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Assignment One:

① Seat the boys & girls such that no boys are next to each other in a line:
If there are $b=15$ boys & $g=12$ girls:
Find the number of ways to make no boys next to each other:
We start by first finding the number of ways to arrange the girls in the row:

$-g_1-g_2-g_3-\cdots-g_{12}-$ | $-$: represents gap for a boy

expand

this means there are exactly $12!$ ways of arranging the girls.

$\underset{①}{-}g_1\underset{②}{-}g_2\underset{③}{-}g_3\underset{④}{-}g_4\underset{⑤}{-}g_5\underset{⑥}{-}g_6\underset{⑦}{-}g_7\underset{⑧}{-}g_8\underset{⑨}{-}g_9\underset{⑩}{-}g_{10}\underset{⑪}{-}g_{11}\underset{⑫}{-}g_{12}\underset{⑬}{-}$

there are 13 gaps (the "$-$"), yet there's exactly **15** boys thus there is no way we can arrange the boys such that no two boys are next to each other.

to prove this, we note this is simply a case of the pigeonhole principle:
- each of the gaps in the above chain can be thought as pigeonholes
- each boy can be thought of as a pigeon
- Since no two boys can be next to each other, we may think of this as there being a restriction that there's only one pigeon per hole
- Each girl allows for a new pigeonhole by segmenting space

thus, we end up with 13 pigeonholes (1+ the 12 girls adding space) which can be occupied by only 1 boy each.
- Thus, we have an injection (each pigeon maps to exactly one hole, yet some holes can stay empty):

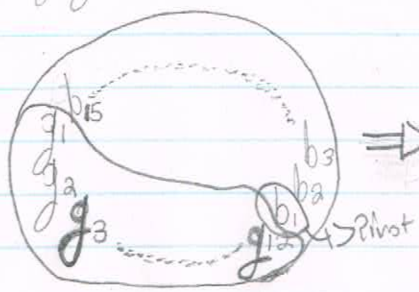
Ⓘ if $N_b \to N_{g+1}$ is injection, thus $b \leq g+1$ is a condition for b-boys & g-girls

we see that $b=15$ & $g=12 \Rightarrow$ if $b \leq g+1$ then we see $15 \nleq 12+1 \to 15 \nleq 13$
thus, in this case $b > g+1$ which is the contrapositive of Ⓘ
$\therefore N_b \to N_{g+1}$ is not an injection meaning there's no way to map it so no two boys are next to each other

$\therefore$ there is no way to arrange the class in a line such that no two boys are next to each other (By Pigeonhole Principle)

Seat boys & girls in Circle Such that all girls are together:

When arranging in a Circle, we See the following Pattern emerge:



$\Rightarrow$ Since all girls are together, they function almost as a Single unit.

Let G be the Set of all girls, & B be the set of all boys

We see there's 12 girls thus G has $12!$ orderings

We See there's 15 boys thus B has $15!$ orderings

↳ there's 27 ways to rotate a table around pivot b_1

thus there are 27 Combinations for each arrangment

$\therefore$ there are $12! \cdot 15! \cdot 27$ different ways to seat the boys & girls

(2)

a) Select 3 from $\{1,\dots,17\}$ to get even, odd, & total Sum:

we Know that:

2 odd numbers plus an even number is even

3 even numbers are even

& that 3-odds, even+even+odd, are both odd

thus, without repitition we Know:

there are exactly 8 even numbers in the Set

Namly: $\{2,4,6,8,10,12,14,16\}$

there's exactly 9 odd numbers, namly: $\{1,3,5,7,9,11,13,15,17\}$

the number of Sums from 3-integers is then:

the number of Sums with 2 odd integers & an even & the number of Sums w/ 3 even numbers: (even)

Note: $\boxed{C(n,k)=\frac{n!}{k!(n-k)!}}$ finds the number of Combinations without repition.

