4) Find the area of the inner loop of the Cardioid: $r=1-\sqrt{2}\sin\theta$

Sketch:

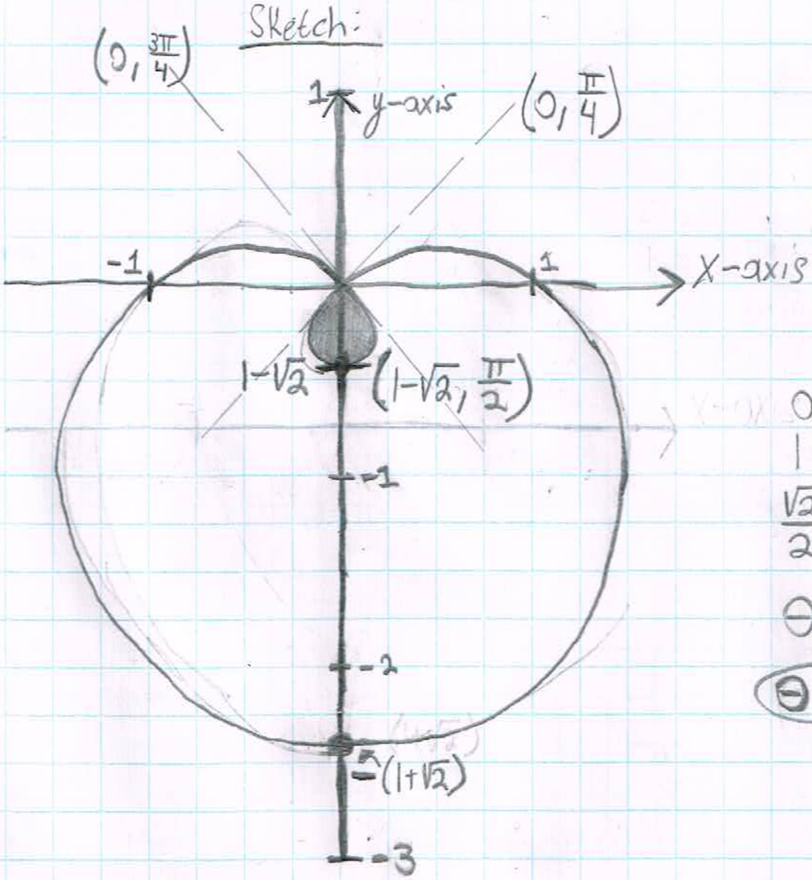


table of values:

| $\theta$ | $r$ |
|---|---|
| 0 | 1 |
| $\pi/2$ | $1-\sqrt{2}$ |
| $\pi$ | 1 |
| $3\pi/2$ | $1+\sqrt{2}$ |
| $2\pi$ | 1 |

$0=1-\sqrt{2}\sin\theta$

$1=\sqrt{2}\sin\theta$

$\frac{\sqrt{2}}{2}=\sin\theta$

$\theta=\mathrm{ArcSin}\left(\frac{\sqrt{2}}{2}\right)$

$\theta=\pi/4$ OR $\theta=3\pi/4$

to find the Area, we take the integration of $\theta$ from $\pi/4$ to $3\pi/4$ via the following:

$$A=\int_{\alpha}^{B}\frac{1}{2}r^2\,d\theta \quad \text{Where } r=1-\sqrt{2}\sin\theta,\ B=\frac{3\pi}{4},\ \&\ \alpha=\frac{\pi}{4}$$

thus: plug in the values

$$A=\int_{\pi/4}^{3\pi/4}\left(\frac{1}{2}\right)(1-\sqrt{2}\sin\theta)^2\,d\theta$$

$\frac{B-\alpha}{2}=\frac{\pi}{4}$

$\alpha+\frac{\pi}{4}=\frac{\pi}{2}$

$\therefore$ the new $B=\pi/2$

We know that the inner loop of the Cardioid (or in this case a limaçon) is symmetrical, thus the area of 2 times the integral over half the range $\theta$ is the same as the above equation.

$$\therefore A=\cancel{2}\int_{\pi/4}^{\pi/2}\cancel{\frac{1}{2}}(1-\sqrt{2}\sin\theta)^2\,d\theta$$

$$=\int_{\pi/4}^{\pi/2}(1-\sqrt{2}\sin\theta)^2\,d\theta=\int_{\pi/4}^{\pi/2}1-2\sqrt{2}\sin\theta+\sqrt{2}^2\sin^2\theta\,d\theta$$

$$=\int_{\pi/4}^{\pi/2}1-2\sqrt{2}\sin\theta+2\sin^2\theta\,d\theta=\Big[\theta+2\sqrt{2}\cos\theta\Big]_{\pi/4}^{\pi/2}+2\int_{\pi/4}^{\pi/2}\sin^2\theta\,d\theta$$

$$=\Big[\theta+2\sqrt{2}\cos\theta\Big]_{\pi/4}^{\pi/2}+\cancel{2}\int_{\pi/4}^{\pi/2}\cancel{\frac{1}{2}}(1-\cos(2\theta))\,d\theta=\Big[\theta+2\sqrt{2}\cos\theta+\theta-\frac{1}{2}\sin 2\theta\Big]_{\pi/4}^{\pi/2}$$

$$=\left(\pi+\cancel{2\sqrt{2}\cos(\tfrac{\pi}{2})}-\cancel{\tfrac{1}{2}\sin(\pi)}\right)-\left(\frac{\pi}{2}+2\sqrt{2}\cos\left(\frac{\pi}{4}\right)-\frac{1}{2}\sin\left(\frac{\pi}{2}\right)\right)=\pi-\pi/2-2+\frac{1}{2}=\frac{\pi}{2}-\frac{3}{2}=\boxed{\frac{\pi-3}{2}}$$

thus: the area of the inner loop of the Cardioid is

$$A=\frac{\pi-3}{2}$$ [or, approx. $\approx 0.070796$]