Series 5

Exercise 1

Check if the condition

$$\int_X |f(x)|\,\mathrm{d}x < \infty$$

ist met for the Riemann integral for the following functions $f : X \to \mathbb{R}$:

(i) $X = (1, \infty)$, $f(x) := x^s, s \in \mathbb{R}$.

Answer. For $s = -1$:

$$\int_X |x^s|\,\mathrm{d}x = \lim_{x\to\infty} \ln(x) - \ln(1) = \infty.$$

For $s \neq 1$:

$$\int_X |x^s|\,\mathrm{d}x = \frac{1}{s+1}(\lim_{x\to\infty} x^{s+1} - 1) = \begin{cases} \infty, & s > -1, \\ -\frac{1}{s+1}, & s < -1. \end{cases}.$$

So 'no' for $s \geq -1$ and 'yes' for $s < -1$.

(ii) $X = \mathbb{R}, f(x) := e^{-x^2}$.

Answer. This'll probably be the wrong way round, but here goes: The probability density function of the standard normal is given by $g : \mathbb{R} \to \mathbb{R}$:

$$g(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}.$$

So $g(x) = \sqrt{\frac{f(x)}{2\pi}}$. So $f(x) = 2\pi(g(x))^2$. Since $0 < g(\mathbb{R}) < 1$, $(g(x))^2 < g(x)$ for all $x \in \mathbb{R}$. Since $\int_{-\infty}^{\infty} g(x)\,\mathrm{d}x = 1$, $\int_{-\infty}^{\infty}(g(x))^2\,\mathrm{d}x \leq 1$. So $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x \leq 2\pi < \infty$. So 'yes'.

Exercise 2

(a) Let (X, τ) be a topological space. Show that $C_c(X)$ is a vector subspace of $C(X)$. Further show that $f, g \in C_c(X) \Rightarrow fg \in C_c(X)$.

Beweis. $C_c(X)$ is a vector subspace:

- $0 \in C(X)$ is continuous. supp $0 = \overline{\emptyset} = \emptyset$, which is compact. So $0 \in C_c(X)$.
- $f, g \in C_c(X)$. Then $f + g$ is continuous. If $f(x) = g(x) = 0$, then $(f+g)(x) = 0$. Further, $\overline{A \cup B} = \overline{A} \cup \overline{B}$. Hence:

$$\begin{aligned} \text{supp } (f+g) &= \overline{\{x \in X : f(x) + g(x) \neq 0\}} \\ &\subset \overline{\{x \in X : f(x) \neq 0 \vee g(x) \neq 0\}} \\ &= \overline{\{x \in X : f(x) \neq 0\} \cup \{x \in X : g(x) \neq 0\}} \\ &= \text{supp } f \cup \text{supp } g, \end{aligned}$$

which is compact as the union of two compact sets. Closed subsets of compact sets are compact, too, hence supp $(f+g)$ is compact. So $f + g \in C_c(X)$.

- $f \in C_c(X), \alpha \in \mathbb{C}$. Then αf is continuous. If $\alpha = 0, \alpha f \equiv 0 \in C_c(X)$. If $\alpha \neq 0$, $\alpha f(x) \neq 0 \Leftrightarrow f(x) \neq 0$. Hence supp $\alpha f =$ supp f, which is compact. So $\alpha f \in C_c(X)$.

For $f, g \in C_c(X)$, fg is continuous. Further: $(fg)(x) \neq 0 \Leftrightarrow f(x) \neq 0 \wedge g(x) \neq 0$. So

$$\text{supp } fg = \overline{\{x \in X : f(x) \neq 0\} \cap \{x \in X : g(x) \neq 0\}} \subset \text{supp } f.$$

Closed subsets of compact sets are compact. So $fg \in C_c(X)$. □

(b) Let $f : \mathbb{R} \to \mathbb{R}, f := \chi_{[-\pi,\pi]}$ and

$$g : \mathbb{R} \to \mathbb{R},$$
$$x \mapsto \begin{cases} \sin(x), & x \geq 0, \\ 0, & x < 0. \end{cases}$$

Determine $\text{supp } f, \text{supp } g, \text{supp } fg$. Check if $f, g, fg \in C_c(\mathbb{R})$.

Further: For $a \in \mathbb{R}, \varepsilon > 0$, define $f_{(\varepsilon,a)} : \mathbb{R} \to \mathbb{R}, f_{(\varepsilon,a)}(x) := f(\varepsilon x - a)$. Determine $\text{supp } f_{(\varepsilon,a)}$.

Answer. $f \notin C_c(\mathbb{R})$ since it's not continuous.

$$\text{supp } f = \overline{\{x \in \mathbb{R} : f(x) \neq 0\}} = \overline{[-\pi, \pi]} = [-\pi, \pi].$$

$g \notin C_c(\mathbb{R})$ since its support isn't bounded.

$$\text{supp } g = \overline{\{x \in \mathbb{R} : g(x) \neq 0\}} = \overline{\mathbb{R}_{\geq 0} \setminus \pi\mathbb{Z}} = \mathbb{R}_{\geq 0}.$$

$fg \in C_c(\mathbb{R})$ since it's continuous ($(fg)|_{[0,\pi]} \equiv \sin(x)|_{[0,\pi]}, (fg)|_{[0,\pi]^c} \equiv 0|_{[0,\pi]^c}$, and $\sin(0) = \sin(\pi) = 0$) and its support is compact.

$$\begin{aligned} \text{supp } fg &= \overline{\{x \in \mathbb{R} : f(x)g(x) \neq 0\}} \\ &= \overline{\{x \in [-\pi, \pi] : g(x) \neq 0\}} \\ &= \overline{\{x \in [0, \pi] : \sin(x) \neq 0\}} \\ &= \overline{(0, \pi)} = [0, \pi]. \end{aligned}$$

Further:

$$\begin{aligned} \text{supp } f_{(\varepsilon,a)} &= \overline{\{x \in \mathbb{R} : f(\varepsilon x - a) \neq 0\}} \\ &= \overline{\{x \in \mathbb{R} : \varepsilon x - a \in [-\pi, \pi]\}} \\ &= \overline{\{x \in \mathbb{R} : x \in [\frac{a - \pi}{\varepsilon}, \frac{a + \pi}{\varepsilon}]\}} \\ &= [\frac{a - \pi}{\varepsilon}, \frac{a + \pi}{\varepsilon}]. \end{aligned}$$

Exercise 3

(a) Let the hypotheses of Riesz representation theorem be met. Prove the following:

- $E \in \mathfrak{M}$ is σ-compact $\Rightarrow$ E has σ-finite measure.

  Beweis. Since E is σ-compact, there exist compact $K_i \subset X, i \in \mathbb{N}$, such that $E = \bigcup_{i=1}^{\infty} K_i$. By Theorem 4.7(i), $\mu(K_i) < \infty$ for all $i \in \mathbb{N}$. □

- $E \in \mathfrak{M}$ has σ-finite measure $\Rightarrow$ E is inner regular.

  Beweis. For E with $\mu(E) < \infty$, Theorem 4.7 suffices. For $\mu(E) = \infty$, we may write $E = \bigcup_{i=1}^{\infty} E_i$, for E_i with $\mu(E_i) < \infty$ for all $i \in \mathbb{N}$. Note that

$$\mu(\bigcup_{i=1}^{n} E_i) \leq \sum_{i=1}^{n} \mu(E_i) < \infty.$$

Hence, by Theorem 4.7,

$$\mu(\bigcup_{i=1}^{n} E_i) = \sup\{\mu(K) : K \subset \bigcup_{i=1}^{n} E_i, K \text{ compakt}\}.$$

As $n \to \infty$, we obtain

$$\mu(\bigcup_{i=1}^{\infty} E_i) = \mu(E) = \sup\{\mu(K) : K \subset E, K \text{ compakt}\}. \qquad \square$$

(b) Let (X, d) be a metric space, λ a Borel measure on X with $\lambda(K) < \infty$ for all compact $K \subset X$. Prove that

$$\Lambda : C_c(X, \mathbb{R}) \to \mathbb{R},$$
$$f \mapsto \int_X f \, \mathrm{d}\lambda$$

is a monotonous positive linear form.

