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MATH 3801 Problem Set Two:

① In Statistics, one Sometimes wants to fit a linear function $f(X)=\alpha X+\beta$ to a set of data points (x_i, y_i), $i=1,\dots,n$ by finding α & β that the least absolute deviation $\sum_{i=1}^{n}|y_i-f(x_i)|$ is minimized. In other words, one wants to solve the optimization problem $\min\sum_{i=1}^{n}|y_i-\alpha x_i-\beta|$ over the variables α & β. It turns out that the optimization problem can be formulated as the linear programming Problem:

$$\min \sum_{i=1}^{n} Z_i$$

s.t.

$$Z_i+f(X_i)\geq y_i \quad i=1,\dots,n$$
$$Z_i-f(X_i)\geq -y_i \quad i=1,\dots,n$$

Note:
Variables in the linear Programming problem are $Z_1,\dots,Z_n,\alpha,\beta$

a) Use the excel Solver to find α & β for the dataset:

First derive an actual linear Program from the optimization Problem:

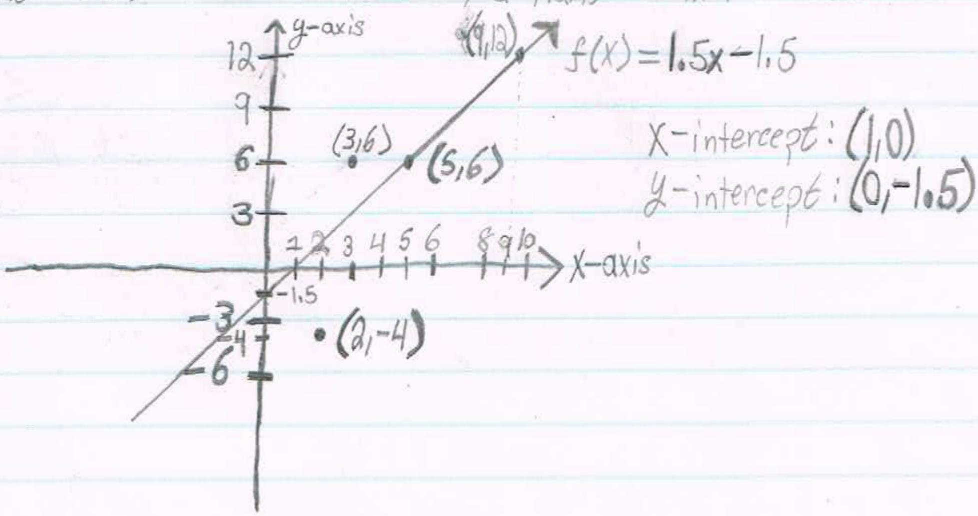
$$\min\ Z_1+Z_2+Z_3+Z_4$$

s.t.

$$Z_1+2\alpha+\beta\geq -4$$
$$Z_1-2\alpha-\beta\geq 4$$
$$Z_2+3\alpha+\beta\geq 6$$
$$Z_2-3\alpha-\beta\geq -6$$
$$Z_3+5\alpha+\beta\geq 6$$
$$Z_3-5\alpha-\beta\geq -6$$
$$Z_4+9\alpha+\beta\geq 12$$
$$Z_4-9\alpha-\beta\geq -12$$

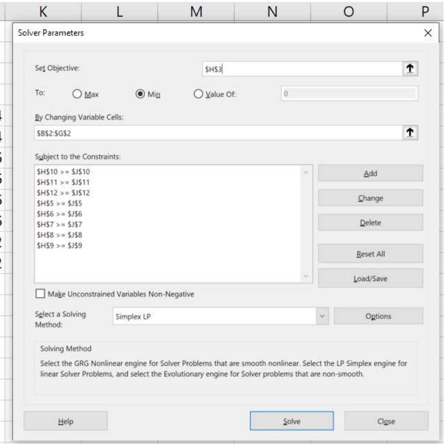
the excel Solver returns a value of 8.5 for the Problem
See attachments

b) We see that $f(X)=1.5X-1.5$ thus, a hand drawn Sketch Shows:



Question One Excel Solver Output Attachment

	A	B	C	D	E	F	G	H	I	J
1	Variables	z1	z2	z3	z4	a	B	value		
2	Solution	5.5	3	-8.9E-16	0	1.5	-1.5			
3	Objective	1	1	1	1	0	0	8.5		
4										
5	Constraint 1	1	0	0	0	2	1	7	>=	-4
6	Constraint 2	1	0	0	0	-2	-1	4	>=	4
7	Constraint 3	0	1	0	0	3	1	6	>=	6
8	Constraint 4	0	1	0	0	-3	-1	-1.8E-15	>=	-6
9	Constraint 5	0	0	1	0	5	1	6	>=	6
10	Constraint 6	0	0	1	0	-5	-1	-6	>=	-6
11	Constraint 7	0	0	0	1	9	1	12	>=	12
12	Constraint 8	0	0	0	1	-9	-1	-12	>=	-12



Microsoft Excel 16.0 Answer Report

Worksheet: [New Microsoft Excel Worksheet.xlsx]Sheet1

Report Created: 10/5/2020 10:27:26 PM

Result: Solver found a solution. All Constraints and optimality conditions are satisfied.

