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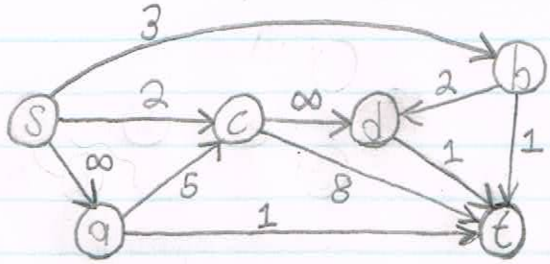
Acknowledgement: No Help Receaved

MATH 3802 Assigment Five:

① Consider s-t network depicted below:



The numbers next to the arcs denote the arc capacities:

a) (2 Points) For each of the following S, give the capacity of the s-t cut $\delta^+(S)$:

i) $S=\{s,a,b,d\}$: let u_i denote the capacity for all arcs e

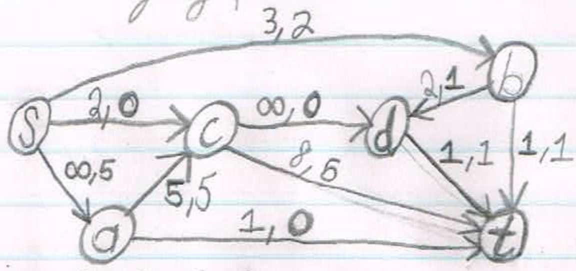
$u(\delta^+(S)) = 2+5+1+1+1 = \boxed{10}$

ii) $S=\{s,a,d\}$:

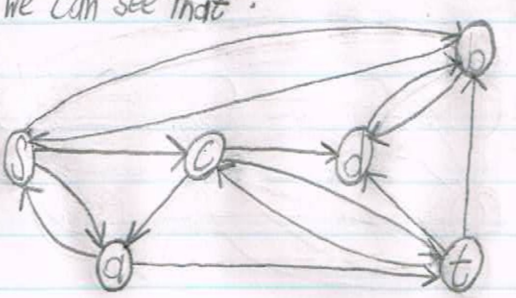
$u(\delta^+(S)) = 3+2+5+1+1 = \boxed{12}$

b) (2 Points) Consider the s-t flow x^* given by $x_{sa}=x_{ac}=x_{ct}=5$, $x_{sb}=2$, $x_{bd}=x_{bt}=x_{dt}=1$, & $x_{sc}=x_{cd}=x_{at}=0$

form the auxiliary digraph $G(x)$:



thus, We Can See that:



When at capacity, reverse arc
When less then capacity, double arc
When zero, Keep arc

c) (5 Point) Obtain a maximum flow and a minimum-capacity s-t cut by applying the Ford-Fulkerson augmenting path algorithm starting with the s-t flow in Part b. Show your work:

First, we obtain a dipath from the auxilary graph:

I) s, sa, a, at, t

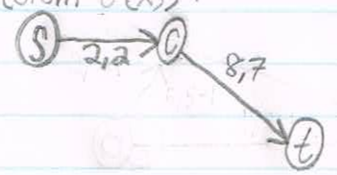
II) s, sc, c, ct, t

III) s, sc, c, ca, a, at, t

Second, we augment the flow along the augmenting path corrosponding to a dipath above:

Select dipath II

thus (from G(x)):



We add two to the arcs moving to t.

Remove 2 from arcs moving away (Note: None exist)

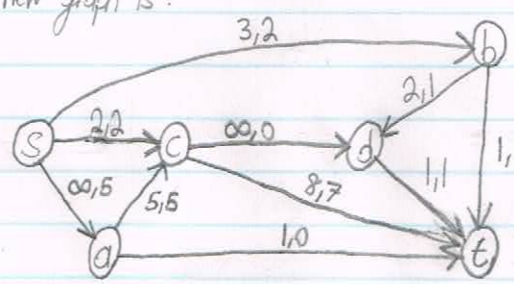
Such that:

$sc = 2, ct = 7$

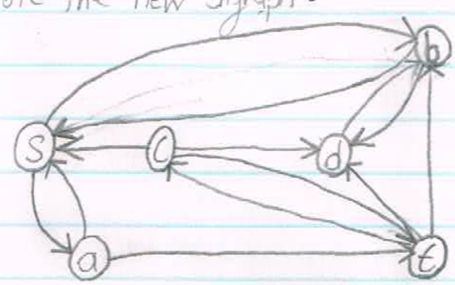
thus, the net flow into t is now:

