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MATH 365
Assignment Two:

1) let $r(t) = e^t i - \ln(t) j + t^2 K$. Give the following derivative:

$$\frac{d}{dt}\left[r \cdot (r' \times r'')\right]$$

in two different ways:

a) Apply the rules of Theorem 12.2.6 & (7) (pg. 851–852) of the textbook:
- We are Allowed to use (6) as was instructed by the tutor (pg. 852)

Using (6):

$$(6): \frac{d}{dt}\left[r_1(t) \cdot r_2(t)\right] = r_1(t) \cdot \frac{dr_2}{dt} + \frac{dr_1}{dt} \cdot r_2(t)$$

$\to$ So:

$$\frac{d}{dt}\left[r \cdot (r' \times r'')\right] = r(t) \cdot \frac{d(r' \times r'')}{dt} + \frac{dr}{dt} \cdot (r' \times r'')$$

$$= r(t) \cdot \frac{d}{dt}\left[r' \times r''\right] + r'(t) \cdot (r' \times r'')$$

Using (7):

$$\frac{d}{dt}\left[r_1(t) \times r_2(t)\right] = r_1(t) \times \frac{dr_2}{dt} + \frac{dr_1}{dt} \times r_2(t)$$

So:

$$= r(t) \cdot \left(r'(t) \times r'''(t) + r''(t) \times r''(t)\right) + r'(t) \cdot (r'(t) \times r''(t))$$

Using $u \cdot (v + w) = u \cdot v + u \cdot w$:

$$= r(t) \cdot (r'(t) \times r'''(t)) + r(t) \cdot (\cancel{r''(t) \times} r''(t)) + r'(t) \cdot (r'(t) \times r''(t))$$

Now we may Simplify: $(r''(t) \times r''(t) = 0$ & $v \cdot 0 = 0)$:

$$= r(t) \cdot \left(r'(t) \times r'''(t)\right) + \cancel{r(t) \cdot 0} + r'(t) \cdot (r'(t) \times r''(t))$$

Simplify w/ $u \cdot (v \times w) = (u \times v) \cdot w$:

$\to v \times v = 0$ & $0 \cdot v = 0$

$$= r(t) \cdot (r'(t) \times r'''(t)) + \cancel{(r'(t) \times r'(t)) \cdot r''(t)}$$

$$= r(t) \cdot \left(r'(t) \times r'''(t)\right) + 0$$

$$= r(t) \cdot \left(r'(t) \times r'''(t)\right)$$

We can now Compute the Sol^{n}:

$$\frac{d}{dt}\left[r(t)\cdot(r'(t)\times r''(t))\right] = r(t)\cdot(r'(t)\times r'''(t))$$

$$r'(t) = e^{t}i - 1/t\, j + 2tK$$

$$r''(t) = e^{t}i + \frac{1}{t^2}j + 2K$$

$$r'''(t) = e^{t}i - \frac{2}{t^3}j$$

$$= (e^{t}i - \ln(t)j + t^2K)\cdot\left([e^{t}i - \tfrac{1}{t}j + 2tK]\times[e^{t}i - \tfrac{2}{t^3}j]\right)$$

$$\begin{vmatrix} i & j & K & i & j \\ e^t & -1/t & 2t & e^t & -1/t \\ e^t & -2/t^3 & 0 & e^t & -2/t^3 \end{vmatrix} = 2te^{t}j - \frac{2e^t}{t^3}K + \frac{e^t}{t}K + 4/t^2\, i$$

$$= (e^{t}i - \ln(t)j + t^2K)\cdot\left(\frac{4}{t^2}i + 2te^{t}j + e^{t}\left(\frac{1}{t} - \frac{2}{t^3}\right)K\right)$$

$$\boxed{= \frac{4e^t}{t^2} - 2te^{t}\ln(t) + t^2e^{t}\left(\frac{1}{t} - \frac{2}{t^3}\right)}$$

Simplify

$$= \frac{4e^t - 2t^3e^t\ln(t)}{t^2} + e^t\left(t - \frac{2}{t}\right)$$

$$= \frac{4e^t - 2t^3e^t\ln(t)}{t^2} + te^t - \frac{2e^t}{t} = \frac{4e^t - 2t^3e^t\ln(t) + t^3e^t - 2te^t}{t^2}$$

$$= \frac{4e^t - 2t^3e^t\ln(t) + (-2e^t + e^tt^2)t}{t^2}$$

thus, we can obtain the Sol^{n} w/out differentiation via. the above methods.

b) Find the function $r\cdot(r'\times r'')$, and then differentiate:

Find r':

$$r'(t) = e^t i - \frac{1}{t} j + 2tk$$

Find r'':

$$r''(t) = e^t i + \frac{1}{t^2} j + 2k$$

Calculate $r'\times r''$:

$$\begin{vmatrix} i & j & k & i & j \\ e^t & -\frac{1}{t} & 2t & e^t & -\frac{1}{t} \\ e^t & \frac{1}{t^2} & 2 & e^t & \frac{1}{t^2} \end{vmatrix}$$

$$- \quad - \quad - \quad + \quad + \quad +$$

thus:

$$r'\times r'' = \cancel{(2)\left(-\frac{1}{t}\right)i} + \cancel{(2t)(e^t)j} + \cancel{(1/t^2)(e^t)k}$$
$$-(e^t)\left(-\frac{1}{t}\right)k - \cancel{\left(\frac{1}{t^2}\right)(2t)i} - \cancel{(2)(e^t)j}$$
$$= -\frac{4}{t} i + 2(te^t - e^t)j + e^t\left(\frac{1}{t^2} + \frac{1}{t}\right)$$

Calculate $r\cdot(r'\times r'')$:

$$r\cdot(r'\times r'') = (e^t i - \ln(t) + t^2 k)\cdot\left(-\frac{4}{t} i + 2(te^t - e^t)j + e^t\left(\frac{1}{t^2}+\frac{1}{t}\right)k\right)$$
$$= -\frac{4e^t}{t} - 2\ln(t)(te^t - e^t) + t^2e^t\left(\frac{1}{t^2} + \frac{1}{t}\right)$$

Thus: $\frac{dr}{dt}(r\cdot(r'\times r''))$:

$$= -\frac{(4e^t)'(t) - (t)'(4e^t)}{t^2} - 2\left(\ln(t)'(te^t - e^t) + \ln(t)(te^t - e^t)'\right) + (e^t)' + (te^t)'$$
$$= -\frac{4te^t - 4e^t}{t^2} - 2\left(\frac{te^t - e^t}{t} + \ln(t)(te^t + \cancel{e^t} - \cancel{e^t})\right) + e^t + (e^t + te^t)$$
$$= \frac{-\cancel{4te^t} + \cancel{4e^t} - \cancel{2t^2e^t} + \cancel{2te^t} - 2\ln(t)t^3e^t + \cancel{2t^3e^t} + \cancel{t^3e^t}}{t^2}$$

$$= \frac{-2te^t + 4e^t + t^3e^t - 2\ln(t)t^3e^t}{t^2}$$

$$= \frac{4e^t - 2t^3e^t\ln(t) + (-2e^t + e^tt^2)t}{t^2}$$

thus, we get the following result from differentiation:

Both a and b result in the Same Solution (Circled above)

2) Give the arc length Parametrization of $r(t) = e^t i + e^t \sin t j + e^t \cos t K$ from the Point $(1,0,1)$ in the direction of increasing t:

$r(t) = \langle 1,0,1 \rangle$ when $t=0$

thus, the arc length Parameterization is $0 \le t \le t$

$$S = \int_0^t \|r'(t)\| dt$$

$\hookrightarrow r'(t) = e^t i + e^t(\sin t + \cos t) j + e^t(\cos t - \sin t) K$

So:

$$\|r'(t)\| = \sqrt{(e^t)^2 + (e^t(\sin t + \cos t))^2 + (e^t(\cos t - \sin t))^2}$$

$$= \sqrt{e^{2t} + e^{2t}\sin^2 t + e^{2t}\cos^2 t + \cancel{2e^{2t}\cos t \sin t} + e^{2t}\cos^2 t + e^{2t}\sin^2 t - \cancel{2e^{2t}\sin t \cos t}}$$

$$= \sqrt{e^{2t} + 2e^{2t}(\cos^2 t + \sin^2 t)} = \sqrt{3e^{2t}} = \boxed{\sqrt{3}e^t}$$

$$S(t) = \sqrt{3}\int_0^t e^t dt = \sqrt{3}\left[e^t\right]_0^t = \boxed{\sqrt{3}(e^t - 1)}$$

if $S = \sqrt{3}(e^t - 1)$:

$$\frac{S}{\sqrt{3}} = e^t - 1$$

$$e^t = \frac{S}{\sqrt{3}} + 1$$

$$t = \ln\left(\frac{S}{\sqrt{3}} + 1\right)$$

Note:

$\frac{S}{\sqrt{3}} + 1$ Can also be expressed as $\frac{\sqrt{3}S}{3} + 1$ OR $\frac{\sqrt{3}S + 3}{3}$, but i'll use $\frac{S}{\sqrt{3}} + 1$

thus, we can reparameterize $r(t)$ in terms of S as:

$$X = \cancel{e}^{\cancel{\ln}(S/\sqrt{3}+1)} = \frac{S}{\sqrt{3}} + 1$$

$$y = \cancel{e}^{\cancel{\ln}(S/\sqrt{3}+1)} \sin\ln\left(\frac{S}{\sqrt{3}} + 1\right) = \left(\frac{S}{\sqrt{3}} + 1\right)\sin\ln\left(\frac{S}{\sqrt{3}} + 1\right)$$

$$Z = \cancel{e}^{\cancel{\ln}(S/\sqrt{3}+1)} \cos\ln\left(\frac{S}{\sqrt{3}} + 1\right) = \left(\frac{S}{\sqrt{3}} + 1\right)\cos\ln\left(\frac{S}{\sqrt{3}} + 1\right)$$

thus, $r(t)$ in terms of S is:

$$r(t) = \left(\frac{S}{\sqrt{3}} + 1\right) i + \left(\frac{S}{\sqrt{3}} + 1\right)\sin\ln\left(\frac{S}{\sqrt{3}} + 1\right) j + \left(\frac{S}{\sqrt{3}} + 1\right)\cos\ln\left(\frac{S}{\sqrt{3}} + 1\right) K$$

So, if r(t) in terms of S is:

$$r(t) = \left(\frac{s}{\sqrt{3}}+1\right)i + \left(\frac{s}{\sqrt{3}}+1\right)\sin\ln\left(\frac{s}{\sqrt{3}}+1\right)j + \left(\frac{s}{\sqrt{3}}+1\right)\cos\ln\left(\frac{s}{\sqrt{3}}+1\right)K$$

We can Simplify: [When t=s]

$$r(s) = \left(\frac{s}{\sqrt{3}}+1\right)\left(i + \sin\ln\left(\frac{s}{\sqrt{3}}+1\right)j + \cos\ln\left(\frac{s}{\sqrt{3}}+1\right)K\right)$$

OR: $r(t) = \left(\frac{s}{\sqrt{3}}+1\right)\left\langle 1,\ \sin\ln\left(\frac{s}{\sqrt{3}}+1\right),\ \cos\ln\left(\frac{s}{\sqrt{3}}+1\right)K\right\rangle$

[Note: $\frac{s}{\sqrt{3}}+1$ Can be Subbed w/ $\frac{\sqrt{3}s}{3}+1$ OR $\frac{\sqrt{3}s+3}{3}$ if Prefered as a Simplification]

3) Identify the Curve $r(t) = \cos(3t)i + tj + \sin(3t)K$:

a) Give the Curvature function $K(t)$ for the Curve $r(t)$:

$K(t) = \dfrac{\|r'(t) \times r''(t)\|}{\|r'(t)\|^3}$, So:

$r'(t) = -3\sin(3t)i + j + 3\cos(3t)K$

$r''(t) = -9\cos(3t)i - 9\sin(3t)K$

$\|r'(t)\| = \sqrt{(-3\sin(3t))^2 + (1)^2 + (3\cos(3t))^2}$

$= \sqrt{9(\sin^2(3t) + \cos^2(3t)) + 1} = \sqrt{9+1} = \sqrt{10}$

$\|r'(t) \times r''(t)\|$:

$$\left|\begin{array}{ccc|cc} i & j & K & i & j \\ -3\sin(3t) & 1 & 3\cos(3t) & -3\sin(3t) & 1 \\ -9\cos(3t) & 0 & -9\sin(3t) & -9\cos(3t) & 0 \end{array}\right|$$

$= -9\sin(3t)i - 27\cancel{\cos^2(3t)}j - (27\cancel{\sin^2(3t)})j + 9\cos(3t)K$

$= -9\sin(3t)i - 27j + 9\cos(3t)K$, thus:

$= \sqrt{(-9\sin(3t))^2 + (-27)^2 + (9\cos(3t))^2}$

$= \sqrt{81\cancel{\sin^2(3t)} + 81\cancel{\cos^2(3t)} + 729}$

$= \sqrt{81 + 729} = \sqrt{810} = \sqrt{81 \cdot 10} = \sqrt{81}\sqrt{10} = 9\sqrt{10}$

$K(t) = \dfrac{9\sqrt{10}}{(\sqrt{10})^3} = \dfrac{9\cancel{\sqrt{10}}}{10\cancel{\sqrt{10}}} = \boxed{\dfrac{9}{10}}$

b) Give the tangent, normal, & binormal vectors at the Point $(-1, \frac{\pi}{3}, 0)$:

When $P(-1, \frac{\pi}{3}, 0)$, $t = \frac{\pi}{3}$ as $\cos\pi = -1$, $\sin\pi = 0$ $\therefore$ $r(\pi/3) = -i + \frac{\pi}{3}j$

<u>tangent vector:</u>

$T(t) = \dfrac{r'(t)}{\|r'(t)\|}$; in 3a, we calculated r' & $\|r'\|$, so following from that:

