MATH 3101 Assignment One:

1) a) let $a, b, c, \& d$ be integers such that $a|b$ & $c|d$. Prove that $ac|bd$:

$a|b$ thus $b = aZ_1$, $Z_1 \in \mathbb{Z}$

$c|d$ thus $d = cZ_2$, $Z_2 \in \mathbb{Z}$

$ac|bd$ thus $bd = (Z_1 a)(Z_2 c)$

$= (Z_1 Z_2) ac$

For $Z_1 Z_2$:

by the properties of integers, we know $\mathbb{Z}$ is closed under multiplication.
Since $Z_1, Z_2 \in \mathbb{Z}$, & all integers are closed under multiplication:

$Z_1 \cdot Z_2 = Z$, $Z \in \mathbb{Z}$

thus, by the above:

$bd = (Z_1 Z_2)(ac)$

$= Z(ac)$ [$Z_1 \cdot Z_2$ makes integer Z]

Therefore, we can say $ac|bd$ by definition

b) let a be an integer. Prove that $2|a(a+1)$:

firstly, lets state the division algorithm:

$a = bq + r$ w/ $0 \leq r < b$; $r, q, a, b \in \mathbb{Z}$

Since we divide by two, let $b = 2$, thus:

$a = 2q + r$, $0 \leq r < 2$

As r is $[0, 2)$, w/ $r \in \mathbb{Z}$, we know:

$r = 0$ & $r = 1$

Case one ($r = 0$):

If $r = 0$, then $a = 2q + 0 = 2q$, so:

$a(a+1) = 2c$, thus:

$a(a+1) = 2q(a+1) = 2q(2q+1)$

As $q, a \in \mathbb{Z}$ & integers are closed under multiplication & Addition:

$2q(a+1) = 2q(2q+1) = 2c$ then:

$q(a+1), q(2q+1) \in \mathbb{Z}$ thus $2|a(a+1)$ in Case 1

Case Two ($r = 1$):

If $r = 1$, then $a = 2q + 1$, so:

$a(a+1) = (2q+1)(2q+1+1) = (2q+1)(2)(q+1)$ $[= 2a(q+1)]$

So, $2c = 2(2q+1)(q+1)$ $[= 2a(q+1)]$

As $q, a \in \mathbb{Z}$, & $\mathbb{Z}$ is closed under Add./mult., we know:

$a(q+1), (2q+1)(q+1) \in \mathbb{Z}$ thus $2|a(a+1)$ by definition in Case 2

Since both Case 1 & 2 are proven, we can conclude the statement is true by exhaustion:

$\therefore 2|a(a+1)$ is true

2) let $a=-921$ & $b=18$, Find the integers q & r that Satisfy the Conditions given in the division algorithm:

the division algorithm has the following form:

$$\frac{a}{b}=q+\frac{r}{b},$$ we Know $a=-921$ & $b=18$

lower$\left(\frac{-921}{18}\right)$

So:

$-921 \bmod 18 = 15$

$\frac{a}{b}=\frac{-921}{18}$, the modulo of $\frac{a}{b}$ is 15, & the lower division is 52

$\frac{-921}{18} \doteq -51.17$ So, rounding to the lowest integer

we get $-52=q$ & $r=15$, we can Also find this by:

$-921=-51(18)+(-3)$

$$-921=-51(18)+(-1)(18)+(-3)+18$$
$$=-52(18)+(18-3)$$
$$=-52(18)+15 \quad \text{where } \frac{a}{b}=q+\frac{r}{b}$$

thus:

$q=-52, r=15$ in the division Algorithm with $a=-921$ & $b=18$

Notes:

$0 \le r < b \rightarrow 0 \le 15 < 18$ is true, So r is within valid Range

3) In each part, find the greatest common divisor (a,b) as well as integers m & n Such that $(a,b)=am+bn$. Show your work:

a) $a=382$, $b=26$:

$(a,b)=am+bn$; $m,n \in \mathbb{Z}$

use the euclidean Algorithm:

$$a=bq_0+r_1, \quad 0 \le r_1 < b$$
$$b=r_1q_1+r_2, \quad 0 \le r_2 < r_1$$
$$r_1=r_2q_2+r_3, \quad 0 \le r_3 < r_2$$
$$\vdots$$
$$r_k=r_{k+1}q_{k+1}+r_{k+2}, \quad 0 \le r_{k+2} < r_{k+1}$$

So:

$$382=(26)q_0+r_1$$
$$=26(14)+18 \quad (q_0=14, r_1=18)$$
$$26=(18)q_1+r_2$$
$$=(18)(1)+8 \quad (q_1=1, r_2=8)$$

$18 = 8q_2 + r_3$

$\hookrightarrow = 8(2) + 2 \qquad (q_2 = 2, r_3 = 2)$

$\hookrightarrow$ last nonzero remainder

$8 = 2q_3 + r_4$

$\hookrightarrow = 2(4) + 0 \qquad (q_3 = 4, r_4 = 0) \rightarrow$ remainder is zero

thus, the greatest common divisor is 2 (the last non-zero remainder)

So: $(382, 26) = 2$

We know that $(a,b) = am + bn$, thus:

$(382, 26) = 2 = am + bn$; $a = 382$, $b = 26$

$\hookrightarrow 2 = 382m + 26n$

We must now use the euclidean Algorithm:

Since $r_1 = 18, r_2 = 8, r_3 = 2$, the remainders can be shown as:

$$18 = 382(1) + (26)(-14)$$

$$8 = 26(1) + (18)(-1)$$

$$2 = 18(1) + (8)(-2)$$

by letting: $r_{k+2} = r_k(1) + (r_{k+1})(-q_{k+1})$

If we substitute the remainders from the previous questions:

$$\begin{aligned}
2 &= 18 + 8(-2) \\
&= 18 - 8(2) \\
&= 18 - (26(1) + 18(-1))(2) \quad [8 = 26(1) + (18)(-1)] \\
&= 18 - (26 - 18)(2) \\
&= 18 - (26 - \cancel{18})(2) + 18(2) \\
&= 18(1+2) - 26(2) \\
&= 18(3) - 26(2) \\
&= (382 + 26(-14))(3) - 26(2) \quad [18 = 382(1) + 26(-14)] \\
&= 382(3) + (3)(26)(-14) - 2(26) \\
&= 382(3) + (26)(-42) - 2(26) \\
&= 382(3) + 26(-44)
\end{aligned}$$

So, Since: $2 = am + bn = 382m + 26n$ & $2 = 382(3) + 26(-44)$

$m = 3, n = -44$

In Conclusion, we can see that:

$(382, 26) = 2, m = 3, n = -44$

b) $a=382, b=-26$;

Assume: $\gcd(a,b)=\gcd(a,|b|)$:

Proof:

We Suppose that z/b, $z \in \mathbb{Z}$

then: $b=qz$

So:

$|b| = \pm qz = (\pm q)z \Rightarrow z/|b|$

thus, it follows that every divisor of b is also one for |b|

Since $b=\pm|b|$, it also follows every divisor of |b| is one for b

$\therefore$ the Common divisors for (a,b) are also the Same for $(a,|b|)$

Since in a we found $(382,26)=2$, we can conclude w/ the above Proof that $(382,-26)=2$

thus:

$(a,b)=am+bn$ So $2=382m+(-26)n$

In Part a we see that $m=3$ & $n=-44$

Since in b we reverse the Sign value of b turning it to -26, We must do the Same to n in the equation $(am+bn)$:

Since in a $m=3$ & $n=-44$, & as stated above $am+b(-n)$ relative to a, n in b is $-n$ So $n=-(-44)=44$, thus:

$(a,b)=am+bn \Rightarrow 2=3m+44n=3(382)+(44)(-26)=2$

$\therefore m=3, n=44$

In Conclusion:

$(a,b)=(382,-26)=2$

$m=3, n=44$

4) Determine wheather the following Statements are true or false. Justify your responses:

a) The Well-ordering Principle implies that the set of odd integers Contains a least element:

the well ordering Principle requires a nonempty set of positive integers However, the set of all odd integers Contains negative numbers take for example, -1

$-1 \in O$ (the set of odd integers is O)

-1 is not Positive

$\therefore$ We Cannot Say the well ordering Principle implies that the set of odd integers Contains a least element, as the set of odd ints. has negatives which cannot apply to the well ordering Principle by definition.

