt{16u^2\cancel{(\cos^2(v)+\sin^2(v))} + 256u^2} = \sqrt{272}\,u = 4u\sqrt{17}$$

thus:

$$n = \frac{\langle -4\cancel{u}\cos v, -4\cancel{u}\sin v, 16\cancel{u} \rangle}{\cancel{u}\,4\sqrt{17}} = \frac{\langle -4\cos v, -4\sin v, 16 \rangle}{4\sqrt{17}}$$

When $u=1$ & $v=0$:

$$n = \frac{\langle -4, 0, 16 \rangle}{4\sqrt{17}} = \frac{\langle -1, 0, 4 \rangle}{\sqrt{17}}$$

Dot by displacment:

$$0 = \frac{\langle -1, 0, 4 \rangle}{\sqrt{17}} \cdot \langle x-4, y-0, z-1 \rangle = -\frac{1}{\sqrt{17}}(x-4) + \cancel{0(y)} + \frac{4}{\sqrt{17}}(z-1)$$

$$0 \cdot \sqrt{17} = -(x-4) + 4(z-1) \rightarrow -x + \cancel{4} + 4z - \cancel{4} = 0$$

finally:

$-x+4z=0$ so $x=4z$ OR $x-4z=0$

this matches our first solution.

$\therefore$ the tangent plane is:

$$\boxed{x=4z}$$

c) Find the Surface area of the portion of the Surface described in part a:

As Shown in part b, $r_u \times r_v = \langle -4u\cos(v), -4u\sin(v), 16u \rangle$

to find this, use the Scaler Surface Integrel:

$$S = \iint_R \sqrt{\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1}\, dA = \iint_R \left\| \frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v} \right\| dA$$

find the magnitude of the Cross Product

$0 \leq u \leq 1$

$0 \leq v \leq 2\pi$

$$\Rightarrow \|r_u \times r_v\| = \|\langle -4u\cos(v), -4u\sin(v), 16u \rangle\|$$

$$= \sqrt{16u^2\cos^2(v) + 16u^2\sin^2(v) + 256u^2}$$

$$= \sqrt{16u^2(\cancel{\cos^2(v) + \sin^2(v)}) + 256u^2}$$

$$= \sqrt{16u^2 + 256u^2} = \sqrt{272u^2} = u\sqrt{272}$$

$$= u\sqrt{16 \cdot 17} = 4u\sqrt{17}$$

$$S = \int_0^1 \int_0^{2\pi} \|r_u \times r_v\| \, dv\,du = \int_0^1 \int_0^{2\pi} 4u\sqrt{17}\, dv\,du = 4\sqrt{17} \int_0^1 \int_0^{2\pi} u\, dv\,du$$

$$= 4\sqrt{17} \int_0^1 [uv]_0^{2\pi} du = 4\sqrt{17} \int_0^1 (2\pi - 0) u\,du = 8\pi\sqrt{17} \int_0^1 u\,du$$

$$= 8\pi\sqrt{17} \left[\frac{u^2}{2}\right]_0^1 = 8\pi\sqrt{17}\left(\frac{1^2}{2} - \cancel{\frac{0^2}{2}}\right) = \boxed{4\pi\sqrt{17}}$$

thus:

the Surface area of the Portion described in Part a is $4\pi\sqrt{17}$.

③ Choose appropriate Coordinates to evaluate the tripple integrals listed below

a) $\iiint_E (y - 3x)\, dV$ where E is bounded by the parabolic cylinder $y = x^2$ & the Planes $x = z$, $x = y$, & $z = 0$.

— this lets us know the following:

$y = x^2$ & $y = x$ So

- $x^2 = x$ as $y = y$ ∴ $x^2 - x = 0$ & $x(x-1) = 0$ So $x = 0$ & $x = 1$

thus, the limits for x are $0 \leq x \leq 1$

- As $y = x^2$ & $y = x$, we can see that as x approches 1 from 0, $x \geq x^2$

So $x^2 \leq y \leq x$

- Finally, we know $z = 0$ & $z = x$, since x moves from 0 to 1,

$z = x \geq 0$, So $0 \leq z \leq x$

— This lets us generate the following Integrel:

$$\iiint_E (y - 3x)\, dV = \int_0^1 \int_{x^2}^{x} \int_0^x y - 3x \, dz\,dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} [z(y - 3x)]_0^x \, dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} [x(y - 3x) - \cancel{0(y - 3x)}]\, dy\,dx$$

$$= \int_0^1 \int_{x^2}^{x} xy - 3x^2 \, dy\,dx$$

$$= \int_0^1 \left[\frac{y^2 x}{2} - 3x^2 y \right]_{x^2}^{x} dx$$

$$= \int_0^1 \left[\frac{x^2(x)}{2} - 3x^2(x) - \frac{(x^2)^2 x}{2} + 3x^2(x^2) \right] dx$$

$$= \int_0^1 \left[\frac{x^3}{2} - 3x^3 - \frac{x^5}{2} + 3x^4 \right] dx$$

$$= \int_0^1 3x^4 - \frac{x^5}{2} - \frac{5x^3}{2} \, dx$$

$$= \left[\frac{3x^5}{5} - \frac{x^6}{12} - \frac{5x^4}{8} \right]_0^1$$

$$= \frac{3(1)^5}{5} - \frac{1^6}{12} - \frac{5(1)^4}{8} - 0$$

$$= \frac{3}{5} - \frac{1}{12} - \frac{5}{8} = -\frac{13}{120}$$

therefore:

$$\iiint_E (y-3x)\,dV = \int_0^1 \int_{x^2}^{x} \int_0^{x} y - 3x \, dz\,dy\,dx = -\frac{13}{120}$$

b) $\iiint_E y\,dV$ Where E is enclosed by the Planes $z=0$ & $z=x+y+4$ & the cylinders $x^2+y^2=4$ & $x^2+y^2=9$.

