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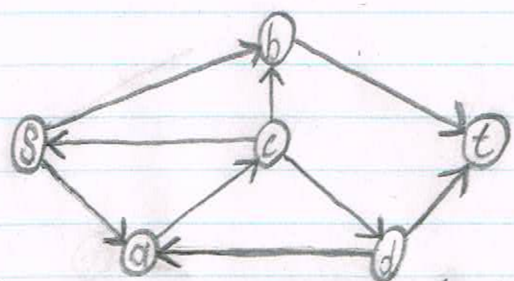
Acknowledgement: No Help Receaved

MATH 3802 Assignment Six:

① Let G denote the digraph depicted below:

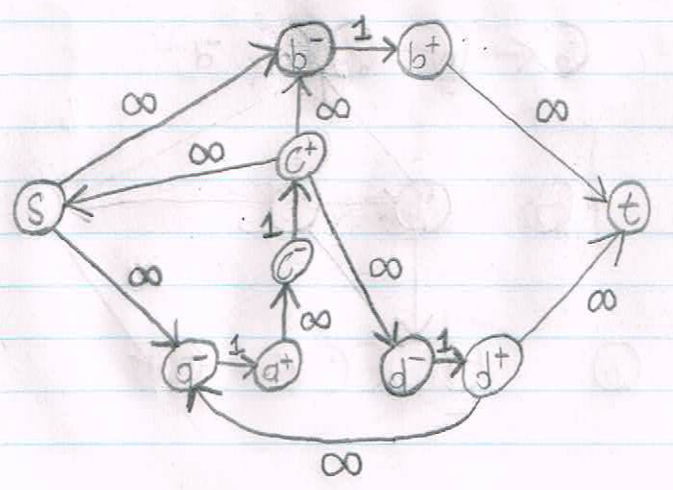


a) (5 Points) Construct the s-t network (N', A', u) from G using the construction described in the Proof of Theorm 6.1 in the notes from Week 6:

We can produce the following s-t network using the info:

- Every Node in N gives rise to a pair of nodes (except s&t)
- All node pairs are connected such that sink-nodes are negative & source-nodes are positive superscripts. The sink-nodes are directed w/ an arc towards the source nodes.
- If an arc connects the source & sink node-pair, its cost-value is 1; all other arcs have a cost-value of infinity.

We can produce the following Network:



b) (5 Points) Find the maximum number of internally node-disjoint s-t dipaths & a minimum-cardinality s-t separating set for G by solving the max-flow min-cut problem on the network in part a:

Internally node-disjoint s-t dipaths:

- A dipath if they do not share any common node other than s & t
- The maximum number of internally node-disjoint s-t dipaths equals the minimum cardinality of an s-t separating set.

Minimum-Cardinality s-t Separating Set:

- A separating set is a set such that:

  $G \setminus U$ has no s-t dipath such that $U \in \mathcal{N} \setminus \{s,t\}$
- Options include: $\{ba, bd, bc\}$, all having a cardinallity of two

Thus:

The maximum number of internally node-disjoint s-t dipaths equals two.

Now, Solving the max-flow min-cut problem on the network:

In General:

$$\begin{aligned} &\max \quad x(\delta^+(s)) - x(\delta^-(s)) \\ (MF) \quad &\text{s.t.} \quad x(\delta^+(v)) - x(\delta^-(v)) = 0 \qquad \forall v \in \mathcal{N} \setminus \{s,t\} \\ &\qquad 0 \leq x_e \leq u_e \qquad\qquad\qquad\quad \forall e \in A \end{aligned}$$

thus, as $v = \{b, c, a, d\}$ & $U = \{v \in \mathcal{N} : v^-v^+ \in \delta^+(S)\}$

thus all $\delta^+(S)$ is in form v^-v^+ for any $v \in \mathcal{N}$

$S = \{s\} \cup \{v^- : v \in \mathcal{N}\}$ & $s \in S$ So $S = \{s, b^-, c^-, a^-, d^-\}$ for all $\mathcal{N}; S \subset \mathcal{N}$

We see that:

I) $S = \{s, b^-\}$ & $x(\delta^+(S)) - x(\delta^-(S)) = 1 - 1 = 0$ OR

II) $S = \{s, a^-, c^-, d^-\}$ & $x(\delta^+(S)) - x(\delta^-(S)) = 3 - 1 = 2$

$$\begin{aligned} \hookrightarrow &\max \quad x(\delta^+(\{(s,b^-),(s,a^-,c^-,d^-)\}) - \delta^-(\{(s,b^-),(s,a^-,c^-,d^-)\})) \\ (MF) \quad &\text{s.t.} \quad x(\delta^+(\{a^-,a^+,b^-,b^+,c^-,c^+,d^-,d^+\}) - \delta^-(\{a^-,a^+,b^-,b^+,c^-,c^+,d^-,d^+\})) \\ &\qquad 0 \leq x_e \leq u_e \qquad \forall e \in A \end{aligned}$$

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Finally:

$$\max \quad x(\delta^+(\{(s,b^-),(s,a^-,c^-,d^-)\}) - \delta^-(\{(s,b^-),(s,a^-,c^-,d^-)\}))$$

$$\text{s.t.} \quad x(\delta^+(\{a^-,a^+\}) - \delta^-(\{a^-,a^+\})) = 0$$

$$x(\delta^+(\{b^-,b^+\}) - \delta^-(\{b^-,b^+\})) = 0$$

$$x(\delta^+(\{c^-,c^+\}) - \delta^-(\{c^-,c^+\}) = 0$$

$$x(\delta^+(\{d^-,d^+\}) - \delta^-(\{d^-,d^+\}) = 0$$

$$0 \le x_e \le \infty \text{ for all arcs in } A$$

Simplified:

$$\max \quad (2+3+2) - (3+2) = 2$$

$$1-1=0$$

$$1-1=0$$

$$1-1=0$$

$$1-1=0$$

$$\text{All arcs: } 0 \le x_e \le \infty \quad \forall e \in A$$

therefore, the maximum number of internally node-disjoint s-t dipaths & a minimum-cardinality s-t separating set is two when solving the max-flow min-cut problem.

Visual: