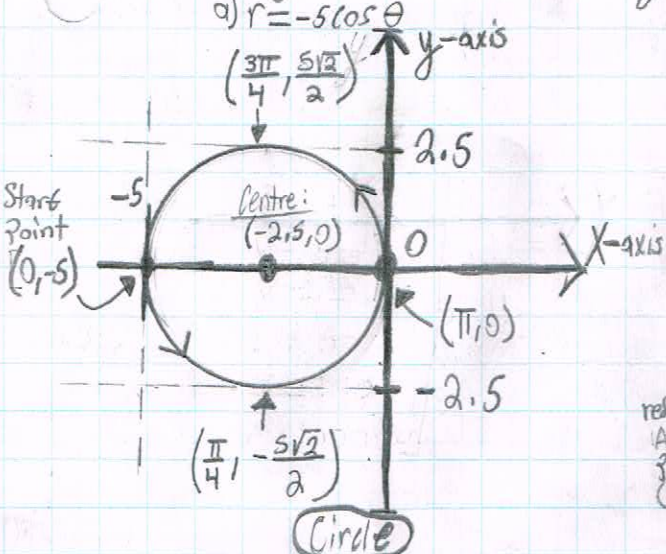
Assignment One:

① Identify & describe the Curves given by the Polar equations listed below:

a) $r = -5\cos\theta$



- this is a Polar equation representing a Circle.
- the distance from the origin is of magnitude of 5, w/ -5 being the furthest point from the origin along the x-axis.
- the Graph Starts at -5, as $-5\cos(0) = -5(1) = -5$

For more values, See below:

θ	r	θ	r
0	-5	$\frac{5\pi}{4}$	$\frac{5\sqrt{2}}{2}$
$\pi/4$	$-\frac{5\sqrt{2}}{2}$	$\frac{3\pi}{2}$	0
$\pi/2$	0	$\frac{7\pi}{4}$	$-\frac{5\sqrt{2}}{2}$
$3\pi/4$	$\frac{5\sqrt{2}}{2}$	2π	-5
π	5		

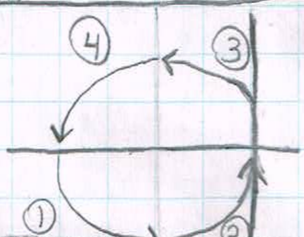
reflected Angle positions $(r<0)$

- As seen, at $\theta = 0$, $r = -5$; thus placing it 5 units to the reflection of angle $\theta = 0$ (aka -5, on the x-axis)
- the graph then rotates (Counter-clockwise) to zero on the X-axis, then from there back to -5 on the X-axis as θ approches π.

↳ one Period occurs in π radians, thus: $-5\cos(0) = -5\cos(\pi) = -5$ (on x-axis)

- the Circle has a radius of 2.5 units, a diameter of 5, & a Center at (-2.5, 0).

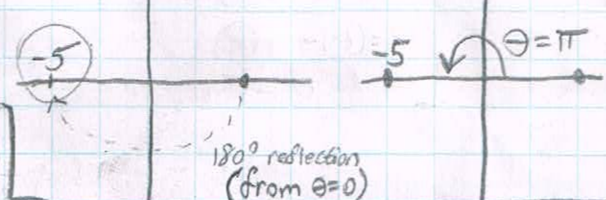
the graph forms in the following order:



① : $0 \leq \theta \leq \frac{\pi}{4}$
② $\frac{\pi}{4} \leq \theta \leq \frac{\pi}{2}$
③ $\frac{\pi}{2} \leq \theta \leq \frac{3\pi}{2}$
④ $\frac{3\pi}{2} \leq \theta \leq \pi$

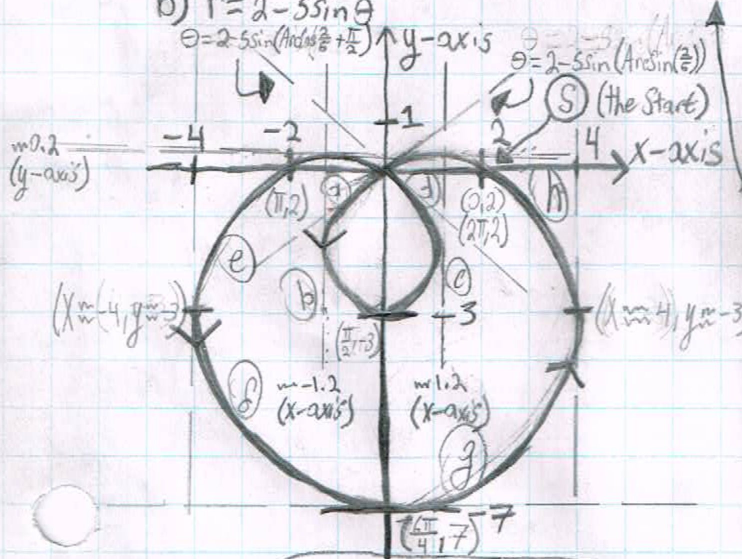
to Show this:

$-5\cos(0) = -5$ $\qquad$ $-5\cos(\pi) = -(-5) = 5$



$\frac{2}{5} < 1$ ∴ limaçon w/ inner loop

b) $r = 2 - 5\sin\theta$



Reflected Angle (θ) Positions $(r<0)$

table of values:

r	θ	
0	2	S
$\pi/4$	$\frac{4-5\sqrt{2}}{2}$	a
$\pi/2$	-3	b
$3\pi/4$	$\frac{4-5\sqrt{2}}{2}$	c
π	2	d
$\frac{5\pi}{4}$	$\frac{4+5\sqrt{2}}{2}$	e
$\frac{6\pi}{4}$	7	f
$\frac{7\pi}{4}$	$\frac{4+5\sqrt{2}}{2}$	g
2π	2	h

- the period is 2π

marks indicate Polar region on Graph

$r=0$ when
$0 = 2 - 5\sin\theta$
thus: $\theta = \mathrm{Arcsin}\left(\frac{2}{5}\right)$
$\doteq 0.4115\ldots$

thus: $2 - 5\sin\left(\mathrm{ArcSin}\frac{2}{5}\right) = 0$

$r=0$:
$\sin\theta = \frac{2}{5}$
$\cos\left(\theta - \frac{\pi}{2}\right) = \frac{2}{5}$
$\theta = \mathrm{Arc}\cos\left(\frac{2}{5}\right) + \frac{\pi}{2}$
$\doteq 2.73\ldots$

