COMP 2804 Assignment 4:

Question 1:

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Question 2:

- Are x & y independent variables ? $\rightarrow$ Let: $X = i+j$, $\rightarrow$ Let: $Y = i-j$, | Let (i,j) where i is the result of the red die & j the result of blue
  - If x & y are independent then:

    $Pr(X=\#_1 \wedge Y=\#_2) = Pr(X=\#_1) \cdot Pr(Y=\#_2)$ otherwise there not independent.
  - $Pr(X \wedge Y)$ — This is the probability that X & Y will equate to their given values taken from the Sample Space of all rolls.
    - For Example, there is only one way to have $X=12$ with 2 dice, by rolling 2 6's, This is because 6 is the largest roll per die & thus only 2 Sixes can Sum to 12. This would give a y of: $Y=6-6=0 \therefore \boxed{y=0}$
    - There are a grand total of 6 faces per dice, rolled independently. The product Rule may model the number of possible face Combinations per both dice rolled independently. Thus the Sample Space (# of possible rolls) $= 6 \cdot 6 = 36$.
    - If only 1 Combo of rolls of 36 possibilities may occur, then the Change of getting 2 Sixes is $\boxed{1/36}$.
  - $Pr(X=\#_1)$ — The probability that Value X may occur. For example, if $X=12$, theres only a $\boxed{1/36}$ Chance of occurence, as stated above.
  - $Pr(Y=\#_2)$ — The probability that Value Y may occur.
    - ☐ In the above example, we must roll 2 6's to draw a X of 12, this would result in a Y of zero. The # of ways to make Y equal zero are: For $(i,j) \Rightarrow (1,1)$ OR $(2,2)$ OR $(3,3)$ OR $(4,4)$ OR $(5,5)$ OR $(6,6)$

      $\therefore 6/36 = Pr(Y=0)$

- Therefore if we wish $Pr(X=\#_1, Y=\#_2)$ to equal $Pr(X=12 \wedge Y=0)$ then $Pr(X=12)$ & $Pr(Y=0)$ must be evaluated.

As already stated: $Pr(Y=0) = 6/36$, $Pr(X=12) = 1/36$, $Pr(X=12 \wedge Y=0) = 1/36$

$\therefore Pr(X=12 \wedge Y=0) = Pr(X=12) \cdot Pr(Y=0)$

$\rightarrow 1/36 = (1/36)(6/36)$

$\boxed{1/36 \neq 1/216}$ ∴ they are $\boxed{\text{not independent}}$ because at least one case breaks the independence equation equation for random variables.

Question 3:

— $Pr(X=1) = Pr(X=-1) = Pr(Y=1) = Pr(Y=-1) = 1/2$ is Given:

- We know that: $Pr(X=1) = Pr(X=-1) = 1/2$, these two events are dependent, because X cannot be both -1 & 1 at the same time. The sum rule can model this: $1/2 + 1/2 = 1$ ∴ X must be either 1 or -1.
  - ☐ The same principle applys to Y: its either 1 or -1.
- There are 4 combos of Z:
  - ☐ Let $X=1$ & $Y=1$ $\rightarrow Z = 1 \cdot 1 = \boxed{1}$
  - ☐ $X=1, Y=-1 \longrightarrow Z = 1 \cdot -1 = \boxed{-1}$
  - ☐ $X=-1, Y=1 \longrightarrow Z = -1 \cdot 1 = \boxed{-1}$
  - ☐ $X=-1, Y=-1 \longrightarrow Z = -1 \cdot -1 = \boxed{1}$

∴ there is a ① $1/2$ chance Z is -1 & a $1/2$ chance Z is 1

— If X & Z are independent then: $Pr(X=\#_1 \wedge Z=\#_2) = Pr(X=\#_1) \cdot Pr(Z=\#_2)$.