$T(t) = \dfrac{-3\sin(3t)i + j + 3\cos(3t)K}{\sqrt{10}}$

<u>Normal vector:</u>

$N(t) = \dfrac{T'(t)}{\|T'(t)\|}$; $T'(t) = \dfrac{-9\cos(3t)i - 9\sin(3t)K}{\sqrt{10}}$

$$\|T'(t)\| = \sqrt{\left(\frac{-9\cos(3t)}{\sqrt{10}}\right)^2 + \left(\frac{-9\sin(3t)}{\sqrt{10}}\right)^2}$$

$$= \sqrt{\frac{81\cos^2(3t)}{10} + \frac{81\sin^2(3t)}{10}} = \sqrt{\frac{81}{10}} = \boxed{\frac{9}{\sqrt{10}}}$$

thus:

$$N(t) = \frac{T'(t)}{\|T'(t)\|} = \left(\frac{-9\cos(3t)i - 9\sin(3t)K}{\sqrt{10}}\right) / \left(\frac{9}{\sqrt{10}}\right)$$

$$= \frac{-\cancel{9}(\cos(3t)i + \sin(3t)K)}{\cancel{\sqrt{10}}} \cdot \frac{\cancel{\sqrt{10}}}{\cancel{9}}$$

$$N(t) = -(\cos(3t)i + \sin(3t)K)$$

Binormal vector:

$B(t) = T(t) \times N(t)$; we Know from above that $N = -(\cos(3t)i + \sin(3t)K)$ & $T(t) = \frac{-3\sin(3t)i + j + 3\cos(3t)K}{\sqrt{10}}$

$T(t) \times N(t)$:

$$\left|\begin{array}{ccc|cc} i & j & K & i & j \\ \frac{-3\sin(3t)}{\sqrt{10}} & \frac{1}{\sqrt{10}} & \frac{3\cos(3t)}{\sqrt{10}} & \frac{-3\sin(3t)}{\sqrt{10}} & \frac{1}{\sqrt{10}} \\ -\cos(3t) & 0 & -\sin(3t) & -\cos(3t) & 0 \end{array}\right|$$

$$= -\frac{\sin(3t)}{\sqrt{10}} i - \frac{3\cos^2(3t)}{\sqrt{10}} j + \frac{\cos(3t)}{\sqrt{10}} K - \frac{3\sin^2(3t)}{\sqrt{10}} j$$

$$= -\frac{1}{\sqrt{10}}\left(\sin(3t)i + 3\cancel{\cos^2(3t)}j + 3\cancel{\sin^2(3t)}j - \cos(3t)K\right)$$

$$B(t) = -\frac{1}{\sqrt{10}}\left(\sin(3t)i + 3j - \cos(3t)K\right)$$

When we evaluate for $t = \pi/3$:

$$T\left(\frac{\pi}{3}\right) = \frac{\cancel{-3\sin(\pi)i} + j + 3\cos(\pi)K}{\sqrt{10}} = \boxed{\frac{j-3K}{\sqrt{10}}}$$

$$N\left(\frac{\pi}{3}\right) = -\left(\cos(\pi)i + \cancel{\sin(\pi)j}\right) = -(-1)i = \boxed{i}$$

$$B\left(\frac{\pi}{3}\right) = -\frac{1}{\sqrt{10}}\left(\cancel{\sin(\pi)i} + 3j - \cos(\pi)K\right) = -\frac{1}{\sqrt{10}}(3j+K) = \boxed{-\frac{3j+K}{\sqrt{10}}}$$

c) Give the equation of the osculating plain at the point $(-1, \pi/3, 0)$:
the osculating plain is the TN-plain, & is perpendicular to the equation; & passes through the point:

$$B(t) = T(t) \times N(t)$$

$P(-1, \pi/3, 0)$ occurs when $t = \pi/3$ as was shown above in 3b.

As was shown in 3b,

$$B(\pi/3) = -(3j+k)/\sqrt{10}$$

thus:

$$\langle 0, -3/\sqrt{10}, -1/\sqrt{10} \rangle \cdot \langle x, y, z \rangle$$

$$= \left\langle 0, -\frac{3}{\sqrt{10}}, -\frac{1}{\sqrt{10}} \right\rangle \cdot \left\langle -1, \frac{\pi}{3}, 0 \right\rangle = -\frac{\pi}{\sqrt{10}}$$

$$\therefore -\frac{3}{\sqrt{10}}y - \frac{1}{\sqrt{10}}z = -\frac{\pi}{\sqrt{10}} \quad \left[\text{Multiply by } -\frac{1}{\sqrt{10}}\right]$$

$3y + z = \pi$ is the equation of the osculating plain.

4) Find the tangential & normal scalar components of acceleration of the curve:

$$r(t) = e^t i + \sqrt{2}\, t j + e^{-t} k$$

thus, at time t:

$$v(t) = r'(t) = e^t i + \sqrt{2} j - e^{-t} k$$

$$a(t) = v'(t) = e^t i + e^{-t} k$$

$$\|v(t)\| = \sqrt{(e^t)^2 + (\sqrt{2})^2 + (-e^{-t})^2} = \sqrt{e^{2t} + 2 + e^{-2t}}$$

$$= \sqrt{(e^t + e^{-t})^2} = \boxed{e^t + e^{-t}}$$

$$v(t) \cdot a(t) = (e^t i + \sqrt{2} j - e^{-t} k) \cdot (e^t i + e^{-t} k)$$

$$= (e^t)(e^t) + \cancel{(\sqrt{2})(0)} + (-e^{-t})(e^{-t})$$

$$= e^{2t} - e^{-2t} = (e^t - e^{-t})(e^t + e^{-t})$$

$v(t) \times a(t)$:

$$\left|\begin{array}{ccc|cc} i & j & k & i & j \\ e^t & \sqrt{2} & -e^{-t} & e^t & \sqrt{2} \\ e^t & 0 & e^{-t} & e^t & 0 \end{array}\right|$$

$$= \sqrt{2} e^{-t} i - (e^{-t})(e^t) j - \sqrt{2}(e^t) k - (e^{-t})(e^t) j$$

$$= (\sqrt{2} e^{-t}) i - 2j - (\sqrt{2} e^t) k$$

$$\|v \times a\| = \sqrt{(\sqrt{2}e^{-t})^2+(-2)^2+(-\sqrt{2}e^{t})^2}$$
$$= \sqrt{2e^{-2t}+4+2e^{2t}} = \sqrt{2}\left(\cancel{\sqrt{(e^{-t}+e^{t})^2}}\right)$$
$$= \sqrt{2}\left(e^{t}+e^{-t}\right)$$
$$a_T = \frac{v \cdot a}{\|v\|} = \frac{(e^{t}-e^{-t})\cancel{(e^{t}+e^{-t})}}{\cancel{(e^{t}+e^{-t})}} = e^{t}-e^{-t}$$
$$a_N = \frac{\|v \times a\|}{\|v\|} = \frac{\sqrt{2}\cancel{(e^{t}+e^{-t})}}{\cancel{(e^{t}+e^{-t})}} = \boxed{\sqrt{2}}$$

thus:

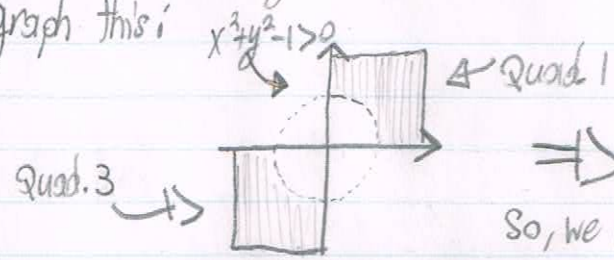
the tangential Scalar (a_T) Component of acceleration is $e^{t}-e^{-t}$ for the Curve.
the normal Scalar (a_N) Component of acceleration is $\boxed{\sqrt{2}}$ for the Curve.

