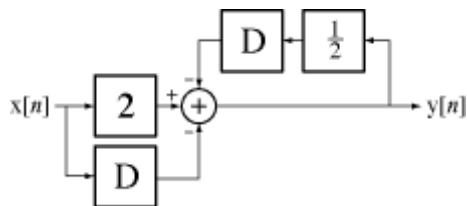


# **$z$ Transform Signal and System Analysis**

## Chapter 12

### Block Diagrams and Transfer Functions

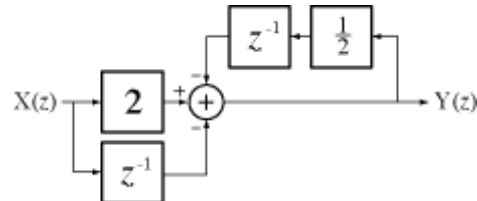
Just as with CT systems, DT systems are conveniently described by block diagrams and transfer functions can be determined from them. For example, from this DT system block diagram the difference equation can be determined.



$$y[n] = 2x[n] - x[n-1] - \frac{1}{2}y[n-1]$$

# Block Diagrams and Transfer Functions

From a z-domain block diagram the transfer function can be determined.

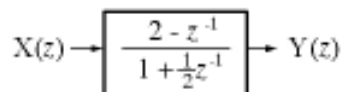
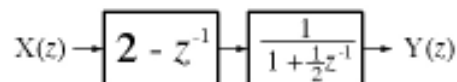
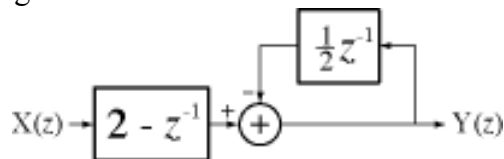


$$Y(z) = 2X(z) - z^{-1}X(z) - \frac{1}{2}z^{-1}Y(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{2 - z^{-1}}{1 + \frac{1}{2}z^{-1}} = \frac{2z - 1}{z + \frac{1}{2}}$$

## Block Diagram Reduction

All the techniques for block diagram reduction introduced with the Laplace transform apply exactly to z transform block diagrams.



# System Stability

A DT system is stable if its impulse response is absolutely summable. That requirement translates into the  $z$ -domain requirement that all the poles of the transfer function must lie in the open interior of the unit circle.

# System Interconnections

## Cascade

$$X(z) \rightarrow [H_1(z)] \rightarrow X(z)H_1(z) \rightarrow [H_2(z)] \rightarrow Y(z) = X(z)H_1(z)H_2(z)$$

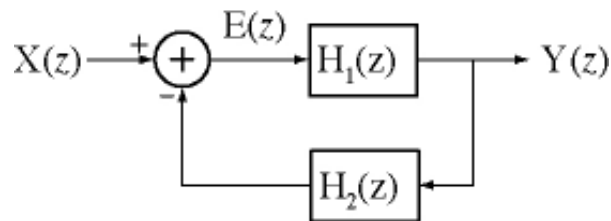
$$X(z) \rightarrow [H_1(z)H_2(z)] \rightarrow Y(z)$$

## Parallel

$$X(z) \rightarrow \begin{cases} [H_1(z)] \\ [H_2(z)] \end{cases} \rightarrow \begin{matrix} X(z)H_1(z) \\ X(z)H_2(z) \end{matrix} \rightarrow \oplus \rightarrow Y(z) = X(z)H_1(z) + X(z)H_2(z) = X(z)[H_1(z) + H_2(z)]$$

$$X(z) \rightarrow [H_1(z) + H_2(z)] \rightarrow Y(z)$$

## System Interconnections



$$H(z) = \frac{Y(z)}{X(z)} = \frac{H_1(z)}{1 + H_1(z)H_2(z)} = \frac{H_1(z)}{1 + T(z)}$$

$$T(z) = H_1(z)H_2(z)$$

## Responses to Standard Signals

If the system transfer function is  $H(z) = \frac{N(z)}{D(z)}$  the  $z$  transform of the unit step sequence response is  $Y(z) = \frac{z}{z-1} \frac{N(z)}{D(z)}$

which can be written in partial-fraction form as

$$Y(z) = z \frac{N_1(z)}{D(z)} + H(1) \frac{z}{z-1}$$

If the system is stable the transient term,  $z \frac{N_1(z)}{D(z)}$ , dies out and the steady-state response is  $H(1) \frac{z}{z-1}$ .

## Responses to Standard Signals

Let the system transfer function be  $H(z) = \frac{Kz}{z-p}$

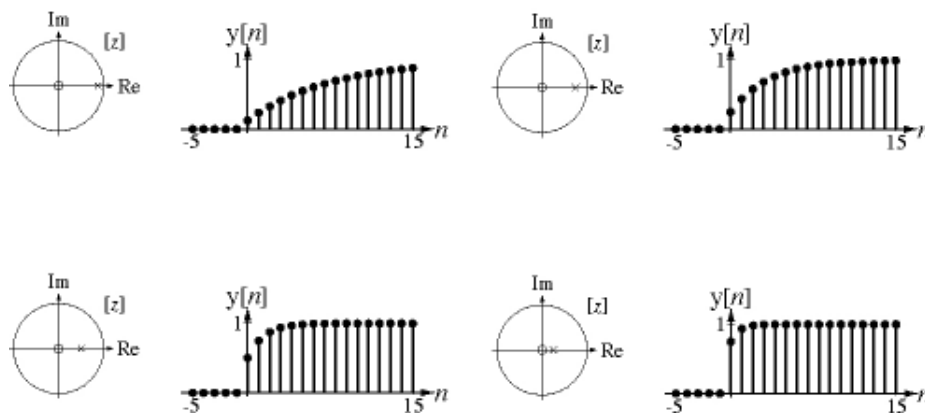
$$\text{Then } Y(z) = \frac{z}{z-1} \frac{Kz}{z-p} = \frac{Kz}{1-p} \left( \frac{z}{z-1} - \frac{pz}{z-p} \right)$$

$$\text{and } y[n] = \frac{K}{1-p} (1 - p^{n+1}) u[n]$$

Let the constant  $K$  be  $1-p$ . Then  $y[n] = (1 - p^{n+1}) u[n]$

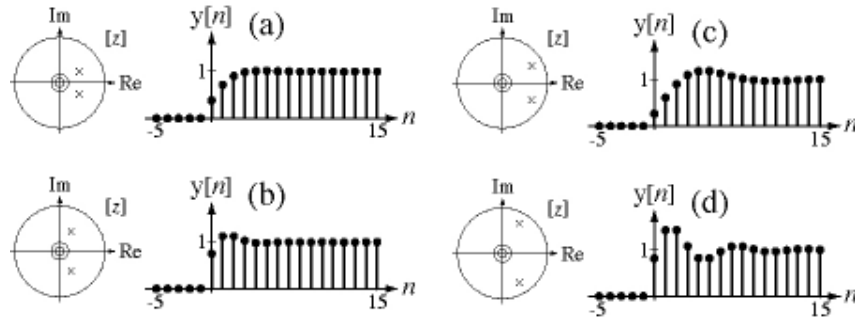
## Responses to Standard Signals

### Unit Sequence Response One-Pole System



## Responses to Standard Signals

### Unit Sequence Response Two-Pole System



Chapter 12

ENSC 380 – Linear Systems

11

## Responses to Standard Signals

If the system transfer function is  $H(z) = \frac{N(z)}{D(z)}$  the  $z$  transform

of the response to a suddenly-applied sinusoid is

$$Y(z) = \frac{N(z)}{D(z)} \frac{z[z - \cos(\Omega_0)]}{z^2 - 2z \cos(\Omega_0) + 1}$$

The system response can be written as

$$y[n] = Z^{-1} \left( z \frac{N_1(z)}{D(z)} \right) + |H(p_1)| \cos(\Omega_0 n + \angle H(p_1)) u[n]$$

and, if the system is stable, the steady-state response is

$$|H(p_1)| \cos(\Omega_0 n + \angle H(p_1)) u[n]$$

a DT sinusoid with, generally, different magnitude and phase.