Substitute the new ranges in Cylindrical Coordinates

- We can find the bounds via Cylindrical Coordinates:

$x = r\cos\theta$ & $y = r\sin\theta$ So $z = x+y+4 \rightarrow z = r\cos\theta + r\sin\theta + 4$

$\hookrightarrow$ So we know z ranges from: $\boxed{0 \le z \le r\cos\theta + r\sin\theta + 4}$

the cylinders change to:

$r^2\cos^2\theta + r^2\sin^2\theta = r^2(\sin^2\theta + \cos^2\theta) = r^2$ So

$\hookrightarrow r^2 = 4 \rightarrow r = 2$ & $r^2 = 9 \rightarrow r = 3$

$\hookrightarrow$ We know r ranges from: $\boxed{2 \le r \le 3}$

the cylinders span all four quadrents, So we know:

$\hookrightarrow \boxed{0 \le \theta \le 2\pi}$

$$\int_0^{2\pi} \int_2^3 \int_0^{r\cos\theta + r\sin\theta + 4} (r\sin\theta)\,dV = \int_0^{2\pi} \int_2^3 \int_0^{r\cos\theta + r\sin\theta + 4} (r)(r\sin\theta)\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_2^3 \left[z r^2 \sin\theta \right]_0^{r\cos\theta + r\sin\theta + 4} dr\,d\theta$$

$$= \int_0^{2\pi} \int_2^3 (r\cos\theta + r\sin\theta + 4)(r^2\sin\theta) - \cancel{0}\,dr\,d\theta$$

$$= \int_0^{2\pi} \int_2^3 r^3\cos\theta\sin\theta + r^3\sin^2\theta + 4r^2\sin\theta\,dr\,d\theta$$

$$= \int_0^{2\pi} \left[\frac{r^4\cos\theta\sin\theta}{4} + \frac{r^4\sin^2\theta}{4} + \frac{4r^3\sin\theta}{3} \right]_2^3 d\theta$$

$$= \int_0^{2\pi} \cancel{\frac{81\cos\theta\sin\theta}{4}} + \cancel{\frac{81\sin^2\theta}{4}} + \cancel{\frac{108\sin\theta}{3}} - \cancel{\frac{16\cos\theta\sin\theta}{4}} - \cancel{\frac{16\sin^2\theta}{4}} - \cancel{\frac{32\sin\theta}{3}} \Big] d\theta$$

$$= \int_0^{2\pi} \frac{65\cos\theta\sin\theta}{4} + \frac{65\sin^2\theta}{4} + \frac{76\sin\theta}{3} \, d\theta$$

$$= \int_0^{2\pi} \frac{65\sin 2\theta}{4\cdot 2} + \frac{65(1-\cos 2\theta)}{2\cdot 4} + \frac{76\sin\theta}{3} \, d\theta$$

$$= \left[\frac{65}{8}\cdot\frac{1}{2}\cdot(-\cos 2\theta)\right]_0^{2\pi} + \frac{65}{8}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi} + \frac{76}{3}\left[-\cos\theta\right]_0^{2\pi}$$

$$= \left[\frac{65}{16}\cdot(-\cos 2\pi + \cos 0)\right] + \frac{65}{8}\left(2\pi - \cancel{\frac{\sin 4\pi}{2}} - \cancel{0}\right) + \frac{76}{3}(-\cos 2\pi + \cos 0)$$

$$= \cancel{\frac{65}{16}(-1+1)} + \cancel{\frac{76}{3}(-1+1)} + \frac{65}{8}(2\pi) = \frac{130\pi}{8} = \boxed{\frac{65\pi}{4}}$$

therefore:

$$\iiint_E y\,dV = \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta + 4} r^2\sin\theta \, dz\,dr\,d\theta = \boxed{\frac{65\pi}{4}}$$

c) $\iiint_E z\,dV$ Where E lies between the Spheres $x^2+y^2+z^2=1$ & $x^2+y^2+z^2=9$ in the first octant

- We can find the bounds via. Spherical Coordinates!