5) Give the largest region on the Cartesion Plane where the function $f(x,y) = \frac{\ln(xy)}{\sqrt{x^2+y^2-1}}$ is Continuous:

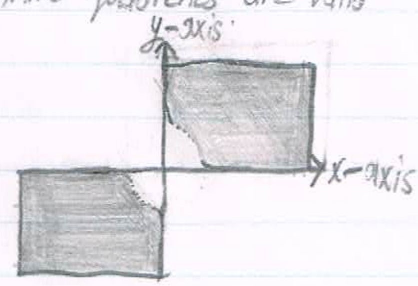
→ $x^2+y^2-1 > 0$ as $\sqrt{x^2+y^2-1}$ Can't be negative & we Cannot divide by zero.

→ $xy > 0$ as ln is only defined for Positive numbers

- If $x^2+y^2-1>0$ then all Points lie outside the circle $x^2+y^2=1$
- If $xy>0$ then only Points in the first & third quadrents are valid
- To graph this:



⇒ So, we see



thus, the largest region where f is Continuous is:

→ All the points outside the Circle $x^2+y^2=1$ in the first & third quadrents, where $xy > 0$.

6) Evaluate the limit $\lim_{(x,y)\to(1,0)} \frac{xy}{e^x - y^2}$ along the following:

a) line $y=x-1$:

the line $y=x-1$ has Parametric equations $x=t, y=t-1$, with Point $(1,0)$ Corresponding to $t=1$, So:

$$\lim_{\substack{(x,y)\to(1,0)\\(\text{along } y=x-1)}} f(x,y) = \lim_{t\to 1} f(t,t-1)$$

$$= \lim_{t\to 1} \frac{(t)(t-1)}{e^t-(t-1)^2} = \lim_{t\to 1} \frac{t^2-t}{e^t-t^2+2t-1}$$

$$= \frac{1^2-1}{e^1-\cancel{1^2}+2\cancel{-1}} = \frac{0}{e+2} = 0$$

b) the Curve $y=\ln x$:

the Curve $y=\ln x$ has Parametric equations $x=t, y=\ln(t)$, with Point $(1,0)$ Corresponding to $t=1$, So:

$$\lim_{\substack{(x,y)\to(1,0)\\(\text{Along } y=\ln(x))}} f(x,y) = \lim_{t\to 1} f(t,\ln(t))$$

$$= \lim_{t\to 1} \frac{t(\ln(t))}{e^t-\ln(t)^2} = \frac{1(\cancel{\ln(1)})}{e^1-\cancel{\ln(1)^2}} = \frac{1(0)}{e} = \frac{0}{e} = \boxed{0}$$

c) the Curve $r(t)=\sin(t)i+\cos(t)j$; So $x=\sin(t), y=\cos(t)$

the Curve $r(t)$ has Point $(1,0)$ Corresponding to $t=\pi/2$, So:

$$\lim_{\substack{(x,y)\to(1,0)\\(\text{Along } r(t)=\sin t i+\cos t j)}} f(x,y) = \lim_{t\to\frac{\pi}{2}} \frac{\sin(t)\cos(t)}{e^{\sin(t)}-\cos^2(t)}$$

$$= \frac{\sin(\pi/2)\cos(\pi/2)}{e^{\sin(\pi/2)}-\cos^2(\pi/2)} = \frac{(1)(0)}{e^1-\cancel{(0)^2}} = \frac{0}{e} = \boxed{0}$$

7) Evaluate the limit $\lim_{(x,y)\to(0,1)} \frac{xy}{e^x - y^2}$ along the following:

a) line $y=x+1$:

the line $y=x+1$ has Parametric equations $x=t, y=t+1$, w/ Point $(0,1)$ Corrosponding to $t=0$, So:

$$\lim_{\substack{(x,y)\to(0,1)\\ (\text{along } y=x+1)}} f(x,y) = \lim_{t\to 0} f(t, t+1)$$

$$= \lim_{t\to 0} \frac{(t)(t+1)}{e^t - (t+1)^2} = \lim_{t\to 0} \frac{t^2+t}{e^t - t^2 - 2t - 1}$$

Apply l'hopital's rule:

$$\overset{H}{=} \lim_{t\to 0} \frac{2t+1}{e^t - 2t - 2} = \frac{\cancel{2(0)}+1}{e^0 - \cancel{2(0)} - 2} = \frac{1}{1-2} = \frac{1}{-1} = \boxed{-1}$$

b) The Curve $y=e^x$:

the line $y=e^x$ has Parametric equations $x=t, y=e^t$, w/ Point $(0,1)$ Corrosponding to $t=0$, So:

$$\lim_{\substack{(x,y)\to(0,1)\\ (\text{Along } y=e^x)}} f(x,y) = \lim_{t\to 0} f(t, e^t)$$

$$= \lim_{t\to 0} \frac{(t)(e^t)}{e^t - (e^t)^2} = \lim_{t\to 0} \frac{t\cancel{(e^t)}}{\cancel{e^t}(1-e^t)} = \lim_{t\to 0} \frac{t}{1-e^t}$$

Apply l'hopital's rule:

$$\overset{H}{=} \lim_{t\to 0} \frac{1}{-e^t} = \frac{1}{-e^0} = \frac{1}{-1} = -1$$

c) The Curve $r(t) = \sin(t)i + \cos(t)j$:

the Curve $r(t) = \sin(t)i + \cos(t)j$ has Parametric equations $x=\sin(t), y=\cos(t)$, w/ Point $(0,1)$ Corrosponding to $t=0$, So:

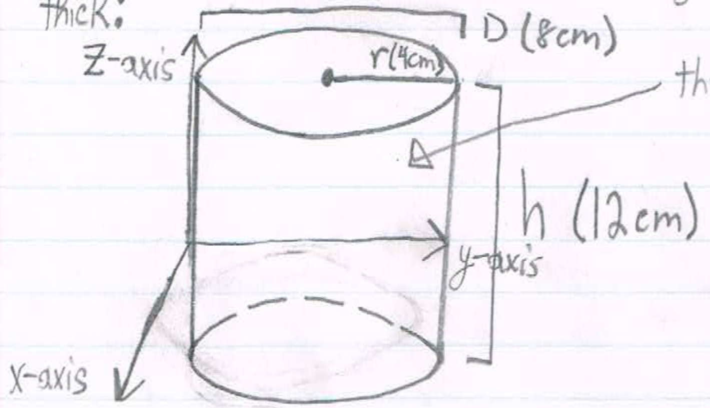
$$\lim_{\substack{(x,y)\to(0,1)\\ (\text{Along } r(t)=\sin(t)i+\cos(t)j)}} f(x,y) = \lim_{t\to 0} f(\sin(t), \cos(t))$$