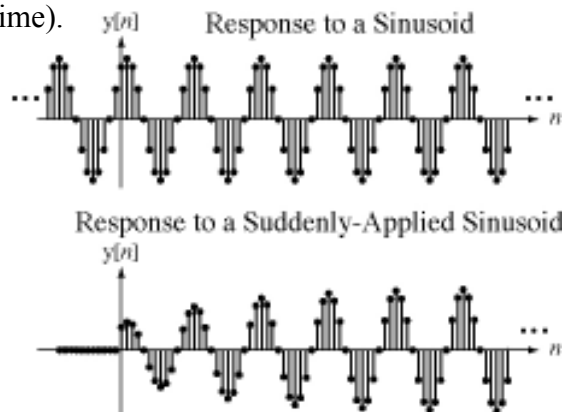
Chapter 12

ENSC 380 – Linear Systems

12

## Pole-Zero Diagrams and Frequency Response

For a stable system, the response to a suddenly-applied sinusoid approaches the response to a true sinusoid (applied for all time).



Chapter 12

ENSC 380 – Linear Systems

13

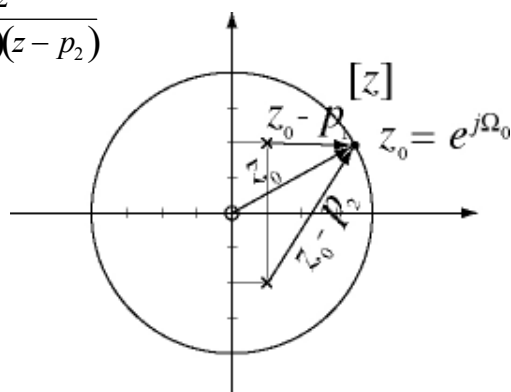
## Pole-Zero Diagrams and Frequency Response

Let the transfer function of a DT system be

$$H(z) = \frac{z}{z^2 - \frac{z}{2} + \frac{5}{16}} = \frac{z}{(z - p_1)(z - p_2)}$$

$$p_1 = \frac{1 + j2}{4} \quad p_2 = \frac{1 - j2}{4}$$

$$|H(e^{j\Omega})| = \frac{|e^{j\Omega}|}{|e^{j\Omega} - p_1| |e^{j\Omega} - p_2|}$$

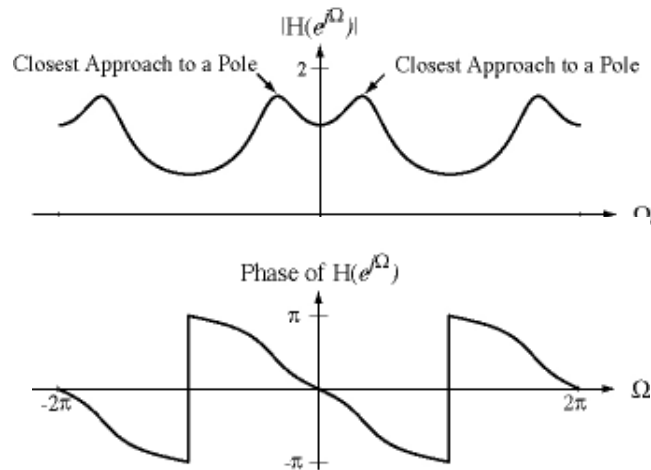


Chapter 12

ENSC 380 – Linear Systems

14

# Pole-Zero Diagrams and Frequency Response



Chapter 12

ENSC 380 – Linear Systems

15

# The Jury Stability Test

Let a transfer function be in the form,  $H(z) = \frac{N(z)}{D(z)}$

where  $D(z) = a_D z^D + a_{D-1} z^{D-1} + \dots + a_1 z + a_0$

Form the “Jury” array

|          |           |           |           |     |           |           |       |
|----------|-----------|-----------|-----------|-----|-----------|-----------|-------|
| 1        | $a_0$     | $a_1$     | $a_2$     | ... | $a_{D-2}$ | $a_{D-1}$ | $a_D$ |
| 2        | $a_D$     | $a_{D-1}$ | $a_{D-2}$ | ... | $a_2$     | $a_1$     | $a_0$ |
| 3        | $b_0$     | $b_1$     | $b_2$     | ... | $b_{D-2}$ | $b_{D-1}$ |       |
| 4        | $b_{D-1}$ | $b_{D-2}$ | $b_{D-3}$ | ... | $b_1$     | $b_0$     |       |
| 5        | $c_0$     | $c_1$     | $c_2$     | ... | $c_{D-2}$ |           |       |
| 6        | $c_{D-2}$ | $c_{D-3}$ | $c_{D-4}$ | ... | $c_0$     |           |       |
| $\vdots$ | $\vdots$  | $\vdots$  | $\vdots$  | ... |           |           |       |
| $2D-3$   | $s_0$     | $s_1$     | $s_2$     |     |           |           |       |

Chapter 12

ENSC 380 – Linear Systems

16



## The Jury Stability Test

The third row is computed from the first two by

$$b_0 = \begin{vmatrix} a_0 & a_D \\ a_D & a_0 \end{vmatrix}, b_1 = \begin{vmatrix} a_0 & a_{D-1} \\ a_D & a_1 \end{vmatrix}, b_2 = \begin{vmatrix} a_0 & a_{D-2} \\ a_D & a_2 \end{vmatrix}, \dots, b_{D-1} = \begin{vmatrix} a_0 & a_1 \\ a_D & a_{D-1} \end{vmatrix}$$

The fourth row is the same set as the third row except in reverse order. Then the  $c$ 's are computed from the  $b$ 's in the same way the  $b$ 's are computed from the  $a$ 's. This continues until only three entries appear. Then the system is stable if

$$D(1) > 0 \quad (-1)^D D(-1) > 0$$

$$a_D > |a_0|, |b_0| > |b_{D-1}|, |c_0| > |c_{D-2}|, \dots, |s_0| > |s_2|$$

## Root Locus

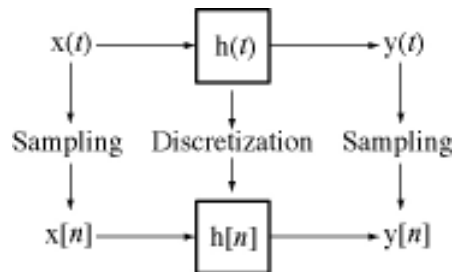
Root locus methods for DT systems are like root locus methods for CT systems except that the interpretation of the result is different.

CT systems: If the root locus crosses into the right half-plane the system goes unstable at that gain.

DT systems: If the root locus goes outside the unit circle the system goes unstable at that gain.

## Simulating CT Systems with DT Systems

The ideal simulation of a CT system by a DT system would have the DT system's excitation and response be samples from the CT system's excitation and response. But that design goal is never achieved exactly in real systems at finite sampling rates.



## Simulating CT Systems with DT Systems

One approach to simulation is to make the impulse response of the DT system be a sampled version of the impulse response of the CT system.

$$h[n] = h(nT_s)$$

With this choice, the response of the DT system to a DT unit impulse consists of samples of the response of the CT system to a CT unit impulse. This technique is called *impulse-invariant* design.

