Milestone 2 AST5220

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This milestone builds upon the background evolution of the universal constants and energy densities derived in the first assignment. Here we analyze the behavior of the simplest, earliest baryonic material and its interaction with radiation during the recombination period at a redshift of $z = 1100$. Due to factors characteristic of expansion that we have modeled here, the free electrons recombined with protons, Thomson scattering by free electrons then dropped, and a great number of photons were able to travel unhindered and exist today as the CMB. To quantify this effect, we model the fractional electron density X_e, and use this value to calculate the optical depth $\tau(x)$ for arbitrary values of x, where as before $x = log(a)$, and a is the scale factor of universal expansion. Finally, τ is used to calculate the visibility function $g(x)$, which is a probability density function for scattering time of a photon.

First we calculate the fractional electron density well before recombination can be characterized by the Saha equation given as

$$\frac{X_e^2}{1-X_e} = \frac{1}{n_b}\left(\frac{m_e k_b T_b}{2\pi\hbar^2}\right)^{3/2} e^{-\frac{\epsilon_0}{k_b T_b}} \tag{1}$$

This equation utilizes the time dependent number density of baryons, and the baryonic temperature. The Saha equation is valid for early times when X_e is close to 1 and is a standard quadratic equation that can be solved analytically. For values less than 0.99, the Peebles equation better models the evolution of X_e. Here is the equation as I have entered it into my code:

$$\frac{dX_e}{dx} = \frac{C_r(T_b)}{H}\left[\beta(T_b)(1-X_e)e^{-\epsilon_0/k_B T_b} - n_{\mathrm{H}}\alpha^{(2)}(T_b)X_e^2\right], \tag{2}$$

where the right hand side is composed of the following equations:

$$\begin{aligned}
C_r(T_b) &= \frac{\Lambda_{2s\to 1s}+\Lambda_\alpha}{\Lambda_{2s\to 1s}+\Lambda_\alpha+\beta^{(2)}(T_b)}\,,\\
\Lambda_{2s\to 1s} &= 8.227\,\mathrm{s}^{-1}\,, \qquad \Lambda_\alpha = H\frac{(3\epsilon_0)^3}{(8\pi)^2 n_{1s}\hbar^3 c^3}\,,\\
n_{1s} &\simeq (1-X_e)n_{\mathrm{H}}\,, \qquad \beta^{(2)}(T_b) = \beta(T_b)e^{-\epsilon_0/4k_B T_b}\,,\\
\beta(T_b) &= \alpha^{(2)}(T_b)\left(\frac{m_e k_B T_b}{2\pi\hbar^2}\right)^{3/2}\,,\\
\alpha^{(2)}(T_b) &= \frac{\hbar^2 64\pi}{c\sqrt{27\pi}}\frac{\alpha^2}{m_e^2}\sqrt{\frac{\epsilon_0}{T_b}}\phi_2(T_b)\,,\\
\phi_2(T_b) &\simeq 0.448\ln(\epsilon_0/k_B T_b)\,.
\end{aligned} \tag{3}$$

The physical constants have been added back into the equations and the exponential component that was originally part of $\beta(T_b)$ was moved to $\beta^{(2)}(T_b)$ in order to avoid a numerical error in the calculations and was accounted for in equation (2). Equation (2) was solved using a differential equation solver and splined so that the equation can be calculated for arbitrary values of the time variable x. The results are shown in the figure.

