Name: Connor Raymond Stewart
ID: 101041125

Acknowledgement: Received help for question 3c from Kevin Cheung & Amanda Chafee.
No help received for other questions

MATH 3802 Assignment #2:

① let $A=\begin{bmatrix}1&0&1\\-1&1&-1\\0&1&0\end{bmatrix}$. Is A totally unimodular? Justify your answer:

$\boxed{I}$ – we note that given positive integers m & n, a matrix $A \in \mathbb{Z}^{m\times n}$ is said to be totally unimodular if every square submatrix of A has determinant 0, 1, or −1.

– we see that A is a 3x3 matrix, thus the square-submatrices are of dimentions 1x1, 2x2, & 3x3. This leaves us w/ the following submatrixes:

1x1:

$[1],[0],[1],[-1],[1],[-1],[0],[1],[0]$

2x2:

$\begin{bmatrix}1&0\\-1&1\end{bmatrix},\begin{bmatrix}1&0\\0&1\end{bmatrix},\begin{bmatrix}-1&1\\0&1\end{bmatrix},\begin{bmatrix}1&1\\-1&-1\end{bmatrix},\begin{bmatrix}1&1\\0&0\end{bmatrix},\begin{bmatrix}-1&-1\\0&0\end{bmatrix},$

$\begin{bmatrix}0&1\\1&-1\end{bmatrix},\begin{bmatrix}0&1\\1&0\end{bmatrix},\begin{bmatrix}1&-1\\1&0\end{bmatrix}$

3x3:

$\begin{bmatrix}1&0&1\\-1&1&-1\\0&1&0\end{bmatrix}$

– We know that:

1x1: Det = # in matrix

Since all matricies are from 1, 0, −1, this satisfies $\boxed{I}$

2x2: Det = ad − bc where $A=\begin{bmatrix}a&b\\c&d\end{bmatrix}$

$$\begin{bmatrix}1&-1\\1&0\end{bmatrix}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}=\begin{bmatrix}1&0\\0&1\end{bmatrix}=1,\quad \begin{bmatrix}-1&1\\0&1\end{bmatrix}=\begin{bmatrix}0&1\\1&-1\end{bmatrix}=\begin{bmatrix}0&1\\1&0\end{bmatrix}=-1,$$

$$\begin{bmatrix}1&1\\-1&-1\end{bmatrix}=\begin{bmatrix}1&1\\0&0\end{bmatrix}=\begin{bmatrix}-1&-1\\0&0\end{bmatrix}=0,$$ this satisfies $\boxed{I}$

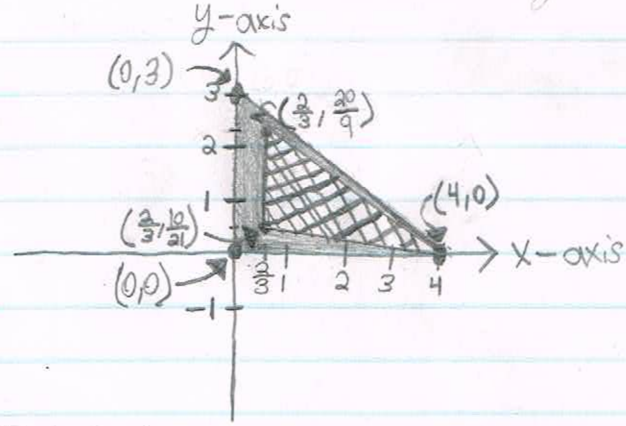
3x3: we can find the determinent as follows

$$\left[\begin{array}{ccc|cc}1&0&1&1&0\\-1&1&-1&-1&1\\0&1&0&0&1\end{array}\right]=(1)^2(-1)-(1)^2(-1)=-1-(-1)=0,$$ this satisfies $\boxed{I}$

thus, as all square-submatracies of A have determinents of 1, 0, −1, we can conclude that A is totally unimodular.

② (3 Points) Given a Polyhedron $P \subseteq \mathbb{R}^n$, let P_I denote the Convex hull of $P \cap \mathbb{Z}^n$. Suppose that $P = \left\{ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 : 2x_1 + 3x_2 \leq 8,\ x_1 + 7x_2 \geq 4,\ 3x_1 \geq 2 \right\}$. Give a sketch of both P & P_I on the same set of axes:

This can be converted to the following:



Intersects:

$2x_1 + 3x_2 \leq 8,\ x_1 + 7x_2 \geq 4,\ 3x_1 \geq 2$ $\quad [2x_1 + 3x_2 \leq 8$ thus $-2x_1 - 3x_2 \geq -8]$

I) let $3x_1 \geq 2$ be $x_1 \geq \frac{2}{3}$ So that $x_1 + 7x_2 \geq 4$ is:

$\frac{2}{3} + 7x_2 \geq 4 \rightarrow 7x_2 \geq \frac{10}{3} \rightarrow x_2 \geq \frac{10}{21}$

thus: $x_1 \geq \frac{2}{3}$ & $x_2 \geq \frac{10}{21}$

II) let $x_1 \geq \frac{2}{3}$ & $2x_1 + 3x_2 \leq 8$ So $3x_2 \leq 8 - 2(\frac{2}{3})$

$3x_2 \geq \frac{20}{3}$ & $x_2 \geq \frac{20}{9}$

thus: $x_1 \geq \frac{2}{3}$ & $x_2 \geq \frac{20}{9}$

III) let $2x_1 + 3x_2 \leq 8$ & $x_1 + 7x_2 \geq 4$ So $-x_1 - 7x_2 \leq -4$:

$x_1 \leq 4 - \frac{3}{2}x_2$ thus $-(4 - \frac{3}{2}x_2) - 7x_2 \leq -4$

$-4 + \frac{3}{2}x_2 - 7x_2 \leq -4$

$\frac{3}{2}x_2 - \frac{7}{2}\cancel{x_2} \leq 0$

$3x_2 - 7x_2 \leq 0$

$-4x_2 \leq 0$ thus $x_2 \geq 0$

Since $x_2 \geq 0$ then $x_1 + 7x_2 \geq 4$ is $x_1 \geq 4 - 7x_2$:

$x_1 \geq 4 - 0 \rightarrow x_1 \geq 4$

thus, boundary points lie at $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 2/3 \\ 10/21 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 2/3 \\ 20/9 \end{bmatrix}$, $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \geq \begin{bmatrix} 4 \\ 0 \end{bmatrix}$

AS all of P must be in P_I, & P_I is of Integers, we know that we must take the value of all verticies in P's tryangle to include all P in P_I:

$(\frac{2}{3}, \frac{20}{9}) \rightarrow (0,3)$
$(\frac{2}{3}, \frac{10}{21}) \rightarrow (0,0)$
$(4,0) \rightarrow (4,0)$

Note: the fraction values in P must be adjusted Such that the Point in P_I is moved away from P's vertex

(3) let $G=\{N,A\}$ be a directed graph such that N contains distinct nodes r,s & $A \subseteq N \times N$. let $c \in \mathbb{R}^A$, we call $y \in \mathbb{R}^N$ a **potential** if $y_w - y_u \leq c_{uw}$.
We define the cost of an r-v dipath P, where $v \in N$, to be $\sum_{e \in P} c_e$:

a) (1 Point) Prove that if y is a potential & P is an r-s dipath, then the cost of P is at least $y_s - y_r$:

- $y_w - y_u \leq c_{uw}$ as y is a Potential (for $y \in \mathbb{R}^N$)
- the cost of P is $\sum_{e \in P} c_e$, Since P is an r-s dipath ($S \in N$)

the cost to move from r to s is $\sum_{e \in P} c_e$:

$c_i + c_j + c_k + \ldots = c_{rs}$, where $i,j,k,\ldots$ are e's along the path from r to s.

this is because c_{rs} is the cost from r to s

So, $\sum_{e \in P} c_e = c_{rs}$

We note that:

$y_s - y_r \leq c_{rs}$ by the definition of y being a Potential

thus, as $\sum_{e \in P} c_e$ is c_{rs}, we see:

$$y_s - y_r \leq c_{rs} = \sum_{e \in P} c_e$$

So, P is at least $y_s - y_r$

b) (1 Point) Prove that no potential exists if there exists a dicycle C such that $\sum_{e \in C} c_e < 0$:

Suppose the following G exists:

$$r \xrightarrow{-2} x \xrightarrow{-3} s$$

then $\sum_{e \in C} c_e = -2-3 = -5$

Yet, we see that: $-3-(-2) = -1 \nleq -5$. Thus, it follows no potential exists.

We can follow this logic to see that $y_s - y_r$ results in a value greater then $\sum_{e \in C} c_e$ Since: (let us assume that the Statement is true)

if $y_s - y_r \leq c_{rs}$ & $c_{rs} < 0$ then $y_s + y_r$ is negative

(I) If $|y_r| > |y_s|$ then $y_s - y_r \ngtr 0$

(II) If $|y_r| < |y_s|$ & $y_r < 0$ then $y_s - y_r > \sum_{e \in C} c_e$ (Contradicts $y_s - y_r \leq c_{rs}$)

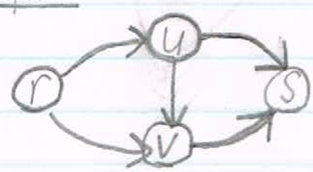
(I) Contradicts the given $\sum_{e \in C} c_e > 0$ & (II) Contradicts the definition of Potential

$\hookrightarrow$ ∴ It is False by Contradiction

c) (1 Point) Write down a linear Programming problem whose duel problem is:

$$\max\ y_s - y_r$$
$$\text{s.t. } y_w - y_u \le c_{uw} \quad \forall uw \in A$$

Example Graph:



We can formulate the following: (As a Concrete example)

$$\begin{array}{ll} & \max\ y_s - y_r \\ x_{ru}: & \text{s.t. } y_u - y_r \le c_{ru} \\ x_{rv}: & \quad y_v - y_r \le c_{rv} \\ x_{uv}: & \quad y_v - y_u \le c_{uv} \\ x_{us}: & \quad y_s - y_u \le c_{us} \\ x_{vs}: & \quad y_s - y_v \le c_{vs} \\ & \quad y_s, y_s, y_u, y_v \ge 0 \end{array}$$

$\Rightarrow$ thus duel is $\Rightarrow$

$$\begin{array}{ll} \min\ c_{ru}x_{ru} + c_{rv}x_{rv} + c_{uv}x_{uv} + c_{us}x_{us} & \\ \quad + c_{vs}x_{vs} & \\ \text{s.t. } x_{us} + x_{vs} = 1 & : y_s \\ \quad -x_{ru} - x_{rv} = -1 & : y_r \\ \quad x_{ru} - x_{uv} - x_{us} = 0 & : y_u \\ \quad x_{rv} + x_{uv} - x_{vs} = 0 & : y_v \\ \quad x_{us}, x_{rv}, x_{uv}, x_{us}, x_{vs} \ge 0 & \end{array}$$

thus, we converted the program to:

$$\boxed{\begin{array}{ll} \min \sum c_{wu}x_{wu} & \\ \text{s.t. } \sum c_{sr} \sum x_{sr} \ge d & \forall sr \in N \end{array}}$$

Where d is a vector such that $d_s = 1, d_r = -1$, & $d_u = 0$ for all u NOT Equal to s or r.