## Simulating CT Systems with DT Systems

When  $h[n] = h(nT_s)$  the impulse response of the DT system is a sampled version of the impulse response of the CT system *but the unit DT impulse  $\delta[n]$  is not a sampled version of the unit CT impulse  $\delta(t)$ .*

A CT impulse cannot be sampled. If the CT impulse were sampled at its time of occurrence what would the sample value be? The functional value of the impulse is not defined at its time of occurrence *because the impulse is not an ordinary function.*

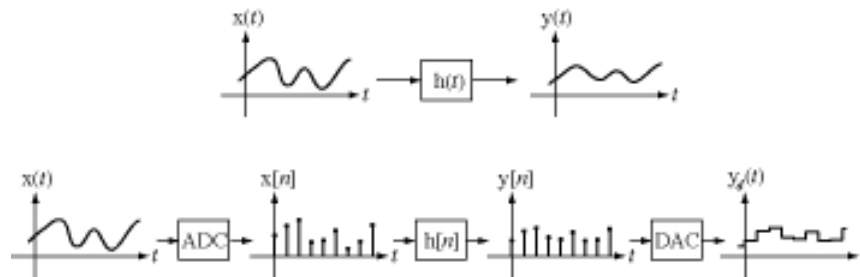
## Simulating CT Systems with DT Systems

In impulse-invariant design, even though the impulse response is a sampled version of the CT system's impulse response *that does not mean that the response to samples from any arbitrary excitation will be a sampled version of the CT system's response to that excitation.*

All design methods for simulating CT systems with DT systems are approximations and whether or not the approximation is a good one depends on the design goals.

# Sampled-Data Systems

In a practical simulation of CT systems by DT systems, we usually sample the excitation with an ADC, process the samples, and then produce a CT signal with a DAC.



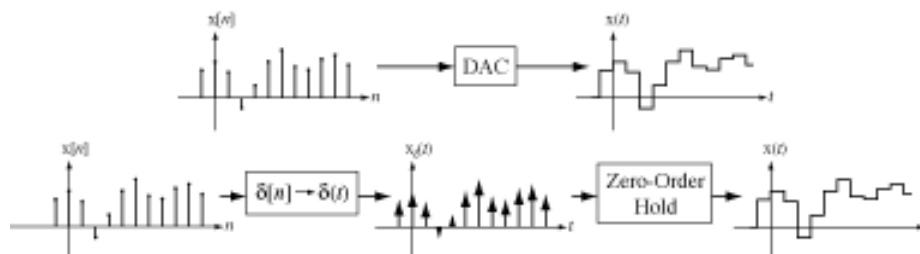
Chapter 12

ENSC 380 – Linear Systems

23

# Sampled-Data Systems

An ADC simply samples a signal and produces numbers. A common way of modeling the action of a DAC is to imagine the DT impulses in the DT signal which drive the DAC are instead CT impulses of the same strength and that the DAC has the impulse response of a zero-order hold.



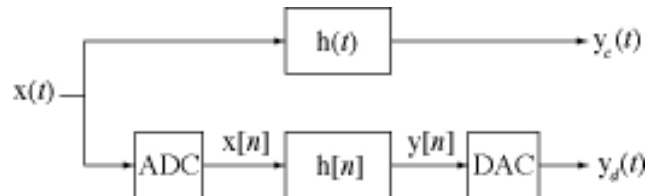
Chapter 12

ENSC 380 – Linear Systems

24

## Sampled-Data Systems

The desired equivalence between a CT and a DT system is illustrated below.



The design goal is to make  $y_d(t)$  look as much like  $y_c(t)$  as possible by choosing  $h[n]$  appropriately.

## Sampled-Data Systems

Consider the response of the CT system *not to the actual signal*,  $x(t)$ , but rather to an impulse-sampled version of it,

$$x_\delta(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s) = x(t) * f_s \text{comb}(f_s t)$$

The response is

$$y(t) = h(t) * x_\delta(t) = h(t) * \sum_{m=-\infty}^{\infty} x[m] \delta(t - mT_s) = \sum_{m=-\infty}^{\infty} x[m] h(t - mT_s)$$

where  $x[n] = x(nT_s)$  and the response at the  $n$ -th multiple of  $T_s$  is

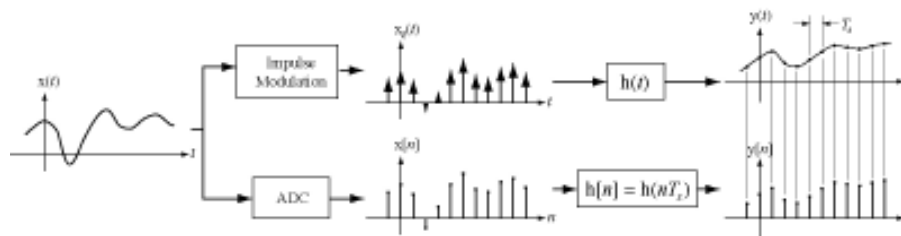
$$y(nT_s) = \sum_{m=-\infty}^{\infty} x[m] h((n - m)T_s)$$

The response of a DT system with  $h[n] = h(nT_s)$  to the excitation,  $x[n] = x(nT_s)$  is

$$y[n] = x[n] * h[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m]$$

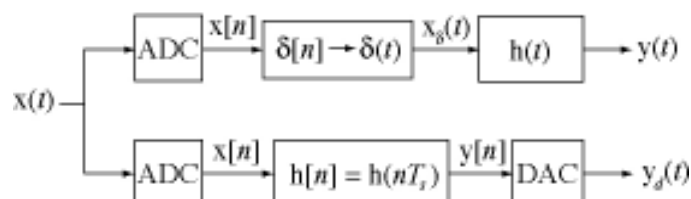
# Sampled-Data Systems

The two responses are equivalent in the sense that the values at corresponding DT and CT times are the same.

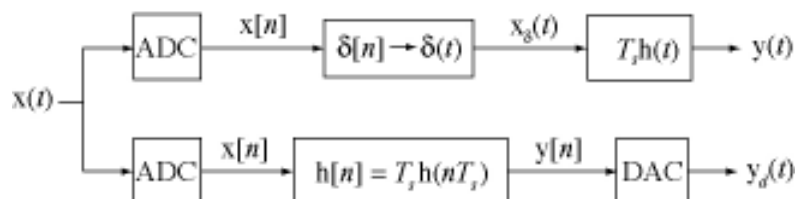


# Sampled-Data Systems

Modify the CT system to reflect the last analysis.



Then multiply the impulse responses of both systems by  $T_s$



## Sampled-Data Systems

In the modified CT system,

$$y(t) = x_s(t) * T_s h(t) = \left[ \sum_{n=-\infty}^{\infty} x(nT_s) \delta(t - nT_s) \right] * h(t) T_s = \sum_{n=-\infty}^{\infty} x(nT_s) h(t - nT_s) T_s$$

In the modified DT system,

$$y[n] = \sum_{m=-\infty}^{\infty} x[m] h[n - m] = \sum_{m=-\infty}^{\infty} x[m] T_s h((n - m)T_s)$$

where  $h[n] = T_s h(nT_s)$  and  $h(t)$  still represents the impulse response of the original CT system. Now let  $T_s$  approach zero.

$$\lim_{T_s \rightarrow 0} y(t) = \lim_{T_s \rightarrow 0} \sum_{n=-\infty}^{\infty} x(nT_s) h(t - nT_s) T_s = \int_{-\infty}^{\infty} x(\tau) h(t - \tau) d\tau$$

This is the response,  $y_c(t)$ , of the original CT system.

## Sampled-Data Systems

Summarizing, if the impulse response of the DT system is chosen to be  $T_s h(nT_s)$  then, in the limit as the sampling rate approaches infinity, the response of the DT system is exactly the same as the response of the CT system.

Of course the sampling rate can never be infinite in practice. Therefore this design is an approximation which gets better as the sampling rate is increased.