For $X=1; Z=1$:

Independent

$Z = X \cdot Y$

$1 = (1)(1) \rightarrow 1 = 1$

$\therefore X=1, Y=1, Z=1$

LS: $Pr(X=1 \wedge Z=1) = Pr(X=1 \wedge Y=1) = Pr(X=1) \cdot Pr(Y=1)$

$\hookrightarrow 1/2 \quad \hookrightarrow 1/2$

$LS = (1/2)(1/2) = 1/4$

RS: $Pr(X=1) \cdot Pr(Z=1) = (1/2)(1/2) = 1/4$

$\hookrightarrow 1/2 \quad \hookrightarrow 1/2$

∴ $LS = RS \rightarrow 1/4 = 1/4$

For $X=1; Z=-1$:

$Z = X \cdot Y$

$-1 = (1)(-1)$

$-1 = -1$

$\therefore X=1, Y=-1, Z=-1$

independent

LS: $Pr(X=1 \wedge Z=-1) = Pr(X=1 \wedge Y=-1) = \overset{\nearrow 1/2}{Pr(X=1)} \cdot \overset{\nearrow 1/2}{Pr(Y=-1)}$

$LS = (1/2)(1/2) = 1/4$

∴ $LS = RS \rightarrow 1/4 = 1/4$

RS: $\overset{\rightarrow 1/2}{Pr(X=1)} \cdot \overset{\rightarrow 1/2}{Pr(Z=-1)} = (1/2)(1/2) = 1/4$

$Z = X \cdot Y$
$1 = (-1)(-1)$
$\boxed{1 = 1}$ $\therefore X=-1, Y=-1, Z=1$

For $X=-1; Z=1$:

independent

LS: $Pr(X=-1 \wedge Z=1) = Pr(X=-1 \wedge Y=-1) = \overset{1/2}{Pr(X=-1)} \cdot \overset{1/2}{Pr(Y=-1)}$

$LS = (1/2)(1/2) = \boxed{1/4}$ $\therefore LS = RS \longrightarrow \underline{1/4 = 1/4}$

RS: $\overset{1/2}{Pr(X=-1)} \cdot \overset{1/2}{Pr(Z=1)} = (1/2)(1/2) = \boxed{1/4}$

$Z = X \cdot Y$
$-1 = (-1)(1)$
$\boxed{-1 = -1}$

For $X=-1, Z=-1$:

independent

LS: $Pr(X=-1 \wedge Z=-1) = Pr(X=-1 \wedge Y=1) = \overset{1/2}{Pr(X=-1)} \cdot \overset{1/2}{Pr(Y=1)}$

$LS = (1/2)(1/2) = \boxed{1/4}$ $\therefore LS = RS \longrightarrow \underline{1/4 = 1/4}$

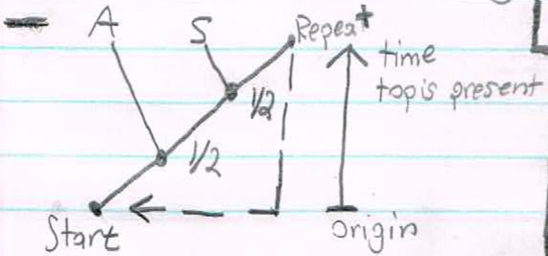
$\boxed{\therefore X=-1, Y=1, Z=-1}$ RS: $Pr(X=-1) \cdot Pr(Z=-1) = (1/2)(1/2) = \boxed{1/4}$

— All four Cases of X & Z result in the left Side equalling the right Side of the independence equation for random variables. With all Possibilities exhaused, we can Conclude that X and Z are $\boxed{\text{independent}}$ random variables

- Thus: $Pr(X=\#_1 \wedge Z=\#_2) = Pr(X=\#_1) \cdot Pr(Z=\#_2)$ is True for the given Conditions of random variables $\therefore$ $\boxed{X \text{ and } Y \text{ are independent random Variables.}}$

Question 4:

— Alexa needs to loose if She flips tails & win if She flips heads, She has a fair Coin, ∴ She will have a 1/2 Chance of winning if She flips heads & a 1/2 Chance of loosing if She flips tails.



— the left diagram Shows that as time Progresses in the game, Alexa flips ; if She wins, then the game ends, if She looses then its Shellys turn. If Shelly wins, the game ends, otherwise if She looses, the game repeats back to Alexa. Each flip has a 1/2 Chance of winning or loosing for both players.