(Apply l'Hopital's Rule)

$$= \lim_{t\to 0} \frac{\sin(t)\cos(t)}{e^{\sin(t)} - (\cos(t))^2} \overset{H}{=} \lim_{t\to 0} \frac{\cos^2(t) + (-\sin^2(t))}{e^{\sin(t)}\cos(t) + 2\cos(t)\sin(t)}$$

$$= \frac{\cos^2(0) - \sin^2(0)}{e^{\sin(0)}\cos(0) + 2\cos(0)\sin(0)} = \frac{1^2 - \cancel{0^2}}{e^0(1) + \cancel{2(1)(0)}} = \frac{1}{1^2} = 1$$

8) use differentials to estimate the amount of tin used in a closed cylindrical tin w/ diameter 8cm & height 12cm if the tin is 0.05cm thick:



- if we reduce thickness to nearly nothing, the height drops by $2 \times 0.05cm = 0.1cm$, & the radius drops by 0.05cm.

We know the volume of a cylinder is $V=\pi r^2 h$, so if: $r=4cm$ & $h=12cm$, we can use the definition of the differential along the z-axis (dz):

$$dz = f_x(x,y)dz + f_y(x,y)dy$$
$$= \frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy$$

When applied along the cylinder:

$$dV = \frac{\partial V}{\partial r}dr + \frac{\partial V}{\partial h}dh \Bigg\} \text{ Where: } dr = 0.05cm,\ r=4cm \quad dh = 0.10\,cm,\ h=12cm$$

So, the differential of the volume is:

$$dV = (\pi r^2 h)\frac{d}{dr}(0.05) + (\pi r^2 h)\frac{d}{dh}(0.1)$$
$$= (2\pi r h)(0.05) + (\pi r^2)(0.1)$$
$$= (2\pi)(4)(12)(0.05) + (\pi)(4)^2(0.1)$$
$$= 4.8\pi + 1.6\pi = 6.4\pi \doteq 20.11\ cm^3$$

thus, the amount of tin used is estimated to be $6.4\pi\ cm^3$ or $\sim 20.11 cm^3$.

9) let $f(x,y) = \sin(3x)i - e^{y^2}j$ & $g(x,y) = \cos(xy)i + y\ln(x)j - x\ln(y)k$:

a) Give the differential matrices of f & g:

$\frac{\partial f_m}{\partial x_i}$ for $1 \le i \le n$ where x_i is an entrie for the function, $1 \le j \le m$ where f_m is an the derivative

thus: $\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \\ & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix}$

$$Df = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial z} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial z} \end{pmatrix}$$

$$= \begin{pmatrix} 3\cos(3x) & 0 & 0 \\ 0 & -2ye^{y^2} & 0 \end{pmatrix}$$

$$Dg = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{pmatrix} = \begin{pmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \\ \frac{\partial f_3}{\partial x} & \frac{\partial f_3}{\partial y} \end{pmatrix}$$

$$= \begin{pmatrix} -y\sin(xy) & -x\sin(xy) \\ \frac{y}{x} & \ln(x) \\ -\ln(y) & -\frac{x}{y} \end{pmatrix}$$

b) use the matrices in part a. to find $\frac{\partial^2 (g\circ f)_2}{\partial x \partial y}$:
to find the Partial derivative:

$$\frac{\partial^2(g\circ f)}{\partial x\partial y} = \left|Df \circ Dg\right| = \left|Df\cdot Dg\right|$$

$$= \left| \begin{pmatrix} -y\sin(xy) & -x\sin(xy)\\ y/x & \ln(x) \\ -\ln(y) & -x/y\end{pmatrix}\cdot\begin{pmatrix}3\cos(3x) & 0 & 0\\ 0 & -2ye^{y^2} & 0\end{pmatrix}\right|$$

$$= \left| \begin{pmatrix} (-y\sin(xy))(3\cos(3x)) + \cancel{(0)(-x\sin(xy))} & \cancel{(-y\sin(xy))(0)} + (-x\sin(xy))(-2ye^{y^2}) & \cancel{(-y\sin(xy))(0) + (-x\sin(xy))(0)} \\ (y/x)(3\cos(3x)) + \cancel{\ln(x)(0)} & \cancel{(0)(y/x)} + (\ln(x))(-2ye^{y^2}) & \cancel{(-\ln(x))(0)+(-y/x)(0)} \\ (-\ln(y))(3\cos(3x)) + \cancel{(-x/y)(0)} & \cancel{(-\ln(y))(0)} + (-x/y)(-2ye^{y^2}) & \cancel{(-\ln(y))(0)+(-x/y)(0)} \end{pmatrix}\right|$$

$$= \left|\begin{pmatrix} -3y\sin(xy)\cos(3x) & 2xy\sin(xy)e^{y^2} & 0 \\ (3y\cos(3x))/x & -2y\ln(x)e^{y^2} & 0 \\ -3\ln(y)\cos(3x) & 2xe^{y^2} & 0 \end{pmatrix}\right|$$

Now we must apply the determinent:

$$= \left|\begin{array}{ccc|cc} -3y\sin(xy)\cos(3x) & 2xy\sin(xy)e^{y^2} & 0 & -3y\sin(xy)\cos(3x) & 2xye^{y^2}\sin(xy) \\ (3y\cos(3x))/x & -2y\ln(x)e^{y^2} & 0 & (3y\cos(3x))/x & -2y\ln(x)e^{y^2} \\ -3\ln(y)\cos(3x) & 2xe^{y^2} & 0 & -3\ln(y)\cos(3x) & 2xe^{y^2} \end{array}\right|$$

$$- \quad - \quad - \quad + \quad + \quad +$$

$$= \cancel{(-3y\sin(xy)\cos(3x))(-2y\ln(x)e^{y^2})(0)} + \cancel{(2xy\sin(xy)e^{y^2})(0)(-3\ln(y)\cos(3x))}$$
$$- \cancel{(-3\ln(y)\cos(3x))(2y\ln(x)e^{y^2})(0)} - \cancel{(2xe^{y^2})(0)(-3y\sin(xy)\cos(3x))}$$
$$- \cancel{(0)(3y\cos(3x)/x)(2xye^{y^2}\sin(xy))} + \cancel{(0)(3y\cos(3x)/x)(2xe^{y^2})}$$

$= 0$

So, the final differential matrix is:

$$g\circ f = \left|\begin{matrix} -3y\sin(xy)\cos(3x) & 2xy\sin(xy)e^{y^2} & 0 \\ (3y\cos(3x))/x & -2y\ln(x)e^{y^2} & 0 \\ -3\ln(y)\cos(3x) & 2xe^{y^2} & 0 \end{matrix}\right|$$

This matrix has a determinent of:

$$|g\circ f| = 0$$

∴ the matrix result of $g\circ f$ is circled above, & it has a determinent of 0.