## Digital Filters

- Digital filter design is simply DT system design applied to filtering signals
- A popular method of digital filter design is to simulate a proven CT filter design
- There are many design approaches each of which yields a better approximation to the ideal as the sampling rate is increased

## Digital Filters

- Practical CT filters have *infinite-duration impulse responses*, impulse responses which never actually go to zero and stay there
- Some digital filter designs produce DT filters with infinite-duration impulse responses and these are called *IIR* filters
- Some digital filter designs produce DT filters with finite-duration impulse responses and these are called *FIR* filters

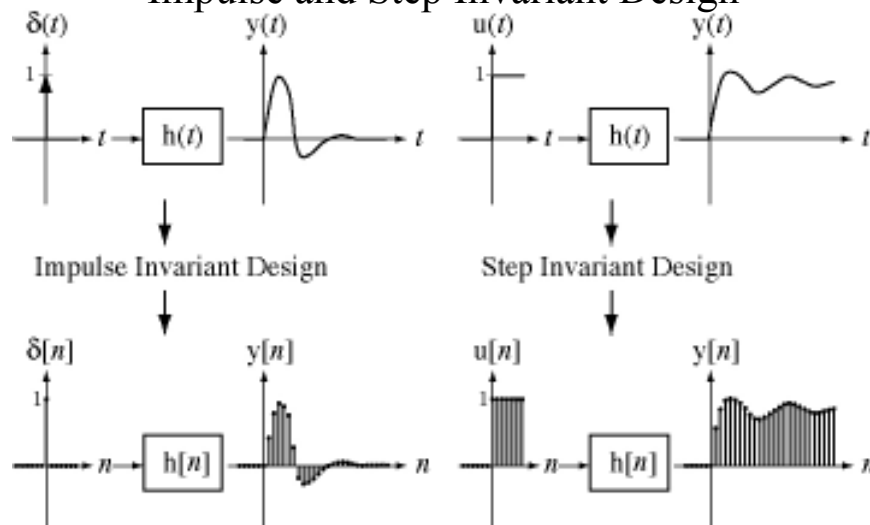


# Digital Filters

- Some digital filter design methods use time-domain approximation techniques
- Some digital filter design methods use frequency-domain approximation techniques

## Digital Filters

### Impulse and Step Invariant Design



# Digital Filters

## Impulse and Step Invariant Design

Impulse invariant:

$$H_s(s) \xrightarrow{L^{-1}} h(t) \xrightarrow{\text{Sample}} h[n] \xrightarrow{Z} H_z(z)$$

Step invariant:

$$H_s(s) \xrightarrow{\times \frac{1}{s}} \frac{H_s(s)}{s} \xrightarrow{L^{-1}} h_{-1}(t) \xrightarrow{\text{Sample}} h_{-1}[n] \xrightarrow{Z} \frac{z}{z-1} H_z(z) \xrightarrow{\times \frac{z-1}{z}} H_z(z)$$

Chapter 12

ENSC 380 – Linear Systems

35

# Digital Filters

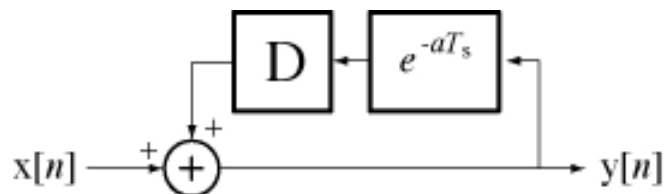
## Impulse and Step Invariant Design

Impulse invariant approximation of the one-pole system,

$$H_s(s) = \frac{1}{s+a}$$

yields

$$H_z(z) = \frac{1}{1 - e^{-aT_s} z^{-1}}$$



Chapter 12

ENSC 380 – Linear Systems

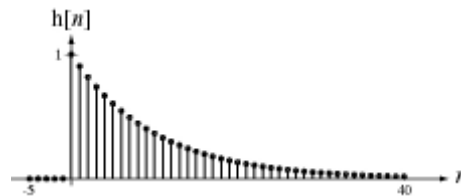
36

# Digital Filters

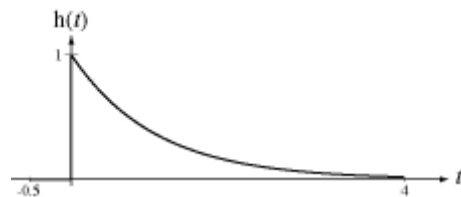
## Impulse and Step Invariant Design

Let  $a$  be one and let  $T_s = 0.1$  in  $H_z(z) = \frac{1}{1 - e^{-aT_s} z^{-1}}$

Digital Filter  
Impulse Response



CT Filter  
Impulse Response



Chapter 12

ENSC 380 – Linear Systems

37

# Digital Filters

## Impulse and Step Invariant Design

Step response of  $H_z(z) = \frac{1}{1 - e^{-aT_s} z^{-1}}$  Notice scale difference

Digital Filter  
Step Response



CT Filter  
Step Response



Chapter 12

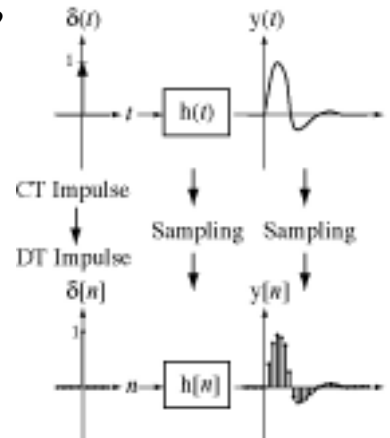
ENSC 380 – Linear Systems

38

# Digital Filters

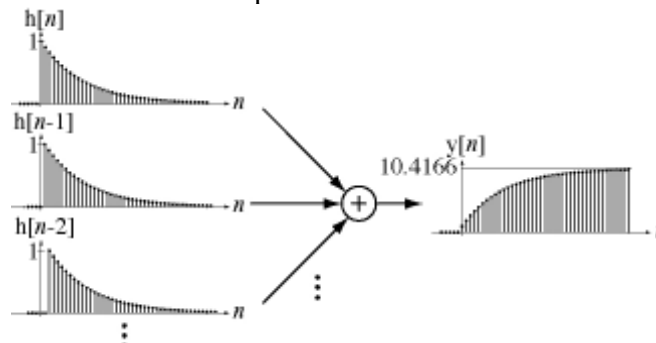
Why is the impulse response exactly right while the step response is wrong?

This design method forces an equality between the impulse strength of a CT excitation, a unit CT impulse at zero, and the impulse strength of the corresponding DT signal, a unit DT impulse at zero. It also makes the impulse response of the DT system,  $h[n]$ , be samples from the impulse response of the CT system,  $h(t)$ .



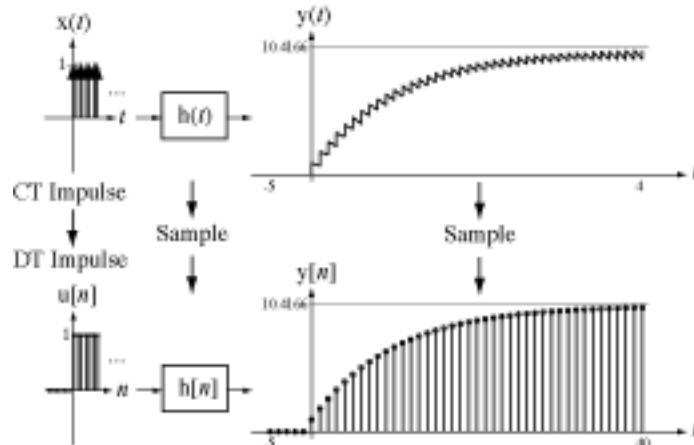
# Digital Filters

A CT step excitation is not an impulse. So what should the correspondence between the CT and DT excitations be now? If the step excitation is sampled at the same rate as the impulse response was sampled, the resulting DT signal is the excitation of the DT system and the response of the DT system is the sum of the responses to all those DT impulses.



# Digital Filters

If the excitation of the CT system were a sequence of CT unit impulses, occurring at the same sampling rate used to form  $h[n]$ , then the response of the DT system would be samples of the response of the CT system.



Chapter 12

ENSC 380 – Linear Systems

41

# Digital Filters

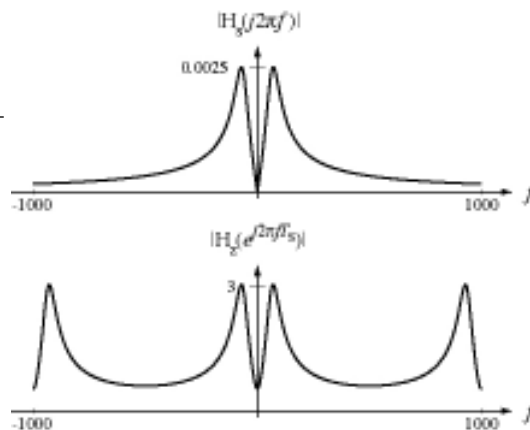
## Impulse and Step Invariant Design

Impulse invariant approximation of

$$H_s(s) = \frac{s}{s^2 + 400s + 2 \times 10^5}$$

with a 1 kHz sampling rate  
yields

$$H(z) = \frac{z(z - 0.9135)}{z^2 - 1.508z + 0.6703}$$



Chapter 12

ENSC 380 – Linear Systems

42

# Digital Filters

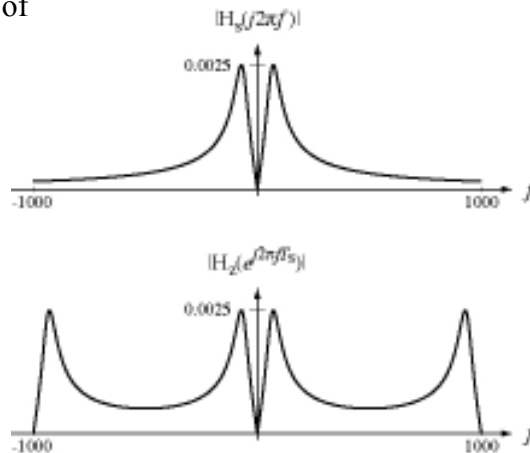
## Impulse and Step Invariant Design

Step invariant approximation of

$$H_s(s) = \frac{s}{s^2 + 400s + 2 \times 10^5}$$

with a 1 kHz sampling rate  
yields

$$H_z(z) = \frac{7.97 \times 10^{-4}(z-1)}{z^2 - 1.509z + 0.6708}$$



# Digital Filters

## Finite Difference Design

Every CT transfer function implies a corresponding differential equation. For example,

$$H_s(s) = \frac{1}{s+a} \Rightarrow \frac{d}{dt}(y(t)) + ay(t) = x(t)$$

Derivatives can be approximated by finite differences.

Forward

$$\frac{d}{dt}(y(t)) \cong \frac{y[n+1] - y[n]}{T_s}$$

Backward

$$\frac{d}{dt}(y(t)) \cong \frac{y[n] - y[n-1]}{T_s}$$

Central

$$\frac{d}{dt}(y(t)) \cong \frac{y[n+1] - y[n-1]}{2T_s}$$

# Digital Filters

## Finite Difference Design

For example, using a forward difference to approximate the derivative,

$$H_s(s) = \frac{1}{s+a} \Rightarrow \frac{y[n+1]-y[n]}{T_s} + ay[n] = x[n]$$

A more systematic method is to realize that every  $s$  in a CT transfer function corresponds to a differentiation in the time domain which can be approximated by a finite difference.

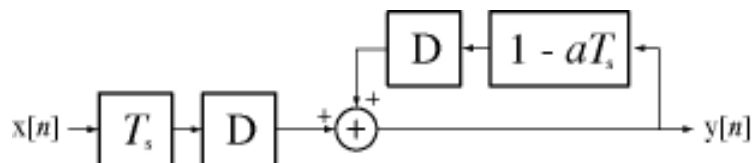
|                                      |   |  |
|--------------------------------------|---|--|
| Forward                              | Backward                                  | Central                                    |
| $s \rightarrow \frac{z-1}{T_s} X(z)$ | $s \rightarrow \frac{1-z^{-1}}{T_s} X(z)$ | $s \rightarrow \frac{z-z^{-1}}{2T_s} X(z)$ |

# Digital Filters

## Finite Difference Design

Then, forward difference approximation is

$$H_s(s) = \frac{1}{s+a} \Rightarrow H_z(z) = \left[ \frac{1}{s+a} \right]_{s \rightarrow \frac{z-1}{T_s}} = \frac{T_s}{z - (1 - aT_s)}$$



# Digital Filters

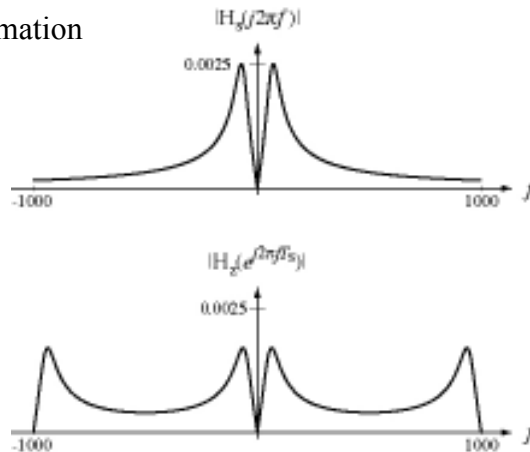
## Finite Difference Design

Backward difference approximation  
of

$$H_s(s) = \frac{s}{s^2 + 400s + 2 \times 10^5}$$

with a 1 kHz sampling rate  
yields

$$H(z) = \frac{6.25 \times 10^{-4} z(z-1)}{z^2 - 1.5z + 0.625}$$



Chapter 12

ENSC 380 – Linear Systems

47

# Digital Filters

## Direct Substitution and Matched z-Transform Design

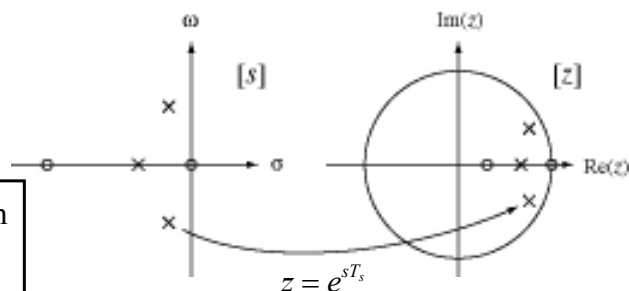
Direct substitution and matched z-transform design use the relationship  $z = e^{sT_s}$  to map the poles and zeros of an s-domain transfer function into corresponding poles and zeros of a z-domain transfer function. If there is an s-domain pole at  $a$ , the z-domain pole will be at  $e^{aT_s}$ .

Direct Substitution

$$s - a \rightarrow z - e^{aT_s}$$

Matched z-Transform

$$s - a \rightarrow 1 - e^{aT_s} z^{-1}$$



Chapter 12

ENSC 380 – Linear Systems

48



# Digital Filters

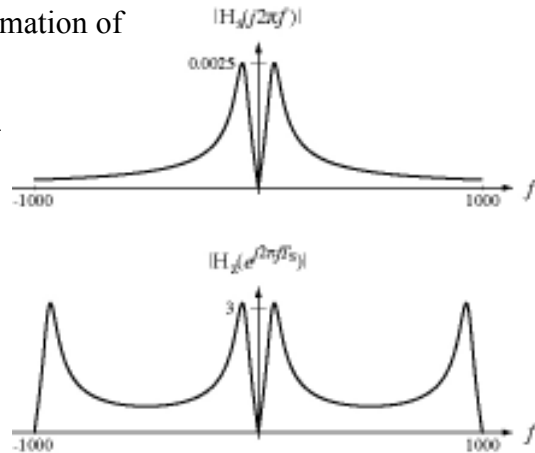
## Direct Substitution and Matched z-Transform Design

Matched z-transform approximation of

$$H_s(s) = \frac{s}{s^2 + 400s + 2 \times 10^5}$$

with a 1 kHz sampling rate  
yields

$$H_z(z) = \frac{z(z-1)}{z^2 - 1.509z + 0.6708}$$



# Digital Filters

## Bilinear Transformation

This method is based on trying to match the frequency response of a digital filter to that of the CT filter. As a practical matter it is impossible to match exactly because a digital filter has a periodic frequency response but a good approximation can be made over a range of frequencies which can include all the expected signal power.