— As Stated, if Shelly looses, then Alexa goes again, this gives her another 1/2 Chance of wining.

Alexas Chance of loosing is ∴ (Product rule)

$(1/2)(1/2) = \boxed{1/4}$ meaning her Chance of winning is $1 - 1/4 = \boxed{3/4}$ on her Second roll. This gives alexa a greater overall Chance of winning, as She will have a greater chance of winning then Shelly on each Subsequent flip.

— To prevent a repeat back to Alexa, Shelly must always win ∴ $\boxed{q=1}$.

— Assume that $0<q<1$:

- The probability that Alexa wins the game:

☐ $P(A) = \frac{1}{2} + \frac{1}{2}(1-q)(P(A))$

↳ Recursion (if game repeats from both players loosing)

↳ Shelly flips T

↳ Alexa flips T

↳ Alexa flips H

— As can be seen, P(A) is very similar to the diagram from the start of the question.

☐ $P(A) = \frac{1}{2} + \frac{1}{2}(1-q)(P(A))$ [Algabra]

$P(A) - 1/2(1-q)(P(A)) = 1/2$

$P(A) - (1/2 - 1/2q)(P(A)) = 1/2$

$P(A) - 1/2P(A) + 1/2qP(A) = 1/2$

$1/2P(A) + 1/2qP(A) = 1/2$

$P(A)[1/2 + 1/2q] = 1/2$

$P(A) = \frac{(1/2)}{(1/2 + 1/2q)}$

$\boxed{P(A) = \frac{1}{1+q}}$ [final Sol^n]

- Proove: for any q with $0<q<1$, Pr. Alexa wins is greater then 1/2.

☐ $Pr(A) = p + (1-p)(1-q)P(A)$

↳ Recursion for repeated games

↳ probability Shelly flips T

↳ Probability Alexa flips T

↳ Probability Alexa flips H

☐ $Pr(A) = p + (1-p)(1-q)Pr(A)$ [Algabra]

$Pr(A) - (1-p)(1-q)Pr(A) = p$

$Pr(A) - (1-q-p+pq)(Pr(A)) = p$

$Pr(A) - (Pr(A) - Pr(A)q - Pr(A)p + Pr(A)pq) = p$

$\cancel{Pr(A)} - \cancel{Pr(A)} + Pr(A)q + Pr(A)p - Pr(A)pq = p$

$Pr(A)q + Pr(A)p - Pr(A)pq = p$

$Pr(A)(q + p - pq) = p$

$Pr(A) = (p)/(q + p - pq)$ [Convert to Square w/ Algabra]

$$Pr(A) = \frac{p}{1-(1-p)(1-q)}$$

If Alexas win rate (p) is greater then 1/2:

$$\frac{p}{1-(1-p)(1-q)} > \frac{(1/2)}{1-(1-1/2)(1-q)}$$ [Algabra]

$$= \frac{1/2}{1-(1/2)(1-q)}$$

$$= \frac{1/2}{1-(1/2 - 1/2q)}$$

$$= \frac{1/2}{1 - 1/2 + 1/2q}$$

$$= \frac{1/2}{1/2 + 1/2q}$$

$$= \frac{\cancel{1/2}}{\cancel{1/2}(1+q)}$$

$$= \frac{1}{1+q}$$

$$\frac{p}{1-(1-p)(1-q)} > \frac{1}{1+q}$$ [Substitute in Simplified Value]

— $\frac{1}{1+q}$ must ∴ alway be within the range of 1 & 1/2 as $0 < q < 1$. To Prove this, we can test when $q = 0 + 10^{-8}$ or $1 - 10^{-8}$:

$$\frac{1}{1+10^{-8}} \doteq 0.9999999.... \quad \& \quad \frac{1}{1+(1-10^{-8})} \doteq 0.500000002$$