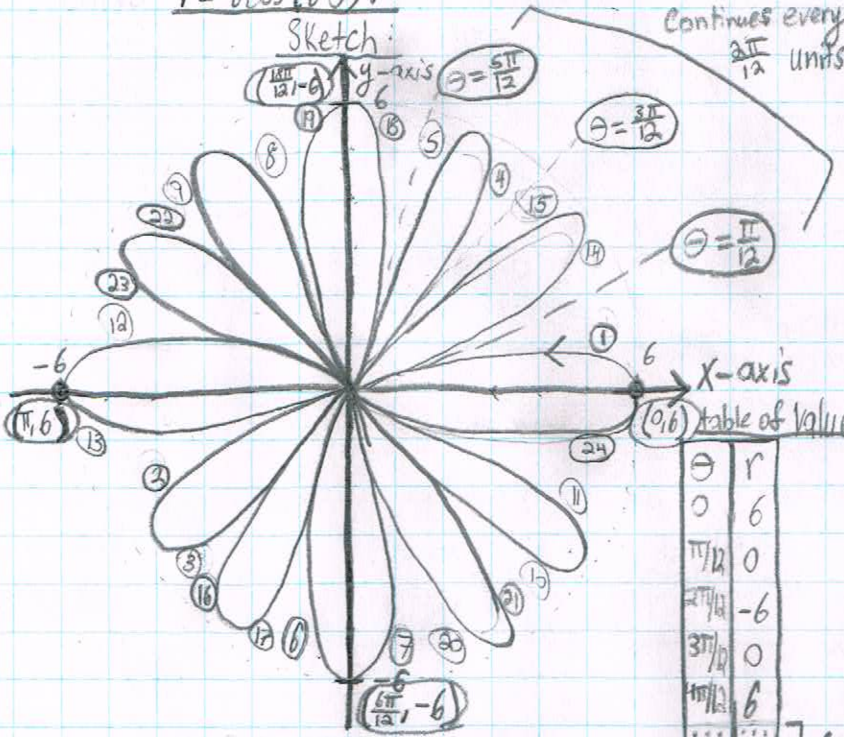
thus: $2 - 5\sin\left(\mathrm{Arc}\cos\left(\frac{2}{5}\right) + \frac{\pi}{2}\right) = 0$

- This is a Limaçon w/ inner loop
- the rotation is Counter-ClockWise

c) $r=6\cos(6\theta)$

$r=6\cos(6\theta)$:

Sketch:



12-Petal Rose

table of values:

θ	r
0	6
$\pi/12$	0
$2\pi/12$	-6
$3\pi/12$	0
$4\pi/12$	6
⋮	⋮
2π	6

} Continue Pattern...

When $6\cos(6\theta)=0$:

$\theta=\frac{\operatorname{Arccos}(0)}{6}$

$\theta=\pi/12$

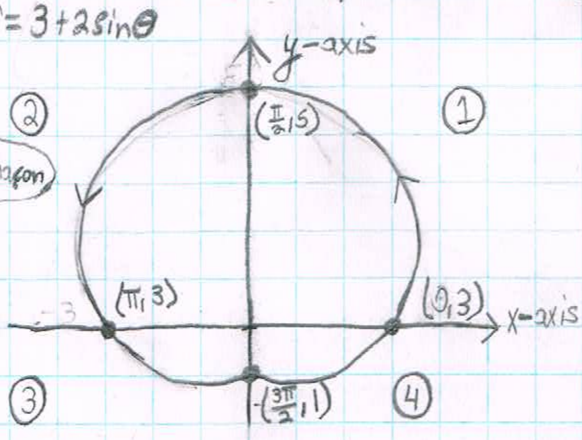
When $\theta=3\pi/12$, $6\cos(6\theta)=0$

- ∴ Each Petal occupies $2\pi/12$ of the graph.
- the Petals are Spaced evenly every $2\pi/12$ radians, thus there are 12 Petals as 2π divided by $2\pi/12$ equals 12. When θ is multiplied by even numbers, the Petals don't overlap, meaning all 12 Petals appear on the graph as θ changes from 0 to 2π.
- The order that the leaves appear when θ Shifts from 0 to 2π are marked as ① to ㉔ on the graph

d) $r=3+2\sin\theta$

$1<\frac{3}{2}<2$

∴ Dimpled limaçon

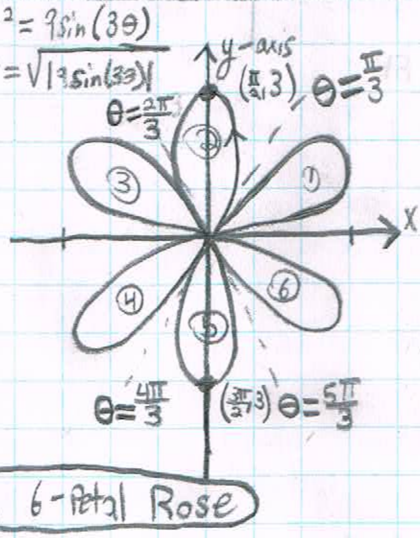


- When $\theta=0$:
  $r=3+\cancel{2\sin(0)}$
  $r=3$
- When $\theta=2\pi$
  $r=3+\cancel{2\sin(2\pi)}$
  $r=3$
- When $\theta=\frac{3\pi}{2}$:
  $r=3+2\sin(3\pi/2)$
  $r=3-2$
  $r=1$
- When $\theta=\pi$:
  $r=3+\cancel{2\sin(\pi)}$
  $r=3$
- When $\theta=\pi/2$:
  $r=3+2\sin(\pi/2)$
  $r=5$

- As θ increases, the graph Progresses from ① to ④
  ①: 0 to $\frac{\pi}{2}$ ③: π to $\frac{3\pi}{2}$
  ②: $\frac{\pi}{2}$ to π ④: $\frac{3\pi}{2}$ to 2π

e) $r^2=9\sin(3\theta)$

$r=\sqrt{|9\sin(3\theta)|}$



∴ 6-Petal Rose

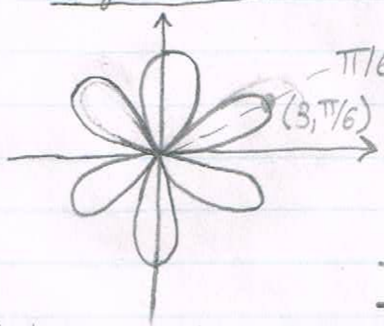
θ	r
0	0
$\pi/6$	3
$2\pi/3$	0
$\pi/3$	0
$\pi/2$	3
$5\pi/3$	3
π	0
⋮	⋮
2π	0

} Same Pattern for π to 2π

- the leafs appear every $\frac{\pi}{3}$ radians in the order listed in the graph. The leafs appear from ① to ⑥ as the value θ changes from 0 to 2π

2) Give the Slope of the tangent line to the Polar Curve $r^2 = 9\sin(3\theta)$ at the Point $(3, \frac{\pi}{6})$:

Rough Sketch:



When $\theta = \pi/6$:

$r = 3$ & r of 3 is max.

