Serie 4, Aufgabe 4

Let $(X, \mathfrak{M}, \mu)$ be a measure space, $f_n, g_n, f, g \in \mathscr{L}_1(\mu), n \in \mathbb{N}$, with $f_n \to f, g_n \to g, n \to \infty$ pointwise nearly everywhere, $|f_n| \leq g_n, n \in \mathbb{N}$, and $\int_X g_n \, \mathrm{d}\mu \to \int_X g \, \mathrm{d}\mu, n \to \infty$. Show that

$$\int_X f_n \, \mathrm{d}\mu \to \int_X f \, \mathrm{d}\mu, n \to \infty.$$

Proof. For real-valued f, g, f_n, g_n, we have $g_n - f_n \geq 0$ for all n. The sequence $(g_n - f_n)_{n \in \mathbb{N}}$ converges to $g - f$ nearly everywhere. By Fatou's Lemma, we obtain

$$\begin{aligned}
\int_X g \, \mathrm{d}\mu - \int_X f \, \mathrm{d}\mu &= \int_X g - f \, \mathrm{d}\mu \\
&= \int_X \lim_{n \to \infty} g_n - f_n \, \mathrm{d}\mu \\
&\leq \liminf_{n \to \infty} \int_X g_n - f_n \, \mathrm{d}\mu \\
&= \liminf_{n \to \infty} \int_X g_n \, \mathrm{d}\mu + \int_X -f_n \, \mathrm{d}\mu \\
&= \int_X g \, \mathrm{d}\mu + \liminf_{n \to \infty} \int_X -f_n \, \mathrm{d}\mu \\
&= \int_X g \, \mathrm{d}\mu - \limsup_{n \to \infty} \int_X f_n \, \mathrm{d}\mu.
\end{aligned}$$

Hence $\int_X f \, \mathrm{d}\mu \geq \limsup_{n \to \infty} \int_X f_n \, \mathrm{d}\mu$.

Similarly, $g_n + f_n \geq g - |f_n| \geq 0$ for all n. By the same logic, we obtain $\int_X f \, \mathrm{d}\mu \leq \liminf_{n \to \infty} \int_X f_n \, \mathrm{d}\mu$. Hence $\int_X f \, \mathrm{d}\mu = \lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu$.

For complex-valued functions, note that $g_n \pm \mathrm{Re}(f_n) \geq 0$ and $g_n \pm \mathrm{Im}(f_n) \geq 0$. From the above, it follows that

$$\int_X \mathrm{Re}(f) \, \mathrm{d}\mu = \lim_{n \to \infty} \int_X \mathrm{Re}(f_n) \, \mathrm{d}\mu$$

and

$$\int_X \mathrm{Im}(f) \, \mathrm{d}\mu = \lim_{n \to \infty} \int_X \mathrm{Im}(f_n) \, \mathrm{d}\mu.$$

So $\int_X f \, \mathrm{d}\mu = \lim_{n \to \infty} \int_X f_n \, \mathrm{d}\mu$. $\square$