— Assuming that $P=1/2$, Determine the value of q for which Alexa & Shelly have the same probability of winning the game:

$$Pr(A) = \frac{p}{1-(1-p)(1-q)} \quad \text{When } p<1/2 \text{ \& } P(A)=1/2$$

$$1/2 = p/(1-(1-p)(1-q))$$

$$2 = (1-(1-p)(1-q))/p$$

$$2p = 1-(1-p)(1-q)$$

$$2p = 1-(1-q-p+pq)$$

$$2p-p = \cancel{1}-\cancel{1}+q+\cancel{p}-pq$$

$$p = q-pq$$

$$p = q(1-p)$$

$$\boxed{q = \frac{p}{1-p}}$$

→ • To Prove $p<1/2$ & q is between 0 & 1:

$$q = \frac{10^{-8}}{1-10^{-8}} \doteq 0.00000001 \quad \& \quad \frac{1/2-10^{-8}}{1-(1/2-10^{-8})} \doteq 0.99999976$$

∴ the value for q for which Alex & Shelly have the same probability of winning the game is $\boxed{q=\frac{p}{1-p}}$.

Question 5:

— I roll a fair dice 5 times (All rolls independent):

'Let x = the largest value in the 5 rolls

• Prove: $E(x) = \frac{14077}{2592}$

☐ the sample space for this problem is the total number of dice face combinations. We know we roll 1 fair die 5 times. A die has 6 faces, each roll is independent ∴ the product of the number of possible faces per roll is the sample space. Theres always 6 faces & 5 rolls ∴ $(6)(6)(6)(6)(6) = 6^5 = \boxed{7776}$

☐ Let S be the Set of 5 rolls & D_x be the roll number

☐ If $X=1$ then there is only 1 way to make this occurence, if all 5 rolls equal 1:

$$D_1=1, D_2=2, D_3=3, D_4=4, D_5=5$$

☐ If $X=2$ then the number of occurences can be expressed as: $S=\{D_1, D_2, D_3, D_4, D_5\}$, where $0 < D_x \leq 2$ & $0 < x \leq 6$, thus for any of 6 faces (x), no die can result in a # other then 1 or 2.

There are 2 valid values, 5 rolls, & a six faced die $\therefore$

$2^5 = \textcircled{32}$ is the # of faces possible, because for all 5 rolls theres only 2 possible valid numbers between 1 & 2 per dice. We must also subtract 1, as theres 1 occurence where all 5 rolls result in 1.

$\therefore 32-1=\textcircled{31}$.

☐ This type of problem can be modeled with the equation:

- $E(X) = \sum_{w \in S} x(w) \cdot Pr(w)$

↳ Probability the largest roll is that value

↳ value were rolling for (largest value)

$$= \left(1 \cdot \frac{1^5}{6^5}\right) + \left(2 \cdot \frac{(2^5-1^5)}{6^5}\right) + \left(3 \cdot \frac{3^5-2^5}{6^5}\right) + \left(4 \cdot \frac{4^5-3^5}{6^5}\right) + \left(5 \cdot \frac{5^5-4^5}{6^5}\right) + \left(6 \cdot \frac{6^5-5^5}{6^5}\right)$$

- Explaning this: take $\left(5 \cdot \frac{5^5-4^5}{6^5}\right)$ as an example

6^5: As explaned earlyer, this is the sample space

5^5: there are 5 Rolls, & any combo of values up to 5 can allow X to equal 5.

4^5: The number of die combo's resulting with no 5, for example occurences such as all 4's being rolled must be removed.

5: This is the weight of the expected value, its multiplied by the probability.

- Computing E(X) from the above formula results in:

$$E(X) = \frac{1}{7776} + \frac{31}{3888} + \frac{211}{2592} + \frac{781}{1944} + \frac{10505}{7776} + \frac{4651}{1296}$$

$$\boxed{E(X) = \frac{14077}{2592}}$$

∴ Its proven that the expected value (E(X)) of the random variable X equals $\boxed{\frac{14077}{2592}}$.

