Milestone 3 AST5220

Evan Markel

This milestone focuses on the implementation of the first order Einstein-Boltzmann equations which model the perturbations of matter, cold dark matter, and radiation throughout most of the universe's history. The evolution module in the code sets the initial conditions of these equations just after inflation, at $a = 10^{-8}$ s^{-1}, and evolves through the radiation dominated universe, matter-radiation equality, recombination, the matter dominated, and through til the present day.

There are seven quantities comprising the linear Einstein-Boltzmann equations that the module measures in this way: $\Psi(x,k)$ and $\Phi(x,k)$ - the Newtonian gravitational potential and the perturbation to the spatial curvature, $\delta(x,k)$ and $\delta_b(x,k)$ - the density of dark matter and baryonic matter respectively, $v(x,k)$ and $v_b(x,k)$ - the perturbations to the velocity of cold dark matter and baryons respectively, and $\Theta(x,l,k)$ - which represents the perturbations to the photon distribution. The full equation set is given here:

All these equations rely on time - the logarithmic $x = log(a)$ time component utilized in the previous two milestones, as well as the Fourier modes k. The k values correspond to the Fourier transformed positions of the CMB photons. Additionally, the Θ functions also depend on the multipole l as the photons are relativistic. Only the first two moments are needed for the non-relativistic baryons and cold dark matter, and these terms are given as the density and velocity perturbed equations. For the multipole moments, only the monopole and the first 6 multipoles are necessary for the model with the line of sight integration method. There are several other mathematical processes that must be done in order to significantly reduce the computational load of these complex coupled equations. By Fourier transforming the equations, the resulting amplitudes are able to be calculated as a set of an uncoupled ordinary differential equations that evolve separately for each of the different k-modes. All of the equations with the exception of Ψ are ordinary differential equations that I solve using the Bulirsch-Stoer integration method and save the values for each value in the x-grid, which runs from x_{init} through the present day $x = 0$. Ψ is a straightforward algebraic equation which is calculated at each grid point of x. This iteration is performed for each k-mode on the grid $ks(0 : n_k - 1)$ where $n_k = 100$, $k_{min} = 0.1\frac{H_0}{c}$, and $k_{max} = 1000\frac{H_0}{c}$. As recommended by Callin, the values between k_{min} and k_{max} are quadratically distributed by

$$k_i = kmin + (k_{max} - k_{min})\cdot\Big(\frac{i - 1.0}{n_k - 1.0}\Big)^2 \tag{1}$$

Within each k-mode, there are 3 different equation sets that are necessary for smooth numerical calculations. The initial conditions are derived from inflationary theory and are directly calculated prior to integration. Note that Φ below is set equal to 1 in this milestone.

$$\begin{aligned}
\Theta_0 &= \frac{1}{2}\Phi\,,\\
\delta &= \delta_b = \frac{3}{2}\Phi\,,\\
\Theta_1 &= -\frac{ck}{6\mathcal{H}}\Phi\,,\\
v &= v_b = \frac{ck}{2\mathcal{H}}\Phi\,,\\
\Theta_2 = -&\frac{8ck}{15\mathcal{H}\tau'}\Theta_1\,,\\
\Theta_l &= -\frac{l}{2l+1}\frac{ck}{\mathcal{H}\tau'}\Theta_{l-1}\,.
\end{aligned} \tag{2}$$

The first set of differential equations correspond to the tight coupling regime, when the universe was optically very thick. This period was optically thick so only the first two photon moments - the monopole and the dipole - are needed for the calculations at this time. Since this time period corresponds to large τ' and the radiation values Θ_1 are very small, this leads to numerical instability. Therefore, a modified equation set must be used for the terms containing the product of τ' and Θ_1. Tight coupling ends at recombination for the majority of the k-modes -roughly the first 77- and ends slightly before recombination for the remaining largest k-modes where either $\tau' < 10$ or $\frac{ck}{\mathcal{H}\tau'} > 0.1$ before the time-grid x-before-rec reaches the time of recombination. Once the x-value closest to the end of tight coupling is reached in the integration loop, the code then switches to the full Einstein-Boltzmann equation set, which integrates for each post tight coupling time step until the present day. The full equation set is given here:

$$\begin{aligned}
\Theta_0' &= -\frac{ck}{\mathcal{H}}\Theta_1 - \Phi' \,, \\
\Theta_1' &= \frac{ck}{3\mathcal{H}}\Theta_0 - \frac{2ck}{3\mathcal{H}}\Theta_2 + \frac{ck}{3\mathcal{H}}\Psi + \tau'\left[\Theta_1 + \frac{1}{3}v_b\right] , \\
\Theta_l' &= \frac{lck}{(2l+1)\mathcal{H}}\Theta_{l-1} - \frac{(l+1)ck}{(2l+1)\mathcal{H}}\Theta_{l+1} \\
&\quad +\tau'\left[\Theta_l - \frac{1}{10}\Theta_l\,\delta_{l,2}\right] , \qquad 2 \leq l < l_{max} \,, \\
\delta' &= \frac{ck}{\mathcal{H}}v - 3\Phi' \,, \\
v' &= -v - \frac{ck}{\mathcal{H}}\Psi \,, \\
\delta_b' &= \frac{ck}{\mathcal{H}}v_b - 3\Phi' \,, \\
v_b' &= -v_b - \frac{ck}{\mathcal{H}}\Psi + \tau' R\,(3\Theta_1 + v_b) \,, \\
\Phi' &= \Psi - \frac{c^2k^2}{3\mathcal{H}^2}\Phi + \frac{H_0^2}{2\mathcal{H}^2}[\Omega_m a^{-1}\delta + \Omega_b a^{-1}\delta_b + 4\Omega_r a^{-2}\Theta_0], \\
\Psi &= -\Phi - \frac{12H_0^2}{c^2k^2a^2}\Omega_r\Theta_2 \,, \\
R &= \frac{4\Omega_r}{3\Omega_b a}.
\end{aligned} \tag{3}$$

In the code, Ψ -an algebraic expression- and Φ' are calculated first in order to calculated all the others at each integration step. This method is used to calculate all the parameter values in one iteration, and the corresponding values are saved at each grid step of of time and Fourier modes and stored in a two-dimensional array. The plots shown below are for 6 different k modes, illustrating the perturbation effects on the low and high regimes. The 6 represented k are $[0, 8, 23, 42, 67, 98]\cdot\frac{H_0}{c}$. $k = 0, 8$ corresponds to the largest wavelength regime, which are much less affected by the perturbations than higher k-value regimes. The intermediate 2 k-values correspond roughly to the intermediate regime, where some oscillations are observed. The last 2 k-values shown correspond to the smallest wavelength, which exhibit much greater oscillations with damping seen until present day in many of the plotted parameters.

The code itself is attached after the figures.

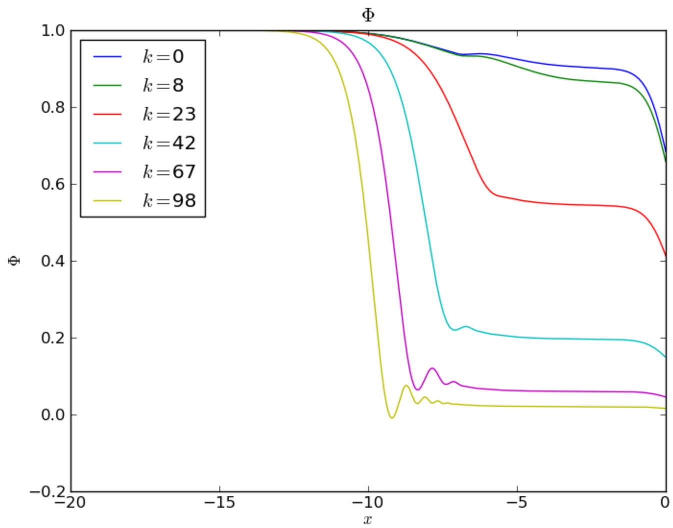

Figure 1: The gravitational potential modeled from x = -18.42 until the present day. The small k-values are smooth functions and show no oscillation. There is a slight bump indicating increase in Φ at recombination. The $k = 42$ value shows a slight oscillation and quick damping while the two largest k-modes-and smallest λ show several oscillations before recombination and subsequent damping.