c) Give the value of $\frac{\partial^2 (g\circ f)_2}{\partial x \partial y}(1,\sqrt{\ln 3})$:

to find the Partial derivative:

$$\frac{\partial^2 (g\circ f)_2}{\partial x\partial y}(1,\sqrt{\ln 3}) = |g\circ f|(1,\sqrt{\ln 3})$$

thus: $(g\circ f)(1,\sqrt{3})$:

$$= \begin{bmatrix} -3(\sqrt{\ln 3})\sin((1)(\sqrt{\ln 3}))\cos(3(1)) & 2(1)(\sqrt{\ln 3})\sin((1)(\sqrt{3}))e^{(\sqrt{\ln 3})^2} & 0 \\ (3(\sqrt{\ln 3})\cos(3(1)))/(1) & \cancel{-2(\sqrt{\ln 3})\ln(1)e^{(\sqrt{\ln 3})^2}} & 0 \\ -3\ln(\sqrt{\ln 3})\cos(3(1)) & (2)(1)e^{(\sqrt{\ln 3})^2} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3\sqrt{\ln(3)}\sin(\sqrt{\ln(3)})\cos(3) & 2\sqrt{\ln(3)}\,e^{(\sqrt{\ln(3)})^2}\sin(\sqrt{\ln(3)}) & 0 \\ 3(\sqrt{\ln(3)})\cos(3) & \cancel{-2\sqrt{\ln(3)}(0)e^{\ln(3)}} & 0 \\ -3\ln(\sqrt{\ln(3)})\cos(3) & 2e^{\ln(3)} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -3\sqrt{\ln(3)}\cos(3)\sin(\sqrt{\ln(3)}) & 6(\sqrt{\ln(3)})\sin(\sqrt{\ln(3)}) & 0 \\ 3\sqrt{\ln(3)}\cos(3) & 0 & 0 \\ -3\ln(\sqrt{\ln(3)})\cos(3) & 6 & 0 \end{bmatrix}$$

This matrix has a determinent of:

$|g\circ f|$, thus:

$$|g\circ f| = \left[\begin{array}{ccc|cc} -3\sqrt{\ln(3)}\cos(3)\sin(\sqrt{\ln(3)}) & 6(\sqrt{\ln(3)})\sin(\sqrt{\ln(3)}) & 0 & -3\sqrt{\ln(3)}\cos(3)\sin(\sqrt{\ln(3)}) & 6(\sqrt{\ln(3)})\sin(\sqrt{\ln(3)}) \\ 3\sqrt{\ln(3)}\cos(3) & 0 & 0 & 3\sqrt{\ln(3)}\cos(3) & 0 \\ -3\ln(\sqrt{\ln(3)})\cos(3) & 6 & 0 & -3\ln(\sqrt{\ln(3)})\cos(3) & 0 \end{array}\right]$$

$-\quad -\quad -\quad +\quad +\quad +$

$$= (-3\sqrt{\ln(3)}\cos(3)\sin(\sqrt{\ln(3)}))(0)(0) + (6(\sqrt{\ln(3)})\sin(\sqrt{\ln(3)}))(0)(-3\ln(\sqrt{\ln(3)})\cos(3))$$
$$+(0)(3\sqrt{\ln(3)}\cos(3))(0) - (0)(0)(-3\ln(\sqrt{\ln(3)})\cos(3))$$
$$-(6)(0)(-3\sqrt{\ln(3)}\cos(3)\sin(\sqrt{\ln(3)})) - (0)(3\sqrt{\ln(3)}\cos(3))(6(\sqrt{\ln(3)})(\sin(\sqrt{\ln(3)})))$$
$$= 0+0+0-0-0-0 = 0$$

So, as shown in 9.b, the determinent works out to zero.

∴ the result of $(g\circ f)(1,\sqrt{\ln(3)})$ is the circled matrix above, & it has a determinent of 0.

10) let $f(X,Y,Z) = X^2yz + xy\cos(z)$:

a) Find the directional derivative at the point $(2,1,0)$ in the direction of $V = \langle -1,-2,1 \rangle$:

$$\nabla f(X,Y,Z) = (2xyz + y\cos(z))i + (X^2Z + X\cos(Z))j + (X^2y - xy\sin(Z))K$$

$$\nabla f(2,1,0) = (\cancel{2(2)(1)(0)} + (1)\cos(0))i + (\cancel{(2)^2(0)} + 2\cos(0))j + (2^2(1) - \cancel{(2)(1)\sin(0)})K$$

$$= (1)(1)i + 2(1)j + (4)(1)K$$

$$= i + 2j + 4K$$

$$u = \frac{(-i-2j+K)}{\sqrt{(-1)^2+(-2)^2+(1)^2}} = \frac{-i-2j+K}{\sqrt{6}}$$

$$D_u f = \nabla f(2,1,0) \cdot u$$

$$= (i+2j+4K) \cdot \left(\frac{-i-2j+K}{\sqrt{6}}\right)$$

$$= \frac{(1)(-1)+(2)(-2)+(4)(1)}{\sqrt{6}} = \frac{-1-\cancel{4}+\cancel{4}}{\sqrt{6}} = \boxed{-\frac{1}{\sqrt{6}}}$$

thus, the directional derivative is $-\frac{1}{\sqrt{6}}$ (Alternatively written $-\frac{\sqrt{6}}{6}$) at the Point $(2,1,0)$ in the direction $v = \langle -1,-2,1 \rangle$ for function f.

b) Find the maximum rate of change of f at the point $(2,1,0)$:

- the max. rate of Change occurs in the direction of the gradient vector, $\nabla f(X,Y,Z)$.
- The max. rate of change is the is the norm of the gradient vector, $\|\nabla f(X,Y,Z)\|$

So, if $\nabla f(X,Y,Z) = f_x(X,Y,Z)i + f_y(X,Y,Z)j + f_z(X,Y,Z)K$, then:

$$\nabla f(X,Y,Z) = (2xyz + y\cos(z))i + (x^2z + x\cos(z))j + (x^2y - xy\sin(z))K$$

$$\nabla f(2,1,0) = (\cancel{2(2)(1)(0)} + (1)\cos(0))i + (\cancel{(2)^2(0)} + (2)\cos(0))j + ((2)^2(1) - \cancel{(2)(1)\sin(0)})K$$

$$= (1)(1)i + (2)(1)j + (2)^2(1)K$$

$$= \boxed{i + 2j + 4K}$$

$$\|i+2j+4K\| = \sqrt{(1)^2+(2)^2+(4)^2} = \sqrt{1+4+16} = \boxed{\sqrt{21}}$$

thus the max. rate of change is $\sqrt{21}$

(Also: the direction of the max. rate of Change is $\frac{i+2j+4K}{\sqrt{21}}$)

c) Give the Parametric equation of the normal line to the level curve of $f(x,y,z)$ Passing through $(-1,-2,0)$:

- if $\nabla f(x,y,z) \neq 0$, then $\nabla f(x,y,z)$ should be normal to the level curve $f(x,y,z)=c$ at any point (x,y,z) on the curve.

