Discrete Structures Assignment I:

Question 1:

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Question 2:

$S \in Z > 6543$ $P = \{0,1,2,3,4,5,6\}$

$X_1 > 4$ digits

- The Set must be larger then 6543 & Contain no repeating digits. The Set Cant Contain 7, 8, or 9.
  ∴ no 4 digit number is greater then 6543
- For the first option, you Cant pick zero, for the second you Can
  |highest digit| = |Second highest digit|
- 5 digit: Start with 6 options (1-6 without 0), after (0-6 excluding 1 per digit)
  $6 \cdot 6 \cdot 5 \cdot 4 \cdot 3$
- 6 digit:
  $6 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2$
- 7 digit:
  $6 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$
- 8 digit:
  Cant be done, not enough digits in Set
- 4 digit:
  Cant be done; distinct digit num. excluding 7, 8, 9 > 6543

- We Can multiply Possibilities Via the product Rule.
- ∴ total = 10800

Question 3:

Without Complement Rule →

- There are 9 numbers, a zero, & 4 is excluded from 9:
  - Zero Can't be the first number
  - There are 8 numbers without 4, & 9 without 4 & with zero

∴ — First digit: 8 numbers (Set dosn't 0)

Product Rule → — Second digit: $8 \cdot 9$ numbers (first num. Cant have zero but Second Can)

— Third digit: 1, the only 3rd digit num. is 100 in the Set.

Awnser:

$8 \cdot 9 + 8 + 1 = \boxed{81}$

∴ theres 81 possibilities

With Comp. Rule →

- must pick all possibities (1 through 100) & Subtract it by the number of digits that have 4.
- $|A \cup B| = |A| + |B| - |A \cap B|$

(1) — Theres 100 possibilities (+100)

(A) — there 1 num (4) with 1 digit (-1)

(B) — All 2 digit nums. with 4 as Second digit (-10)

$100 - 1 - 10 - 9 + 1 = \boxed{81}$

(1) (2) (3) (4) (5) (6)

(4) — All 2 digit nums with 4 as first digit (-9)

(5) · We double Count 44 between 4 & 3 (+1)

— ∴ total Possibilities equals 81

Nxt. Pg. →

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Question 3 Explenation Continued:

- the Compement is all numbers that have 4
- I find the Cardinality of the Set & remove the Complement
- thus, All numbers with no 4 = the Cardinality of the Set Subtracted by All numbers with at least 1 4.

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- They must win 4 games to win a Series, there are 7 in total.
- If only 4 games are played, there is only 1 way to win, winning all 4 games.
- To find the number of winning games of a set of games, using Combinations can model it.

For 4 games:
$$\binom{3}{3} = \frac{3!}{3!(3-3)!} = \frac{3!}{3!(0)!} = \frac{3!}{3!(1)} = 1$$

For 5 games:
$$\binom{4}{3} = \frac{4!}{3!(4-3)!} = \frac{4!}{3!(1)!} = \frac{4 \cdot \cancel{3 \cdot 2 \cdot 1}}{\cancel{3 \cdot 2 \cdot 1} \cdot 1} = 4$$

For 6 Games:
$$\binom{5}{3} = \frac{5!}{3!(5-3)!} = \frac{5!}{3!(2)!} = \frac{5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{\cancel{3 \cdot 2 \cdot 1} \cdot 2 \cdot 1} = \frac{20}{2} = 10$$

For 7 Games:
$$\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6!}{3!(3)!} = \frac{6 \cdot 5 \cdot 4 \cdot \cancel{3 \cdot 2 \cdot 1}}{\cancel{3 \cdot 2 \cdot 1} \cdot 3 \cdot 2 \cdot 1} = \frac{120}{6} = 20$$

total games:
$1+4+10+20=\underline{35}$

∴ there are 35 ways that they can win a Series of 7 games.

- As seen above, the Sens can only win a game if they win the last game & have already won 3.
  - For 4 games: Theres only 1 way, winning all 4
    W, W, W, W
  - For 5 games: last game is win, the first 4 are variable "X"
    X, X, X, X, W
  - For 6 games: 5 Variable
    X, X, X, X, X, W
  - For 7 games: 6 Variable
    X, X, X, X, X, X, W

∴ There are 35 ways they can win the Series of 7 games.

Question 6:

Shelf

$m/2$	$(m+n)/2$	$n/2$

— the leftmost $m/2$ are all beer bottles.

— The rightmost $n/2$ are all Cider bottles.

— The remaining middle are Constitutes of the remaining $(m+n)/2$ bottles.

— Out of all the beer & Cider bottles, we must choose half to put allong the ends. (use choose formula)

$\therefore \binom{m}{m/2}$ & $\binom{n}{n/2}$ model this Selection

① — The remaing bottles (Selected from the total) are placed uniqely in the middle of the Shelf.

$\therefore$ Since we alredy chose the bottles for the left & right Side, we simply need to order the remaining bottles. (use factoral function)

thus: $\left(\frac{m+n}{2}\right)!$ models this Sol$^{\underline{n}}$

② — The product rule can thus model this, because the Selection of the beer & cider bottles affects the Subsequent Selection of the middle Shelf bottles. This represents a Conjoined, Series of Placements.

③ — Finally, we must find the order of the chosen bottles on the left & right Side.

$$\text{left}: \binom{m}{m/2}(m/2)! \quad \text{Right}: \binom{n}{n/2}(n/2)!$$

· as seen above, we pick half the bottles, then choose the order of them. the product of the Choice & order makes Sense as the Selection of half the bottles is Conjoined with there order on the left/right side. This is representative of a product rule problem.

— Now we must apply the product Rule on the left, right, & middle Selves; This was already described at ②.

$$\left[\binom{m}{m/2}(m/2)!\right] \cdot \left[\binom{n}{n/2}(n/2)!\right] \cdot \left(\frac{m+n}{2}\right)!$$

? Question 6:

— There are 40 places to choose from, this can be modeled as $\binom{40}{8}$ Places to choose from within the set.

— Diagram:

a

$\downarrow$

_ _ _ _

$(40-8)=32$

$\uparrow$

b/c

— There are $40-8=32$ remaining places for b & c to appear, this can be modeled as a bitstring with b/c as a 1/0 pair.

$\therefore$ there are 2^{32} different choices for b/c.

— This means the total number of choices is the product of the placements for a by the choices of a & b.

$\therefore \binom{40}{8} \cdot (2^{32}) \longrightarrow = |A|$

— b has a similar problem as a, except we choose 7 & there are 33 remaining places for a/c.

double counting

$\therefore \binom{40}{7} \cdot (2^{33}) \longrightarrow = |B|$

— There is one problem however, there are certain strings within the set of 8 a's that contain 7 many b's. This intersection isn't empty, & thus should be removed.

- This can be modeled as: $|A \cup B| = |A| + |B| - |A \cap B|$ where a is the strings of 8 a's & B is the strings with 7 b's.
- the total cool strings is the sum of 8 a's, 7 b's, excluding the intersection of 8 a's with 7 b's to prevent double counting.
- Thus we should place exactly 8 a's then 7 b's & find the product of them, & multiply by the remaining c's.
- Place 8 A's: $40-8=32$, Place 7 b's: $32-7=25$, remaining c's: $\binom{25}{25}=1^{25}$

$\therefore |A \cap B| = \binom{40}{8} \cdot \binom{32}{7} \cdot \binom{25}{25}^{25} = \binom{40}{8} \cdot \binom{32}{7} \cdot 1^{25} = \binom{40}{8}\binom{32}{7}$

— the total number of cool strings is $\therefore |A \cup B| = |A| + |B| - |A \cap B|$

$$= \binom{40}{8}(2^{32}) + \binom{40}{7}(2^{33}) - \binom{40}{8}\binom{32}{7}$$

Connor [illegible]

Question 7:

— 13 Students like trump, 25 like bieber, & 8 like both, from a pool of 100 Students.

