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MATH 3101 Assignment 5:

① let $Q^* \times \mathbb{Z}_5^* := \{(x,[y]) \mid x \in Q^*, [y] \in \mathbb{Z}_5^*\}$. Define equality & multiplication in $Q^* \times \mathbb{Z}_5^*$ as follows:

$(x,[y]) = (x',[y'])$ if and only if $x=x'$ and $[y]=[y']$

$(x,[y]) \cdot (w,[z]) := (xw, [y][z])$

Show that $Q^* \times \mathbb{Z}_5^*$ is a group:

Note: I to IV are the four conditions for a group.

I) G is closed under $*$. That is, $x \in G$ & $y \in G$ imply that $x * y$ is in G:

$Q^* \times \mathbb{Z}_5^*$ clearly has the binary operation of multiplication the integers & rationals are both closed under multiplication, thus for all $x,y \in G$, $x \times y \in G$ where G is $Q^* \times \mathbb{Z}_5^*$

thus, closure is true

II) $*$ is Associative. For all x,y,z in G, $x*(y*z) = (x*y)*z$:

let $(a,[b]), (c,[d]), (e,[f]) \in Q^* \times \mathbb{Z}_5^*$, then:

$((a,[b]) \times (c,[d])) \times (e,[f])$
$= (ac, [b][d]) \times (e,[f])$ [definition of multiplication in group]
$= (ace, [b][d][f])$ [definition of multiplication in group]
$= (a \times ce, [b] \times [d][f])$ [Associativity in both Q & $\mathbb{Z}_5$]
$= (a,[b]) \times (ce, [d][f])$ [definition of multiplication in group]
$= (a,[b]) \times ((c,[d]) \times (e,[f]))$ [definition of multiplication in group]

thus, associativity is true

III) G has identity element e. There is an e in G such that $x*e = e*y = x$ for all $x \in G$:

let $e = (1,[1])$ then for any $(a,[b]) \in Q^* \times \mathbb{Z}_5^*$,

$(a,[b]) \times (1,[1]) = (a(1), [b][1]) = (a,[b]) = ((1)a, [b][1]) = (1,[1]) \times (a,[b])$

↳ operations made under the definition of multiplication in group

thus, identities existence is true

IV) G contains inverses. For each $a \in G$, there exists $b \in G$ such that $a*b = b*a = e$:

for $(x,[y])$ we note that, for $a,b \in Q$:

x has inverse: $a \times b = b \times a = 1 \rightarrow b = 1/a$ thus, since $1/a \in Q$ each a has an inverse b where b is $1/a$

We also note that, for $[a],[b] \in \mathbb{Z}_5^*$:

remove zero as $(0,5)=5$

$[y]$ has inverse: $[a] \times [b] = [b] * [a] = [1][a]^{-1} \rightarrow \mathbb{Z}_5^* = \{\cancel{[0]}, [1], [2], [3], [4]\}$

for $\{[1],[2],[3],[4]\} = a$, the multiplicative inverse $[a]^{-1}$ exists since $(a,5)=1$

thus, there's an inverse where $[a]^{-1}$ is the inverse

thus, $(x,[y])$ clearly has an inverse in the group

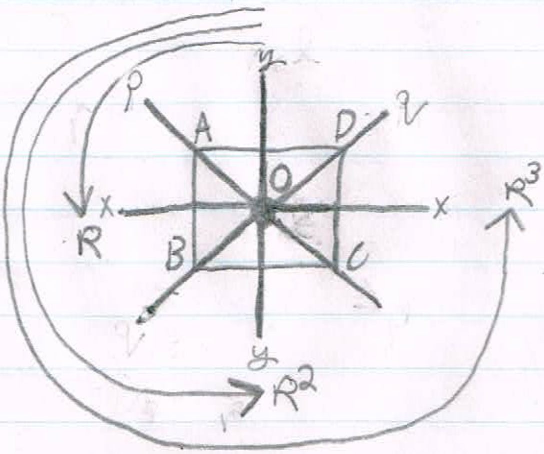
thus, the inverses existence is true

Thus, by the definition of groups, $Q^* \times \mathbb{Z}_5^*$ is a group

② Write out the elements of D_4, & demonstrate that D_4 is non-Abelian:

D_4 is the dihedral group, a Subgroup of S_4

We Start with a Sketch:



let x,y – Axis'

let O – Centroid

let $\Delta ABCD$ – Square

let p,q – Reflection lines

Mappings:

- R, R^2, R^3 are $90°, 180°, 270°$ rotations Counterclockwise about the Centroid O
- ε is the identity mapping
- let L_K be the reflections about the line l_K for $K = p, y, q, x$

