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MATH 3801 Assignment #6:

① Is $x^* = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ an extreme point of $P = \{x \in \mathbb{R}^3 : Ax \geq b\}$ where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 0 & 2 & 2 \\ 2 & 0 & 1 \end{bmatrix} \ \& \ b = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$? Justify your answer:

$$Ax = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -2 \\ 0 & 2 & 2 \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 + x_2 + x_3 \\ x_1 - x_2 - 2x_3 \\ 2x_2 + 2x_3 \\ 2x_1 + x_3 \end{bmatrix}$$ thus, since $Ax \geq b$ we know:

$$\left.\begin{array}{l} \text{I } x_1 + x_2 + x_3 \geq 1 \\ \text{II } x_1 - x_2 - 2x_3 \geq 2 \\ \text{III } 2x_2 + 2x_3 \geq 0 \\ \text{IV } 2x_1 + x_3 \geq 1 \end{array}\right\}$$

Since P is the intersection of halfspaces, it's a polyhedron. P is bounded if & only if the linear programming problems are bounded for each $i = 1, 2, 3$.

– By elimination, we note that: (Remove x_2)

Ⓐ $2x_1 - x_3 \geq 3$ (from I+II)
Ⓑ $x_1 \geq 1$ (from I $-\frac{1}{2}$III)
Ⓒ $2x_1 + x_3 \geq 1$

Next, eliminate x_3:

$$\left.\begin{array}{ll} 4x_1 \geq 4 & \text{(from A+C)} \\ x_1 \geq 1 & \end{array}\right\} \Rightarrow \text{thus } x_1 \geq 1$$

– By eliminating x_1:

Ⓐ $2x_2 + 3x_3 \geq -1$ (from I−II)
Ⓑ $x_2 + \frac{1}{2}x_3 \geq 1/2$ (from I $-\frac{1}{2}$IV)
Ⓒ $2x_2 + 2x_3 \geq 0$

Next, eliminate x_2:

$$\left.\begin{array}{ll} 2x_3 \geq -2 & \text{(from A−2B)} \\ x_3 \geq -1 & \text{(from A−C)} \end{array}\right\} \Rightarrow \text{thus } x_3 \geq -1$$

– By eliminating x_3:

Ⓐ $\frac{3}{2}x_1 + \frac{1}{2}x_2 \geq 2$ (from I $-\frac{1}{2}$II)
Ⓑ $x_1 \geq 1$ (from I $-\frac{1}{2}$III)
Ⓒ $-x_1 + x_2 \geq 0$ (from I−IV)

Next, eliminate x_1:

$$\left.\begin{array}{ll} \frac{1}{2}x_2 \geq \frac{1}{2} & \text{(from A} - \frac{3}{2}\text{B)} \\ 2x_2 \geq 2 & \text{(from A} + \frac{3}{2}\text{C)} \end{array}\right\} \text{thus } x_2 \geq 1$$

thus:

Since $x_1 \geq 1$, $x_2 \geq 1$, & $x_3 \geq -1$ we see that $\boxed{x^* \text{ is an extreme point of } P}$

Further, when Ax^* we see that:

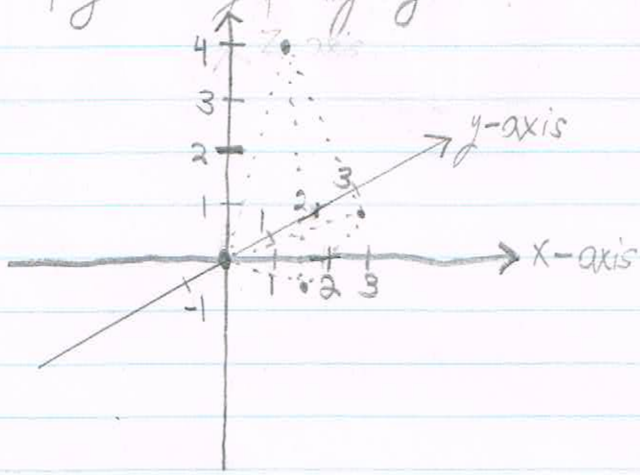
$1+1-1=1$
$1-1+2=2$
$0+2-2=0$
$2+0-1=1$

Meaning $Ax^* = b$, which supports the above conclusion.

(2) Obtain a list of inequalities that define the convex hull of $\left\{ \begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\4\end{bmatrix} \right\}$;

Hint: there should be four inequalities

We can display this graphically by:



thus, the tetrahedron connecting these 4 points can be represented as being the area contained within 4 faces.