Question 6:

$Y = \min(A)$ $Z = \max(A)$ $X = Z - y$

- first we must find the range of X.
  - We must find the range of Z & Y first
    - ☐ Range of Z: 0 to n
    - ☐ Range of Y: 0 to n

    ] both Z & Y can be any value from n to ∅, where ∅ results in a 0.
  - If we flip heads, an element ends up in the subset, the probability that any single element gets added is 1/2.
  - This can be modeled by the linearity of expectations:
    - ☐ We will flip tails up till the first heads flipped
    - ☐ The # of time the Alg. is run equals that value:
      - Let K = i where i is the iteration # of n in the Algorithm.
      - Let X = 1/2, this is the chance of rolling a head (or tail) on a fair coin
      - Let n = n (from the Alg.) where n is the length of the list being iterated.
      - Plug these values into the formula for linearity:

☐ $$\sum_{K=1}^{n} K \cdot X^K = \frac{X(n \cdot X^{n+1} - (n+1) \cdot X^n + 1)}{(X-1)^2}$$

- This is the min

- $$\sum_{K=1}^{n} K \cdot (1/2)^K = \frac{(1/2)(n \cdot (1/2)^{n+1} - (n+1)(1/2)^n + 1)}{(1/2 - 1)^2}$$

$$= \frac{(1/2)((n)(1/2)^{n+1}) - (1/2)(n+1)(1/2)^n + 1/2}{(-1/2)^2}$$

min(A) = y

$$= \frac{(1/2)((n)(1/2)^{n+1}) - (1/2)(n+1)(1/2)^n + 1/2}{1/4}$$

$$= 2(n)(1/2)^{n+1} - (2)(n+1)(1/2)^n + 2$$

$\therefore$ min(A) = $y = 2(n)(1/2)^{n+1} - (2)(n+1)(1/2)^n + 2$

- To find the max(A):

☐ We Count from back to front to find first head fliped: H T T T

H	T	T	T
5	6	7	8
K			n

→ ☐ to find index Difference we must find: 4 where n=8 (the length) & K=5 (the index of the head).

☐ H is on index of 5

☐ The Space between n & K is ④

☐ $\therefore n - K + 1 = 5$ where: $8 - 5 + 1 = 3 + 1 = ④$

☐ $$\sum_{K=1}^{n} K \cdot (1/2)^{n-K+1} = \sum_{K=1}^{n} K \cdot (1/2)^n \cdot (1/2)^{-K} \cdot (1/2)^1$$

rule = $x^{-k} = \left(\frac{1}{x}\right)^k$

$$= (1/2)^n (1/2) \sum_{K=1}^{n} K \cdot (1/2)^{-K}$$

$$= (1/2)^n (1/2) \sum_{K=1}^{n} K \cdot (2)^K$$

$$(1/2)^n (1/2) \sum_{K=1}^{n} K \cdot (2)^K = \left(\frac{(2)(n \cdot (2)^{n+1} - (n+1)(2)^n + 1)}{(2-1)^2}\right)(1/2)^n (1/2)$$

max(A) = Z

$$= ((2)(n \cdot \cancel{(2)^{n+1}} - (n+1)(2)^{\cancel{n}} + 1)\cancel{(1/2)^{n+1}}$$

$$= 2n - \cancel{(2)(1/2)}(n+1) + (2)(1/2)^{n+1}$$

$$= 2n - (n+1) + (2)(1/2)^{n+1}$$

$$= n - 1 + (2)(1/2)^{n+1}$$

$\therefore$ Max(A) = $Z = n - 1 + (2)(1/2)^{n+1}$

- $X = Z - Y$

$$= \left(n-1+(2)(1/2)^{n+1}\right) - \left(\left(2(n)(1/2)^{n+1}\right) - (2)(n+1)(1/2)^{n} + 2\right)$$

$$= n-1+(2)(1/2)^{n+1} - (2)(n)(1/2)^{n+1} + (2)(n+1)(1/2)^{n} - 2$$

$$= n-3+2\left((1/2)^{n+1} - (n)(1/2)^{n+1} + (n+1)(1/2)^{n}\right)$$

$$= n-3+(2)(1/2)^{n+1}\left(1-n+(n+1)(1/2)^{-1}\right)$$

$$= n-3+(2)(1/2)^{n+1}\left(1-n+(2)(n+1)\right)$$

$$= n-3+(2)(1/2)^{n+1}(n+3)$$

$$= n-3+(2)(2)^{-(n+1)}(n+3) \qquad \left[\text{rule} = x^{-k} = \left(\tfrac{1}{x}\right)^{k}\right]$$