$y = r\sin\theta$ ∴ $dy/d\theta = r\cos\theta + \sin\theta \frac{dr}{d\theta}$

$x = r\cos\theta$ ∴ $dx/d\theta = -r\sin\theta + \cos\theta \frac{dr}{d\theta}$

thus: $\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{r\cos\theta + \sin\theta \frac{dr}{d\theta}}{-r\sin\theta + \cos\theta \frac{dr}{d\theta}}$

find rectangular Coordinates: ($\theta = \pi/6$ & $r = 3$)

$x = 3\cos(\pi/6) = \frac{3\sqrt{3}}{2}$

$y = 3\sin(\pi/6) = \frac{3}{2}$

Next, we find the Slope of the tangent line, ∴ We must find the derivative at the point:

$$m = \frac{dy}{dx} \text{ when } t = \frac{\pi}{6}$$

We must also derive r:

$$\frac{dr}{dt} = \left(\sqrt{|9\sin(3\theta)|}\right)' = 3 \cdot \frac{1}{2} |\sin(3\theta)|^{-1/2} \cdot 3\sin(3\theta)\cos(3\theta)$$

$$= \frac{9\left(\frac{\sin 3\theta}{|\sin 3\theta|}\right)\cos(3\theta)}{2|\sin(3\theta)|^{1/2}} = \frac{9\sin(3\theta)\cos(3\theta)}{2|\sin(3\theta)|^{3/2}}$$

Or, Simply $\Rightarrow \frac{9\cos(3\theta)}{2\sqrt{\sin(3\theta)}}$ Such that $9\sin(3\theta) \geq 0$

thus; $\frac{dy}{dx} = \frac{r\cos\theta + \sin\theta\, r'}{-r\sin\theta + \cos\theta\, r'} = \frac{\left(\sqrt{|9\sin(3\theta)|}\right)\cos\theta + \sin\theta\left(\frac{9\cos(3\theta)}{2\sqrt{\sin(3\theta)}}\right)}{\left(\frac{9\cos(3\theta)}{2\sqrt{\sin(3\theta)}}\right)\sin\theta - \left(\sqrt{|9\sin(3\theta)|}\right)\sin\theta}$

let $t = \pi/6$:

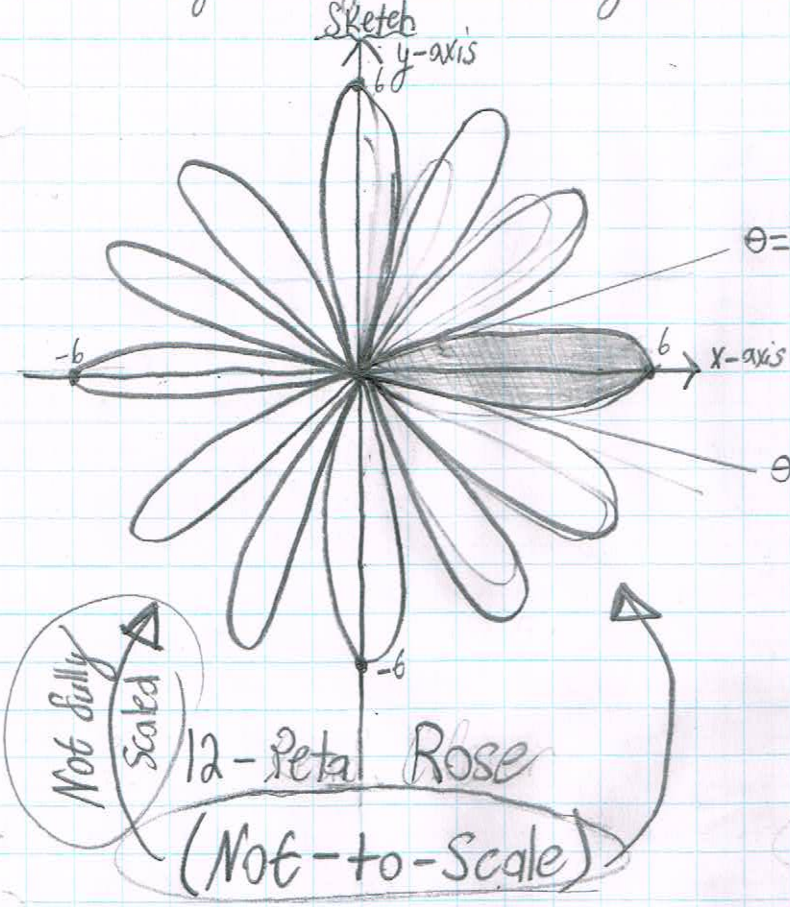
$$m = \frac{\left(\sqrt{|9\sin(\pi/2)|}\right)\cos\pi/6 + \sin\pi/6\left(\frac{9\cos(\pi/2)}{2\sqrt{\sin(\pi/2)}}\right)}{\left(\frac{9\cos(\pi/2)}{2\sqrt{\sin(\pi/2)}}\right)\sin\pi/6 - \left(\sqrt{|9\sin(\pi/2)|}\right)\sin\pi/6} = \frac{3\sqrt{3}/2}{-3/2} = \frac{\cancel{3}\sqrt{3}}{\cancel{2}} \cdot -\frac{\cancel{2}}{\cancel{3}}$$

$m = -\sqrt{3}$

∴ the Slope of the tangent line is $-\sqrt{3}$.

3) the integrel that gives the arc length of one of the leaves of the polar curve $r=6\cos(6\theta)$:

Sketch



$\alpha=\pi/12$, $\beta=-\pi/12$, $r=6\cos(6\theta)$

$$\int_\alpha^\beta \frac{1}{2}r^2 d\theta$$

$$=\int_{-\pi/12}^{\pi/12}\frac{1}{2}(6\cos(6\theta))^2 d\theta$$

$$=\cancel{(2)}\cancel{\left(\tfrac{1}{2}\right)}\int_0^{\pi/12} 36\cos^2(6\theta)\,d\theta$$

↳ we convert α to 0 & multiply by two since the integrel of 0 to $\frac{\pi}{12}$ is exactly half the area; thus the results are equivilent

$$= 36\int_0^{\frac{\pi}{12}}\cos^2(6\theta)\,d\theta$$

$$\left[\int_{-\pi/12}^{\pi/12}\frac{1}{2}(6\cos(6\theta))^2 d\theta \text{ unsimplified}\right]$$

4) Find the area of the inner loop of the Cardioid: $r = 1-\sqrt{2}\sin\theta$

Sketch:

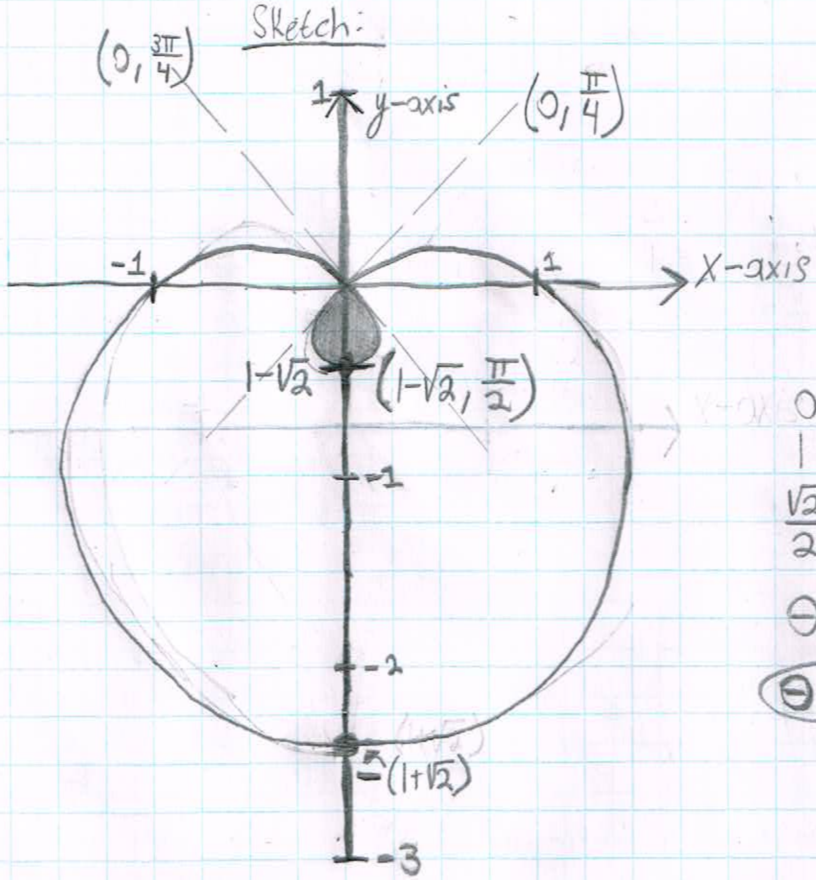


table of values:

θ	r
0	1
$\pi/2$	$1-\sqrt{2}$
π	1
$3\pi/2$	$1+\sqrt{2}$
2π	1

$0 = 1-\sqrt{2}\sin\theta$

$1 = \sqrt{2}\sin\theta$

$\frac{\sqrt{2}}{2} = \sin\theta$

$\theta = \text{ArcSin}\left(\frac{\sqrt{2}}{2}\right)$

$\theta = \pi/4$ OR $\theta = 3\pi/4$

to find the Area, we take the integration of θ from $\pi/4$ to $3\pi/4$ via the following:

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta \quad \text{Where } r = 1-\sqrt{2}\sin\theta,\ \beta = \frac{3\pi}{4},\ \&\ \alpha = \frac{\pi}{4}$$

thus: Plug in the Values

$$A = \int_{\pi/4}^{3\pi/4} \left(\tfrac{1}{2}\right)(1-\sqrt{2}\sin\theta)^2 \, d\theta$$

$\frac{\beta-\alpha}{2} = \frac{\pi}{4}$

$\alpha + \frac{\pi}{4} = \frac{\pi}{2}$

$\therefore$ the new $\beta = \pi/2$

We Know that the inner loop of the Cardioid (or in this Case a limaçon) is Symmetrical, thus the area of 2 times the integral over half the range θ is the Same as the above equation.

$$\therefore A = \cancel{2}\int_{\pi/4}^{\pi/2} \cancel{\tfrac{1}{2}}(1-\sqrt{2}\sin\theta)^2 \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} (1-\sqrt{2}\sin\theta)^2 \, d\theta = \int_{\pi/4}^{\pi/2} 1 - 2\sqrt{2}\sin\theta + \sqrt{2}^2\sin^2\theta \, d\theta$$

$$= \int_{\pi/4}^{\pi/2} 1 - 2\sqrt{2}\sin\theta + 2\sin^2\theta \, d\theta = \Big[\theta + 2\sqrt{2}\cos\theta\Big]_{\pi/4}^{\pi/2} + 2\int_{\pi/4}^{\pi/2} \sin^2\theta \, d\theta$$

$$= \Big[\theta + 2\sqrt{2}\cos\theta\Big]_{\pi/4}^{\pi/2} + \cancel{2}\int_{\pi/4}^{\pi/2} \cancel{\tfrac{1}{2}}(1-\cos(2\theta))\, d\theta = \Big[\theta + 2\sqrt{2}\cos\theta + \theta - \tfrac{1}{2}\sin 2\theta\Big]$$

$$= \left(\pi + \cancel{2\sqrt{2}\cos(\tfrac{\pi}{2})} - \cancel{\tfrac{1}{2}\sin(\pi)}\right) - \left(\tfrac{\pi}{2} + 2\sqrt{2}\cos(\tfrac{\pi}{4}) - \tfrac{1}{2}\sin(\tfrac{\pi}{2})\right) = \pi - \pi/2 - 2 + \tfrac{1}{2} = \tfrac{\pi}{2} - \tfrac{3}{2} = \boxed{\tfrac{\pi-3}{2}}$$

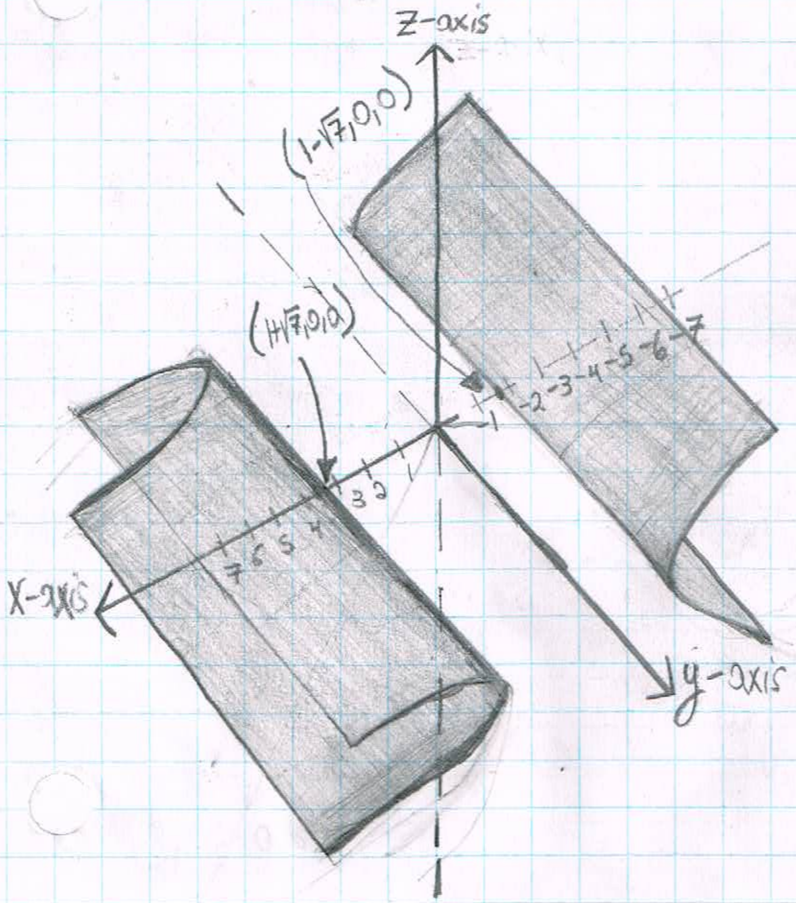
thus: the area of the inner loop of the Cardioid is

$A = \frac{\pi - 3}{2}$ [or, approx. ≈ 0.070796]

5) Provide a Sketch & description of the Surfaces in three dimensions listed below:

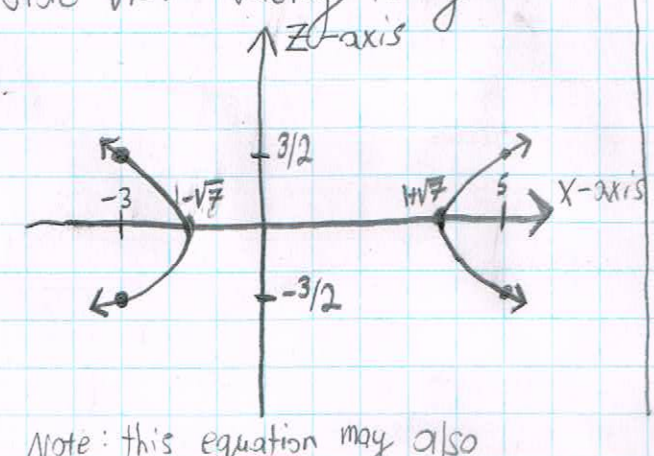
a) $x^2 - 2x - 4z^2 = 6$

Sketch:



Shape: Hyperbolic Cylander

Side view: facing the y-axis



Note: this equation may also be interpreted as: $x^2-2x-4y^2=6$ on a 2D graph (when the 3D version is "viewed down" the y-axis).