$x=\rho\sin\phi\cos\theta$, $y=\rho\sin\phi\sin\theta$, $z=\rho\cos\phi$

Use the Spherical Coordinate Conversion

So:

$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi = 9$

$\rho^2\sin^2\phi\cancel{(\cos^2\theta+\sin^2\theta)} + \rho^2\cos^2\phi = 9$

$\rho^2\cancel{(\sin^2\phi+\cos^2\phi)} = 9$

$\rho^2 = 9$ So $\rho = 3$

And

$\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi = 1$

[use the same steps as above]

$\rho^2 = 1$ So $\rho = 1$

$1 \le \rho \le 3$

- We are restricted to the first octant:

the first octant lies w/in the first quadrent $\therefore$ $\boxed{0 \le \theta \le \frac{\pi}{2}}$

the first octant opens up along the Positive z-axis $\therefore$ $\boxed{0 \le \phi \le \frac{\pi}{2}}$

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_1^3 z\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_1^3 (\rho\cos\phi)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_1^3 \rho^3\cos\phi\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left[\frac{\rho^4\cos\phi\sin\phi}{4}\right]_1^3 d\phi\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{81\cos\phi\sin\phi}{4} - \frac{\cos\phi\sin\phi}{4}\,d\phi\,d\theta$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} 20\cos\phi\sin\phi\,d\phi\,d\theta = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{20\sin 2\phi}{2}\,d\phi\,d\theta$$

$$= 10\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \sin(2\phi)\,d\phi\,d\theta = 10\int_0^{\frac{\pi}{2}}\left[\frac{-\cos(2\phi)}{2}\right]_0^{\frac{\pi}{2}} d\theta$$

$$= 5\int_0^{\frac{\pi}{2}} \left[-\cos(2\cdot\pi/2) + \cos(\tfrac{\pi}{2}\cdot 0)\right] d\theta$$

$$= 5\int_0^{\frac{\pi}{2}} \left(-(-1)+1\right) d\theta = 10\int_0^{\frac{\pi}{2}} d\theta = 10\left[\theta\right]_0^{\frac{\pi}{2}}$$

$$= 10\left(\tfrac{\pi}{2} - 0\right) = \boxed{5\pi}$$

therefore:

$$\iiint_E z\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_1^3 (\rho\cos\phi)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta = 5\pi$$

(4) Express the integrel $\iiint_E f(x,y,z)\,dV$ as an iterated integrel in six different ways, where E is the solid bounded by the surfaces $x^2+z^2=9$, $y=0$, $y=6$

- We know that y is bounded by: $0 \le y \le 6$
- We know $x^2+z^2=9$, so:
  - $x=\sqrt{9-z^2}$
  - $z=\sqrt{9-x^2}$
- If we let $x=\pm\sqrt{9-z^2}$ then $z^2=9$ so $z=\pm 3$ or $-3\le z\le 3$:

$$\boxed{\text{I}}\ \int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}}\int_0^6 f(x,y,z)\,dy\,dx\,dz$$

- If $z=\pm\sqrt{9-x^2}$ &, $-3\le x\le 3$:

$$\boxed{\text{II}}\ \int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}\int_0^6 f(x,y,z)\,dy\,dz\,dx$$

- For the remaining integrels, we can simply rearrange the order of integration:

$$\left.\begin{aligned}&\boxed{\text{III}}\ \int_{-3}^{3}\int_0^6\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dx\,dy\,dz\\ &\boxed{\text{IV}}\ \int_0^6\int_{-3}^{3}\int_{-\sqrt{9-z^2}}^{\sqrt{9-z^2}} f(x,y,z)\,dx\,dz\,dy\end{aligned}\right\}\ \text{for } x=\pm\sqrt{9-z^2},\ z=\pm 3$$

$$\left.\begin{aligned}&\boxed{\text{V}}\ \int_{-3}^{3}\int_0^6\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x,y,z)\,dz\,dy\,dx\\ &\boxed{\text{VI}}\ \int_0^6\int_{-3}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} f(x,y,z)\,dz\,dx\,dy\end{aligned}\right\}\ \text{for } z=\pm\sqrt{9-z^2},\ x=\pm 3$$

As can be seen from integrels I to VI, the solid bounded by the surfaces $x^2+y^2=9$, $y=0$, $y=6$. can be expressed as an iterated integrel six different ways.

⑤ Let E be the Solid that lies within the Sphere $x^2+y^2+z^2=4$, above the xy-plane, & below the Cone $z=\sqrt{x^2+y^2}$:

a) Find the volume of the Solid E:

— this Can be naturally expressed in Spherical Coordinates:

$$x^2+y^2+z^2=4$$
$$(\rho\sin\phi\cos\theta)^2+(\rho\sin\phi\sin\theta)^2+(\rho\cos\phi)^2=4$$
$$\rho^2\sin^2\phi\cos^2\theta+\rho^2\sin^2\phi\sin^2\theta+\rho^2\cos^2\phi=4$$
$$\rho^2\sin^2\phi\,\cancel{(\cos^2\theta+\sin^2\theta)}+\rho^2\cos^2\phi=4$$
$$\rho^2\cancel{(\sin^2\phi+\cos^2\phi)}=4 \rightarrow \rho^2=4 \rightarrow \rho=2$$

— we Know $z=\sqrt{x^2+y^2}$ So:

$$\rho^2\cos^2\phi=\rho^2\sin^2\phi(\sin^2\theta+\cos^2\theta)$$
$$\cos^2\phi=\sin^2\phi$$
$$\tfrac{1}{2}(\cos(2\phi)+1)=\tfrac{1}{2}(1-\cos(2\phi))$$
$$\cos 2\phi+1=1-\cos 2\phi$$
$$\cos(2\phi)=-\cos(2\phi)$$
$$2\cos(2\phi)=0$$
$$\cos(2\phi)=0$$
$$\text{ArcCos}(\cos 2\phi)=\text{ArcCos}(0)$$
$$2\phi=\pi/2$$

$\therefore \phi=\pi/4$, So $\phi \geq \pi/4$ when the Solid is below $z=\sqrt{x^2+y^2}$

— We Know $z>0$ as the region lies above the xy-plain:

thus: $0=\rho\cos\phi$

$0=\cos\phi$

$\text{ArcCos}(0)=\phi$

$\therefore \phi=\pi/2$

So this means that $\phi\leq\pi/2$ if the Solid is above xy-plain.