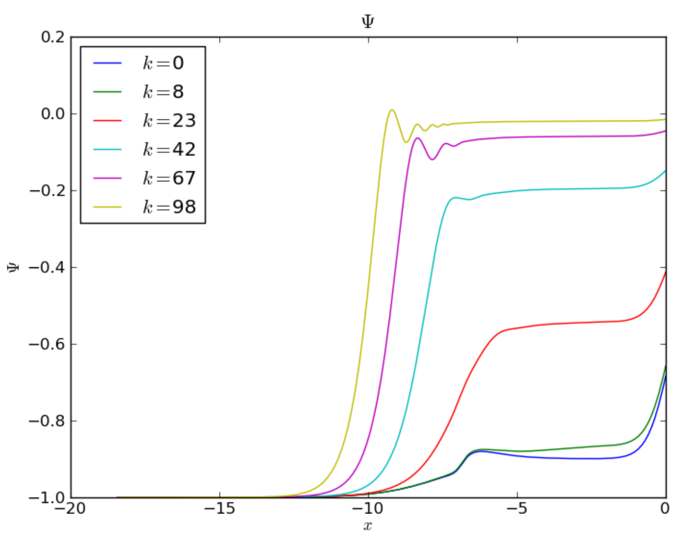

Figure 2: Ψ- the Newtonian gravitational potential from the conformal Newtonian gauge mirrors the behavior of Φ as $\Psi = -\Phi$ for most of the x-grid points.

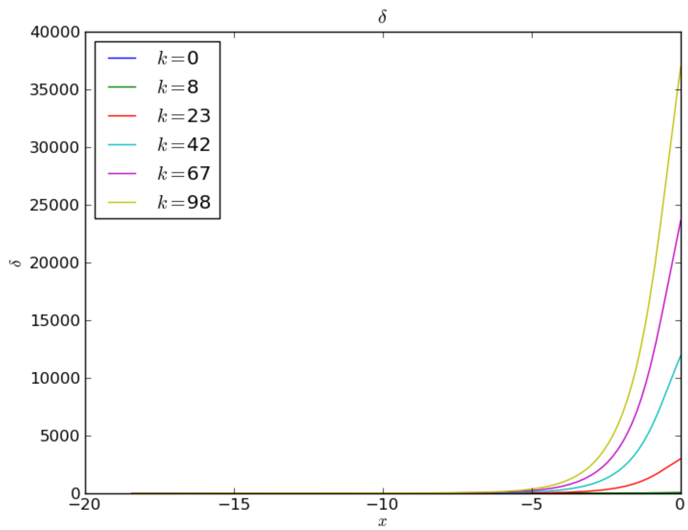

Figure 3: δ shows the density of cold dark matter over the time period. The functions are smooth for all k-modes without much oscillation. The density values rise exponentially in recent periods for all but the largest scales as the universe enters the dark-matter dominated epoch.