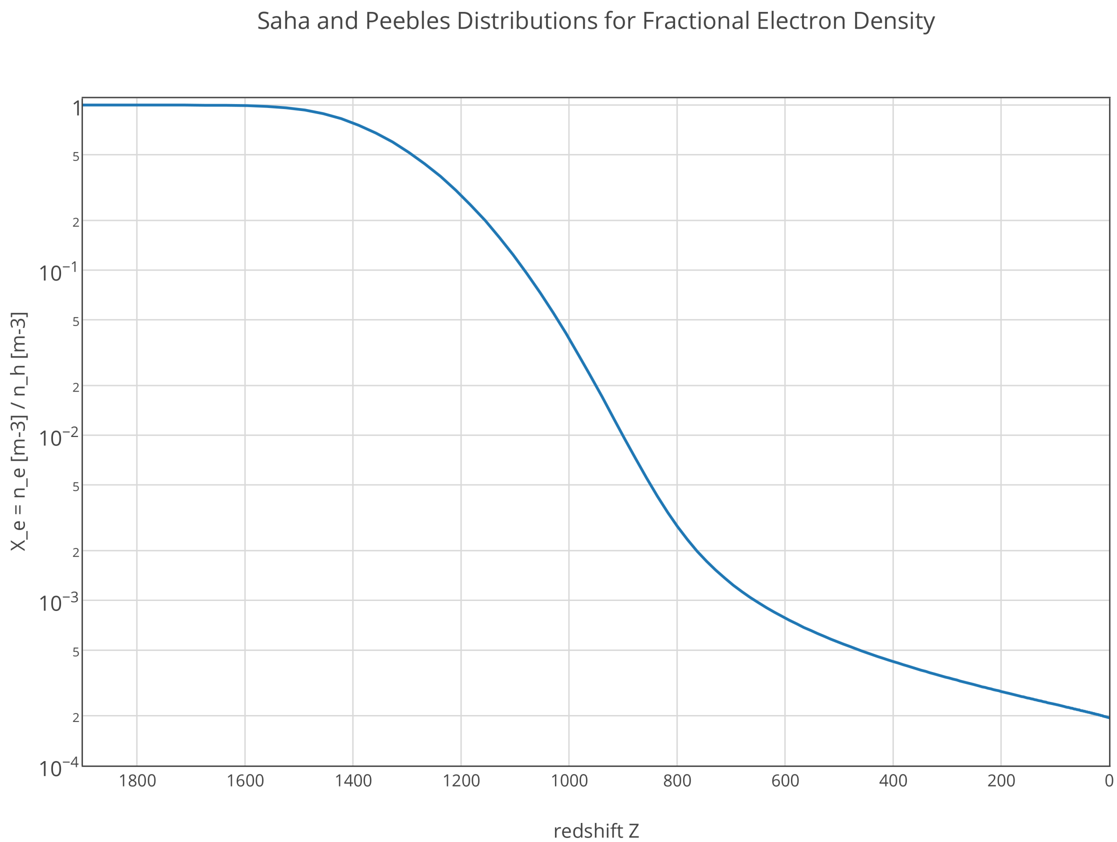

Figure 1: The plot shows the evolution of the fractional free electron density through recombination. The universe is essentially free electrons up until redshift near 1600 where the ratio begins to drop. The algorithm switches from Saha to Peebles at $z = 1595$. In cosmological time, it is then a very sudden drop to bound electrons. The ratio is equal at z 1100.

After obtaining the electron density given by $X_e \cdot n_h = n_e$, I then calculated the optical depth, given by the first order differential equation:

$$\tau' = -\frac{n_e \sigma_T a \cdot c}{\mathcal{H}} \tag{4}$$

Since the electron density n_e varies so much over the timescale $a = [10^{-10}, 1]$, we spline $\log n_e$ for easier computation. We then exponentiate the resulting spline to be used in the calculation of τ. Since we are focusing on an epoch where the universe went from very opaque to very transparent, τ also changes many orders of magnitude, so we will use the spline and interpolation on its logarithm. This complicates the first and second derivatives as we have to compensate for the terms that arise from the chain rule when differentiating the $\log \tau$ as opposed to τ.

$$\frac{d}{dx} \log \tau(x) = \frac{1}{\tau} \tau' , \frac{d^2}{dx^2} \log \tau(x) = \frac{1}{\tau} \tau'' - \tau' \frac{1}{\tau^2} \tag{5}$$

In the code, I then compensated for the added terms so that only the first and second derivatives remained. For example, for τ' i had to multiply the result by τ and do the same for τ" and add the value of the second term to cancel it out from the calculation. Note that the values of τ and derivatives in the present time is 0 so the logarithmic splines had to avoid calculations at $n = 1000$ to avoid $\log 0$ errors.