— The total number of Students who don't like trump & don't like bieber, Can be modeled by inclusion/exclution.

— $|A|$ = Students who like trump, $|B|$ = Students who like bieber, $|A \cap B|$ = Students who like both.

— The total Students who like nither trump nor Beiber Can modeled as $|A \cup B| = |A| + |B| - |A \cap B|$. We must remove the intersection of A & B because the Sum of A & B includes Students in both, resulting in double Counting.

— Let $|A| = 13$, $|B| = 25$, $|A \cap B| = 8$

$$|A| + |B| - |A \cap B| = |A \cup B|$$
$$13 + 25 - 8 = |A \cup B|$$
$$38 - 8 = |A \cup B|$$
$$30 = |A \cup B|$$

— We must find the Students who don't like trump or Beiber.

— Let $|C|$ = the Class of 100 people

— $||C| - |A \cup B||$ = the difference (Students who like nither)

thus if (As Shown above) $|A \cup B| = 30$ & $|C| = 100$ then

$$||C| - |A \cap B|| = |100 - 30| = |70| = \boxed{70}$$

$\therefore$ $\boxed{70}$ Students in the class don't like Trump & don't like Beiber.

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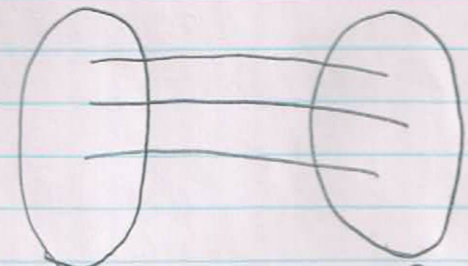
Question 8:

— A Common Combinatorial proof Counts how many elements are to the left & right.

— Diagram:



S S

$f: S \to S$
$\{1, 2, \dots, n\}$
$f(1) = 1, 2, 3, \dots, n$
$f(2) = 1, \dots, n$
$\vdots$
$f(n) = 1, \dots, n$

— there are n things

Diagram 1 — the total number of function that map n to n:

- All functions have of any Value, have n choices

$\therefore \underset{f(1)}{n} \cdot \underset{f(2)}{n} \cdot n \cdot \ldots \cdot \underset{f(n)}{n} = \boxed{n^n}$

— A is a Subset of S: Let $A \subseteq S$; K is a fixed integer: Let $|A| = K$, $K \in \mathbb{Z}$

— $f(x) = x$ if $x \in A$. Say if $A = \{1, \dots, K\}$

- $f(1) = 1$
- $f(2) = 2$
- $\vdots$
- $f(K) = K$

only 1 Choice, the Value itself

K+1 Cant map to itself because its not in A

$n-K$ $\left[\begin{array}{l} f(K+1) = 1, 2, 3, \dots, K, K+2, \dots, n \longrightarrow n-1 \\ \vdots \\ f(n) = 1, 2, \dots, n-1 \longrightarrow n-1 \end{array}\right.$

— the number of options can be represented as:

$\begin{bmatrix} 1 \cdot 1 \cdot \ldots \cdot 1 & (n-1) \longrightarrow (n-1) \\ f(n) \cdot \ldots \cdot f(K) & f(K+1) \quad f(n) \end{bmatrix}$ — $1^K = 1$ is the number of K terms of 1

— $n-K$ terms, because there are K elements in A & $n-K$ elements Subsequently not in A.

— the function we are Counting can be represented as:

$$|\{f \mid f(x) = x, \forall x \in A\}|$$

$$= (n-1)^{n-K}$$

$$\sum_{K=0}^{n} \binom{n}{K} (n-1)^{n-K} = n^n \longrightarrow \text{All functions } f: S \to S$$