Solver Engine

Solver Options

Objective Cell (Min)

Cell	Name	Original Value	Final Value
$H$3	Objective value	0	8.5

Variable Cells

Cell	Name	Original Value	Final Value	Integer
$B$2:$G$2				

Constraints

Cell	Name	Cell Value	Formula	Status	Slack
$H$10	Constraint 1 value	-6	$H$10>=$J$10	Binding	0
$H$11	Constraint 1 value	12	$H$11>=$J$11	Binding	0
$H$12	Constraint 1 value	-12	$H$12>=$J$12	Binding	0
$H$5	Constraint 1 value	7	$H$5>=$J$5	Not Binding	11
$H$6	Constraint 2 value	4	$H$6>=$J$6	Binding	0
$H$7	Constraint 1 value	6	$H$7>=$J$7	Binding	0
$H$8	Constraint 1 value	-1.77636E-15	$H$8>=$J$8	Not Binding	6
$H$9	Constraint 1 value	6	$H$9>=$J$9	Binding	0

② Using Fourier-Motzkin Elimination Method, obtain a certificate of infeasibility for the system $Ax \geq b$ where:

$$A = \begin{bmatrix} 1 & 5 \\ 1 & -12 \\ -3 & 2 \end{bmatrix}, b = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$

Show your Work:

Since the system $y^TA=0$ is homogeneous, we can find without loss of generality fix $y^Tb=1$:

$$y^TA=0$$
$$y^Tb=1$$
$$y \geq 0$$

Using fourier-Motzkin Elimination

① $x_1+5x_2 \geq -1$

② $x_1-12x_2 \geq 2$ $\quad$ $(②-①)=⑤$

③ $-3x_1+2x_2 \geq 1$ $\quad$ $(③+3①)=④$

eliminate x_1:

④ $17x_2 \geq -2$

⑤ $-17x_2 \geq 3$ $\quad$ $(⑤+④)=⑥$

eliminate x_2 to get:

⑥ $0 \geq 1 \longrightarrow$ this is a contradiction

retracing computations gives:

$$⑤+④ = ②-①+③+3① = 2①+②+③$$

therefore:

$y = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ is a certificate of infeasibility

We can verify this using the following: (Check $y^TA=0$ & $y^Tb>0$)

$$y^TA = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} 1 & 5 \\ 1 & -12 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 1 & -12 \\ -3 & 2 \end{bmatrix} = 2(1)+(1)(1)+(1)(-3) + 2(5)+1(-12)+1(2)$$

$$= 2+1-3+10-12+2 = 3-3+12-12 = 0+0=0$$

$$y^Tb = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \cancel{(2)(-1)+(1)(2)}+(1)(1)=1$$

thus, $y^TA=0$ & $y^Tb>0$ thus the solution's valid

③ Determine all values of α Such that the linear programming Problem:

$$\min \ \alpha x + 3y$$
$$\text{S.t. } 4x + y \geq 2$$
$$x + 6y \geq 1$$

has at least one optimal Solution. Justify your answer:

We can See that:

$4x+y \geq 2$ is at minimum $4x+y=2$

$x+6y \geq 1$ is at minimum $x+6y=1$

thus:

$4x+y=2 \rightarrow 12x+3y=6$ meaning that for $\alpha x+3y$ to be feasible, α must be at most 12.

Since $12x+3y=6$ is the min. if $\alpha > 12$ we cannot have a Solution.

$X+6y=1 \rightarrow \frac{1}{2}x+6y=\frac{1}{2}$ meaning that for $\alpha x+3y$ to be feasible, α must be greater then $1/2$

Since $\frac{1}{2}x+6y=\frac{1}{2}$ is the min. if $\alpha < \frac{1}{2}$ we cannot have a Solution

therefore;

α Can't be either less then $1/2$ or greater then 12 thus

$1/2 \leq \alpha \leq 12$ as a result

Since $1/2 \leq \alpha \leq 12$, we can Prove this by:

take the value of α:

$$4x+y \geq 2 \rightarrow 12x+3y \geq 6$$
$$X+6y \geq 1 \rightarrow \frac{1}{2}x+3y \geq 1/2$$

Plug in $\alpha x+3y$ through elimination:

$$12x-\alpha x \geq 6 \Rightarrow x(12-\alpha) \geq 6$$
$$\frac{1}{2}x-\alpha x \geq 1/2 \Rightarrow x(1/2-\alpha) \geq 1/2$$

If $\alpha > 12$ then both $12-\alpha$ & $1/2-\alpha$ become negative, thus the objective becomes unbounded along negative x (decreases forever)

If $\alpha < 0.5$ then both $12-\alpha$ & $1/2-\alpha$ become Positive, thus the Solution becomes unbounded as well Since the objective Keeps decreasing forever.

At $\alpha=12$ & $\alpha=1/2$ the Slope of $\alpha x+3y$ is Parrallel to the feasible region $\therefore$ At least one Solution exists given the Constraints.

Finally, as $\frac{1}{2} \leq \alpha \leq 12$ has bounds with the feasible region there must be at least one optimal value for α's in that range over the objective function.