So:

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla f = \left\langle \frac{\partial}{\partial x}\left(x^2yz + xy\cos(z)\right), \frac{\partial}{\partial y}\left(x^2yz + xy\cos(z)\right), \frac{\partial}{\partial z}\left(x^2yz + xy\cos(z)\right)\right\rangle$$

$$\nabla f = \left\langle 2xyz + y\cos(z), x^2z + x\cos(z), x^2y - xy\sin(z)\right\rangle$$

$$\nabla f(-1,-2,0) = \left\langle \cancel{2(-1)(-2)(0)} + (-2)\cos(0), \cancel{(-1)^2(0)} + (-1)\cos(0), (-1)^2(-2) - \cancel{(-1)(-2)\sin(0)}\right\rangle$$

$$= \left\langle -2(1), -1(1), \cancel{(1)}(-2)\right\rangle$$

$$= \langle -2, -1, -2\rangle$$

∴ the normal line to $f(x,y,z)$ at $(-1,-2,0)$ is the line w/ Parametric equation:

$$x = x_0 + f_x(x_0,y_0,z_0)t = -1 + (-2)t = -1-2t$$
$$y = y_0 + f_y(x_0,y_0,z_0)t = -2 + (-1)t = -2-t$$
$$z = z_0 + f_z(x_0,y_0,z_0)t = 0 + (-2)t = -2t$$

thus:

$$\boxed{\begin{aligned} x &= -1-2t \\ y &= -2-t \\ z &= -2t \end{aligned}}$$

11) Give the equation of the tangent plain to the surface $z=\sqrt{4-x^2+4y^2}$ at the point $(2,3,6)$:

- the tangent plain to the surface $z=f(x,y)$ at the point $P_0(x_0,y_0,f(x_0,y_0))$ is the plain: $z=f(x_0,y_0)+f_x(x_0,y_0)(x-x_0)+f_y(x_0,y_0)(y-y_0)$

thus:

if $(x_0,y_0,z_0)=(2,3,6)$

$$f_x=\frac{(4-x^2+4y^2)'}{2\sqrt{4-x^2+4y^2}}=\boxed{-\frac{x}{\sqrt{4-x^2+4y^2}}}$$

$$f_y=\frac{(4-x^2+4y^2)'}{2\sqrt{4-x^2+4y^2}}=\boxed{\frac{4y}{\sqrt{4-x^2+4y^2}}}$$

then:

$$z=f(2,3)+f_x(2,3)(x-2)+f_y(2,3)(y-3)$$

$$z=\sqrt{4-2^2+4(3)^2}-\frac{2}{\sqrt{4-2^2+4(3)^2}}(x-2)+\frac{4(3)}{\sqrt{4-2^2+4(3)^2}}(y-3)$$

$$z=\sqrt{36}-\frac{2}{\sqrt{36}}(x-2)+\frac{12}{\sqrt{36}}(y-3)$$

$$z=6-\frac{1}{3}(x-2)+\frac{12}{6}(y-3)$$

therefore:

$$z=6-\frac{1}{3}(x-2)+2(y-3)$$

OR, Alternativly:

$$0=-\frac{1}{3}(x-2)+2(y-3)-(z-6)$$

We can simplify this more:

$$z=\cancel{6}-\frac{1}{3}x+\frac{2}{3}+2y-\cancel{6}=-\frac{1}{3}x+2y+\frac{2}{3}$$

therefore, when simplified:

$$\boxed{z=-\frac{1}{3}x+2y+\frac{2}{3}}$$

12) Find the absolute minimum & maximum of $f(x,y) = xy - x - 2y$ on the triangular region with vertices $(1,0)$, $(5,0)$, & $(1,4)$:

- f is polynomial, so its continuous & there is both an absolute maximum & minimum on the closed bounded triangle.
- Find the values of f at the crit. points: (When $f_x = f_y = 0$)

$$\left.\begin{aligned} f_x &= y-1 \\ f_y &= x-2 \end{aligned}\right]$$

$f_x = 0$ So $y-1=0$ $\therefore$ $y=1$

$f_y = 0$ So $x-2=0$ $\therefore$ $x=2$

So, the only critical point is: $(2,1)$

the value of f at that point is:

$$f(2,1) = 2(1) - 2 - 2(1)$$
$$= \cancel{2} - \cancel{2} - 2 = -2$$

- Next, look for values on the boundary:

① Point $(1,0)$ to $(5,0)$ → on this line, $y=0$

② Point $(1,0)$ to $(1,4)$ → on this line, $x=1$

③ Point $(5,0)$ to $(1,4)$ → this line is modeled by the function: $\boxed{y=-x+5}$

$$\left(m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-4}{5-1} = -\frac{4}{4} = -1 \right.$$
$$\left. (1,4) \text{ to } (0,y) \text{ w/ } m \text{ is } (0,5), \therefore b=5 \right)$$

① - Along $y=0$: (endpoints $x=1, x=5$) correspond to $(1,0)$ & $(5,0)$

$f_1(x,y) = f_1(x,0) = \cancel{(x)(0)} - x - \cancel{2(0)} = -x$, $1 \leq x \leq 5$

the line decreases by a factor of $-x$ from $(5,0)$ to $(1,0)$

$f_1' = -1$ $\therefore$ No crit. points $\therefore$ extreme values occur at endpoints

② - Along $x=0$: (endpoints $y=0, y=4$) corr. to $(1,0)$ & $(1,4)$

$f_2(x,y) = f_2(0,y) = \cancel{(0)(y)} - \cancel{(0)} - 2(y) = -2y$, $0 \leq y \leq 4$

the line decreases by a factor of $-2y$ from $(1,4)$ to $(1,0)$

$f_2' = -2$ $\therefore$ No crit points $\therefore$ extreme values occur at endpoints

③ - Along $y=-x+5$: (endpoints $y=0, y=4$) corr. to $(5,0)$ & $(1,4)$

$f_3(x,y) = f_3(x,-x+5) = (x)(-x+5) - x - 2(-x+5) = -x^2 + 5x - x + 2x - 10$

$= -x^2 + 6x - 10$, $1 \leq x \leq 5$

the line decreases by a factor of above from $(1,4)$ to $(5,0)$

$f_3' = -2x + 6$ $\therefore$ $f_3' = 0$ when $x=3$ So Crit. Point $(3,2)$ [from $y=-x+5$]

- Now, we may list $f(x,y)$ at the interior crit. Point & at the points on the boundary where an absolute extremum can occur:

(x,y)	$(2,1)$	$(3,2)$	$(1,0)$	$(1,4)$	$(5,0)$
$f(x,y)$	-2	-1	-1	-5	-5

thus, we conclude the absolute max. of f can occur at:

$$f(3,2)=f(1,0)=-1$$

And the absolute min. of f can occur at:

$$f(1,4)=f(5,0)=-5$$

$\therefore$ Absolute min. $= -5$
Absolute max. $= -1$

13) Find the volume of the largest rectangular box in the first octant with three faces in the coordinate plane & one vertex in the plane $x+2y+3z=6$:

a) Using the Second Partial test (Theorem 13.8.6):

<u>Theorem 13.8.6: The Second Partials test</u>

Let f be a function of two variables with continuous second-order partial derivatives in some disk centred at a critical point (x_0, y_0), & let:

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}^2(x_0, y_0)$$

<u>So, if $f(x,y,z) = x+2y+3z-6$:</u>

$$z = \frac{6-x-2y}{3} \; ; \; V = f(x,y) = xyz$$

$$V = f(x,y) = xy\left(\frac{6-x-2y}{3}\right)$$

<u>A local max occurs if $f_x = f_y = 0$:</u>

$$f_x = y\frac{(6-x-2y)}{3} + xy(6-x-2y)'/3$$

$$= y\frac{6-x-2y}{3} - \frac{xy}{3} = y\frac{6-x-2y}{3} - y\frac{x}{3}$$

$$= y\frac{6-2x-2y}{3} = \frac{2y(3-x-y)}{3}$$

<u>thus: $f_x = 0$ when:</u>

$$0 = \frac{2y(3-x-y)}{3} \rightarrow 0 = 2y(3-x-y) \Rightarrow 0 = y(3-x-y)$$

So: $y=0$ or $y=3-x$ when $f_x = 0$

$$f_y = x\left(\frac{6-x-2y}{3}\right) + \left(\frac{xy}{3}\right)(6-x-2y)'$$

$$= x\frac{6-x-2y}{3} - xy\frac{2}{3} = x\frac{6-x-2y-2y}{3}$$

$$= x\frac{6-x-4y}{3}$$

thus: $f_y = 0$ when:

$$0 = x\frac{6-x-4y}{3}$$

At $y=0$:

$$0 = x\frac{6-x-\cancel{4(0)}}{3} = \frac{1}{3}(x)(6-x)$$

thus, $x=0$ OR $x=6$; Crit. Points $(0,0),(6,0)$

At $y=3-x$:

$$0 = x\frac{6-x-4(3-x)}{3} = x\frac{6-x-12+4x}{3}$$

$$= x\frac{-6+3x}{3} = x(x-2)$$

thus, $x=0$ OR $x=2$

$\hookrightarrow$ So: $y=3-0=3$
$y=3-2=1$

Crit. Points $(0,3),(2,1)$

thus, we have 4 crit. Points: $(0,0),(6,0),(0,3),(2,1)$

We may now use the second Partial test: (use f_x & f_y from above)

$$D = f_{xx}(a,b)f_{yy}(a,b) - [f_{xy}(a,b)]^2$$

$$f_{xx} = -\frac{2y}{3} ; f_{yy} = -\frac{4x}{3}$$

$$f_{xy} = \frac{\partial}{\partial y}\frac{6y-2xy-2y^2}{3} = \frac{6-2x-4y}{3} = 2-\frac{2x}{3}-\frac{4y}{3}$$

At $(0,0)$:

$$D = \cancel{\left(-\frac{2(0)}{3}\right)\left(-\frac{4(0)}{3}\right)} - \left(2-\cancel{\frac{2(0)}{3}}-\cancel{\frac{4(0)}{3}}\right)^2 = -2^2 = -4$$

At $(6,0)$:

$$D = \cancel{\left(-\frac{2(0)}{3}\right)\left(-\frac{4(6)}{3}\right)} - \left(2-\frac{2(6)}{3}-\cancel{\frac{4(0)}{3}}\right)^2$$

$$= -\left(2-\frac{12}{3}\right)^2 = -(2-4)^2 = -(-2)^2 = -4$$

At (0,3):

$$D=\cancel{\left(-\frac{2(3)}{3}\right)\left(-\frac{4(0)}{3}\right)}-\left(2-\cancel{\frac{2(0)}{3}}-\frac{4(3)}{3}\right)^2$$

$$=-\left(2-\frac{4\cancel{(3)}}{\cancel{3}}\right)^2=-(2-4)^2=-(-2)^2$$

$$=-4$$

At (2,1):

$$D=\left(-\frac{2(1)}{3}\right)\left(-\frac{4(2)}{3}\right)-\left(2-\frac{2(2)}{3}-\frac{4(1)}{3}\right)^2$$

$$=\left(-\frac{2}{3}\right)\left(-\frac{8}{3}\right)-\left(2-\frac{4}{3}-\frac{4}{3}\right)^2$$

$$=\left(\frac{16}{9}\right)-\left(\frac{6}{3}-\frac{8}{3}\right)^2=\frac{16}{9}-\left(-\frac{2}{3}\right)^2$$

$$=\frac{16}{9}-\frac{4}{9}=\frac{12}{9}=\boxed{\frac{4}{3}}$$

thus:

Points (0,0), (6,0), (0,3) have $D<0$ meaning their saddle points
Point (2,1) has $D=\frac{4}{3}>0$ & $f_{xx}=-\frac{2}{3}<0$, thus the point is a relative maximum.

Thus, if (2,1) is a relative maximum, then the largest rectangular box in the first octant w/ three faces in the coordinate plane & one vertex in the plane $x+2y+3z=6$:

$$V=(2)(1)z,\quad z=\frac{6-x-2y}{3}$$

$$\hookrightarrow z=\frac{6-2-2(1)}{3}=\boxed{\frac{2}{3}}$$; thus max. point is $\left(2,1,\frac{2}{3}\right)$

$$V=(2)(1)\left(\frac{2}{3}\right)=\boxed{\frac{4}{3}}$$

∴ the largest box is 4/3 units

b) Using Lagrange Multipliers:

we are using the larange multiplier to optimize the function to find the max.:

$f(x,y,z) = xyz$

using the function $g(x,y,z) = x+2y+3z = 6$

the maximum value occurs when:

$$\nabla f(x,y,z) = \lambda \nabla g(x,y,z) \ \& \ g(x,y,z) = 6 = K$$

$$\left.\begin{aligned} f_x &= yz = \lambda \\ f_y &= xz = 2y\lambda \\ f_z &= xy = 3z\lambda \end{aligned}\right\} \quad x+2y+3z=6$$

$\rightarrow$ thus: $xyz = x\lambda = 2y\lambda = 3z\lambda$

$\lambda \neq 0$ as the box must have a volume > 0 to exist

thus:

$$x\lambda = 2y\lambda = 3z\lambda$$
$$x = 2y = 3z$$

Sub for x: $2y = 3z = x$ (Set all to x & evaluate $x+2y+3z=6$)

$3x = 6$ So $x = 2$

Sub for y:

$2y = x$: thus: $y = \frac{x}{2} = \frac{2}{2} = 1$ So $y = 1$

Sub for z:

$3z = x$: thus: $z = \frac{x}{3} = \frac{2}{3}$ So $z = \frac{2}{3}$

thus: the point $(2, 1, \frac{2}{3})$ is the point for the largest rectangular box in the first octant w/ three faces in the coordinate plane & one vertex in the plane $x+2y+3z=6$:

$\rightarrow$ thus, if $V = f(x,y,z) = xyz$ & $(x,y,z) = (2,1,\frac{2}{3})$

$$V = (2)(1)\left(\frac{2}{3}\right) = 2\left(\frac{2}{3}\right) = \boxed{\frac{4}{3}}$$

$\therefore$ the largest box is 4/3 units