$\therefore D_4 = \{\varepsilon, R, R^2, R^3, L_y, L_yR, L_yR^2, L_yR^3\}$

the Reflections in D_4 are:

$L_A = (y,x)(y,x)$

$L_B = (x,y)(x,y)$

$L_C = (y,x)(y,x)$

$L_D = (x,y)(x,y)$

So:

$L_AR = (x,y)(x,y) = L_D$

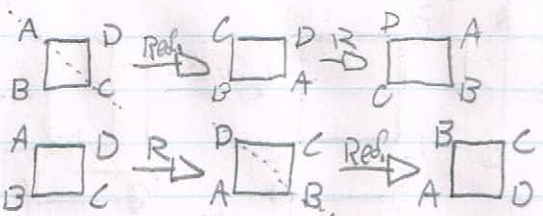
$L_AR^2 = (y,x)(y,x) = L_C$

$L_AR^3 = (x,y)(x,y) = L_B$

element K increases by 3.

So, Symmetry occurs along the x, y, q, p lines as can be seen above.

To Show that D_4 is non-Abelian, we must find a Single pair of elements Which don't Commute:



if b is p (the reflection line from A to C) then we See that $Rb \neq bR$, thus the operation isn't Commutative.

$\therefore D_4$ is Non-Abelian

③ Complete the Cayley table:

	Col	1	2	3	4
Row	•	a	b	c	d
1	a	b	a	d	c
2	b	a	b	c	d
3	c	d	c	b	a
4	d	c	d	a	b

We Note that:

$b \cdot a = a$ thus b is the identity

$\therefore a \cdot b = a$ onto the figure, Solves Row 1 col. 2

Note:

— A Row/Column must have exactly One copy for each of a, b, c, d

— Row one has a, d So it can have either b or c in Row one, Column 4. We see Column 4 already has a b. there cannot be two b's in Col 4 So Row 1 Col. 4 must be c, as the only two options were b or c for that spot & b is taken by the Col.

$\therefore$ Row 1, Col 4 must be either b or c it cannot be b, thus it's c

— Now, Row 1 col. 1 Can only be b since it is the only letter not in the row.

Row 2, Col. 4:

Row two has an a

Col 4 has a, b & c

↳ a, b, c are taken

$\therefore$ use d

Row 3, Col 4:

c, d, & b are taken

↳ $\therefore$ use a

Note:

Row 2 col. 4 shows:

$b \cdot d = d$ So b is identity

↳ $\therefore d \cdot b = d$

this Solves Row 4, Col. 2

Row 4, Col 1:

Row 4 has d & b so a & c are left

Col. 1 has a, so Row 4 Col 1 must be c

↳ $\therefore$ use c

Row 4, Col 3:

c, b, & d are used

↳ $\therefore$ use a

Row 3, col 1:

a, b, c are used

↳ $\therefore$ use d

Rows 2 & 3 Columns 2 & 3:

Can be Solved by Plugging in b to Row 2 Col 2 & c in Row 2 Col 3. Next, Row 3 col 2 must be c & Row 3 Col 3 must be b.

↳ This lets all Col./rows have exactly one copy of a, b, c, d

↳ by letting $c \cdot b = c$ & $b \cdot b = b$, we let all elements have identities with b.

Remarks:

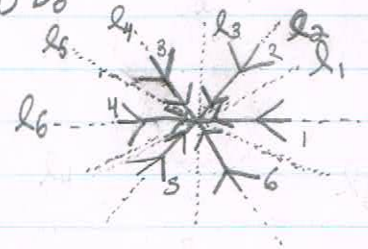
Tables Complete Since:

- All elements have identity element b
- Every element has an inverse
- Operation is associative

↳ Thus, there is an Abelian Group in the table

④ For Parts a, b, & c, determine the Symmetry groups of the given Plane figures:

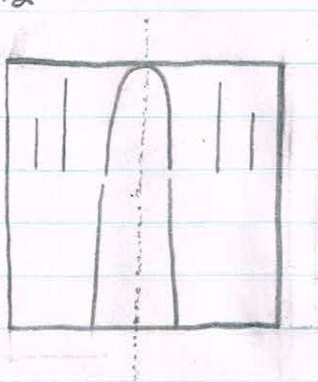
a) D_6



$\Rightarrow$ Both Reflexive & rotational Symmetry thus, this is the Same Symmetry as a hexagon therefore, the Symmetry is a dihedral group of 6 points.

