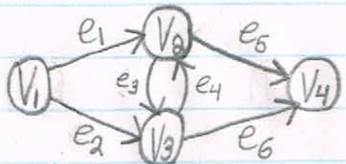
Name: Connor Raymond Stewart
ID: 101041125

MATH 3802 Assignment #3:

Throughout this assignment $G = (N, A)$ denotes the digraph depicted in the figure below:



Let $b = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 0 \end{bmatrix}$ & $c = \begin{bmatrix} 0 \\ 5 \\ 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$

① (2 Points) Give the node-arc incidence matrix of G:

Note that:

$$a_{ij} = \begin{cases} 1 & \text{if } V_i \text{ is the tail of } e_j \\ -1 & \text{if } V_i \text{ is the head of } e_j \\ 0 & \text{otherwise} \end{cases}$$

thus, the node-arc incidence matrix is:

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & -1 & 1 & 0 \\ 0 & -1 & -1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & -1 \end{bmatrix}$$

② We Consider the Minimum-Cost Flow Problem on G, b, c:

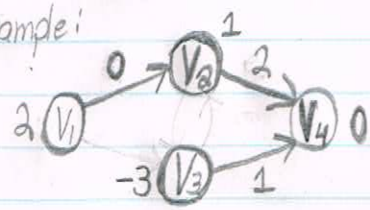
a) (1 Point) Give a tree solution that is not a feasible tree solution:

- A tree solution is feasible if & only if it has a feasible flow
  - ↳ Find a Sol^{n} without a feasible flow:
    - To connect the tree we can get $e_1 + e_5 + e_6$ such that: $V_1 e_1 V_2 e_5 V_4 e_6 V_3$
    - Not only is this a minimum tree since e_1, e_5, e_6 are the 3-lowest edges in C, the flow's also not feasible
    - the number of arcs in a tree is $n-1$, thus $4-1=3$ (So our Sol^{n} is valid)

$$X^* = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 3 \\ -3 \end{bmatrix}$$

thus choose paths $\{e_1, e_5, e_6\}$ such that $\{V_1 e_1 V_2 e_5 V_4 e_6 V_3\}$

- Example:



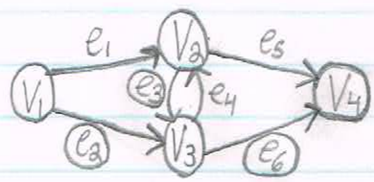
$$x^* = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 3 \\ -3 \end{bmatrix}$$

$x_i \ngeq 0$ for all i from 1 to 6
↳ not feasible flow as negative flow occurs

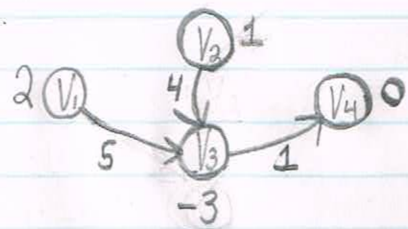
thus, we can see that the flow can follow the flow conversion constraints the flow from the vertexes equals their supply, but it's infeasible since not all elements in x^* are ≥ 0 (namely, $x_6^* = -3 \ngeq 0$)

b) (1 Point) Show that the tree with arc-set $T = \{e_2, e_3, e_6\}$ determines a feasible tree Solution:

- A tree has is feasible if & only if it has a feasible flow
- A tree must have $n-1$ arcs
- Example of flow T:



thus, we see:



$$\Rightarrow x^* = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow x_i \geq 0 \text{ for all } i \text{ from } 1 \text{ to } 6 \Rightarrow \therefore \text{feasible flow}$$

x^* Works Since V_3 - Sinks 3 units, with V_1 & V_2 Supplying 2 units & 1 unit respectivly.

thus;

the tree has $n-1=3$ arcs $\therefore$ It's a valid tree

the tree has only 1 arc between any two nodes $\therefore$ Valid

the tree Satisfies the flow Conversion Constraints Since the net flow out of nodes V_1 & V_2 equals the amount of Supply at those nodes. Since this results in V_3 having a total of $-3+2+1=0$, the flow from V_3 to V_4 is valid since V_4 requires nothing as input.