$\Rightarrow$ Get the four faces as inequalities

to get three planes:

ABC: $\begin{bmatrix}0\\0\\0\end{bmatrix} - \begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}-1\\-2\\0\end{bmatrix}, \begin{bmatrix}2\\-1\\0\end{bmatrix} - \begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}1\\-3\\0\end{bmatrix}, \begin{bmatrix}-1\\-2\\0\end{bmatrix} \times \begin{bmatrix}1\\-3\\0\end{bmatrix} \Rightarrow$

$$\left[\begin{array}{ccc|cc} i & j & k & i & j \\ -1 & -2 & 0 & -1 & -2 \\ 1 & -3 & 0 & 1 & -3 \end{array}\right] = 0i + 0j + 3k + 2k + 0i - 0j = 5k = \begin{bmatrix}0\\0\\5\end{bmatrix}$$

thus, $5z \Rightarrow$ at $z=0$ is $5(0)=0$ so $5z = 0 = z$

thus, $z=0$ at intercept & $z \geq 0$ (See diagram)

BCD: $\begin{bmatrix}1\\2\\0\end{bmatrix} - \begin{bmatrix}2\\-1\\0\end{bmatrix} = \begin{bmatrix}-1\\3\\0\end{bmatrix}, \begin{bmatrix}1\\0\\4\end{bmatrix} - \begin{bmatrix}2\\-1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\4\end{bmatrix}, \begin{bmatrix}-1\\3\\0\end{bmatrix} \times \begin{bmatrix}-1\\1\\4\end{bmatrix} \Rightarrow$

$$\left[\begin{array}{ccc|cc} i & j & k & i & j \\ -1 & 3 & 0 & -1 & 3 \\ -1 & 1 & 4 & -1 & 1 \end{array}\right] = 12i + \cancel{0j} - k + 3k + \cancel{0i} + 4j = 12i + 4j + 2k$$

let $x, y, z = B$ then $12(1) + 4(2) + 2(0) = 20$

thus, $12x + 4y + 2z = 20 \rightarrow 6x + 2y + z = 10$

$\therefore$ $6x + 2y + z = 10$ at intercept & $6x + 2y + z \leq 10$ (See diagram)

ACD: $\begin{bmatrix}0\\0\\0\end{bmatrix} - \begin{bmatrix}2\\-1\\0\end{bmatrix} = \begin{bmatrix}-2\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\4\end{bmatrix} - \begin{bmatrix}2\\-1\\0\end{bmatrix} = \begin{bmatrix}-1\\1\\4\end{bmatrix}, \begin{bmatrix}-2\\1\\0\end{bmatrix} \times \begin{bmatrix}-1\\1\\4\end{bmatrix} \Rightarrow$

$$\begin{bmatrix} i & j & k & i & j \\ -2 & 1 & 0 & -2 & 1 \\ -1 & 1 & 4 & -1 & 1 \end{bmatrix} = 4i - 2K + K + 8j = 4i + 8j - K$$

let $x, y, z = A$ so $4(0) + 8(0) - (0) = 0$

thus, $4x + 8y - z = 0$ $\therefore$ $4x + 8y - z \geq 0$ (by diagram)

ABD: $\begin{bmatrix}0\\0\\0\end{bmatrix} - \begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}-1\\-2\\0\end{bmatrix}, \begin{bmatrix}1\\0\\4\end{bmatrix} - \begin{bmatrix}1\\2\\0\end{bmatrix} = \begin{bmatrix}0\\-2\\4\end{bmatrix}, \begin{bmatrix}-1\\-2\\0\end{bmatrix} \times \begin{bmatrix}0\\-2\\4\end{bmatrix} \Rightarrow$

$$\begin{bmatrix} i & j & k & i & j \\ -1 & -2 & 0 & -1 & -2 \\ 0 & -2 & 4 & 0 & -2 \end{bmatrix} = -8i + 2K + 4j$$

let $x, y, z = A$ so $-8(0) + 4(0) + 2(0) = 0$

thus, $-8x + 4y + 2z = 0 \Rightarrow -4x + 2y + z = 0$

$\therefore$ $-4x + 2y + z \leq 0$ (see diagram)

thus, since the planes must move into the tetrahedron & we have the four planes, we conclude:

the four inequalities are:

$$6x + 2y + z \leq 10$$
$$4x + 8y - z \geq 0$$
$$-4x + 2y + z \leq 0$$
$$z \geq 0$$

& they define the convex hull of the listed points in the question