X-intercept:

When $z=0$: $[x^2-2x-\cancel{4(0)^2}=6]$

$x^2-2x=6$

$x^2-2x-6=0$

$\Rightarrow \dfrac{-(-2)\pm\sqrt{(-2)^2-4(1)(-6)}}{2} = \dfrac{2\pm\sqrt{28}}{2}$

$= \dfrac{2\pm 2\sqrt{7}}{2} = \boxed{1\pm\sqrt{7}}$

thus $x=1-\sqrt{7}$ OR $x=1+\sqrt{7}$ at the x-intercept

Z-intercept:

When $x=0$: $[(0)^2-2(0)-4z^2=6]$

$-4z^2=6$

$z^2 = 6/-4$

$z=\sqrt{-3/2}$

$z=\frac{\sqrt{6}}{2}\sqrt{-1}$

$z=\frac{\sqrt{6}}{2}i$

thus, z is imaginary ∴ there's no z-axis intercept

y-axis:

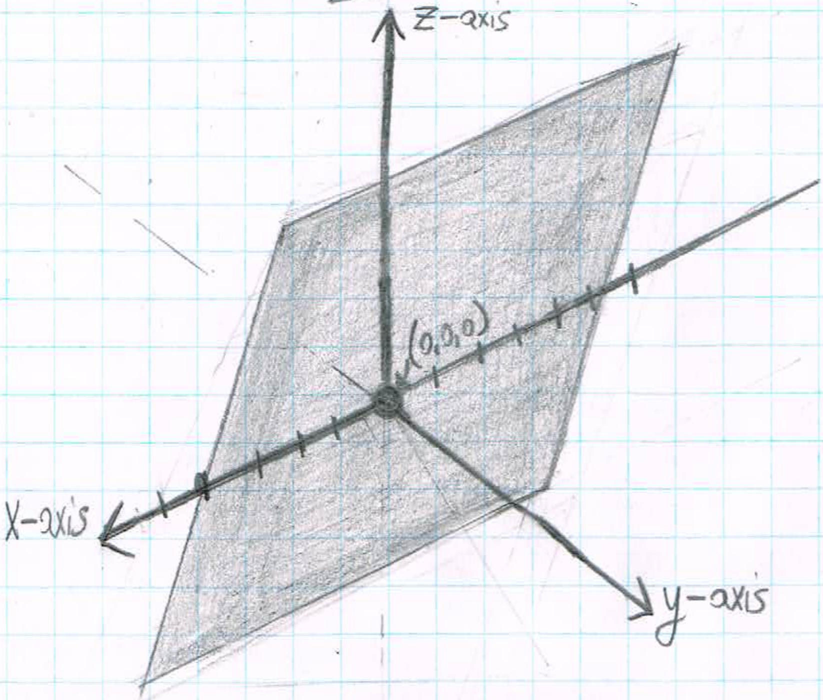
- The y-axis is not in the equation & is therefore free. Thus $y \in \mathbb{R}$.

Shape:

- the equation is of the form: $\dfrac{(x-1)^2}{a} - \dfrac{1}{a} + \dfrac{z^2}{b} = c$, thus the Shape is a Hyperbolic Cylinder which exists in the x & z axis & is free allong the y-axis.
- This Can be Seen in the Sketch

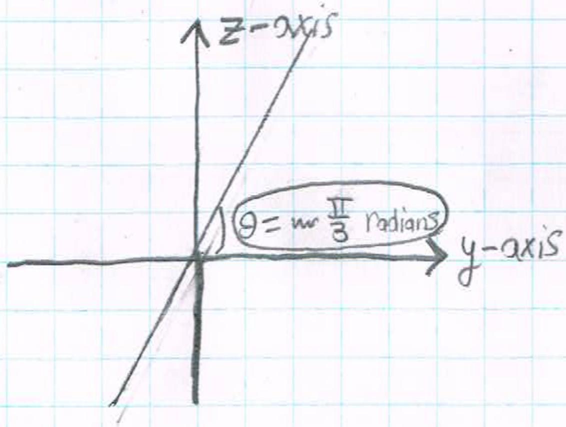
b) $2y - z = 0$

Sketch:



Shape: Tilted Plain

(Side view: facing the X-axis)



Note:

this line can be seen as: $2x = y$ on a 2D graph (i.e. when the 3D version is viewed down the y-axis).

Z-intercept:

When $y=0$:

$z=0$

y-intercept:

When $z=0$:

$y=0$

X-axis:

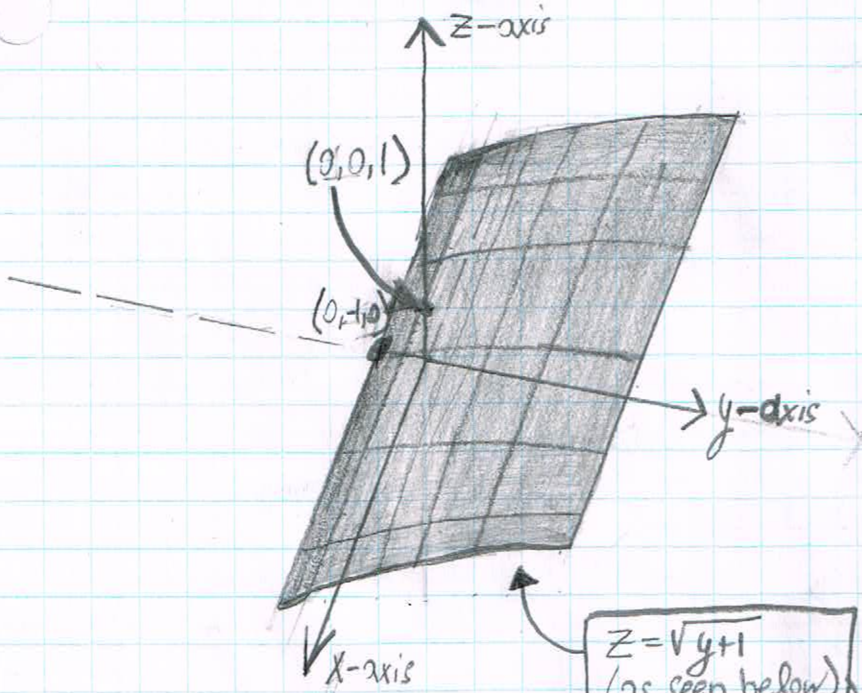
X is not in equation, thus $x \in \mathbb{R}$