$\therefore$ The Statement is False

b) let $a,b,c \in \mathbb{Z}$. If $a|(b-c)$, then $a|b$ or $a|c$:

Since we Know that $a|b$ means $b=ac$, we Know:

$a|(b-c) \rightarrow (b-c)=aZ_0,\ Z_0 \in \mathbb{Z}$

$a|b \rightarrow b=aZ_1,\ Z_1 \in \mathbb{Z}$

$a|c \rightarrow c=aZ_2,\ Z_2 \in \mathbb{Z}$

when we let $a=2, b=3, c=3$, we Can See:

$a|(b-c) \rightarrow 2|(3-3) \rightarrow 2|0$ $\therefore$ True

$\hookrightarrow \ell|0,\ \ell \in \mathbb{Z}$ is always true, So $2|0$ is true

However, we See:

$a|b = a|c = 2|3$, & 2 dosn't divide 3 to any integer

$\therefore$ there's a Contradiction as $a|(b-c)$ is true yet $a|b$ & $a|c$ is false

$\therefore$ the Statement is False

c) let a and b be integers, **not** both zero, **Such** that $1=ax+by$, for Some integers x & y. Then $(a,b)=1$:

we can take the following example:

by the euclidean Algorithm, there is integers x & y Such:

$1=ax+by$

Case one: $x=0$

If $x=0$ then $by=1$ thus

$y=b=\pm 1$ as $(0,\pm 1)=1$ [for (a,b)]

$\therefore$ Case one's True

Case Two: $y=0$

If $y=0$ then $ax=1$ thus

$a=x=\pm 1$ as $(\pm 1, 0)=1$ [for (a,b)]

$\therefore$ Case two's True

Case Three: $x \neq 0$ & $y \neq 0$

let $(a,b)=z$, then:

$$\left.\begin{array}{l} z \mid a \Rightarrow a = k_1 z \\ z \mid b \Rightarrow b = k_2 z \end{array}\right\} k_1, k_2 \in \mathbb{Z}$$

then:

If $ax+by=1 \rightarrow 1 = k_1 z x + k_2 z y = z(xk_1 + yk_2)$

$\therefore z = \pm 1$

Since z must be the greatest common divisor, we know $z=1$, not -1.

$\therefore (a,b) = z = 1$

$\therefore$ Case three's True

Since we have shown all cases to be true, we know that the statements True:

The statement outlined in c is true

d) let $a, b, \& c$ be integers. If $c|ab$, then $c|a$ or $c|b$:

Since:

$c|ab \rightarrow ab = cZ_0,\ Z_0 \in \mathbb{Z}$

$c|a \rightarrow a = cZ_1,\ Z_1 \in \mathbb{Z}$

$c|b \rightarrow b = cZ_2,\ Z_2 \in \mathbb{Z}$

take the following case:

let $a=7, b=2, \& c=7$ then:

$c|ab \rightarrow 7|(7)(2) = 7|14$ so $14 = 7c$, true if $c=2$

$c|a \rightarrow 7|7$ so $7 = 7c$, true if $c=1$

$c|b \rightarrow 7|2$ so $2 = 7c$, false as c must be $\frac{2}{7} \notin \mathbb{Z}$

Since $c \nmid b$ when $a=c=7, b=2$ yet $c|ab$ under the same conditions, we know that this is false by contradiction

$\therefore$ The Statement is False