Name: Connor Raymond Stewart
ID: 101041125

MATH 3802 Assignment #1:

① let P denote the Polytope given by $\{x \in \mathbb{R}^3 : Ax = b, x \geq 0\}$ where $A = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix}$ & $b = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$. Write $\begin{bmatrix} 2/3 \\ 1 \\ 2/3 \end{bmatrix}$ as a convex combination of the extreme points of P:

$$Ax = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + 2x_3 \\ 3x_1 + x_2 \end{bmatrix}$$ thus, Since $Ax \leq b$ we note:

$$\left.\begin{matrix} \text{I} & x_1 + x_2 + 2x_3 \leq 3 \\ \text{II} & 3x_1 + x_2 \leq 3 \end{matrix}\right]$$ Since, P's the intersection of halfspaces, it is a polyhedron P's bounded iff the linear programming Problems are bounded for all $i \in \{1,2,3\}$

- Setting $x_1, x_2, x_3 \geq 0$ to equalities gives the unique Solution $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ which is in P. Hence, $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is an extreme Point in P.
- Setting $x_1 + x_2 + 2x_3 \leq 3$; $3x_1 + x_2 \leq 3$; $x_1, x_2 \geq 0$ gives $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ Which is in P. Hence, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an extreme Point in P.
- Setting $x_1 + x_2 + 2x_3 \leq 3$; $3x_1 + x_2 \leq 3$; $x_1, x_3 \geq 0$ gives $\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$ Which is in P. Hence, $\begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$ is an extreme Point in P.
- Setting $x_1 + x_2 + 2x_3 \leq 3$; $3x_1 + x_2 \leq 3$; $x_2, x_3 \geq 0$ gives $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ which is in P, Hence, $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ is an extreme Point in P.

Since a Convex-Combination Requires $\lambda_1, \dots, \lambda_K \geq 0$ & $\lambda_1 + \dots + \lambda_K = 1$:

$$\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \lambda_2 \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 3 \\ 2/3 \end{bmatrix}$$ when $\lambda_1 = \frac{2}{3}$ & $\lambda_2 = \frac{1}{3}$ Since $\lambda_1 + \lambda_2 = \frac{2}{3} + \frac{1}{3} = 1$

& Since $$\frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1 \\ 2/3 \end{bmatrix}$$

$\therefore$ $\begin{bmatrix} 2/3 \\ 1 \\ 2/3 \end{bmatrix}$ Can be represented by the Convex-Combination of $\frac{2}{3} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ & $\frac{1}{3} \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}$

② Let S denote a set of subsets of $\{1,\dots,n\}$ where n is a positive integer. The Minimum-Cardinality Set-Covering Problem is to select as few sets from S as possible so that every $i \in \{1,\dots,n\}$ is in at least one of the selected sets. The problem can be formulated as the following integer linear programming problem:

$$\min \sum_{s \in S} X_s$$

$$\text{(SC) s.t.} \sum_{s \in S: i \in s} X_s \geq 1 \qquad \forall i \in \{1,\dots,n\}$$

$$0 \leq X_s \leq 1 \qquad \forall s \in S$$

$$X_s \in \mathbb{Z} \qquad \forall s \in S$$

Hence, there is a binary variable for each set s in S such that a set s is selected if & only if $X_s = 1$:

a) (3 points) Write down explicitly the integer linear programming the problem where $n=5$ & S containing the sets $\{1,3,4\}, \{1,4,5\}, \{2,5\}, \{3,5\}$:

Note: each row forces at least one i from 1 to 5 to be set.

$$\text{minimize } X_{\{1,3,4\}} + X_{\{1,4,5\}} + X_{\{2,5\}} + X_{\{3,5\}}$$

$$\text{(SC) s.t. } X_{\{1,3,4\}} + X_{\{1,4,5\}} \geq 1$$

$$X_{\{2,5\}} \geq 1$$

$$X_{\{1,3,4\}} + X_{\{3,5\}} \geq 1$$

$$\cancel{X_{\{1,3,4\}} + X_{\{1,4,5\}} \geq 1}$$ Duplicate for $i=1$ & $i=4$ (Removed)

$$X_{\{2,5\}} + X_{\{3,5\}} \geq 1$$

$$0 \leq X_{\{1,3,4\}}, X_{\{1,4,5\}}, X_{\{2,5\}}, X_{\{3,5\}} \leq 1$$

$$X_{\{1,3,4\}}, X_{\{1,4,5\}}, X_{\{2,5\}}, X_{\{3,5\}} \in \mathbb{Z}$$

b) (2 Points) Solve the problem in part a:

$$\text{Minimize } 1+0+1+0$$

$$\text{s.t. } 1+0 \geq 1$$

$$1 \geq 1$$

$$1+0 \geq 1$$

$$1+0 \geq 1$$

$$1+0 \geq 1$$

$$0 \leq 1,0,1,0 \leq 1$$

$$1,0,1,0 \in \mathbb{Z}$$

Note:
- By setting $X_{\{1,3,4\}} = X_{\{2,5\}} = 1$ we see that all constraints are satisfied.
- As the objective $=2$, we see that it's minimized as no single subset of S contains all i's from 1 to 5. Therefore the objective $\neq 1$ & an objective of 2 must be the optimal solution.

thus: the optimal solution is an objective function of 2 with $X_{\{1,3,4\}} = X_{\{2,5\}} = 1$ for a set of $\{\{1,3,4\},\{2,5\}\}$, which contains numbers 1 to 5.

c) let (FSC) denote the linear programming relaxation of (SC) obtained by removing the integrality constraints. Write down the duel problem of (FSC), associating the dual variable y_i with the constraint $\sum_{S\in S: i\in S} X_S \geq 1$ for each $i \in \{1,\dots,n\}$, & the dual variable U_S with the constraint $X_S \leq 1$ for each $S \in S$:

Duel: maximize $y_1+y_2+y_3+y_4$

$$
\text{(D-FSC) s.t.} \quad
\begin{aligned}
y_1 \quad + y_3 \quad &\geq X_{\{1,3,4\}} \\
y_1 \quad\quad\quad &\geq X_{\{1,4,5\}} \\
y_2 \quad + y_4 &\geq X_{\{2,5\}} \\
y_3 + y_4 &\geq X_{\{3,5\}} \\
y_1, y_2, y_3, y_4 &\geq 1
\end{aligned}
$$

Note:

- from 2a we note that:

$$b=\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\ A=\begin{bmatrix}1&1&0&0\\0&0&1&0\\1&0&0&1\\0&0&1&1\end{bmatrix}$$

we get the duel by taking columbs from A & setting b to the objective.

- from 2a's objective, we set b in the Duel Problem
- Since $\sum_{S\in S: i\in S} X_S \geq 1$, y's 1 to 4 are: ≥ 1
- Since it's stated $X_S \leq 1$, we flag Rows 1 to 4 as: $\geq X_S$

(original Primal): from Question

$$
\begin{aligned}
&\text{minimize } \textstyle\sum_{S\in S: i\in S} X_S \\
\text{(FSC) s.t. } &\textstyle\sum_{S\in S: i\in S} X_S \geq 1 \quad \forall i\in\{1,\dots,n\} \\
&X_S \leq 1 \quad \forall S \in S
\end{aligned}
$$