Shape:

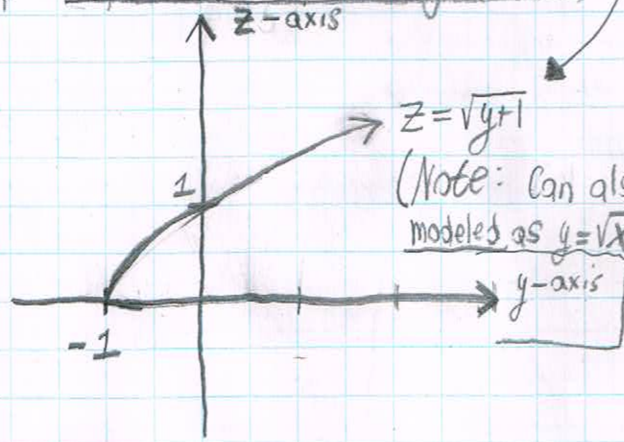
- The equation is of the form: $\frac{2y}{a} - \frac{z}{a} = 0$, thus the shape is a tilted plain which travels through the y & z dimensions, & is free along the x dimension.
- thus, the plain can be seen as infinitly many lines parallel along X-axis traveling through the z & y axises.
- The plain is tilted towards the Z-axis, which is hard to see in the main sketch.
  - The plain itself is tilted $\approx \frac{1}{3}$ towards the Z-axis & $\approx \frac{2}{3}$ towards the y-axis!

c) $z=\sqrt{y+1}$ thus $z^2=|y+1|,\ y+1\geq 0$

Sketch:



Shape: Upper Parabolic Cylinder



(Note: Can also be modeled as $y=\sqrt{x+1}$ on a 2D graph)

Side View: facing down the x-axis

z-intercept:
When $y=0$:
$z=\sqrt{1}$
$\therefore z=1$

y-intercept:
When $z=0$:
$0=\sqrt{y+1}$
$0^2=y+1$
$\therefore y=-1$

X-axis:
- x is not included in the equation $\therefore x\in\mathbb{R}$ for any value of z or y within the equation.

Shape:
- this equation is of the form:
$$z^2-y=1 \text{ where } y+1\geq 0$$
- thus:
  - The equation results in a Parabolic Cylinder forming through the z & y axis, & traveling along the x-axis.
    - the Par. cyl.'s hyperbolic curve forms along the z & y axis.
    - the curve stretches forever allong the x-axis.
  - The restriction that $y+1\geq 0$ results the Parabolic cylinder only having its upper segment.
- Therefore, the shape is an upper Parabolic Cylinder. This is reflected in the sketch.

6) $a = \langle -2, 4, 1 \rangle$, $b = \langle 2, -3, 0 \rangle$, & $c = \langle 3, -4, 2 \rangle$

a) The angle between the vectors b & c:

First use the dot Product to find:

$$b \cdot c = 2(3) + (-3)(-4) + \cancel{0(2)} = 6 + 12 = 18$$

now find the Norm of b & c:

$$\|b\| = \sqrt{2^2 + (-3)^2 + 0^2} = \sqrt{13}$$

$$\|c\| = \sqrt{3^2 + (-4)^2 + 2^2} = \sqrt{29}$$

Now find the ArcCos of the angle:

$$\frac{b \cdot c}{\|b\|\|c\|} = \frac{18}{\sqrt{13}\sqrt{29}} = \frac{18}{\sqrt{377}}$$

use the inverse Cos to find theta:

$$\theta = \operatorname{ArcCos}\left(\frac{b \cdot c}{\|b\|\|c\|}\right) = \operatorname{ArcCos}\left(\frac{18}{\sqrt{377}}\right) \doteq 22.02^\circ$$

thus: $\theta \doteq 22.02^\circ$ [or, $\theta \doteq 0.384337$ radians]

b) $\operatorname{Proj}_b a = \frac{a \cdot b}{\|b\|^2} b$

$$= \frac{\langle -2, 4, 1 \rangle \cdot \langle 2, -3, 0 \rangle}{\|\langle 2, -3, 0 \rangle\|^2} \langle 2, -3, 0 \rangle$$

$$= \frac{(-2)(2) + (4)(-3) + \cancel{(1)(0)}}{\left(\sqrt{2^2 + (-3)^2 + 0^2}\right)^2} \langle 2, -3, 0 \rangle$$

$$= \frac{-4 + (-12)}{\sqrt{13}^2} \langle 2, -3, 0 \rangle$$

$$= -\frac{16}{13} \langle 2, -3, 0 \rangle$$

$$= \left\langle -\frac{32}{13}, \frac{48}{13}, 0 \right\rangle$$

thus:

$$\operatorname{Proj}_b a = -\frac{16}{13} \langle 2, -3, 0 \rangle = \left\langle -\frac{32}{13}, \frac{48}{13}, 0 \right\rangle$$

c) $(a \cdot b)(c - b)$

$= (\langle -2,4,1\rangle \cdot \langle 2,-3,0\rangle)(\langle 3,-4,2\rangle - \langle 2,-3,0\rangle)$

$= (-2(2) + 4(-3) + \cancel{1(0)})(\langle 3-2, -4-(-3), 2-0\rangle)$

$= (-4-12)\langle 1,-1,2\rangle$

$= -16\langle 1,-1,2\rangle$

$= \langle -16,16,-32\rangle$

thus:

$(a \cdot b)(c-b) = -16\langle 1,-1,2\rangle = \langle -16,16,-32\rangle$

d) $(a-b)\times(b-c)$

$= (\langle -2,4,1\rangle - \langle 2,-3,0\rangle) \times (\langle 2,-3,0\rangle - \langle 3,-4,2\rangle)$

$= \langle -2-2, 4-(-3), 1-0\rangle \times \langle 2-3, -3-(-4), 0-2\rangle$

$= \langle -4,7,1\rangle \times \langle -1,1,-2\rangle$

$$\left|\begin{array}{ccc|cc} i & j & k & i & j \\ -4 & 7 & 1 & -4 & 1 \\ -1 & 1 & -2 & -1 & 1 \end{array}\right| = \cancel{-14i} - \cancel{j} - \cancel{4k} + \cancel{7k} \cancel{-i} - \cancel{8j}$$

$= -15i - 9j + 3k$

$= \langle -15,-9,3\rangle$

7) Consider the Parallepiped w/ the following edges:

$u=\langle 3,-2,5\rangle,\ v=\langle 2,1,2\rangle,\ w=\langle 2,5,5\rangle$

a) Find the volume of the Parallelepiped: (use a Scaler triple Product)

$$V=|u\cdot(v\times w)|$$

$$V=|\langle 3,-2,5\rangle\cdot(\langle 2,1,2\rangle\times\langle 2,5,5\rangle)|$$