All functions from $f: S \to S$

Question 8 – Diagram 1:

$$\sum_{k=0}^{n} \binom{n}{k} x^{n-k} = (x+y)^n$$

$x = n-1$

$y = 1$

$$\sum_{k=0}^{n} \binom{n}{k} (n-1)^{n-k} = n^n$$

$S = \{1, 2, \ldots, n\}$

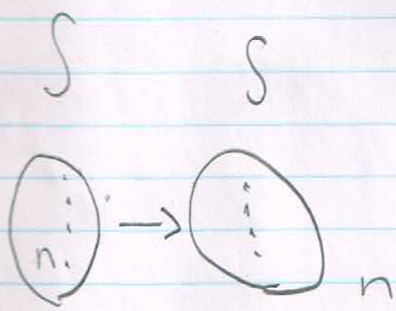
$|S|$

n

$\lvert A \rvert = k$	$S/A = n-k$
$f(x) = x$	$f(x) \neq x$
1	$n-1$

– As seen to the left, within the set of S, there are x elements to map to.

– As seen in the right, there $n-1$ elements to map to. This is due to the fact that S cannot map to A; as such, there are $n-1$ things to map to. Everything can be maped to but the value x itself.



$\underline{n^n}$

Question 9:

- The matrix has 2n things in it, as there are 2 rows per n Columbs.
- Eatch Columb has 4 possible combo's, $\begin{smallmatrix}0\\0\end{smallmatrix}, \begin{smallmatrix}0\\1\end{smallmatrix}, \begin{smallmatrix}1\\0\end{smallmatrix}, \begin{smallmatrix}1\\1\end{smallmatrix}$.
- That means the number of 1's in the matrix are:
  $2n-K$, as there are K awsome matrix, removing the number of awsome matrixs (which are the matrixs with 0's in them) would give the number of matracies with 1's in the 2n matrix.
  $\therefore |1\text{'s}| = 2n-K$ when $K = |n \subseteq 0|$
- There are K many 0's, So we must choose exactly K many elements to get an awsome array.
  - Thus, $K = |0\text{'s in matrix}|$
  - There are $\therefore$ K many awsome 2n matrices there,
- K is the num. of zero's & i is columbs with 0's
  - $\begin{matrix}0 & 1 & 0\\1 & 0 & 0\end{matrix}$ is within the range of i

  $\therefore$ i can never be greater then K in a set of elements with 0
  - $2n-K$ & $n-i$ model the problem:
    # 1's in $\begin{smallmatrix}0\\1\end{smallmatrix}$ & $\begin{smallmatrix}1\\0\end{smallmatrix}$ is: $2n-K$
    # 1's in $\begin{smallmatrix}1\\1\end{smallmatrix}$ is: $(n-i)(2)$, as there are 2 1's

    $\therefore 2n-2i \leq 2n-K$ — logic, you can't have more 1's in a set of K 0's, given the size of the set.

Math:

$\cancel{2n}-2i \leq \cancel{2n}-K$, remove 2n on both sides

$\cancel{-}2i \leq \cancel{-}K$, remove (-) from both sides

$\frac{\cancel{2}i}{\cancel{2}} \geq \frac{K}{2}$, Divide 2 out from both sides

$i \geq K/2$ —> flip around: $K/2 \leq i$

Lemma 1:

$\therefore K/2 \leq i \leq K$

Lemma 1: logic, in a set of 2n elements with i many Colums with 0's, whereas a Columb is greater then 1, there can never be more Columbs with a 0's then there are zeros as a Columb can have 2 0's.

— The number of Columns with 1 0 & 1 is equivelent to :

· We Know $n-i$ is all $\frac{1}{1}$ Columns, ∴ i is all non $\frac{1}{1}$ Columns

· ∴ We need the number of $\frac{0}{0}$ Colums :

□ Theres K zeros, Let $g = \frac{0}{1}$ & $\frac{1}{0}$ Columns

Let $i - g = \frac{0}{0}$ Columns

∴ $i - (i-g) = \frac{1}{0}$ & $\frac{0}{1}$ Columns

$K = 2(i-g) + (i - [i-g])$

$K = 2i - \cancel{2g} + \cancel{i} - \cancel{i} + g$

$K = 2i - g$

$-(K - 2i) = g$

$2i - K = g$

∴ the number of $\frac{1}{0}$ & $\frac{0}{1}$ Columns is equal to g, Which is $2i-K$.