The basic idea is to use the transformation,

$$s \rightarrow \frac{1}{T_s} \ln(z) \quad \text{or} \quad e^{sT_s} \rightarrow z$$

to convert from the  $s$  to  $z$  domain.

## Digital Filters

### Bilinear Transformation

The straightforward application of the transformation,  $s \rightarrow \frac{1}{T_s} \ln(z)$  would be the substitution,

$$H_z(z) = H_s(s) \Big|_{s \rightarrow \frac{1}{T_s} \ln(z)}$$

But that yields a  $z$ -domain function that is a transcendental function of  $z$  with infinitely many poles. The exponential function can be expressed as the infinite series,

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

and then approximated by truncating the series.

## Digital Filters

### Bilinear Transformation

Truncating the exponential series at two terms yields the transformation,

$$1 + sT_s \rightarrow z$$

or

$$s \rightarrow \frac{z-1}{T_s}$$

(This approximation is identical to the finite difference method using forward differences to approximate derivatives.) But, this method has a problem. It is possible to transform a stable  $s$ -domain function into an unstable  $z$ -domain function.

# Digital Filters

## Bilinear Transformation

The stability problem can be solved by a very clever modification of the idea of truncating the series. Express the exponential as

$$e^{sT_s} = \frac{e^{\frac{sT_s}{2}}}{e^{-\frac{sT_s}{2}}} \rightarrow z$$

Then approximate both numerator and denominator with a truncated series.

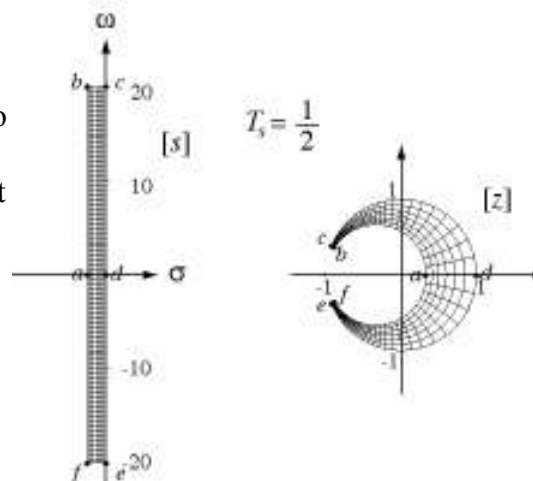
$$\frac{1 + \frac{sT_s}{2}}{1 - \frac{sT_s}{2}} \rightarrow z \quad \longrightarrow \quad s \rightarrow \frac{2}{T_s} \frac{z-1}{z+1}$$

This is called the *bilinear* transformation because both numerator and denominator are linear functions of  $z$ .

# Digital Filters

## Bilinear Transformation

The bilinear transformation has the quality that every point in the  $s$  plane maps into a unique point in the  $z$  plane, *and vice versa*. Also, the left half of the  $s$  plane maps into the interior of the unit circle in the  $z$  plane so a stable  $s$ -domain system is transformed into a stable  $z$ -domain system.



# Digital Filters

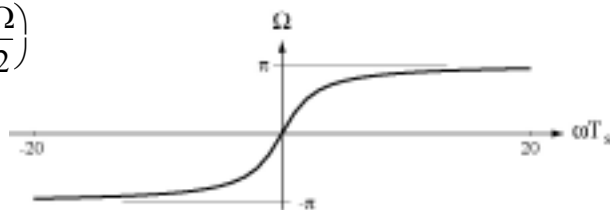
## Bilinear Transformation

The bilinear transformation is unique among the digital filter design methods because of the unique mapping of points between the two complex planes. There is however a “warping” effect. It can be seen by mapping real frequencies in the  $z$  plane (the unit circle) into corresponding points in the  $s$  plane. Letting  $z = e^{j\Omega}$  with  $\Omega$  real, the corresponding contour in the  $s$  plane is

$$s = \frac{2}{T_s} \frac{e^{j\Omega} - 1}{e^{j\Omega} + 1} = j \frac{2}{T_s} \tan\left(\frac{\Omega}{2}\right)$$

or

$$\Omega = 2 \tan^{-1}\left(\frac{\omega T_s}{2}\right)$$

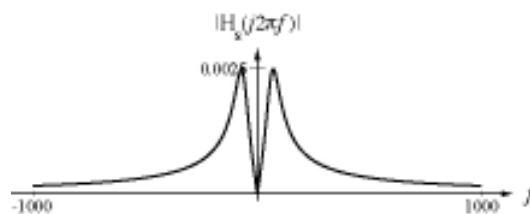


# Digital Filters

## Bilinear Transformation

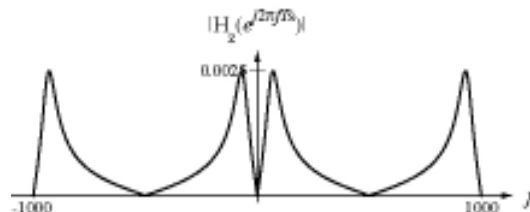
Bilinear approximation of

$$H_s(s) = \frac{s}{s^2 + 400s + 2 \times 10^5}$$



with a 1 kHz sampling rate  
yields

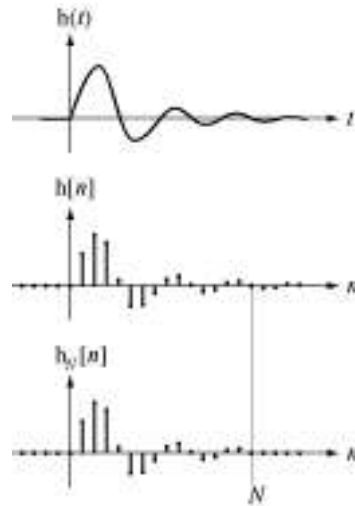
$$H(z) = \frac{z^2 - 1}{z^2 - 1.52z + 0.68}$$



# Digital Filters

## FIR Filters

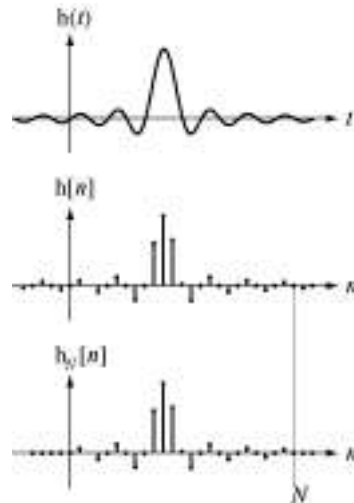
FIR digital filters are based on the idea of approximating an ideal impulse response. Practical CT filters have infinite-duration impulse response. The FIR filter approximates this impulse by sampling it and then *truncating* it to a finite time ( $N$  impulses in the illustration).



# Digital Filters

## FIR Filters

FIR digital filters can also approximate *non-causal* filters by truncating the impulse response both before time  $t = 0$  and after some later time which includes most of the signal energy of the ideal impulse response.

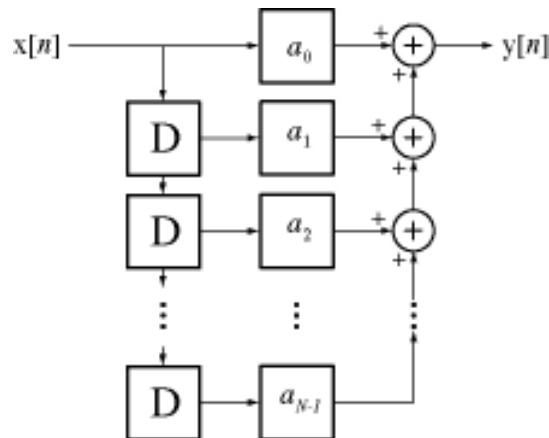


# Digital Filters

## FIR Filters

The design of an FIR filter is the essence of simplicity. It consists of multiple *feedforward* paths, each with a different delay and weighting factor and all of which are summed to form the response.

$$h_N[n] = \sum_{m=0}^{N-1} a_m \delta[n-m]$$



# Digital Filters

## FIR Filters

Since this filter has no feedback paths its transfer function is of the form,

$$H_N(z) = \sum_{m=0}^{N-1} a_m z^{-m}$$

and it is guaranteed stable because it has  $N - 1$  poles, all of which are located at  $z = 0$ .