— Finally, the Cone occupies all four quadrents:

$0\leq\theta\leq2\pi$

— thus: $0\leq\rho\leq2$, $\pi/4\leq\phi\leq\pi/2$, $0\leq\theta\leq2\pi$

$$V=\iiint_G dA=\int_0^{2\pi}\int_{\pi/4}^{\frac{\pi}{2}}\int_0^2(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$$
$$=\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\left[\frac{\rho^3\sin\phi}{3}\right]_0^2 d\phi\,d\theta=\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{8}{3}\sin\phi\,d\phi\,d\theta$$
$$=\frac{8}{3}\int_0^{2\pi}\left[-\cos\phi\right]_{\pi/4}^{\pi/2}d\theta=\frac{8}{3}\int_0^{2\pi}\left(\cancel{-\cos\left(\frac{\pi}{2}\right)}+\cos\left(\frac{\pi}{4}\right)\right)d\theta$$
$$=\left(\frac{8}{3}\right)\left(\frac{\sqrt{2}}{2}\right)\int_0^{2\pi}d\theta=\frac{4\sqrt{2}}{3}\left[2\pi-\cancel{0}\right]=\boxed{\frac{8\sqrt{2}\pi}{3}}$$

$\therefore$ the Volumn of the Solid E is $\frac{8\sqrt{2}\pi}{3}$

b) If the density of E is Constant, Find the Centroid of E:

When the density of a Region is Constant, the dencity function δ Can be moved through the integrel & Canceled; for example:

$$\bar{x}=\frac{\iint_R x\delta(x,y)\,dA}{\iint_R\delta(x,y)\,dA}=\frac{\delta(x,y)}{\delta(x,y)}\cdot\frac{\iint_R x\,dA}{\iint_R dA}=\frac{\iint_R x\,dA}{\iint_R dA}=\frac{1}{\text{area of }R}\iint_R x\,dA$$

— This Concept extends 3D-Space

thus: (By Symmetry, $\bar{x}=\bar{y}=0$ as a Sphere & Cone are balanced along the x&y axis)

$$\bar{x}=\frac{1}{V}\iiint_G x\,dV;\quad \bar{y}=\frac{1}{V}\iiint_G y\,dV;\quad \bar{z}=\frac{1}{V}\iiint_G z\,dV$$

$\bar{X}=\bar{y}=0$ Since the surface within $x^2+y^2+z^2=4$, above the xy-plane, & below the cone $z=\sqrt{x^2+y^2}$ is fully balanced on the x & y axis.

↳ the Sphere is balanced on x,y. Since $z>0$ (above xy-plane), the z-axis is not balanced, as it is not symmetric.

the Cone is balanced only on x & y. Since $z>0$, the z-axis is not balanced since it lacks symmetry.

$\therefore \bar{x}=\bar{y}=0$

to Calculate $\bar{z}$:

$$\bar{Z}=\iiint_G z\,dV=\frac{1}{\left(\frac{8\sqrt{2}\pi}{3}\right)}\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 (\rho^2\sin\phi)(\rho\cos\phi)\,d\rho\,d\phi\,d\theta$$

$$=\frac{3}{8\sqrt{2}\pi}\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\int_0^2 \rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta$$

$$=\frac{3}{8\sqrt{2}\pi}\int_0^{2\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left[\frac{\rho^4\sin\phi\cos\phi}{4}\right]_0^2 d\phi\,d\theta=\frac{3}{8\sqrt{2}\pi}\int_0^{2\pi}\int_{\pi/4}^{\pi/2}\frac{16\sin\phi\cos\phi}{4}-\cancel{0}\,d\phi\,d\theta$$

let $u=\sin\phi$ so $du=\cos\phi\,d\phi$

When $\phi=\frac{\pi}{4}$, $u=\frac{\sqrt{2}}{2}$; $\phi=\frac{\pi}{2}$, $u=1$

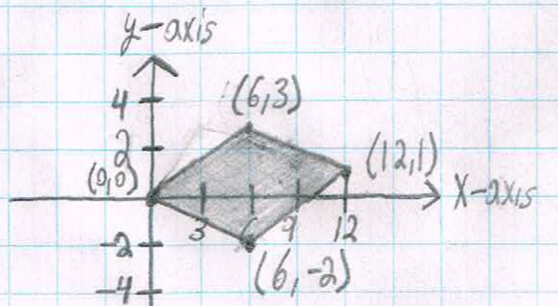
$$=\frac{4\cdot3}{8\sqrt{2}\pi}\int_0^{2\pi}\int_{\frac{\sqrt{2}}{2}}^{1}u\,du\,d\theta=\frac{3}{2\sqrt{2}\pi}\int_0^{2\pi}\left[\frac{u^2}{2}\right]_{\frac{\sqrt{2}}{2}}^{1}d\theta=\frac{3}{2\sqrt{2}\pi}\int_0^{2\pi}\frac{1^2}{2}-\frac{\left(\frac{\sqrt{2}}{2}\right)^2}{2}\,d\theta$$