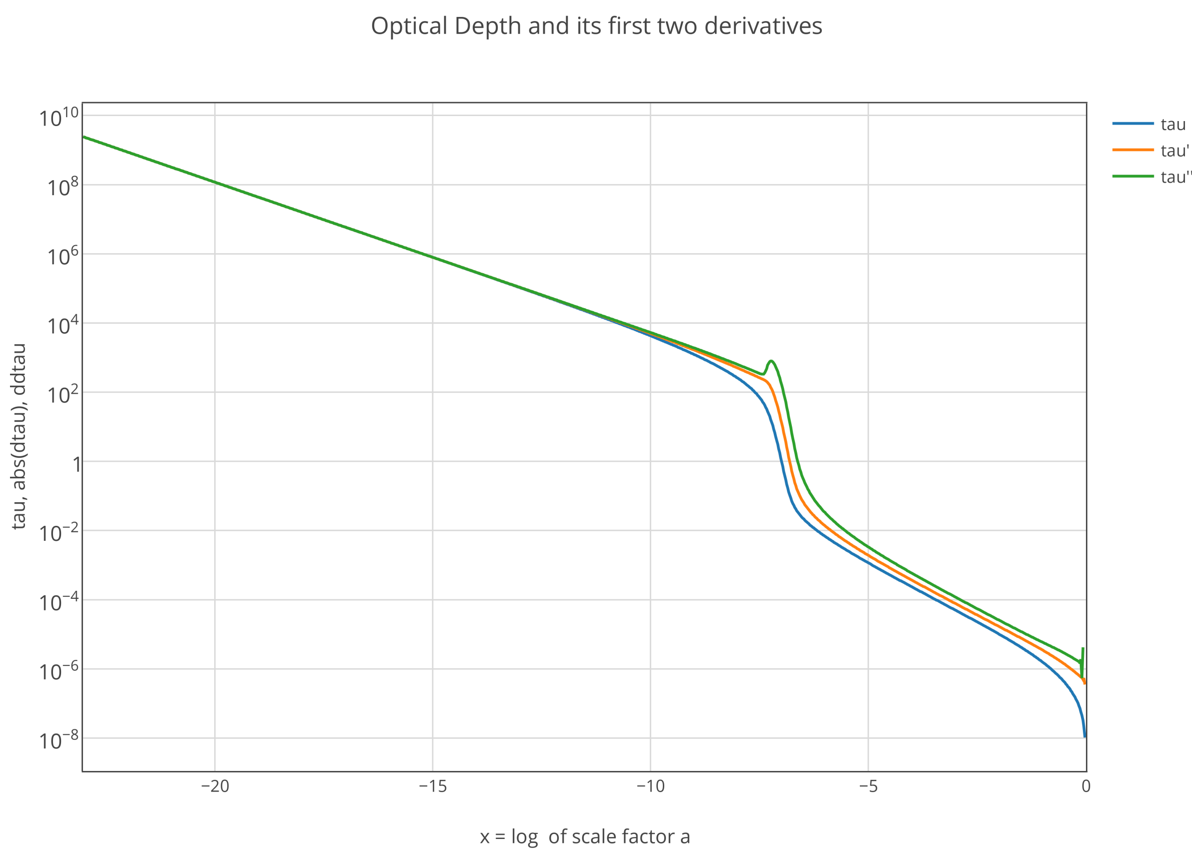

Figure 2: Here τ is shown with its first two derivatives. The steep slope in the graph corresponds to a time of about 7-8 which corresponds to the period of recombination. The drop in τ indicates a significant rapid increase in transparency in the universe. Note that τ' is negative so its absolute value is presented here for comparability.