$v\times w$:

$$\left|\begin{array}{ccc|cc} i & j & k & i & j \\ 2 & 1 & 2 & 2 & 1 \\ 2 & 5 & 5 & 2 & 5 \end{array}\right| = \cancel{5i}+\cancel{4j}+\cancel{10k}-\cancel{2k}-\cancel{10i}-\cancel{10j} = -5i-6j+8k$$

$$\begin{array}{cccccc} - & - & - & + & + & + \end{array}$$

thus: $v\times w: \langle -5,-6,8\rangle$

$$V=|\langle 3,-2,5\rangle\cdot\langle -5,-6,8\rangle|$$

$$V=|3(-5)+(-2)(-6)+5(8)|$$

$$V=|-15+12+40|$$

$$V=|37|$$

$$V=37$$

$\therefore$ the volumn of the Parallelepiped is 37

b) Find the area of the face determined by the vectors u and w:

$$A=\|u\times w\|=\|\langle 3,-2,5\rangle\times\langle 2,5,5\rangle\|$$

$u\times w$:

$$\left|\begin{array}{ccc|cc} i & j & k & i & j \\ 3 & -2 & 5 & 3 & -2 \\ 2 & 5 & 5 & 2 & 5 \end{array}\right| = \cancel{-10i}+\cancel{10j}+15k+4k-\cancel{25i}-\cancel{15j}$$

$$\begin{array}{cccccc} - & - & - & + & + & + \end{array}$$

$$=-35i-5j+19k$$

$$=\langle -35,-5,19\rangle$$

thus: $u\times w: \langle -35,-5,19\rangle$

$$A=\|\langle -35,-5,19\rangle\|$$

$$A=\sqrt{(-35)^2+(-5)^2+19^2}=\sqrt{1611}=\sqrt{9}\sqrt{179}=\boxed{3\sqrt{179}}$$

$\therefore$ the area of the face determined by the vectors u and w is $3\sqrt{179}$ [or, Simply $\sqrt{1611}$ which ≈ 40.137]

c) Find the angle between u & the face determined by the vectors v and w.

Find the Plain Perpendicular to the one determined by v & w:

$v \times w$; $\langle 2,1,2\rangle \times \langle 2,5,5\rangle$:

$$\begin{vmatrix} i & j & k & i & j \\ 2 & 1 & 2 & 2 & 1 \\ 2 & 5 & 5 & 2 & 5 \end{vmatrix} \begin{array}{l} = \cancel{5i+4j+10k-2k-10i-10j} \\ = -5i-6j+8k \\ = \langle -5,-6,8\rangle \end{array}$$

$$- \quad - \quad - \quad + \quad + \quad +$$

We can now find the angle (θ) between u & $v \times w$:

$$\cos\theta = \frac{u\cdot(v\times w)}{\|u\|\,\|v\times w\|} = \frac{\langle 3,-2,5\rangle \cdot \langle -5,-6,8\rangle}{\|\langle 3,-2,5\rangle\|\,\|\langle -5,-6,8\rangle\|}$$

$$\cos\theta = \frac{(3)(-5)+(-2)(-6)+(5)(8)}{\sqrt{3^2+(-2)^2+5^2}\,\sqrt{(-5)^2+(-6)^2+8^2}}$$

$$\cos\theta = \frac{-15+12+40}{\sqrt{38}\,\sqrt{125}} = \frac{37}{5\sqrt{5}\sqrt{38}} = \frac{37}{5\sqrt{190}}$$

$$\cancel{\text{Arccos}\,(\cos\theta)} = \text{Arc}\cos\left(\frac{37}{5\sqrt{190}}\right)$$

$$\theta \doteq 57.53^\circ \;\left[\text{or, } \theta \doteq 1.004 \text{ radians}\right]$$

∴ the angle between u & the face determined by v & w is 57.53° [or simply 1.004 radians]

8) Give the Parametric equations of the line that Passes through the Point $P(1,-2,-3)$ & is Parallel to the vector $\langle 0,-1,5\rangle$.

$r = \langle 1,-2,-3\rangle$ & $v = \langle 0,-1,5\rangle$

vector equation:

$$r = \langle 1,-2,-3\rangle + t\langle 0,-1,5\rangle$$

thus, the Parametric equations of the line are:

$$\boxed{\begin{array}{l} x = 1 \\ y = -2-t \\ z = -3+5t \end{array}}$$

Determine if the points listed below are on this line:

a) $Q(1,-2,-1)$:

① $x=1$ & $x=1$ thus $1=1$ $\therefore t \in \mathbb{R}$ (t is any Real number)

② $y=-2-t$ & $y=-2$ thus $-2=-2-t$ $\therefore t=0$

③ $z=-3+5t$ & $z=-1$ thus $-1=-3+5t$

$$0=-2+5t$$
$$2=5t$$
$$t=\tfrac{2}{5}$$

In ②, we see that t must equal 0, but in ③ we see t must equal $\frac{2}{5}$, thus it is Contridictory to have the point $(1,-2,-1)$ as t can only have one unique value.

$\therefore$ Q is NOT on the line

b) $W(1,5,-38)$:

① $x=1$ & $x=1$ thus $1=1$ $\therefore t \in \mathbb{R}$ (t is any real number)

② $y=-2-t$ & $y=5$ thus $5=-2-t$ $\therefore t=-7$

③ $z=-3+5t$ & $z=-38$ thus $-38=-3+5t$ $\therefore t=-7$

Since both ② & ③ have that $t=-7$, & Since ① States t may be any real number (& since -7 is a real number), the point $(1,5,-38)$ is on the line.

$\therefore$ W IS on the line

9) Give the equation of the Plain that Contains the line $x=3t$, $y=1-2t$, $z=1-t$ & is Perpendicular to the Plane $x-y+3z=3$:

$V=\langle 3,-2,-1\rangle$ is Parallel to the line

$n=\langle 1,-1,3\rangle$ is normal to the given Plain

thus, $V \times n$:

$$\left|\begin{array}{ccc|cc} i & j & k & i & j \\ 3 & -2 & -1 & 3 & -2 \\ 1 & -1 & 3 & 1 & -1 \end{array}\right| = \cancel{-6i-j-3k+2k-i-3j}$$

$$= -7i-10j-k$$

$$= \langle -7,-10,-1\rangle$$

$- \quad - \quad - \quad + \quad + \quad +$

$=\langle -7,-10,-1\rangle$, this line is normal to the Plain were looking for

When $t=0$ on the line:

we get $\langle 0,1,1\rangle$, which also a point on our Plain.