$$= n-3+2^{-n-\cancel{1}+\cancel{1}}(n+3)$$

$$= n-3+2^{-n}(n+3) \qquad \left[\text{rule} = x^{-k} = (1/x)^{k}\right]$$

$$X = n-3+(1/2)^{n}(n+3)$$

$$\boxed{\therefore\ E(X) = n-3+\delta(n) \quad \text{Where} \quad \delta(n) = (1/2)^{n}(n+3)}$$

It is proven that $E(X) = n-3+\delta(n)$ Satisfies the random variable X

— $E(X)$ approximetly equals $n-3$ because as $n \to +\infty$, whereas all values of n are positive (by definition), the result is $\delta(n) \to 0$. Thus, as n reaches infinity, $\delta(n)$ reaches 0, & $\therefore E(X) = n-3+0 = \boxed{n-3}$.

- $\delta(n) \to 0$ as $n \to +\infty$: (Example)

$$\delta(n) = (1/2)^{n}(n+3)$$

$$\delta(1) = (1/2)^{1}(1+3) = (1/2)(4) = 2$$

$$\delta(100) = (1/2)^{100}(100+3)$$

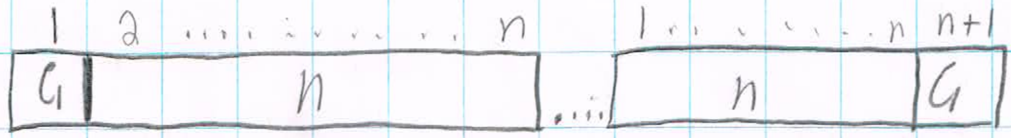
$$= (7.888 \cdot 10^{-31})(103)$$

$$= 8.125 \cdot 10^{-27}$$ (this is insignificant to $n-3$ $[100-3=97]$)

$$\delta(1000) = (1/2)^{1000}(1000+3) \doteq 0$$ (its now unmeasurable)

Question 7:

→ Determine the possible values for x:



☐ As can be seen, the first (& only) cider bottle can be anywhere from the first to the last place on a shelf.

☐ This would mean that the placments of the first cider bottle is anywhere from the first index to 1 place after all the beer bottles.

— ∴ The possible values for x are: $\boxed{1, 2, 3, \ldots, n, n+1}$

→ For any value K that X can take, Prove: $Pr(X=K) = \frac{m}{K} \cdot \frac{\binom{n}{K-1}}{\binom{m+n}{K}}$

$Pr(X=K) =$

- K = any value X can take

this series of events can be modeled by the product rule.

- Choose leftmost cider bottle:

  There's n beer bottles & K−1 spots for beer bottles before the leftmost cider bottle: $\binom{n}{K-1}$

- There are $(K-1)!$ ways of arranging the beer bottles, left of the leftmost cider bottle.
- We must now pick a cider bottle for position K, this is m (the # of cider bottles).
- Now we must find the # of beer & cider bottles to the right of the leftmost cider bottle; this is $(m+n-K)!$, the # of permutations for the remaining bottles.
- We must also divide by the sample space, which is the # of ways to arrange n beer & m cider bottles; this value is $(m+n)!$

- $\dfrac{m\binom{n}{k-1}(k-1)!(m+n-k)}{(m+n)!}$ Models this problem given the above