$$\underbrace{\binom{9}{2}}_{\text{Choose 2 of the 9-odds}}\underbrace{\binom{8}{1}}_{\text{Choose an even}}+\underbrace{\binom{8}{3}}_{\text{Choose 3 evens}}=\left[\frac{9!}{2!(9-2)!}\right]\left[\frac{8!}{1!(8-1)!}\right]+\frac{8!}{3!(8-3)!}$$

$$=\left(\frac{362880}{2(5040)}\right)\left(\frac{40320}{5040}\right)+\frac{40320}{6(120)}$$

$$\boxed{=344 \text{ even Sums of 3 numbers}}$$

Sums w/ 3-odds & 2-evens plus an odd: (odd)

$$\underbrace{\binom{9}{3}}_{\text{Choose 3 odds}}+\underbrace{\binom{8}{2}}_{\text{Choose 2 evens}}\underbrace{\binom{9}{1}}_{\text{Choose an odd}}=\left[\frac{9!}{3!(9-3)!}\right]+\left[\frac{8!}{2!(8-2)!}\right]\left[\frac{9!}{1!(9-1)!}\right]$$

$$=\frac{362880}{6(720)}+\left(\frac{40320}{720(2)}\right)\left(\frac{362880}{40320}\right)$$

$$\boxed{=336 \text{ odd Sums of 3 numbers}}$$

Ways we can Choose 3 numbers from 17: (total)

$$\binom{17}{3}=\frac{17!}{3!(17-3)!}=\boxed{680 \text{ Sums of 3}}$$

thus, the total Sum of even+odd numbers is:

$344+336=680$ which is the same as $\binom{17}{3}=680$

$\Rightarrow$ Results verified

b)

Without repitition, the same logic applies as in a);

2 odd's + even = even

3 even's = even

3 odd's = odd

2 evens + odd = odd

there are 8 evens, namly: {2, 4, 6, 8, 10, 12, 14, 16}

there's 8 odds, namly: {1, 3, 5, 7, 9, 11, 13, 15, 17}

the number of sums is then:

Note: $C(n+k-1, k)$ finds the number of combinations with repition

2 odd + even OR 3 even : (even)

$$= \Big[\underbrace{C(9+2-1,2)}_{\text{finds 2 odd with rep.}}\ \underbrace{C(8+\cancel{1}-\cancel{1},1)}_{\text{find an even}}\Big] + \underbrace{C(8+3-1,3)}_{\text{find 3 evens}}$$

$$= \binom{10}{2}\binom{8}{1} + \binom{10}{3} = \frac{10!}{2!(10-2)!}\cdot\frac{8!}{1!(8-1)!} + \frac{10!}{3!(10-3)!}$$

$$= \frac{3628800}{2(40320)}\cdot\frac{40320}{1(5040)} + \frac{3628800}{6(5040)} = \text{480 even combinations}$$

3 odds OR 2 evens + odd : (odd)

$$= \Big[\underbrace{C(9+3-1,3)}_{\text{finds 3 odds w/ repition}} + \underbrace{C(8+2-1,2)}_{\text{finds 2 evens w/ repition}}\ \underbrace{C(9+\cancel{1}-\cancel{1},1)}_{\text{finds an odd}}\Big]$$

$$= \binom{11}{3} + \binom{9}{2}\binom{9}{1} = \frac{11!}{3!(11-3)!} + \frac{9!}{2!(9-2)!}\cdot\frac{9!}{1!(9-1)!}$$

$$= \frac{39\,916\,800}{6(40320)} + \frac{362880}{2(5040)}\cdot\frac{362880}{40320}$$

$$= \text{489 odd combinations}$$

Ways to choose 3 numbers from 17: (total)

$$C(17+3-1,3) = \binom{19}{3} = \frac{19!}{3!(19-3)!} = 969$$

thus, the total sum of the even+odd numbers is:

$480 + 489 = 969$ which is the same as $\binom{19}{3} = 969$

↳ Results verified

(3) How many integer Solutions to $x+y+z+w+t = 30$ Subject to:

$x \geq 2$

$y \geq 2$

$z \geq 3$

$w \geq 5$

$t \geq 0$

We can convert this to an easier form to work with:

$$(x-2)+(y-2)+(z-3)+(w-5)+(t-0) = 30-[2+2+3+5+0]$$

thus, let:

$$\left.\begin{array}{l} x_1 = x-2 \\ y_1 = y-2 \\ z_1 = z-3 \\ w_1 = w-5 \end{array}\right\} \quad \text{Such that:} \quad \begin{array}{l} x_1+y_1+z_1+w_1+t = 18 \\ x_1, y_1, z_1, w_1, t \geq 0 \end{array}$$