$$=\frac{3}{2\sqrt{2}\pi}\int_0^{2\pi}\frac{1}{2}-\frac{1}{4}\,d\theta=\frac{3}{2\sqrt{2}\pi}\int_0^{2\pi}\frac{1}{4}\,d\theta=\frac{3}{8\sqrt{2}\pi}\int_0^{2\pi}d\theta$$

$$=\frac{3}{8\sqrt{2}\pi}(2\pi-\cancel{0})=\frac{3\cdot2\cancel{\pi}}{8\sqrt{2}\cancel{\pi}}=\boxed{\frac{3}{4\sqrt{2}}}$$

$\therefore$ the Centroid of E is $\left(0,0,\frac{3}{4\sqrt{2}}\right)$

(6) Use the transformation $T(u,v)=(2u+3v,\,u-v)$ to evaluate the integral $\iint_R (x-3y)\,dA$ where R is the rectangle with corners $(0,0),(6,3),(12,1),(6,-2)$:



Convert All points:

If $(x,y)=(0,0)$ then $T(0,0)=(0,0)$

If $(x,y)=(6,3)$ then:

$6=2u+3v$ & $3=u-v$

So: $6=2(3+v)+3v$

$6=6+2v+3v \Rightarrow v=0$

$\therefore$ if $v=0$ then $u=3$

$T(3,0)=(6,3)$

If $(x,y)=(6,-2)$ then:

$6=2u+3v$ & $-2=u-v$

So: $6=2(-2+v)+3v$

$10=5v \rightarrow v=2$

$\therefore u=0$

$T(0,2)=(6,-2)$

If $(x,y)=(12,1)$ then:

$12=2u+3v$ & $1=u-v$

So: $12=2(1+v)+3v$

$12=2+5v \Rightarrow 10=5v \Rightarrow v=2$

$\therefore u=3$

$T(3,2)=(12,1)$

So the Points on the rectangle that R corrasponds to are:

$(0,0), (3,0), (3,2), (0,2)$

these are the Corresponding points along the transformation to the uv-Plane

We Can See that:

$0 \le u \le 3$

$0 \le v \le 2$

based on the Points listed above

So:

$$\iint_R (x-3y)\,dA \text{ Where } 0\le u\le 3, 0\le v\le 2, x=2u+3v, y=u-v$$

$$=\int_0^3\int_0^2 2u+3v-3(u-v)\,dv\,du = \int_0^3\int_0^2 2u+3v-3u+3v\,dv\,du$$

$$=\int_0^3\int_0^2 6v-u\,dv\,du = \int_0^3\left[\frac{6v^2}{2}-uv\right]_0^2 du = \int_0^3\left[3v^2-uv\right]_0^2 du$$

$$=\int_0^3 12-2u-\cancel{(3(0)^2-u(0))}\,du = 2\int_0^3 6-u\,du$$

$$=2\left[6u-\frac{u^2}{2}\right]_0^3 = 2\left[18-\frac{9}{2}\right] = 36-9 = \boxed{27}$$

We must now Calculate the Jacobian of the transformation:

$$J(u,v)=\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v}\end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 1 & -1\end{vmatrix} = 2(-1)-(3)(1) = -2-3 = -5$$

So $|J(u,v)|=5$

to Calculate the Area, use:

$$\iint_R f(x,y)\,dA_{xy} = \iint_S f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dA_{uv}$$

So:

$$=\int_0^3\int_0^2 (x-3y)|-5|\,dv\,du = 5\int_0^3\int_0^2 6v-u\,dv\,du$$

Plug Solution Here.

$$=5(27)=\boxed{135}$$

(7) Evaluate the line integral $\int_C (8x+9z)\,dS$, where C is the curve $x=t$, $y=2t^2$, $z=t^3$, $0 \le t \le 1$:

$$\int_C f(x,y,z)\,dS = \int_a^b f(x(t),y(t),z(t))\,\|r'(t)\|\,dt$$

Where $r(t) = x(t)i + y(t)j + z(t)k$, $a \le t \le b$

So, the above formula may simplify to:

$$= \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

So, when solving for $x=t$, $y=2t^2$, $z=t^3$, $0 \le t \le 1$:

$$\int_C f(x,y,z)\,dS = \int_C (8x+9z)\,dS$$

$$= \int_0^1 (8x(t)+9z(t))\sqrt{(x'(t))^2+(y'(t))^2+(z'(t))^2}\,dt$$

$$= \int_0^1 (8t+9t^3)\sqrt{(1)^2+(4t)^2+(3t^2)^2}\,dt$$

$$= \int_0^1 (8t+9t^3)\sqrt{1+16t^2+9t^4}\,dt$$

let $u = 1+16t^2+9t^4$

then $du = 32t+36t^3$

So $\frac{du}{4} = 8t+9t^3 dt$

When $t=1, u=26$; $t=0, u=1$

$$= \frac{1}{4}\int_1^{26} \sqrt{u}\,du = \frac{1}{4}\left[\frac{u^{3/2}}{3/2}\right]_1^{26}$$