$1+1+7=9$

the new graph is:



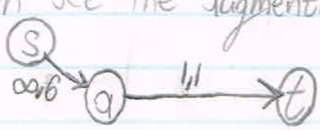
We note the new digraph:



We see the following new paths:

I) s, sa, a, at, t

We then see the augmenting path:



We add 1 to the arcs moving to t

Such that:

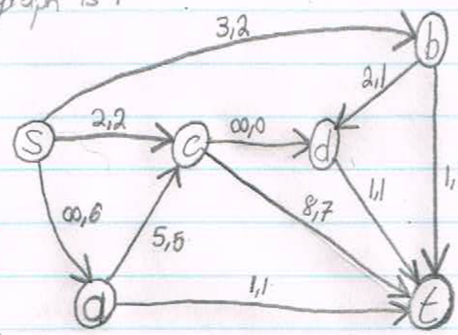
$sa = 6, at = 1$

thus, the new net flow is:

$1+1+1+7=10$

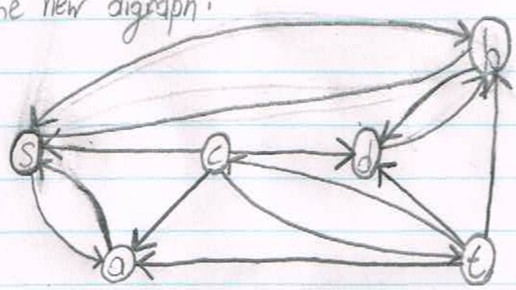
the new graph is:

new Graph



We note the new digraph:

digraph



There are no new augmenting paths from s to t, thus:

$\boxed{10 \text{ is the maximum } s\text{-}t \text{ flow}}$

We can verify the capacity of $S = s, c, a$:

$$u(\delta^+(S)) = X_{sb} + X_{cd} + X_{ct} + X_{at} = 2+0+7+1 = 10$$

thus, the minimum capacity & maximum flow s-t cut is in the graphs above & has a value of 10

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② (1 Point) Let $G=(\mathcal{N},\mathcal{A})$ be a digraph where s,t are distinct nodes in $\mathcal{N}$. Let $u \in \mathbb{R}^4$. Give a linear Programming Problem whose duel problem is:

$$\begin{aligned}
\max\ & x(\delta^+(s)) - x(\delta^-(s)) & \\
\text{s.t. } & x(\delta^+(v)) - x(\delta^-(v)) = 0 & \forall v \in \mathcal{N} \setminus \{s,t\} \\
& x_e \le u_e & \forall e \in \mathcal{A} \\
& x_e \ge 0 & \forall e \in \mathcal{A}
\end{aligned}$$

Given that s is the start node & t is the goal node, we can see that:

$$\min \Sigma_{e\in\mathcal{A}} (u_e y_e)$$

is the objective function as it is associated with:

$$x_e \le u_e \quad \forall e \in \mathcal{A}$$

Note that y_e is the objective above has $e \in \mathcal{A}$

In Constraint one, we see that:

$$x(\delta^+(v)) - x(\delta^-(v)) = 0 \quad \text{thus} \quad x(\delta^+(v)) = x(\delta^-(v))$$

thus, both values equate to the flow of the cut. We can assign u_s & u_t in the duel to reflect this:

$$u_t - u_s = 1$$

As we traverse the graph, we note that:

s moves to node s_1 for δ^+

t_1 moves to node t for δ^-

Since we cannot have an infeasible graph:

$-t_1 + s_1 \ge y_e$, thus for any $k, l \in \mathcal{A}$ where $-u_k + u_l$ is general for G:

$$u_l - u_k - y_e \ge 0, \quad l,k \in \mathcal{A}$$

We know that y_e is greater or equal to 0 by $x_e \ge 0$ (so $y_e \ge 0$ is a const.)

So, we get the following:

$$\boxed{\begin{aligned}
\min\ & \sum_{e\in\mathcal{A}} u_e y_e & \\
\text{s.t. } & u_l - u_k - y_e \ge 0 & \forall e \in \mathcal{A}, \forall (l,k) \in \mathcal{A} \\
& u_t - u_s = 1 & \forall (l,k) \in \mathcal{A} \\
& y_e \ge 0 & \forall e \in \mathcal{A}
\end{aligned}}$$