- to prove this is equal to $\dfrac{m}{K} \cdot \dfrac{\binom{m}{k-1}}{\binom{m+n}{K}}$ we must use binomial coessisients:

$$\frac{m\binom{n}{n-1}(m+n-k)!(k-1)!}{(m+n)!}$$

$$= m\binom{n}{k-1}\frac{(m+n-k)!(k-1)!}{(m+h)!}$$

$$= m\binom{n}{k-1}\frac{\frac{1}{(m+n)!}}{(m+n-k)!} = n\binom{n}{k-1}\left(1/\left(\cancel{K}\left(\frac{m+n!}{(m+n-k)!\cancel{K}(k-1)!}\right)\right)\right)$$

$$= m\binom{n}{k-1} \cdot 1/\left(K\left(\frac{(m+n)!}{(m+n-k)!k!}\right)\right)$$

$$= m\binom{n}{k-1}\left(1/K\binom{m+n}{K}\right)$$

$$= \left(\frac{m}{K}\right) \cdot \binom{n}{k-1} / \binom{m+n}{k}$$

$$= \frac{m}{K} \cdot \frac{\binom{n}{k-1}}{\binom{m+n}{k}}$$

$Pr(X=K) = \dfrac{m}{K} \cdot \dfrac{\binom{n}{k-1}}{\binom{m+n}{k}}$, $\therefore$ Its proven for any Value K that X can take.

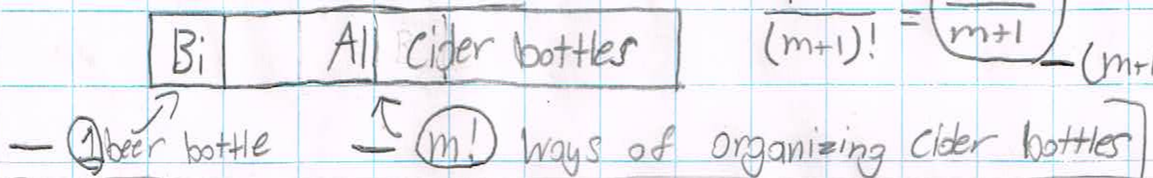
- For each $i = 1, 2, \ldots, n$, define the indicator random variable

$$X_i = \begin{cases} 1 & \text{if } B_i \text{ is to the left of all cider bottles} \\ 0 & \text{otherwise} \end{cases}$$

Prove that: $E(X_i) = \frac{1}{m+1}$

There are $m+n$ cider/beer bottles with bottle B_i. If B_i is the only beer bottle in the sequence then $\underline{m+1}$ bottles in the sequence, the number of all cider bottles plus 1. Thus, the expected value of having that (B_i) beer bottle left of the other (cider) bottles is $\boxed{E(X_i) = \left(\frac{1}{m+1}\right)}$. Theres only $\underline{1}$ beer bottle & $\underline{m+1}$ others,

Diagram:



$$\frac{m!}{(m+1)!} = \boxed{\frac{1}{m+1}}$$

— $(m+1)!$ ways of organizing cider plus 1 beer bottle.

- Express X in terms of $X_1, X_2, \ldots, X_n$:

This can be represented as a summation series:

$$X = \sum_{i=1}^{n} X_i + 1$$

(Note: the +1 outside the sum accounts for the begining cider bottle)

- Use the above expression to determine $E(X)$:

$$E(X) = \sum_{i=1}^{n} E(X_i) + 1 \longrightarrow \left(\sum_{i=1}^{n} \frac{1}{m+1}\right) + 1 = 1 + \frac{n}{m+1} = \frac{n+m+1}{m+1}$$

$$\boxed{E(X) = \frac{n+m+1}{m+1}}$$

- Prove: $\sum_{k=1}^{n+1} \frac{\binom{n}{k-1}}{\binom{m+n}{k}} = \frac{m+n+1}{m(m+1)}$

$$E(X) = \sum_{k=1}^{n+1} k \Pr(X=k) = \sum_{k=1}^{n+1} k \cdot \frac{m}{k} \cdot \frac{\binom{n}{k-1}}{\binom{m+n}{k}} = \sum_{k=1}^{n+1} \frac{m\binom{n}{k-1}}{\binom{m+n}{k}}$$

$$F(X) = \frac{m \sum \frac{\binom{n}{k+1}}{\binom{m+n}{k}}}{m} = \frac{\left(\frac{m+m+1}{m+1}\right)}{m} = \frac{n+m+1}{m(m+1)}$$

Question 8:

- $X =$ # times Die is Rolled
- Roll till 1-6 is landed at least once
- Determine $E(X)$ of the Random variable:
  - First Lets Assign indicator values to each Set of Rolls:

    Let $R_1 =$ # of rolls till a new #'s encountered
  - We must Roll Untill all 6 Numbers have been met, thus theres 6 Stages.
    - ☐ R_0 is the first # Seen thus, $R_0 = 1$.
    - ☐ $R_i =$ # Rolls during Stage i
  - $X = \sum_{i=0}^{5} R_i$
- by Stage i, you have Seen i elements, thus 6-i are left to be Seen.
- Pr(One Roll) = 1, we will find a new element on the first roll.