$$= \left(\frac{2}{12}\right)\left[26\sqrt{26}-1\right]$$

$$= \frac{1}{6}\left(26\sqrt{26}-1\right) = \boxed{\frac{26\sqrt{26}-1}{6}} \doteq 21.929$$

thus:

$$\int_C f(x,y,z)\,dS = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2}\,dt$$

$$= \int_0^1 (8t+9t^3)\sqrt{\left(\frac{d}{dt}(t)\right)^2+\left(\frac{d}{dt}(2t^2)\right)^2+\left(\frac{d}{dt}(t^3)\right)^2}\,dt$$

$$= \frac{26\sqrt{26}-1}{6} \doteq 21.929$$

(8) Find $f(x,y)$ if we know that $\nabla f(x,y) = (ye^{xy}+2xy)i + (xe^{xy}+x^2+3y^2)j$ & $f(0,0)=\frac{3}{2}$:

we know the gradient is:

$$\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle$$

thus:

$$f(x,y) = \int f_x(x,y)\,dx = \int f_y(x,y)\,dy = \int \nabla f(x,y)\,dx = \int \nabla f(x,y)\,dy$$

& since $f(0,0)=\frac{3}{2}$ we can isolate for the constant of integration

So:

$$f(x,y) = \int ye^{xy}+2xy\,dx = \frac{ye^{xy}}{y}+\frac{2x^2y}{2}+C_1 = e^{xy}+x^2y+C_1$$

$$f(x,y) = \int xe^{xy}+x^2+3y^2\,dy = \frac{xe^{xy}}{x}+x^2y+\frac{3y^3}{3}+C_2$$

$$= e^{xy} + x^2y + y^3 + C_2$$

So, we can conclude:

$$f(x,y) = e^{xy} + x^2y + C_1 = e^{xy} + x^2y + y^3 + C_2$$

$$f(0,0) = \frac{3}{2} = e^{(0)(0)} + (0)^2(0) + C_1 = e^{(0)(0)} + (0)^2(0) + 0^3 + C_2$$

$$\frac{3}{2} = 1 + C_1 = 1 + C_2$$

$\frac{1}{2} = C_1 = C_2$, So C_1 & C_2 have a Constant of $\frac{1}{2}$

Also:

$$f(x,y) = \int \nabla f(x,y)\,dx = \int \nabla f(x,y)\,dy$$

$$= \cancel{e^{xy}} + \cancel{x^2y} + C_1 = \cancel{e^{xy}} + \cancel{x^2y} + y^3 + C_2$$

$C_1 = y^3 + C_2$ & $C_2 = \frac{1}{2}$ So $C_1 = y^3 + \frac{1}{2}$

thus:

$$f(x,y) = e^{xy} + x^2y + y^3 + \frac{1}{2}$$

9 Use Green's theorem to evaluate the linear integrel $\int_C xe^{-2x}dx + (x^4 + 2x^2y^2)dy$ along the boundary of the region bounded between the circles $x^2+y^2=1$ & $x^2+y^2=9$ that lies between the planes $z=1$ & $z=4$:

$$\int_C f(x,y)dx + g(x,y)dy = \int_R\int \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\right) dA$$

So: $\oint_C xe^{-2x}dx + (x^4 + 2x^2y^2)\,dy$

$$= \int_R\int \left(\frac{\partial}{\partial x}(x^4 + 2x^2y^2) - \frac{\partial}{\partial y}(xe^{-2x})\right) dA$$

$$= \int_R\int 4x^3 + 4xy^2 - \cancel{0}\,dA = \int_R\int 4x^3 + 4xy^2\,dA$$

if $x^2+y^2=1$ & $x^2+y^2=9$ & $x = r\cos\theta$, $y = r\sin\theta$, then:

$$\left.\begin{aligned} r^2(\cos^2\theta + \sin^2\theta) = 1 &\rightarrow r=1 \\ r^2 = 9 &\rightarrow r=3 \end{aligned}\right\} 1 \le r \le 3$$

Also, the circles both occupy all four quadrents, thus $0 \le \theta \le 2\pi$

So:

$$= \int_0^{2\pi}\int_1^3 (4x^3 + 4xy^2)(r)\,dr\,d\theta = 4\int_0^{2\pi}\int_1^3 x(x^2+y^2)(r)\,dr\,d\theta$$

$$= 4\int_0^{2\pi}\int_1^3 r^3(r\cos\theta)\,dr\,d\theta = 4\int_0^{2\pi}\left[\frac{r^5\cos\theta}{5}\right]_1^3 d\theta$$

$$= \frac{4}{5}\int_0^{2\pi} 243\cos\theta - \cos\theta\,d\theta = \frac{968}{5}\int_0^{2\pi}\cos\theta\,d\theta$$

$$= \frac{968}{5}\Big[\sin\theta\Big]_0^{2\pi} = \frac{968}{5}\Big[\sin 2\pi - \sin 0\Big]$$

$$= \frac{968}{5}(0) = 0$$

thus:

$$\int_C f(x,y)\,dx + g(x,y)\,dy = \int_C xe^{-2x}dx + (x^4+2x^2y^2)dy$$

$$= \int_R\int 4x^3+4xy^2\,dA = 0$$

(10) Evaluate the Surface integral $\iint_S x^2z^2 dS$ where S is the part of the Cone $z^2=x^2+y^2$ that lies between the planes $z=1$ & $z=4$:

$$\iint_S f(x,y,z)\,dS = \iint_R f(x,y,g(x,y))\sqrt{\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2+1}\;dA$$

if $z^2=x^2+y^2$ & if $x=r\cos\theta, y=r\sin\theta$ then:

$z^2=r^2\cos^2\theta+r^2\sin^2\theta$ then $z^2=r^2$ So $z=r$

if $z=1$ or $z=4$ & $z=r$, then $r=1$ or $r=4$

So: $1\le r\le 4$

the cone occupies all four quadrents $\therefore$ $0\le\theta\le 2\pi$

So: let $z=g(x,y)=\sqrt{x^2+y^2}=r$

$$\iint_S x^2z^2dS = \iint_R x^2z^2\sqrt{\left(\frac{2x}{2\sqrt{x^2+y^2}}\right)^2+\left(\frac{2y}{2\sqrt{x^2+y^2}}\right)^2+1}\;dA$$

$$= \int_0^{2\pi}\int_1^4 x^2z^2\sqrt{\frac{x^2}{x^2+y^2}+\frac{y^2}{x^2+y^2}+1}\;dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^4 (r)(x^2)(\sqrt{x^2+y^2}^{\,2})\sqrt{\frac{x^2+y^2}{x^2+y^2}+1}\;dr\,d\theta$$

$$= \int_0^{2\pi}\int_1^4 r(r^2\cos^2\theta)(r^2)\sqrt{2}\;dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\int_1^4 r^5\cos^2\theta\,dr\,d\theta$$

$$= \sqrt{2}\int_0^{2\pi}\cos^2\theta\left[\frac{r^6}{6}\right]_1^4 dr\,d\theta$$

$$= \sqrt{2}\left(\frac{4^6}{6}-\frac{1^6}{6}\right)\int_0^{2\pi}\cos^2\theta\,d\theta$$

$$= \sqrt{2}\left(\frac{4096-1}{6}\right)\int_0^{2\pi}\frac{\cos 2\theta+1}{2}\,d\theta$$

$$= \sqrt{2}\left(\frac{1365}{2}\right)\left(\frac{1}{2}\right)\left[\frac{1}{2}(\sin 2\theta)+\theta\right]_0^{2\pi}$$

$$= \frac{\sqrt{2}\,1365}{4}\left(\cancel{\sin(4\pi)}+2\pi-\cancel{\sin(0)}-\cancel{0}\right)$$

$$= \boxed{\frac{1365\sqrt{2}}{2}\pi} \quad \text{OR Simply: } \frac{1365}{\sqrt{2}}\pi$$

thus:

$$\iint_S x^2 z^2 ds = \sqrt{2}\int_0^{2\pi}\int_1^4 r^5 \cos^2\theta \, dr d\theta = \frac{1365\sqrt{2}}{2}\pi \doteq 3032.268$$

(11) Find the flux of the field $F(x,y,z) = xze^{y}i - xze^{y}j + zK$ across the Surface S Which is the part of the plane $x+y+z=2$ in the first octant & has downward orientation:

$$\Phi = \iint_\sigma F(x,y,z)\cdot n(x,y,z)\,ds$$

$$= \iint_R F\cdot\left(\frac{\partial z}{\partial x}i + \frac{\partial z}{\partial y}j - K\right)dA \quad [\text{oriented down}]$$

If $z = 2-x-y$ then:

$$= \int_R\int F(x,y,2-x-y)\cdot(\langle -1,-1,-1\rangle)\,dA$$

$$= \int_R\int \langle x(2-x-y)e^{y}, -x(2-x-y)e^{y}, 2-x-y\rangle \cdot \langle -1,-1,-1\rangle\,dA$$

$$= \int_R\int -x(2-x-y)e^{y} + x(2-x-y)e^{y} - (2-x-y)\,dA$$

$$= \int_R\int (-2x+x^2+xy)e^{y} + (2x-x^2-xy)e^{y} - 2+x+y\,dA$$

$$= \int_R\int \cancel{-2xe^{y}} + \cancel{x^2e^{y}} + \cancel{xye^{y}} + \cancel{2xe^{y}} - \cancel{x^2e^{y}} - \cancel{xye^{y}} - 2+x+y\,dA$$

$$= \int_R\int x+y-2\,dA$$

if $x+y+z=2$ then; [x,y in the first octant, " starts at zero.] if $z=y=0$, $0 \leq x \leq 2$; if $z=0$ then, $0 \leq y \leq 2-x$

$$= \int_0^2\int_0^{2-x} x+y-2\,dy\,dx = \int_0^2 \left[xy + \frac{y^2}{2} - 2y\right]_0^{2-x} dx$$

$$= \int_0^2 x(2-x) + \frac{(2-x)^2}{2} - 2(2-x)\,dx$$

$$= \int_0^2 2x - x^2 + \frac{(4-4x+x^2)}{2} - 4 + 2x\,dx$$

$$= \int_0^2 \cancel{2x} - x^2 + \cancel{2} - \cancel{2x} + \frac{x^2}{2} - \cancel{4} + 2x\,dx$$