Next, I compute the visibility function:

$$\tilde{g}(x) = -\tau' e^{-\tau} \tag{6}$$

The code utilizes the existing calculations for τ and its derivatives and then splines and interpolates to determine $\tilde{g}'$ and $\tilde{g}''$.

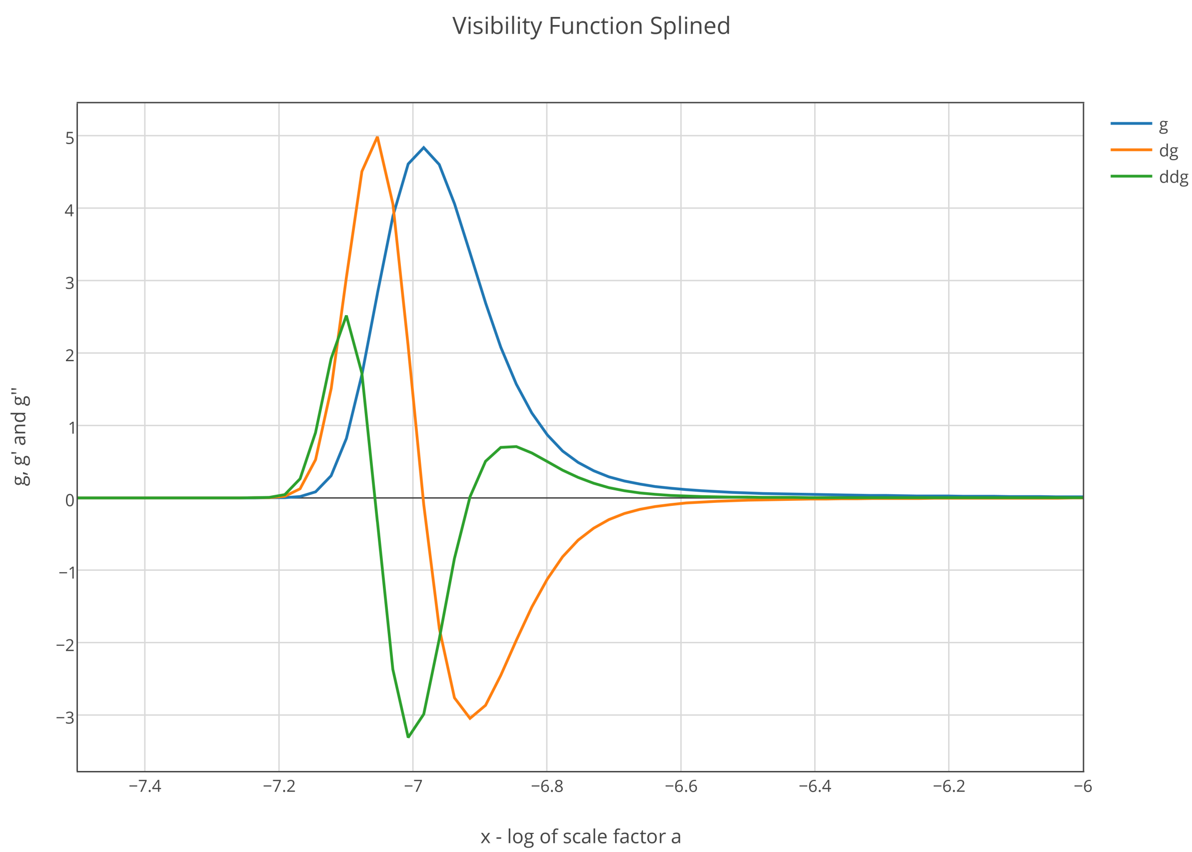

Figure 3: The visibility function describes the probability that a photon was scattered at any given time x. The peak in the graph points to the period of recombination and illustrates the scale of the photon release. Note that since the peak is so sharp, the derivatives have to be scaled in order to present all three in one plot. $\tilde{g}'$ is divided by 10 and $\tilde{g}''$ is divided by 300 for comparability.

The code itself is attached.