Use the point & the normal:

Point $=\langle 0,1,1\rangle$; normal $=\langle -7,-10,-1\rangle$

thus:

$$= -7x-10(y-1)-1(z-1)$$

$$= -7x-10y+10-z+1$$

$$= -7x-10y-z+11$$

$$\therefore -11 = -7x-10y-z$$

or Simply; $11 = 7x+10y+z$

thus, the Plane were looking for is: $\boxed{7x+10y+z=11}$

10) Identify & describe the quadratic surfaces, a sketch isn't necessary:

a) $z^2 = 4x^2 - 3x + y^2 - 2z + 24$

thus:

$$-24 = 4x^2 - 3x + y^2 - z^2 - 2z$$

$$24 = -4x^2 + 3x - y^2 + z^2 + 2z$$

$$24 = -(2x - \tfrac{3}{4})^2 + 9/16 - y^2 + (z+1)^2 - 1$$

$$\frac{391}{16} = -(2x - \tfrac{3}{4})^2 - y^2 + (z+1)^2$$

OR: $$\frac{391}{16} = -4(x - \tfrac{3}{8})^2 - y^2 + (z+1)^2$$

thus:

$$1 = -\frac{64(x - 3/8)^2}{391} - \frac{16y^2}{391} + \frac{16(z+1)^2}{391}$$

this is of the form:

$$1 = \frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2}$$ $\therefore$ This is a hyperboloid of two sheets

the centre of the sheets is: $(\tfrac{3}{8}, 0, -1)$

→ we know this from the values which modify the x, y, z
(i.e. $x - 3/8 \therefore x = 3/8$ at the centre)

the top of the sheet pointing down the z-axis is:

$$\frac{391}{16} = (z+1)^2 \rightarrow z+1 = \pm\frac{\sqrt{391}}{4} \rightarrow z = \frac{-4 - \sqrt{391}}{4}$$

likewise, the bottom of the sheet pointing up the z-axis is:

$$z = \frac{-4 + \sqrt{391}}{4}$$

thus, taking the center into account: (we must make $x = 3/8$ & $y = 0$ for $z = \frac{391}{16}$)

the top sheets local min. is:

$$\left(\frac{3}{8}, 0, \frac{-4 - \sqrt{391}}{4}\right)$$

the bottom sheets local max. is:

$$\left(\frac{3}{8}, 0, \frac{-4 + \sqrt{391}}{4}\right)$$

b) $y = 2x^2 - 6x - z^2 + 5y$

$0 = 2(x^2 - 3x) + 4y - z^2$

$0 = 2(x - \frac{3}{2})^2 - 9/2 + 4y - z^2$

$\frac{9}{2} = 2(x - 3/2)^2 + 4y - z^2$

$9 = 4(x - 3/2)^2 + 8y - 2z^2$

$$1 = \frac{4(x-3/2)^2}{9} + \frac{8y}{9} - \frac{2z^2}{9}$$

this is of the form:

$1 = z - \frac{x^2}{a^2} - \frac{y^2}{b^2}$ as there is one linear term (y above), w/ two quadratic terms. (x & z above) w/ opposite signs.

∴ Hyperbolic Paraboloid

the Graph:

- The Paraboloid opens along the y-axis, in the negative direction
- The local Min. of the Paraboloid (the least of the opening) is:

$0 = 2(x^2 - 3x) + 4y - z^2$ when $z = 0$ & $x = 3/2$

$0 = -4.5 + 4y$

$4.5 = 4y$

$y = 9/8$

thus: the local Min. at the Positive y-axis is $(3/2, 9/8, 0)$

- The Graph forms a Parabolic "void" along the Positive y-axis
- The Graph Grows along the z & x-axis forever, but always opens along the neg. y-axis & has a gap between the Pos. y-axis.

11)

a) Give the equation of an elliptic Cone w/ Centre at Point $(1,-2,1)$, in the direction of the x-axis in rectangular & Cylindrical Coordinates:

– An elliptic Cone is of the form:

$$z^2 - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \xrightarrow[\text{direction}]{\text{X-axis}} x^2 - \frac{z^2}{b^2} - \frac{y^2}{a^2} = 0$$

– Thus, if the Centre is at $P(1,-2,1)$:

$$(x-1)^2 - \frac{(y+2)^2}{a^2} - \frac{(z-1)^2}{b^2} = 0$$ is the equation in Rect. Coord.

– If we let $b=a=1$ then we get a Simple equation:

$$(x-1)^2 - (y+2)^2 - (z-1)^2 = 0$$

– ~~In General terms, the equation~~:

$$0 = \frac{(x-1)^2}{a^2} - \frac{(y+2)^2}{b^2} - \frac{(z-1)^2}{c^2}$$

will Satisfy the question (for rectangular Coordinates)

– to Convert to Cylindrical Coordinates:

let $x = r\cos\theta$, $y = r\sin\theta$, $z = z$; then:

$$(r\cos\theta - 1)^2 - (r\sin\theta + 2)^2 - (z-1)^2 = 0$$

– ~~In General terms, the equation~~:

$$\frac{(r\cos\theta - 1)^2}{a^2} - \frac{(r\sin\theta + 2)^2}{b^2} - \frac{(z-1)^2}{c^2} = 0$$

b) Give the equation of an elliptic Paraboloid with Centre at the point $(0,2,1)$ in the direction of the y-axis in rectangular and Spherical Coordinates:

– An elliptic Paraboloid is of the form:

$$z - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0 \xrightarrow[\text{of the y-axis}]{\text{in the direction}} y - \frac{x^2}{a^2} - \frac{z^2}{b^2} = 0$$

– Thus, if the Centre is at Point $(0,2,1)$:

$$(y-2) - \frac{x^2}{a^2} - \frac{(z-1)^2}{b^2} = 0 \quad \left[(y-2) - x^2 - (z-1)^2 = 0 \text{ When } b=a=1\right]$$

– To Convert to Spherical Coordinates:

let $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$; then:

$$\rho\sin\phi\sin\theta - 2 - (\rho\sin\phi\cos\theta)^2 - (\rho\cos\phi - 1)^2 = 0$$

$$0 = \rho\sin\phi\sin\theta - (\rho\sin\phi\cos\theta)^2 - (\rho\cos\phi - 1)^2 - 2$$

- In General Terms, the equation:

$$0 = \rho\sin\phi\sin\theta - \frac{(\rho\sin\phi\cos\theta)^2}{a^2} - \frac{(\rho\cos\phi - 1)}{b^2}$$

c) Give the equation of a hyperboloid of one sheet with center $(0,0,4)$ in the direction of the z-axis in Cylindrical & Spherical Coordinates:

- A hyperboloid of one sheet has the general form in Cylindrical Coordinates:

$$z^2 = r^2 - 1 \quad [\text{this is already in the direction of the } z\text{-axis}]$$

- Thus, if we let the Centre be $P(0,0,4)$:

$$(z-4)^2 - r^2 + 1 = 0 \quad [\text{When } a=b=c=1]$$

- In General terms, the equation:

$$\frac{(z-4)^2}{d^2} - \frac{r^2}{e^2} + 1 = 0, \text{ when } d, e \in \mathbb{R}$$

- To Convert to Spherical Coordinates:

let $r = \rho\sin\phi$, $\theta = \theta$, $z = \rho\cos\phi$; then:

$$(\rho\cos\phi - 4)^2 - (\rho\sin\phi)^2 + 1 = 0$$

- In General terms, the equation:

$$\frac{(\rho\cos\phi - 4)^2}{d^2} - \frac{(\rho\sin\phi)^2}{e^2} + 1 = 0, \text{ when } d, e \in \mathbb{R}$$