Since replacement is allowed, we use the form:

$$C(n+k-1, k-1) = \binom{n+k-1}{k-1}$$

thus, now $n=18$ & $k=5$, Since all of the variables are ≥ 0:

$$C(18+5-1, 5-1) = \binom{18+5-1}{5-1} = \binom{22}{4}$$

$$= \frac{22!}{4!(22-4)!} = 7315 \text{ Combinations}$$

∴ We can see that there are 7315 Possible Combinations for the given equation.

(4)

a) Use the Sieve Principle to find the number of integer Solutions of $X_1+X_2+X_3+X_4+X_5=14$ with $0 \leq X_1, X_2, X_3, X_4, X_5 \leq 7$:

let P be a non-negative Solution to $X_1+X_2+X_3+X_4+X_5=14$
For $1 \leq i \leq 5$, let ℓ_i be a set of the non-negative integer Solutions to $\sum_{i=1}^{5} x_i = 14$ where $x_i > 7$ (OR: $x_i \geq 8$)
thus, when $X_i \geq 8$ for all i from 1 to 5, we Note:
$0 \leq X_i \nleq 7 \rightarrow \ell_i$ is invalid (Zero) Since we have $x_i > 7$
yet, if $x_i \geq 8$ there are still plenty Solutions to $\sum_{i=1}^{5} X_i = 14$
let P_0 be the integer Solutions to $\sum_{i=1}^{5} X_i = 14$ Such that any $X_i \geq 8$
let P_1 be all Solutions to $\sum_{i=1}^{5} X_i = 14$ Such that $0 \leq X_i \leq 7$
thus P is:

$$P = P_0 + P_1 \quad \therefore \quad P_1 = P - P_0$$

We Can See that:

$$|P| = \binom{n+r-1}{r-1} = \binom{14+5-1}{5-1} = \binom{18}{4}$$
$$= \frac{18!}{4!(18-4)!} = \boxed{3060}$$

thus there's precisely 3060 unique Solutions
We note for all $X_i \geq 8$, we can remove 8 everywhere to obtain:
$\left(\sum_{i=1}^{n} x_i\right) - 8 = 14 - 8$ Such that:
$\sum_{i=1}^{n} X_i = 6$ where $x_i \geq 0$ for i from 1 to 5 inclusive
Clearly, the number of Solutions to $|P_0|$ is $\binom{n+r-1}{r-1} = \binom{6+5-1}{5-1}$
$= \binom{10}{4} = \frac{10!}{4!(10-4)!} = 210$ for each X_i. Thus $(5)(210) = \boxed{1050}$
By the Seive Principle:
$|P| = |P_0| + |P_1|$ Such that $|P_1| = |P| - |P_0|$
thus: $|P_1| = 3060 - 1050 = \boxed{2010}$
$\therefore$ There are 2010 ways to Solve $X_1+X_2+X_3+X_4+X_5=14$ Such that $0 \leq X_1, X_2, X_3, X_4, X_5 \leq 7$ holds true.

b) Use the Sieve Principle to find $\phi(750)$. Then verify your answer by using the formula of $\phi(n)$ in terms of Prime factors:

Eulers function:

$\phi(p) = p-1$ (p Prime)

$\phi(750)$ asks how many integers x in the range $1 \leq x \leq 750$ satisfy $\gcd(x, 750) = 1$

Info:

we Know $750 = 375 \cdot 2 = 75 \cdot 5 \cdot 2 = 15 \cdot 5 \cdot 5 \cdot 2 = 3 \cdot 5 \cdot 5 \cdot 5 \cdot 2$
$= 2 \cdot 3 \cdot 5^3$

thus, we need to find the number of integers x between 1 & 750 that are not divisible by 2, 3, or 5.

