Serie 4, Aufgabe 2

Let $(X, \mathfrak{M}, \mu)$ be a measure space and $f : X \to Y$ a map. Define

$$\mathfrak{N} := \{A \subset Y : f^{-1}(A) \in \mathfrak{M}\}, \quad \nu(A) := \mu(f^{-1}(A)), A \in \mathfrak{N}.$$

Show that $(Y, \mathfrak{N}, \nu)$ is a measure space. Find all $\mathfrak{N}$-measurable maps $g : Y \to \mathbb{C}$ and show that

$$\int_Y g \, \mathrm{d}\nu = \int_X g \circ f \, \mathrm{d}\mu,$$

if either side exists.

Proof. $\mathfrak{N}$ is a σ-algebra (Def. 1.3):

(i) $f^{-1}(Y) = X \in \mathfrak{M} \Rightarrow Y \in \mathfrak{N}$.

(ii) $A \in \mathfrak{N} \Rightarrow f^{-1}(A) \in \mathfrak{M} \Rightarrow (f^{-1}(A))^c = f^{-1}(A^c) \in \mathfrak{M} \Rightarrow A^C \in \mathfrak{N}$.

(iii) $A_i \in \mathfrak{N}, i \in \mathbb{N} \Rightarrow f^{-1}(A_i) \in \mathfrak{M}, i \in \mathbb{N} \Rightarrow \bigcup_{n=1}^{\infty} f^{-1}(A_i) = f^{-1}(\bigcup_{n=1}^{\infty} A_i) \in \mathfrak{M} \Rightarrow \bigcup_{n=1}^{\infty} A_i \in \mathfrak{N}$.

ν is a measure on $\mathfrak{N}$ (Def. 2.1):

(i) Let $A_1, A_2, \dots \in \mathfrak{N}$ be disjoint. Then $f^{-1}(A_1), f^{-1}(A_2), \dots \in \mathfrak{M}$ are also disjoint. Hence:

$$\nu\left(\bigcup_{i=1}^{\infty} A_i\right) = \mu\left(f^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \mu\left(\bigcup_{i=1}^{\infty} f^{-1}(A_i)\right) = \sum_{i=1}^{\infty} \mu(f^{-1}(A_i)) = \sum_{i=1}^{\infty} \nu(A_i).$$

(ii) $f^{-1}(\emptyset_Y) = \emptyset_X \in \mathfrak{M}$. $\mu(\emptyset_X) = 0$. Hence $\nu(\emptyset_Y) = 0 < \infty$.

Let $V \in \mathbb{C}$ be open. Then

$$\begin{aligned} g : Y \to \mathbb{C} \text{ is measurable} &\Leftrightarrow g^{-1}(V) \in \mathfrak{N} \\ &\Leftrightarrow f^{-1}(g^{-1})(V) = (g \circ f)^{-1}(V) \in \mathfrak{M}. \end{aligned}$$

So $g : Y \to \mathbb{C}$ is measurable iff $g \circ f : X \to \mathbb{C}$ is measurable.

If $\int_Y g \, \mathrm{d}\nu$ exists, then

$$\int_Y g \, \mathrm{d}\nu = \int_Y (\mathrm{Re}\ g)^+ \, \mathrm{d}\nu - \int_Y (\mathrm{Re}\ g)^- \, \mathrm{d}\nu + \mathrm{i}\left(\int_Y (\mathrm{Im}\ g)^+ \, \mathrm{d}\nu - \int_Y (\mathrm{Im}\ g)^- \, \mathrm{d}\nu\right).$$

Now consider a simple measurable s with $0 \leq s \leq \mathrm{Re}(g)^+$, $s = \sum_{i=1}^{n} \alpha_i \chi_{A_i}$, $\alpha_i \in \mathbb{R}^+, A_i \in \mathfrak{N}$. Then $0 \leq s \circ f \leq \mathrm{Re}(g \circ f)^+$ is also simple and measurable. Since $\chi_{f^{-1}(A)} = \chi_A \circ f$:

$$\begin{aligned} \int_Y s \, \mathrm{d}\nu = \int_Y \sum_{i=1}^{n} \alpha_i \nu(A_i) &= \sum_{i=1}^{n} \alpha_i \mu(f^{-1}(A_i)) \\ &= \int_{f^{-1}(Y)} \sum_{i=1}^{n} \alpha_i \chi_{f^{-1}(A_i)} \, \mathrm{d}\mu \\ &= \int_X \sum_{i=1}^{n} \alpha_i \chi_{A_i} \circ f \, \mathrm{d}\mu \\ &= \int_X s \circ f \, \mathrm{d}\mu. \end{aligned}$$

From this, $\int_Y (\mathrm{Re}\ g)^+ \,\mathrm{d}\nu \le \int_X (\mathrm{Re}\ g \circ f)^+ \,\mathrm{d}\mu$. Since $Y \supset f(X)$, '$\ge$' holds, too. Hence $\int_Y (\mathrm{Re}\ g)^+ \,\mathrm{d}\nu = \int_X (\mathrm{Re}\ g \circ f)^+ \,\mathrm{d}\mu$. Proceed analogously for the other terms that make up $\int_Y g \,\mathrm{d}\nu$. $\square$