$\hookrightarrow$ $\therefore D_6$

b) R_2



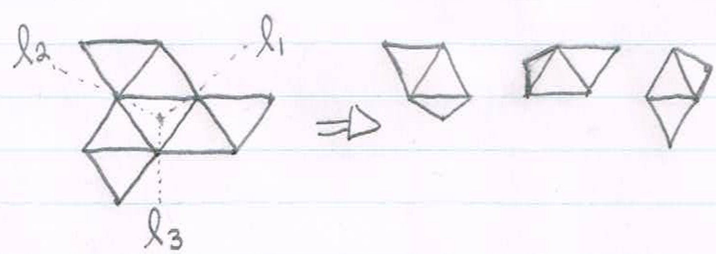
this has Reflexive Symmetry

However, if rotated the image will be inverted

there is mirror Symmetry thus $n=2$

$\hookrightarrow$ $\therefore R_2$

c) C_3



this has rotational Symmetry

this lacks reflexive Symmetry

thus, this is a Cyclic group.

there are three cycle elements.

$\hookrightarrow$ $\therefore C_3$

d) Compute $|\mathbb{Z}^*_{19}|$ & $|\mathbb{Z}^*_{21}|$:

$\underline{|\mathbb{Z}^*_{19}| = 18}$

Since 19 is Prime, every non-zero positive integer less then 19 is relatively Prime to 19 $\therefore |\mathbb{Z}^*_{19}| = 19-1 = 18$

$\underline{|\mathbb{Z}^*_{21}| = 13}$

$\mathbb{Z}^*_{21} = \{[a] \mid 1 \le a < 21,\ (a,21)=1\}$ $\rightarrow$ look for numbers between 1 & 20 that aren't divisible by 3 or by 7.

$= \{[1],[2],[4],[5],[6],[8],[10],$

$[11],[13],[16],[17],[19],[20]\}$ $\rightarrow$ 13 elements

$\hookrightarrow$ $\therefore |\mathbb{Z}^*_{21}| = \boxed{13}$

e) Name a group that is non-abelian & that's not a dihedral group:

the group S_3 is non-abelian Since $(1,3)(2,3)=(1,3,2)$

yet $(2,3)(1,3)=(1,2,3)$.

$\therefore S_3$

⑤ Determine whether the following Statements are true or false. Justify your responses:

a) let G be a group over multiplication & let $x,y,z\in G$.

Then $(xyz)^{-1} \overset{?}{=} x^{-1}y^{-1}z^{-1}$:

$(xyz)^{-1}=(x)^{-1}(yz)^{-1}=x^{-1}y^{-1}z^{-1}$ by the law $(xy)^n=x^ny^n$

However, according to the law of exponents:

If G is abelian, $(xy)^n=x^ny^n$

thus, let G be a non-abelian group

$(xyz)^{-1}=x^{-1}y^{-1}z^{-1}$ if & only if G is abelian

G is not Abelian

$\therefore$ It is false by Contradiction

$\therefore$ False

b) The Set of all nonzero elements of $\mathbb{Z}_8$ is an Abelian group with respect to multiplication:

test for property 4 of groups:

G has an inverse element for each $a\in G$ there's a $b\in G$ Such that $a*b=b*a=e$

$\mathbb{Z}_8$ without Zero elements is $=\{[1],[2],\dots,[6],[7]\}$

let $a_1=[3]$ & $a_2=[7]$:

$$\left.\begin{array}{l} b[3]=[3]b=e \to 3b=e \\ b[7]=[7]b=e \to 7b=e \end{array}\right\} \begin{array}{l} 3b=7b \to 3\neq 7 \\ \text{thus, } e \text{ does not exist} \end{array}$$

$\therefore$ False, theres no identity element in $\mathbb{Z}_8^{\oplus}$ So $\mathbb{Z}_8^{\oplus}\notin G$

$\therefore$ False

c) The identity element in a group G is its own inverse:

let e be the identity of group G

let $a=e^{-1}$, then Since a is the inverse we see:

$a*e=e^{-1}*e=e$ by Condition four of Groups

Since for $a,b\in G$ $a*b=b*a=e$ then if $a=e$ & $b=e^{-1}$

we note $e*e^{-1}=e^{-1}*e=e$, Since $e\in G$

$\therefore$ True by the above

$\therefore$ True

d) let G be a nonAbelian group. Then $xy \neq yx$ for all $x, y \in G$:

let G be a non-Abelian group

let $x, y \in G$

let e be the identity element of G

then by Condition three of groups, there's a $e \in G$ Such that

$x * e = e * x = x$ for all $x \in G$

let $x = e$ & $y \in G$:

$x * y = y * x$ by the above

Since $xy = yx = x$ then it cannot be the case that all $xy \neq yx$ for all $x, y \in G$.

∴ False by Contradiction

∴ False