$$= \int_0^2 -2 + 2x - \frac{x^2}{2}\,dx = \left[-2x + \frac{2x^2}{2} - \frac{x^3}{2\cdot 3}\right]_0^2$$

$$= -2(2) + \frac{2(2)^2}{2} - \frac{(2)^3}{6} = \cancel{-4} + \cancel{4} - \frac{8}{6} = \boxed{-\frac{4}{3}}$$

thus:

$$\Phi = \iint_\sigma F(x,y,z)\cdot n(x,y,z)\,ds = \int_R\int F(x,y,2-x-y)\cdot\langle -1,-1,-1\rangle\,dA$$

$$= \boxed{-\frac{4}{3} \doteq -1.3333}$$

⑫ Show how Stroke's theorem can be used to evaluate $\int_C F \cdot dr$ where $F(x,y,z) = e^{-x}i + e^{x}j + e^{z}k$ & C is the boundary of the part of the plane $x+2y+z=2$ in the first octant, oriented Counterclockwise as viewed from above:

$$\oint_C F \cdot dr = \iint_\sigma (\text{Curl } F) \cdot n\, dS$$

So Curl F is:

$$\text{Curl } F = \left| \begin{array}{ccc|cc} i & j & k & i & j \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} & \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ e^{-x} & e^{x} & e^{z} & e^{-x} & e^{x} \end{array} \right|$$

$$= \cancel{\frac{\partial}{\partial y}(e^{z})i} + \cancel{\frac{\partial}{\partial z}(e^{-x})} + \frac{\partial}{\partial x}(e^{x})k$$
$$+ \cancel{\frac{\partial}{\partial y}(e^{-x})k} + \cancel{\frac{\partial}{\partial z}(e^{x})i} + \cancel{\frac{\partial}{\partial x}(e^{z})j}$$

$$= e^{x}k$$

$\therefore$ Curl $F = \langle 0, 0, e^{x} \rangle$

We must now find the Surface:

if $x+2y+z=2$, then when $y=z=0$, $x=2$

$z=0$, $y = 2-x$

thus, $z = 2-x-2y$

Since the Surface is in the first octant; $x,y,z \geq 0$

$\therefore 0 \leq x \leq 2,\ 0 \leq y \leq \frac{2-x}{2},\ 0 \leq z \leq 2-x-2y$

To verify this is Still the same Surface:

Check: $r_u \times r_v$ = Plane of $x+2y+z=2$

the Plane $x+2y+z$ is $\langle 1,2,1 \rangle$

$r_u = \langle 1, 0, -1 \rangle$, $r_v = \langle 0, 1, -2 \rangle$

$$r_u \times r_v = \left| \begin{array}{ccc|cc} i & j & k & i & j \\ 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & -2 & 0 & 1 \end{array} \right| = k + i + 2j \rightarrow \langle 1,2,1 \rangle$$

(thus we Parameterized r Correctly)

So: the normal vector is $\langle 1,2,1 \rangle$

Find $\int_C F \cdot dr$:

$$\oint_C F \cdot dr = \iint_\sigma (\text{Curl } F) \cdot n\, dS = \iint_\sigma \langle 0,0,e^{x} \rangle \cdot \langle 1,2,1 \rangle\, dS$$

$$= \iint_\sigma e^{x}\, dS$$

thus: $0 \leq x \leq 2,\ 0 \leq y \leq \frac{2-x}{2}$

$$= \int_0^2 \int_0^{1-\frac{x}{2}} e^{x}\, dy\, dx = \int_0^2 \left[y e^{x} \right]_0^{1-\frac{x}{2}} dx$$

$$= \int_0^2 \left(\left(1-\frac{x}{2}\right)e^x - \cancel{(0)e^x} \right) dx$$

$$= \int_0^2 e^x - e^x\left(\frac{x}{2}\right) dx$$

$$= \left[e^x\right]_0^2 - \int_0^2 e^x \frac{x}{2}\, dx$$

$$= e^2 - e^0 - \frac{1}{2}\int_0^2 x e^x\, dx$$

let $u = x$ then $du = dx$
let $dv = e^x dx$ then $v = e^x$

$$= e^2 - 1 - \frac{1}{2}\left[(x)e^x \Big|_0^2 - \int_0^2 e^x dx \right]_0^2$$

$$= e^2 - 1 - \left[\frac{1}{2}\left(2e^2 - \cancel{(0)e^0}\right) - \frac{1}{2}\left[e^x\right]_0^2\right]$$

$$= e^2 - 1 - \left(e^2 - \frac{1}{2}\left(e^2 - e^0\right)\right)$$

$$= \cancel{e^2} - 1 - \cancel{e^2} + \frac{1}{2}e^2 - \frac{1}{2}$$

$$= \boxed{\frac{e^2-3}{2}} \doteq 2.1945$$

Thus:

$$\oint_C F \cdot dr \doteq \iint_\sigma (\text{curl } F) \cdot n\, dS$$

$$= \int_0^2 \int_0^{1-\frac{x}{2}} e^x\, dy\, dx = \boxed{\frac{e^2-3}{2} \approx 2.1945}$$