$\therefore$ It is a feasible tree Solution

c) (6 Points) Solve the Minimum-Cost Flow Problem by applying the network Simplex method Starting at the feasible tree Solution in part b. Whenever there is a Choice for the entering or leaving arc, choose the one with the Smallest index:

I ① $T=\{e_2, e_3, e_6\}$ & $N=\{e_1, e_2, e_4, e_5\}$ & $N=A/T$

② $y_n = y_4 = 0$ $\qquad x^* = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

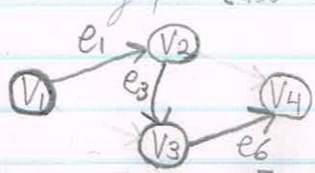
e_6	$y_3 - y_4 = 1 \rightarrow y_3 - 0 = 1 \rightarrow y_3 = 1$
e_3	$y_2 - y_3 = 4 \rightarrow y_2 = 5$
e_2	$y_1 - y_3 = 5 \rightarrow y_1 = 6$

③ $K=1$: $y_1 - y_2 = 6-5 = 1 > 0 = c_1$ thus ④ $C=\{e_1, e_2, e_3\}$ ⑤ C is NOT a dicycle

Add e_1 to form a cycle

⑥ $R=\{e_2, e_3\} \Rightarrow \theta = \min\{2, 3\} = 2 \therefore r=2$

Draw new graph: (add e_1 & drop e_3)



⑦ New $x^* = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

⑧ New $T=\{e_1, e_3, e_6\}$

II ① $T=\{e_1, e_3, e_6\}$ & $N=\{e_2, e_4, e_5\}$ & $N=A/T$

② $y_n = y_4 = 0$

e_6	$y_3 - y_4 = 1 \rightarrow y_3 = 1$
e_3	$y_2 - y_3 = 4 \rightarrow y_2 = 5$
e_1	$y_1 - y_2 = 0 \rightarrow y_1 = 5$

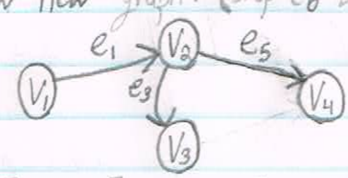
③ $K \neq 2$: $y_1 - y_3 = 5-1 = 4 \not> 5$, $K \neq 4$: $y_3 - y_2 = 1-5 = -4 \not> 3$

$K=5$: $y_2 - y_4 = 5-0 = 5 > 2$ thus ④ $C=\{e_3, e_5, e_6\}$ ⑤ C is NOT a dicycle

Add e_5 to form a cycle

⑥ $R=\{e_3, e_6\} \Rightarrow \theta = \min\{3, 0\} = 0 \therefore r=6$

Draw new graph: (drop e_6 & add e_5)



⑦ New $x^* = \begin{bmatrix} 2 \\ 0 \\ 3 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

⑧ New $T=\{e_1, e_3, e_6\}$

III ① $T=\{e_1, e_3, e_5\}$ & $N=\{e_2, e_4, e_6\}$ & $N=A/T$

② $y_n = y_4 = 0$

e_5	$y_2 - y_4 = 2 \rightarrow y_2 = 2$
e_3	$y_2 - y_3 = 4 \rightarrow y_3 = y_2 - 4 = -2$
e_1	$y_1 - y_2 = 0 \rightarrow y_1 = y_2 \rightarrow y_1 = 2$

③ Since:

$K \neq 2$: $y_1 - y_3 = 2-(-2) = 4 < 5 = c_2$

$K \neq 4$: $y_3 - y_2 = -2-2 = -4 < 3 = c_4$

$K=6$: $y_3 - y_4 = -2-0 = -2 \leq 1 = c_6$

We know $x^* = [2\ 0\ 3\ 0\ 0\ 0]^T$ is a minimum-cost feasible flow.

thus, the min-cost is 12!

$c^T x = \cancel{(0)(2)} + \cancel{(5)(0)} + (4)(3) + \cancel{(3)(0)} + \cancel{(2)(0)} + \cancel{(1)(0)} = \boxed{12}$