— We must put all the terms together to get $2^{2i-K}\binom{n}{n-i}\binom{i}{2i-K}$:

$2i-K$ is the num. of $\frac{1}{0}$ & $\frac{0}{1}$'s in the set, there 2 Choices of 1 & 0 ∴ 2^{2i-K} is the Choices for $\frac{1}{0}$ or $\frac{0}{1}$ in a two number order Column.

· Within n Columns, there are $n-i$ Choices for the $\frac{1}{1}$ Columns.

$n-i$ is the number of $\frac{1}{1}$ Columns in n.

∴ $\binom{n}{n-i}$ is the number of $\frac{1}{1}$ Choices for the Columns in n.

· We must now find the number of $\frac{0}{0}$ Columns in the matrix, & we Know theres $2i-K$ $\frac{1}{0}$ & $\frac{0}{1}$ Colums ∴ we must choose $\frac{1}{0}$ & $\frac{0}{1}$ Columns from all Columns with atleast 1 0 (which is i).

∴ $\binom{i}{2i-K}$ is the choices of $\frac{1}{0}$ & $\frac{0}{1}$ columns in i

· The product Rule models the outcome of these 3 dependent Steps in the problem. The awsome $2n$ matrices with exactly $n-i$ many $\frac{1}{1}$ Columns is the product of the 3 Steps Conjoined

∴ $2^{2i-K}\binom{n}{n-i}\binom{i}{2i-K}$ = # of awsome matricies.

— For the Last Part:

· $\binom{2n}{K}$ — number of 0's(K) in matrix $2n$

· $\binom{n}{n-i}$; identity: $\binom{n}{K} = \binom{n}{n-K}$ ∴ $\binom{n}{n-i} = \binom{n}{i}$

- $\binom{n}{k} = \binom{n}{n-k}$ in $\binom{i}{2i-k}$ $\therefore$ $\binom{i}{2i-k} = \binom{i}{2i-i-(2i-k)}$

$= \binom{i}{i-(2i-k)} = \binom{i}{-i+k}$ or $\binom{i}{k-i}$

- Now we have: $2^{2i-k}\binom{n}{i}\binom{i}{k-i}$

we Should multiply both sides by 2^k as that is the number of zero's, which are needed to make awsome strings.

$$2^{2i-\cancel{k}}\binom{n}{i}\binom{i}{k-i}\cancel{(2^k)} = 2^k$$

$$2^{i}\binom{n}{i}\binom{i}{k-i} = 2^k$$

- I will add $\binom{2n}{k}$ to 2^k as it also counts the # of awsome matracies, but this covers all matraces, meaning we must also add a Summation.

 $\square$ the Sum contains the # of 0's (k) by the valid range of 0's $(k/2)$

$$\therefore \sum_{i=\lceil k/2 \rceil}^{k} 2^{2i}\binom{n}{i}\binom{i}{k-i} = 2^k\binom{2n}{k}$$

Question 10:

$S_1, \ldots, S_{50}$ $S = \{1, \ldots, 55\}$ — There are 50 Subsets of S, which each
$\{1,2,3,4,5,6,7,8\}$, $|S_1| = 7$ Contain the Set $\{1, 2, \ldots, 55\}$.

$\{7, \ldots, 55\}$ — The Smallest possible Set of 7 elements is:
$\{1,2,3,4,5,6,7\}$, this means that 1–6 Can never be the largest element in the Set. Thus, the Set of all largest elements $= \{7, \ldots, 55\}$

— There are thus 48 largest elements $(55-7+1 = 49)$

— this means that the 48 largest elements $(55-7+1=49)$, Cannot fit all 50 different Choices. This Creates the Pigeonhole problem, where more then one element must go into a Single index of elements.

— If the Set of indexes S_i is the Pigeons, & the list of Sets are the holes (i.e. $\{7, \ldots, 55\}$), then at least 2 Pigeons will end up in any 2 holes.