# Digital Filters

## FIR Filters

The effect of truncating an impulse response can be modeled by multiplying the ideal impulse response by a “window” function. If a CT filter’s impulse response is truncated between  $t = 0$  and  $t = T$ , the truncated impulse response is

$$h_T(t) = \begin{cases} h(t) & , \quad 0 < t < T \\ 0 & , \quad \text{otherwise} \end{cases} = h(t)w(t)$$

where, in this case,

$$w(t) = \text{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$$

# Digital Filters

## FIR Filters

The frequency-domain effect of truncating an impulse response is to convolve the ideal frequency response with the transform of the window function.

$$H_T(f) = H(f) * W(f)$$

If the window is a rectangle,

$$W(f) = T \text{sinc}(Tf) e^{-j\pi f T}$$

# Digital Filters

## FIR Filters

Let the ideal transfer function be  $H(f) = \text{rect}\left(\frac{f}{2B}\right)e^{-j\pi fT}$   
 The corresponding impulse response is

$$h(t) = 2B \text{sinc}\left(2B\left(t - \frac{T}{2}\right)\right)$$

The truncated impulse response is

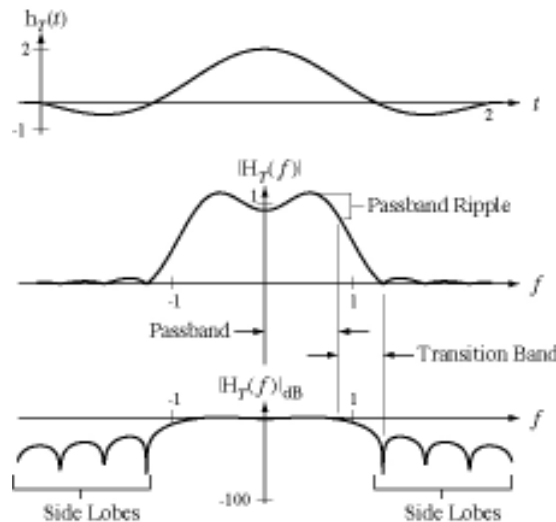
$$h_T(t) = 2B \text{sinc}\left(2B\left(t - \frac{T}{2}\right)\right) \text{rect}\left(\frac{t - \frac{T}{2}}{T}\right)$$

The transfer function for the truncated impulse response is

$$H_T(f) = \text{rect}\left(\frac{f}{2B}\right)e^{-j\pi fT} * T \text{sinc}(Tf)e^{-j\pi fT}$$

# Digital Filters

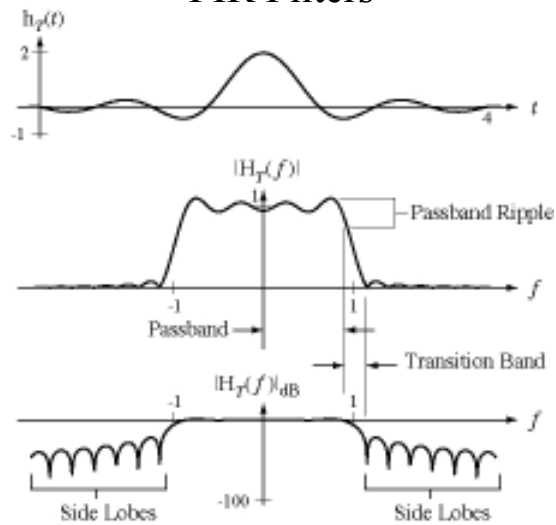
## FIR Filters





# Digital Filters

## FIR Filters



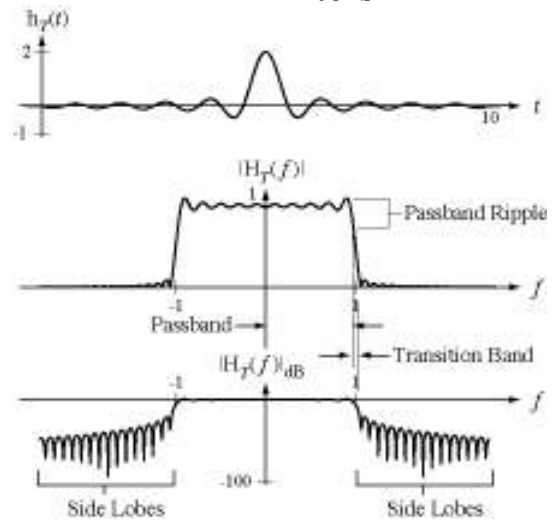
Chapter 12

ENSC 380 – Linear Systems

65

# Digital Filters

## FIR Filters



Chapter 12

ENSC 380 – Linear Systems

66

# Digital Filters

## FIR Filters

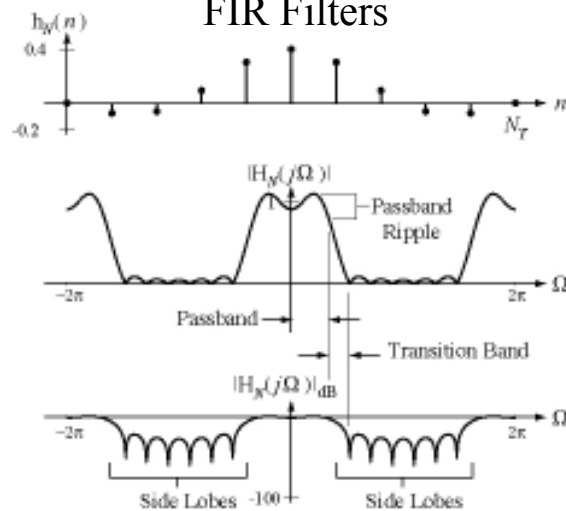
The effects of windowing a DT filter's impulse response are similar to the windowing effects on a CT filter.

$$h_N[n] = \begin{cases} h[n], & 0 \leq n < N \\ 0, & \text{otherwise} \end{cases} = h[n]w[n]$$

$$H_N(j\Omega) = H(j\Omega) \otimes W(j\Omega)$$

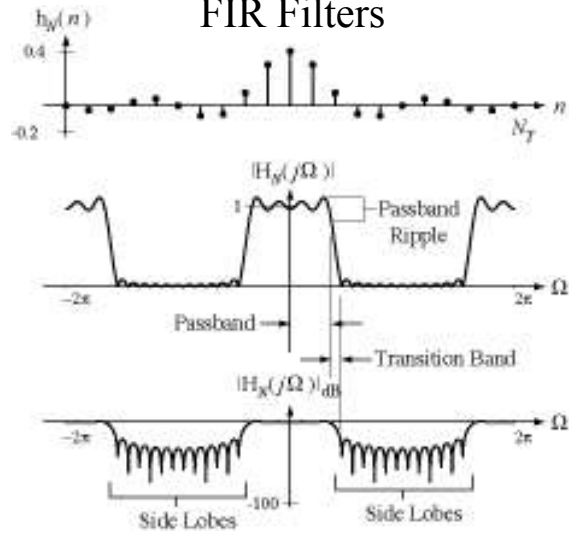
# Digital Filters

## FIR Filters



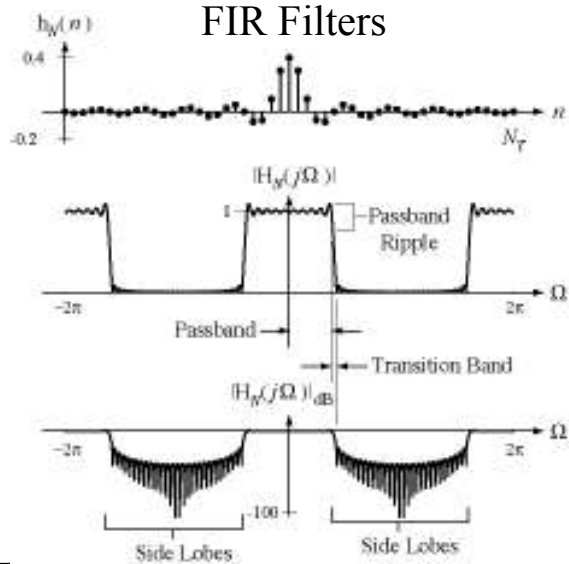
# Digital Filters

## FIR Filters



# Digital Filters

## FIR Filters



# Digital Filters

## FIR Filters

The “ripple” effect in the frequency domain can be reduced by windows of different shapes. The shapes are chosen to have DTFT's which are more confined to a narrow range of frequencies. Some commonly used windows are