So, let $D(2)$ denote the subset N_{750} containing those integers which are divisible by 2

let $D(2,3)$ denote those divisible by 2 & 3

let $D(3)$ & $D(5)$ denote divisability by 3 & 5 respectivly

Continue for $D(3,5)$, $D(2,5)$, & $D(2,3,5)$

Since 2, 3, & 5 are not mutually exclusive prime factors, the addition rule:

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

And since the Complement Rule states $|A| = |B| - |$:

we Know: (for n's w/ 3 Primes $p_1, p_2, \& p_3$)

$$\begin{aligned}\phi(n) &= n - |D(p_1) \cup D(p_2) \cup D(p_3)| \\ &= n - [(|D(p_1)| + |D(p_2)| + |D(p_3)|) \\ &\quad + (|D(2,3)| + |D(2,5)| + |D(3,5)|) \\ &\quad - |D(2,3,5)|\end{aligned}$$

$$\begin{aligned}\phi(750) &= 750 - ([750/2] + [750/3] + [750/5]) \\ &\quad + ([750/(2\cdot3)] + [750/(2\cdot5)] + [750/(3\cdot5)]) \\ &\quad - ([750/(2\cdot3\cdot5)]) \\ &= 750 - (375 + 250 + 150) + (125 + 75 + 50) - 25 \\ &= 750 - 775 + 250 - 25 = 200\end{aligned}$$

verify: $\phi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right)$ & $p|n$ is 2, 3, & 5

$$= 750\left(1-\frac{1}{2}\right)\left(1-\frac{1}{3}\right)\left(1-\frac{1}{5}\right) = 750\left(\frac{4}{15}\right) = 200$$

$\therefore$ We can see that — using both methods — that $\phi(750) = 200$ (using both the Sieve Principle & the Prime factor based methods)

⑤ let $n, m, K \in \mathbb{Z}^+$ with $K \le m$ & $K \le n$, Prove:

$$\binom{m+n}{K} = \binom{m}{0}\binom{n}{K} + \binom{m}{1}\binom{n}{K-1} + \dots + \binom{m}{K}\binom{n}{0}$$

we can use the following identity for this problem:

$$(1+x)^m(1+x)^n = (1+x)^{m+n}$$

we can see that the term:

x^{m+n} is $\binom{m+n}{K}$ in the binomial expantion of $(1+x)^{m+n}$

Since $K \le m,n$ we can convert the forms to:

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \dots + \binom{n}{b}b^n$$

where $a^{n-r}b^r = \binom{n}{r}$

Since: $a=1$ & $b=x$

$$(1+x)^n = \binom{n}{0}\cancel{1^n} + \binom{n}{1}\cancel{1^{1-1}}x + \binom{n}{2}\cancel{1^{1-2}}x^2 + \dots + \binom{n}{x}x^n$$
$$= \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \dots + \binom{n}{x}x^n$$
$$(1+x)^m = \binom{m}{0}\cancel{1^m} + \binom{m}{1}\cancel{1^{1-1}}x + \binom{m}{2}\cancel{1^{1-2}}x^2 + \dots + \binom{m}{x}x^m$$
$$= \binom{m}{0} + \binom{m}{1}x + \binom{m}{2}x^2 + \dots + \binom{m}{x}x^m$$

We multiply these factors to obtain:

a term in x^{n+m} by taking a term $\binom{n}{r}x^r$ from the m-factor & a term $\binom{n}{n-r}x^{n-r}$ from the second factor

thus, we can derive the coefficient of the x^K product as:

$$\left[\binom{m}{p}x^p\right]\left[\binom{n}{n-K+p}x^{K-p}\right]$$

Where we replace r with K & let p range from 0 to K

↳ Since $K \le n$ & p is from 0 to K, this lets us count K to 0

↳ letting p at the bottom allows us to go from 0 to K

When we compare for a term of degree K: (x^K)

$$\left[\binom{m}{0}x^0 + \binom{m}{1}x + \dots + \binom{m}{K}x^K\right] * \left[\binom{n}{n-K}x^K + \binom{n}{n-K+1}x^{K-1} + \dots + \binom{n}{n\cancel{-K+K}}x^{K-K}\right]$$

x^{m+n} is $\binom{m}{K}\binom{n}{n-K}$ for x^K thus:

(expand the binomial):

$$= \binom{m}{0}\binom{n}{K} + \binom{m}{1}\binom{n}{K-1} + \dots + \binom{m}{K}\binom{n}{0}$$

⑥ For the following values of (v,k,r) either construct a design w/ those parameters or explain why such a design d.n.e.:

i) $(v,k,r) = (9,3,4)$:

As we see $3 \mid (9 \times 4)$ $\Rightarrow (k \mid vr)$ as $9 \cdot 4 = 3c$, $c \in \mathbb{Z}$ where $c = 12$
& since $vr/k = 12 < \binom{v}{k} = \binom{9}{3} = \frac{9!}{3!(9-3)!} = 84$ $\therefore$
there must be a design fitting these parameters
A possibility is:

k is 3, so each person runs 3 varieties
↳ Each person has 3 digits
r is 4, so each variety is run by 4 consumers
↳ we have $vr/k = 12$ sets of digits
v is 9, so we count from 1 to 9 with the consumer digits
↳ we can choose digits from 1 to 9

thus, a possibility is:

Person X	Consumer test # C_1	C_2	C_3
1	1	2	3
2	4	5	6
3	7	8	9
4	1	2	6
5	4	5	9
6	7	8	3
7	1	2	9
8	4	5	3
9	7	8	6
10	1	8	3
11	4	2	6
12	7	5	9

$K_n = 4$ when n is 1, 9

therefore: (A possible combination would be)

$\{1,2,3\}, \{4,5,6\}, \{7,8,9\}, \{1,2,6\}, \{4,5,9\}, \{7,8,3\},$
$\{1,2,9\}, \{4,5,3\}, \{7,8,6\}, \{1,8,3\}, \{4,2,6\}, \{7,5,9\}$

ii) $(v,K,r)=(9,6,8)$:

As we see $6 \mid (9\cdot 8) \Rightarrow 9\cdot 8 = 6c,\ c\in\mathbb{Z}$ where $c=12$

Since $vr/K = 12 < \binom{v}{K} = \binom{9}{6} = \frac{9!}{6!(9-6)!} = 84$ $\therefore$

there must be a design fitting these parameters

A possibility is:

K is 6, so each person runs 6 varieties (6 digits)

r is 8, so each variety is run by 8 consumers ($vr/K = 12$ sets)

v is 9, so consumer digits run from 1 to 9

thus, a possibility is:

Person	Consumer test #					
V	C_1	C_2	C_3	C_4	C_5	C_6
1	1	2	3	4	5	6
2	7	8	9	1	2	3
3	4	5	6	7	8	9
4	1	3	5	7	9	2
5	1	3	5	7	9	4
6	1	3	5	7	9	6
7	1	3	5	7	9	8
8	2	4	6	8	1	3
9	2	4	6	8	3	5
10	2	4	6	8	5	7
11	2	4	6	8	7	9
12	2	4	6	8	1	9

therefore: (a possible combination would be)

$\{1,2,3,4,5,6\}, \{7,8,9,1,2,3\}, \{4,5,6,7,8,9\}, \{1,3,5,7,9,2\},$
$\{1,3,5,7,9,4\}, \{1,3,5,7,9,6\}, \{1,3,5,7,9,8\}, \{2,4,6,8,1,3\},$
$\{2,4,6,8,3,5\}, \{2,4,6,8,5,7\}, \{2,4,6,8,7,9\}, \{2,4,6,8,1,9\}$

⑦ let s & n be positive integers. Prove the identity:

$$\binom{s+n}{s} = \binom{s-1}{0} + \binom{s}{1} + \cdots\cdots + \binom{s+n-2}{n-1} + \binom{s+n-1}{n}$$