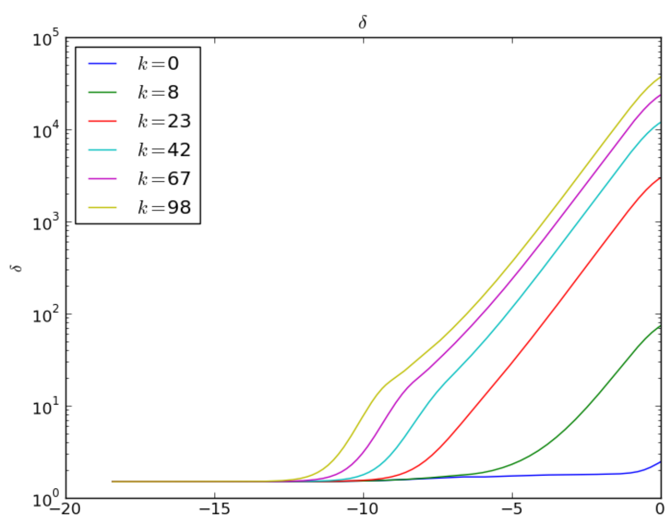

Figure 4: The log plot of δ shows the basically exponential rise in density for all except the $k = 0$ mode, which exhibits a bit of the density increase at very recent time values. For smaller scales dominated by causal physics there is a small perturbation around $x = -10$.

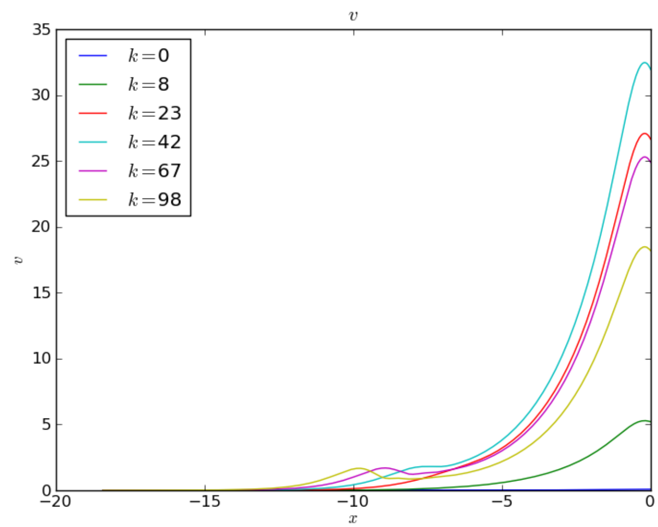

Figure 5: The dark matter velocity is relatively smooth for all regimes, with a slight rise and fall near $x = -10$ and a velocity decrease in the most recent periods.

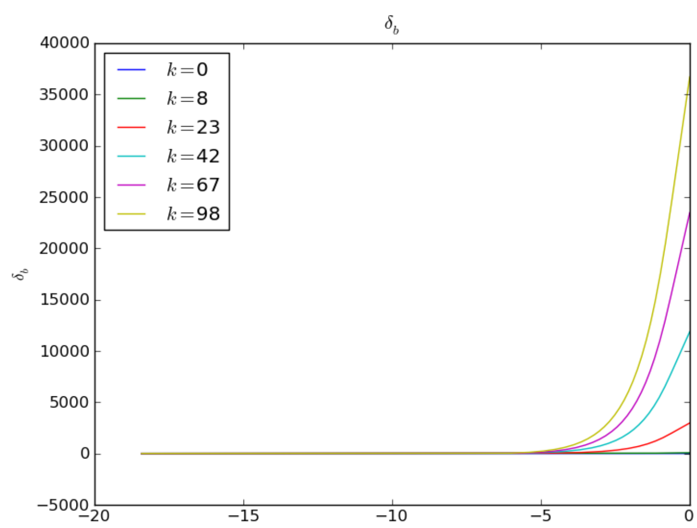

Figure 6: The baryonic density behaves much as the dark matter density except for a fluctuation near $x = -10$.