1. von Hann (Hanning)  $w[n] = \frac{1}{2} \left[ 1 - \cos\left(\frac{2\pi n}{N-1}\right) \right], \quad 0 \leq n < N$

2. Bartlett (triangular)  $w[n] = \begin{cases} \frac{2n}{N-1}, & 0 \leq n \leq \frac{N-1}{2} \\ 2 - \frac{2n}{N-1}, & \frac{N-1}{2} \leq n < N \end{cases}$

# Digital Filters

## FIR Filters

3. Hamming  $w[n] = 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right), \quad 0 \leq n < N$

4. Blackman  $w[n] = 0.42 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right) + 0.08 \cos\left(\frac{4\pi n}{N-1}\right), \quad 0 \leq n < N$

5. Kaiser

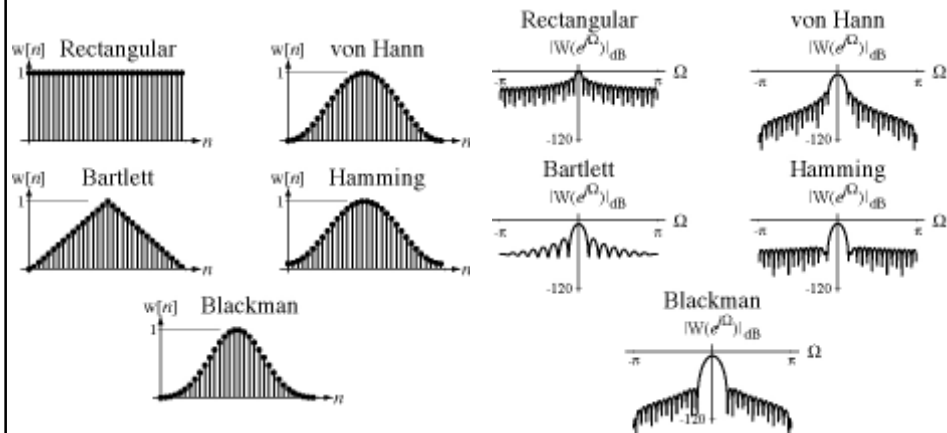
$$w[n] = \frac{I_0\left(\omega_a \sqrt{\left(\frac{N-1}{2}\right)^2 - \left(n - \frac{N-1}{2}\right)^2}\right)}{I_0\left(\omega_a \frac{N-1}{2}\right)}$$

# Digital Filters

## FIR Filters

### Windows

### Window Transforms



Chapter 12

ENSC 380 – Linear Systems

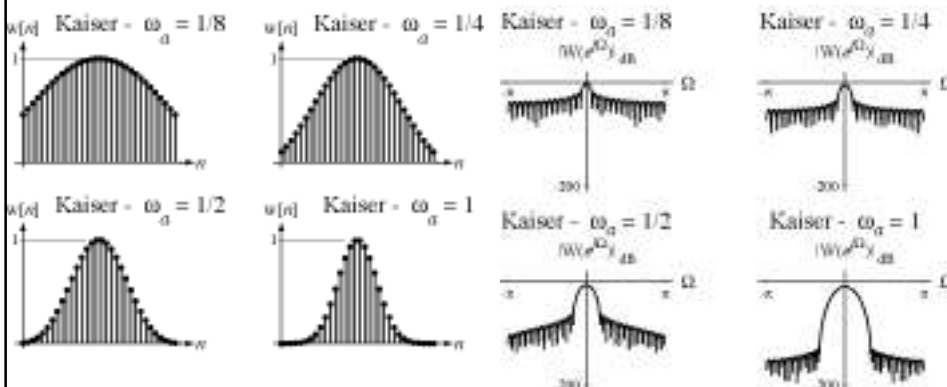
73

# Digital Filters

## FIR Filters

### Windows

### Window Transforms



Chapter 12

ENSC 380 – Linear Systems

74

# Standard Realizations

- Realization of a DT system closely parallels the realization of a CT system
- The basic forms, canonical, cascade and parallel have the same structure
- A CT system can be realized with integrators, summers and multipliers
- A DT system can be realized with delays, summers and multipliers

Chapter 12

ENSC 380 – Linear Systems

75

# Standard Realizations

## Canonical

Diagram illustrating a Canonical Realization of a discrete-time system. The input  $X(z)$  is processed through a series of summer nodes (indicated by  $+$  signs) and delay blocks ( $1/z$ ). The system is composed of two main sections: a feedback path (bottom) and a feedforward path (top).

The feedback path (bottom) consists of a series of summer nodes and delay blocks. The input  $X(z)$  is added to the output of a summer node, which is then delayed by  $1/z$ . This signal is then multiplied by  $a_{N-1}$  and added to the input of the next summer node. This process continues through a series of summer nodes and delay blocks, with the final output being  $Y(z)$ .

The feedforward path (top) consists of a series of summer nodes and delay blocks. The input  $X(z)$  is added to the output of a summer node, which is then delayed by  $1/z$ . This signal is then multiplied by  $b_N$  and added to the input of the next summer node. This process continues through a series of summer nodes and delay blocks, with the final output being  $Y(z)$ .

Labels in the diagram include:

- Summer
- Delay
- Multiplier

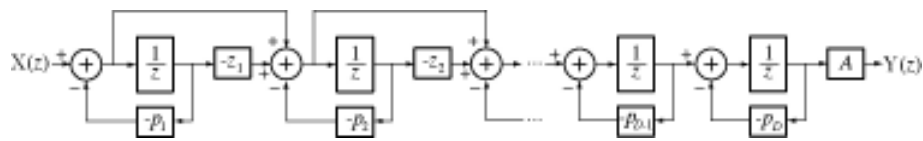
Chapter 12

ENSC 380 – Linear Systems

76

# Standard Realizations

## Cascade



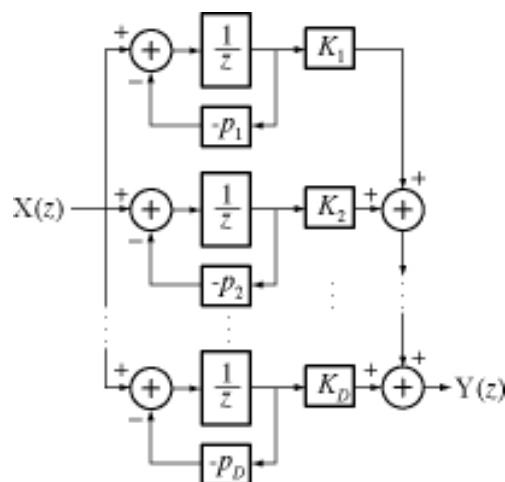
Chapter 12

ENSC 380 – Linear Systems

77

# Standard Realizations

## Parallel



Chapter 12

ENSC 380 – Linear Systems

78