Name: Connor Raymond Stewart
ID: 101041125

MATH 3101 Assignment 8:

① Consider the Permutation $\alpha=(2,4,3,1)(3,4,5)(6,7)$ of $\{1,2,3,4,5,6,7,8\}$:

a) Explicity determine where α maps each of the elements of $\{1,2,3,4,5,6,7,8\}$:

We note the following cycles:

$$6 \rightleftarrows 7 \Rightarrow \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ & & & & & 7 & 6 \end{bmatrix} \quad \begin{matrix} 6 \to 7 \\ 7 \to 6 \end{matrix}$$

$$\begin{matrix} 3 \to 4 \to 5 \\ 2 \to 4 \to 3 \to 1 \end{matrix} \qquad \begin{matrix} 4 \to 4 & 2 \to 5 \\ 5 \to 3 & 1 \to 2 \\ 3 \to 1 & \end{matrix}$$

thus:

$$f=\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 2 & 5 & 1 & 4 & 3 & 7 & 6 \end{bmatrix}$$

the mapping is as follows, for all $a_i \to f(a_i)$:

$1 \to 2$
$2 \to 5$
$3 \to 1$
$4 \to 4$
$5 \to 3$
$6 \to 7$
$7 \to 6$

b) Write α as a Product of disjoint cycles:

We note the following Patterns in f:

$1 \to 2 \to 3 \to 1 \Rightarrow (1,2,3)$
$5 \to 3 \Rightarrow (5,3)$
$6 \to 7 \to 6 \Rightarrow (6,7)$

$\Rightarrow (1,2,3)(5,3)(6,7)(4)$

Alternativly: (Simplified)

$1 \to 2 \to 5 \to 3 \to 1 \quad |4 \to 4| \quad 5 \to 3 \Rightarrow (1,2,5,3)(6,7)(4)$

c) Compute $|\alpha|$:

$|\alpha|$ is the order of element α, with each cycles order being its length:

$\therefore \operatorname{lcm}(4,2)=4$ as $(4)(1)=(2)(2)$

thus, $|\alpha|=4$

d) Find a^{-1}:

Since $a=(1,2,5,3)(6,7)(4)$, we know $f^{-1}(i_{K+1})=i_K$

$a^{-1}=(1,3,5,2)(6,7)(4)$

$\hookrightarrow (3,5,2,1)(7,6)=(1,3,5,2)(6,7) \rightarrow (1,3,5,2)(6,7)(4)$

e) Write a as a product of transpositions, and determine whether a is even or odd:

$(1,2,5,3) \rightarrow (1,2)(1,5)(1,3)$

$(6,7) \rightarrow (6,7)$

$\boxed{(1,2)(1,5)(1,3)(6,7)}$

thus, the transpositions are:

$(1,2)(1,5)(1,3)(6,7)$

there's an even number of transpositions $\therefore$ $\boxed{\text{Even Permutation}}$

② Consider the permutation $B=(1,5)(2,4)(3,7)(6,8)(9,10)$ of $\{1,2,3,4,5,6,7,8,9,10\}$.

Explain why $B^{-1}=B$:

Since $f(i_K)=i_{K+1}$ implies $f^{-1}(i_{K+1})=i_K$, we need to reverse the order of the cycle & move each element up by one index value:

if $f=(1,2,3,4)$ then f^{-1} is:

reverse order $\hookrightarrow (4,3,2,1) \xrightarrow[\text{unit forward}]{\text{move index one}} (\_,4,3,2,1) \xrightarrow[\text{fifth index to the back first index}]{\text{modulo 4, move}} (1,4,3,2)$

However, when f is a transposition:

if $f=(a,b)$ we see:

reverse order $\hookrightarrow (b,a) \xrightarrow[\text{index}]{\text{increment}} (\_,b,a) \xrightarrow{\text{modulo 2}} (a,b)$

thus, it is noted that $f^{-1}=f$ for any $f=(a,b)$

the permutation B consists entirely of disjoint cycles which are also transpositions:

if $f=(a,b)(c,d)\ldots(p,q)$ then $f^{-1}=(a,b)^{-1}(c,d)^{-1}\ldots(p,q)^{-1}$

$\hookrightarrow$ As seen above, transpositions are themselves when inverted so:

$f^{-1}=(a,b)^{-1}(c,d)^{-1}\ldots(p,q)^{-1}=(a,b)(c,d)\ldots(p,q)$

So if $B=(1,5)(2,4)(3,7)(6,8)(9,10)$ then $B^{-1}=B$ as seen above

③ Find a group of permutations that is isomorphic to $\mathbb{Z}_5$. Explicitly define an isomorphism from $\mathbb{Z}_5$ to this group of isomorphisms:

let G be $(\mathbb{Z}_5, +)$:

$\mathbb{Z}_5 = \{[0], [1], [2], [3], [4]\}$

$f(a): \mathbb{Z}_5 \to \mathbb{Z}_5$ where $f_g(x) = g + x$ for each $x \in \mathbb{Z}_5$:

$\therefore$

$f_0(0) = 0$	$f_1(0) = 1$	$f_2(0) = 2$	$f_3(0) = 3$	$f_4(0) = 4$
$f_0(1) = 1$	$f_1(1) = 2$	$f_2(1) = 3$	$f_3(1) = 4$	$f_4(1) = 0$
$f_0(2) = 2$	$f_1(2) = 3$	$f_2(2) = 4$	$f_3(2) = 0$	$f_4(2) = 1$
$f_0(3) = 3$	$f_1(3) = 4$	$f_2(3) = 0$	$f_3(3) = 1$	$f_4(3) = 2$
$f_0(4) = 4$	$f_1(4) = 0$	$f_2(4) = 1$	$f_3(4) = 2$	$f_4(4) = 3$

In a more Compact form, we write:

$f_0 = (0)$	$f_0(x) = 0$
$f_1 = (0,1,2,3,4)$	$f_1(x) = x$
$f_2 = (0,2,4,1,3)$	$f_2(x) = 2x$
$f_3 = (0,3,1,4,2)$	$f_3(x) = 3x$
$f_4 = (0,4,3,2,1)$	$f_4(x) = 4x$

formula for Compact forms

Permutation Can be defined by: $G' = \{f_0, f_1, f_2, f_3, f_4\}$

the mapping $\phi: \mathbb{Z}_5 \to G'$ defined by $\phi(i) = f_i$ for $i = 1,2,3,4,5$ is an isomorphism from $\mathbb{Z}_5$ to G'.

(4) Recall that in Question 2a from Assignment 6, you proved that $H=\{[0],[4],[8]\}$ is a Subgroup of $\mathbb{Z}_{12}$. Write out all of the distinct cosets of H in $\mathbb{Z}_{12}$. Determine $[\mathbb{Z}_{12}:H]$:

$H=\{\boxed{[0]},\boxed{[4]},\boxed{[8]}\}$ is a Subgroup of $\mathbb{Z}_{12}$

the elements of H are in the form $a+H$, $a\in\mathbb{Z}_{12}$ modulo 12

$\mathbb{Z}_{12}=\{[0],[1],\ldots,[10],[11]\}$

There are two types of Cosets, left cosets & right cosets

the group H is generated by $\mathbb{Z}_{12}$ over $(\langle 4\rangle)$:

$$H=\{[0],[4],[8]\}=\langle[4]\rangle=C_3 \text{ [Cyclic group of 3]}$$

for $(\mathbb{Z}_{12},+)$, we Can write the following cosets (additive):

$$x+[4]=\{x+n : n\in[4]\}=\{x, x+[4], x+[8]\}$$

left Cosets: let $x\in\mathbb{Z}_{12}$ modulo 4

four distinct left cosets $[\mathbb{Z}_{12}:H]=4$

$$\begin{cases} \text{let } x=0: & 0+[4]=\{[0],[4],[8]\}=4+[4]=8+[4]=H \\ \text{let } x=1: & 1+[4]=\{[1],[5],[9]\}=5+[4]=9+[4]=[1]+H \\ \text{let } x=2: & 2+[4]=\{[2],[6],[10]\}=6+[4]=10+[4]=[2]+H \\ \text{let } x=3: & 3+[4]=\{[3],[7],[11]\}=7+[4]=11+[4]=[3]+H \end{cases}$$

each of the four cosets has the Same number of elements as H

↳ Since: $|\mathbb{Z}_{12}|/|\langle[4]\rangle|=12/3=4$ Cosets

Each of the four cosets are equal Sized Partions of $\mathbb{Z}_{12}$

Which is necessary by Lagrange's theorm (Theorm 4.15)

$$\mathbb{Z}_{12}=H\cup([1]+H)\cup([2]+H)\cup([3]+H);\ [\mathbb{Z}_{12}:H]=4$$

Right Cosets:

We note $\mathbb{Z}_{12}$ is abelian $\therefore$ the left & right Cosets are the Same

Thus:

$$\boxed{\begin{array}{l}\text{Cosets: } \mathbb{Z}_{12}=H\cup([1]+H)\cup([2]+H)\cup([3]+H) \\ [\mathbb{Z}_{12}:H]=4\end{array}}$$

⑤ Determine whether the following Statements are true or false. Justify your responses:

a) let G be a group, & let H be a Subgroup of G. Then each left Coset of H is a Subgroup of G:

We can use the following example: (See Example 3 on pg. 225)

let $H=\{(1),(1,2)\}$ be a Subgroup of $G=S_3=\{(1),(1,2),(2,3),(1,3),(1,2,3),(1,3,2)\}$

For $a=(1,2,3)$, we have:

$aH=\{(1,2,3)(1),(1,2,3)(1,2)\}=\{(1,2,3),(1,3)\}$

thus, the left coset aH is $\{(1,2,3),(1,3)\}$

We can See that the above left coset aH Cannot be a Subgroup of G Since it does not Contain the identity of G.

↳ (1) is identity of $S_3=G$

↳ $(1)\notin aH$

↳ $\therefore$ aH is not a Subset of $S_3=G$

therefore, we Cannot Say that each left Coset of H is a Subgroup of G, Since we now have the Counterexample above.

$\therefore$ False

b) The Set of all odd permutations in S_n is a Subgroup of S_n:

the identity element $e \in S_n$ is even, So the Set of all odd identities has no identity, which violates the properties of groups/Subgroups.
Also, the product of two odd permutations is even, meaning the Set isn't Closed, which also violates the properties of groups/Subgroups.

Thus, the Set of odd permutations in S_n violates the properties of groups/Subgroups

$\therefore$ False

c) Let G be a group of order n, & let $K|n$. Then there exists $a \in G$ Such that $|a|=K$:

G is of order n So $|G|=n$
$K|n$ So $n=KC$, $C \in \mathbb{Z}$
$\Big\} \; |G|=n=KC$

By Corollary 4.16 on pg. 227:

The order of an element of a finite group must divide the order of the group thus, it follows that: $|a| \mid |G| \rightarrow |G|=|a|d$, $d \in \mathbb{Z}$

For $G=S_3=\{(1),(1,2),(1,3),(2,3),(1,2,3),(1,3,2)\}$

$|G|=n=6$

$n=KC \rightarrow 6=KC$ thus let $K=6$ & $C=1$ then $6|6$ is valid

However, $|a|$ ranges from 1 to 3 for all $a \in G$

$\rightarrow \therefore$ False by Contradiction, as $|a| \neq 6$ in this case

Note:

If this is asking if the order of an element of a finite group divides the order of the group, then the Solution is:

$|G|=KC$ & $|G|=|a|d$ So $KC=|a|d$

if $C=d$ then $K=|a|$ So $|a|$ Can be K, however K need not Correspond to any $|a|$ in G to Satisfy $K|n$.

So, the problems true if: Given $|G|=n$ & $|a|=K \rightarrow K|n$, but not true given: $|G|=n$ & $K|n \longrightarrow |a|=K$

$\therefore$ False

d) Every finite cyclic group is isomorphic to a group of permutations :

By Cayley's Theorm, we know that every group's isomorphic to a group of permutations the set of all finite cyclic groups belongs to the set of all groups

↳ thus, we can infer that the set of all cyclic groups is isomorphic to a group of permutations since all groups are in general

↳ This makes sence, since for any finite group, every row of its Cayley table is just a permutation of the elements of G

i.e. :

finite group $G' = \{e_1, e_2, \dots, e_{n-1}, e_n\}$

group $G = \{g_a g_b\}$ such that $g_a g_b \in \{e_1, e_2, \dots, e_{n-1}, e_n\}$

thus, this is clearly shown to be valid

∴ True