We may use the following theorm:

$$\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$$

Also, note the following theorms:

$$\binom{n}{r} = \binom{n}{n-r} \quad \& \quad \binom{k+1}{r} = \binom{k}{r-1} + \binom{k}{r}$$

thus:

$$\binom{s+n}{s} = \binom{s+n}{\cancel{s}+n-\cancel{s}} = \binom{s+n}{n}$$

So if $\binom{s+n}{s} = \binom{s+n}{n}$:

$$\binom{s+n}{n} = \binom{s+n-1}{n-1} + \binom{s+n-1}{n}$$

if we repeat this on $\binom{s+n-1}{n-1}$ we see:

$$\binom{s+n-1}{n-1} = \binom{s+n-2}{n-2} + \binom{s+n-2}{n-1}$$

if we repeat this on $\binom{s+n-2}{n-2}$ we see:

$$\binom{s+n-2}{n-2} = \binom{s+n-3}{n-3} + \binom{s+n-3}{n-2}$$

if we continue this $n-2$ times:

$$\Rightarrow \binom{s+n-1}{n} + \binom{s+n-2}{n-1} + \binom{s+n-3}{n-2} + \cdots + \binom{s+\cancel{n}-(\cancel{n}-2)}{\cancel{n}-(\cancel{n}-2)}$$

to simplify the end term $\binom{s+2}{2}$, continue like before

$$\Rightarrow \binom{s+2}{2} = \binom{s+2-1}{2-1} + \binom{s+2-1}{2}$$

$$= \binom{s+1}{1} + \binom{s+1}{2}$$

We can simplify $\binom{s+1}{1}$ using $\binom{k+1}{r} = \binom{k}{r-1} + \binom{k}{r}$:

$$= \binom{s}{1-1} + \binom{s}{1} + \binom{s+1}{2}$$

So, we note the following pattern emerges:

$$\Rightarrow \binom{s+n-1}{n} + \binom{s+n-2}{n-1} + \binom{s+n-3}{n-2} + \cdots + \binom{s+1}{2} + \binom{s}{1} + \binom{s}{0}$$

As there's only 1 way to choose 0 objects:

$$\binom{S}{0} = \binom{S-1}{0} = \binom{S-j}{0} = 1 \text{ for all } 0 \leq j \leq S$$

thus, the equation can finally be simplified to: (Since $\binom{S}{0} = \binom{S-1}{0}$)

$$\rightarrow \binom{S+n-1}{n} + \binom{S+n-2}{n-1} + \binom{S+n-3}{n-2} + \dots + \binom{S+1}{2} + \binom{S}{1} + \binom{S-1}{0}$$

this is the same pattern as in the original question:

$$\therefore \text{ it's true that } \binom{S+n}{S} = \binom{S-1}{0} + \binom{S}{1} + \dots + \binom{S+n-2}{n-1} + \binom{S+n-1}{n}$$

$$\text{as } \binom{S+n}{S} = \binom{S+n}{n} = \binom{S+n-1}{n} + \binom{S+n-2}{n-1} + \binom{S+n-3}{n-2} + \dots + \binom{S+1}{2} + \binom{S}{1} + \binom{S-1}{0}$$

So, it is proven.

8)

Since the Probability no two people have the Same birthday in the room is mutually exclusive with the Probability that at least two People do: (Complement Rule)

$P(A) = 1 - P(B)$, A = two people share the Same birthday
B = No one Shares the Same birthday

Since, we are finding B:
by the Product rule:

$(P_1(\text{No Common birthday}) \times (P_2(\cdots)) \times \ldots \times P_{23}(\cdots))$

we must Calculate the Probability no one has a Common birthday

Known:

days of year
No one Shares a birthday
↳ ∴ Probability of finding 23 distinct dates from 365 dates

We Choose a date & remove it from the Set of dates for all 23 People & multiply the Probabitity as per the Product Rule:

$$\therefore \frac{365}{365} \times \frac{364}{365} \times \frac{363}{365} \times \cdots \times \frac{343}{365}$$

$$= \frac{\prod_{i=0}^{22}(365-i)}{365^{23}} \doteq 0.4927$$

there's a 49.27% Chance no one has the Same birthday

↳ By the Complement Rule:

$$1 - \frac{\prod_{i=0}^{22}(365-i)}{365^{23}} = 0.507297234 \doteq 50.73\%$$

∴ The probability a random group of 23 people have at least 2 people w/ the Same birthday is 50.73%, & this is about 50%; therefore, the problem Stated in Q8 is Proven true.