Serie 4, Aufgabe 1

Show that the following holds for each sequence $(E_n)_{n\in\mathbb{N}}$ of measurable sets in a measure space $(X, \mathfrak{M}, \mu)$:

$$\mu(\bigcup_{n=1}^{\infty} E_n) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

Conclude that the countable union of null sets is a null set.

Proof. Define $(A_n)_{n\in\mathbb{N}}$ with $A_1 := E_1$ and $A_{n+1} = E_{n+1} \setminus \bigcup_{k=1}^{n} A_k$ for $n \in \mathbb{N}$. All A_n can be written as finite intersections of measurable sets and are hence measurable, too. Note that $\bigcup_{n=1}^{\infty} E_n = \dot{\bigcup}_{n=1}^{\infty} A_n$ and $0 \leq \mu(A_n) \leq \mu(E_n)$ for all n since $A_n \subset E_n$. Hence

$$\mu(\bigcup_{n=1}^{\infty} E_n) = \mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n) \leq \sum_{n=1}^{\infty} \mu(E_n).$$

If $\mu(E_n) = 0$ for all n, then their union has measure zero, too. $\square$