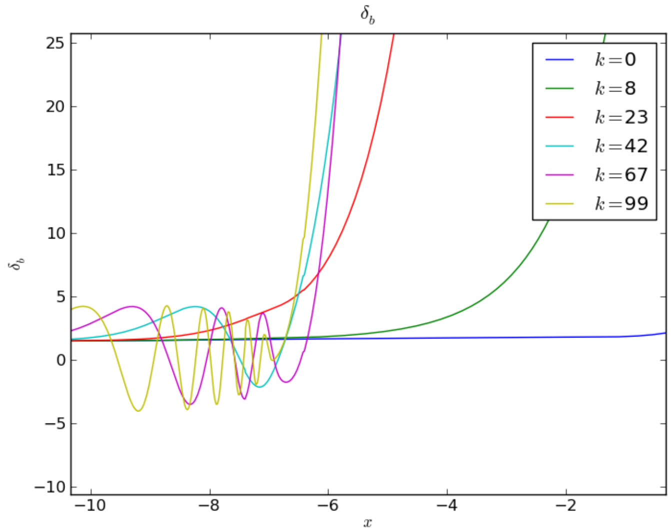

Figure 7: Zooming into the scale of $\delta_b = [0, 25]$, we see that the baryon density undergoes fluctuations for all but the largest scales from near $x = -10$ until the time of recombination, when the density increases exponentially.

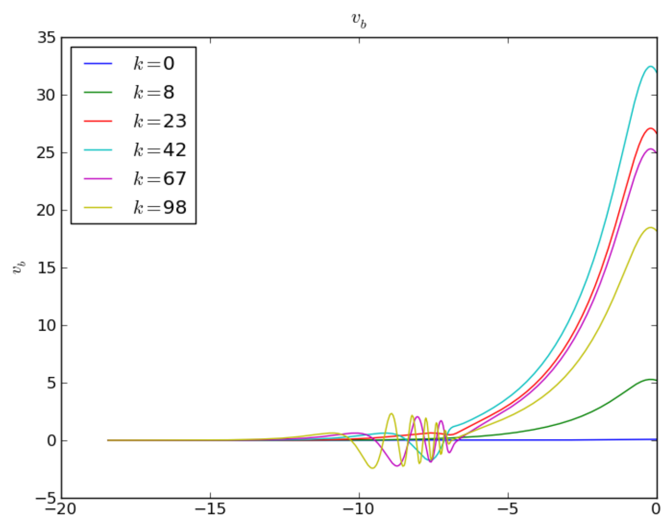

Figure 8: The same fluctuations seen in the baryonic density are also seen in the velocity of the baryons. The largest scales exhibit no oscillations, the intermediate scales show a few oscillations quickly damped, and the smalles scales show even more oscillations for a longer period of time and that causes a greater effect on the proceeding velocity. Note the middle k-scales have the highest velocity at present day.

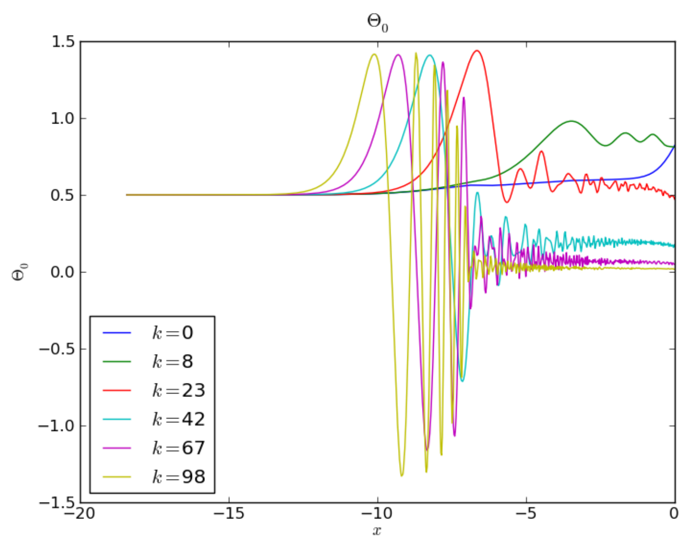

Figure 9: The monopole of the photons exhibits oscillations and damping for all but the largest scale. $k = 0, 8$ seem to converge to the same value while the average radiation temperature for other scales have converged to different values at present day. The medium and small scales have undergone many oscillations with long damping periods and their oscillations appear to be not completely damped at the present day.