  The probability of finding a new element is: $\frac{d-i}{s}$

  When $s =$ Sample Space $= 6$, $d =$ elements not seen $= 6$, i is the Current Stage $= 0$ $\therefore$ $\frac{6-0}{6} = 1$ for first Roll.
  - $Ex(R_i) = \frac{1}{Pr} = \frac{6}{6-i} = 1$
- $Ex(X) = E\left(\sum_{i=0}^{5} R_i\right) = \sum_{i=0}^{5} E(R_i)$

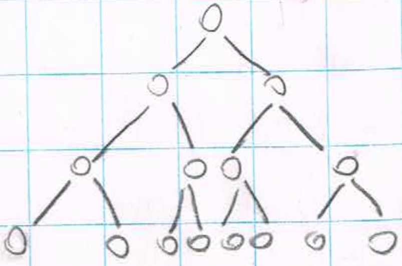
$$= \frac{6}{6-0} + \frac{6}{6-1} + \frac{6}{6-2} + \frac{6}{6-3} + \frac{6}{6-4} + \frac{6}{6-5}$$

$$= \frac{6}{6} + \frac{6}{5} + \frac{6}{4} + \frac{6}{3} + \frac{6}{2} + \frac{6}{1}$$

$$\boxed{= 14.7}$$

Question 9: $0 \leq l \leq K$

- Let n be # of nodes in tree T, Express n in terms of K:
  - X_l = # of nodes in the new graph that are still Connected to the root. $X = \sum_{l=0}^{K} X_l$ = # of nodes Connected to T in T'.
  - If a Section is removed, both the node & its children are no longer Connected to the root.



$2^0 = 1$

$2^1 = 2$

$2^2 = 4$

$2^3 = 8$

Induce:

At level m you have 2^m nodes:

n = # of nodes at level 0 + # Nodes at level 1 + ... + # Nodes level K.

$$= 2^0 + 2^1 + 2^2 + ... + 2^K = \boxed{\sum_{i=0}^{K} 2^i}$$

$$= 2^{K+1} - 1$$

- Prove E(X) of the random variable X equals:

for $i = 1, 2, 3, ..., 2l$

$$X = \begin{cases} 1 & \text{if its Connected} \\ 0 & \text{if it is not} \end{cases}$$

$$X_l = \sum_{i=1}^{l} X_i$$

$$E(X_i) = Pr(X_i = 1)$$
$$= Pr(\text{all edges Survived})$$
$$= (1/2)^l$$

$$E(X_l) = E\left(\sum_{i=1}^{l} X_i\right) = \sum_{i=1}^{2^l} E(X_i) = \sum_{i=1}^{2^l} (1/2)^l$$

If $l=2$: $(1/2)^2 + (1/2)^2 + (1/2)^2 + (1/2)^2 = 1$

$E(X_l) = 1$

$$E(X) = E\left(\sum_{l=0}^{K} X_l\right) = \sum_{l=0}^{K} E(X_l) = \sum_{l=0}^{K} (1) = K+1 = \boxed{\log(n+1)}$$

$n+1 = 2^{K+1} \longrightarrow n = 2^{K+1} - 1$ ∴ $\boxed{\log(n+1)}$

Example: $l = 2$

$1/2 \cdot 1/2 = 1/4$

Sub form:

$= \log((2^{K+1} - 1) + 1)$

$= \log(2^{K+1}) = \boxed{K+1}$

$n+1 = 2^{K+1}$

∴ $\boxed{\log(n+1)}$