Beweis. (1) Λ is well-defined:
$f \in C_c(X, \mathbb{R})$ implies $f(X) \subset \mathbb{R}$ compact (comment re: Definition 4.3), i.e., bounded by a constant function $c(x) = \beta \in \mathbb{R}, x \in X$. $f \in C_c(X, \mathbb{R})$ also implies that f is non-zero on a (subset of) a compact set $K \subset X$. So $\int_X f \, \mathrm{d}\lambda \leq \beta\mu(K) < \infty$.

(2) Λ is a map from a $\mathbb{C}$ vector space to $\mathbb{R} \subset \mathbb{C}$.

(3) Λ is $\mathbb{C}$-linear:
From (1): $f, g \in C_c(X, \mathbb{R}) \Rightarrow f, g \in \mathscr{L}_1(\lambda)$. $\Lambda(f + \alpha g) = \Lambda(f) + \alpha\Lambda(g)$ follows from Theorem 3.2.

(4) Λ is positive since by Proposition 2.6:

$$0 \leq f \Rightarrow 0 = \int_X 0 \, \mathrm{d}\lambda \leq \int_X f \, \mathrm{d}\lambda.$$

So Λ is a positive linear form. All positive linear forms are monotonous (as proved in lecture). $\square$

Exercise 4 (Lebesgue sums)

Let $(X, \mathfrak{M}, \mu)$ be a measure space and let $f : X \to \mathbb{R}$ be positive, bounded and measurable. Define

$$A_j := \{x \in X : \alpha + \frac{(j-1)(\beta - \alpha)}{n} \leq f(x) < \alpha + \frac{j(\beta - \alpha)}{n}\}$$

and

$$A_n := \{x \in X : \alpha + \frac{(n-1)(\beta - \alpha)}{n} \leq f(x) \leq \beta\}$$

for $j = 1, \ldots, n-1, n \in \mathbb{N}$ and with $\alpha := \inf\{f(x) : x \in X\}$, $\beta := \sup\{f(x) : x \in X\}$. Prove that

$$\int_X f \, \mathrm{d}\mu = \lim_{n \to \infty} s_n,$$

where

$$s_n := \sum_{j=1}^{n} \left(\alpha + \frac{(j-1)(\beta - \alpha)}{n}\right) \mu(A_j),$$

$n \in \mathbb{N}$.

Beweis. Note that $s_n = \int_X z_n \, \mathrm{d}\mu$, where $z_n : X \to \mathbb{R}, z_n := \sum_{j=1}^{n} \left(\alpha + \frac{(j-1)(\beta-\alpha)}{n}\right) \chi_{A_j}, n \in \mathbb{N}$, is a simple measurable function.

Further note that $z_n \leq f$: For $i \neq j, A_i \cap A_j = \emptyset$. Hence, $z_n(x) = \alpha + \frac{(k-1)(\beta-\alpha)}{n} \leq f(x)$ for one $k \in \{1, \ldots, n\}$. Since f is bounded, so is z_n.

Moreover, $z_n \to f, n \to \infty$: As $n \to \infty$, $\left(\alpha + \frac{j(\beta-\alpha)}{n}\right) - \left(\alpha + \frac{(j-1)(\beta-\alpha)}{n}\right) \to 0$ for all $j \in \{0, \ldots, j-n\}$. Similarly, $\beta - \left(\alpha + \frac{(n-1)(\beta-\alpha)}{n}\right) \to 0$. So for $n \to \infty$ and for $k = 1, \ldots, n$, $z_n(x) = \frac{(k-1)(\beta-\alpha)}{n} \to f(x)$.

Now assume that there exists a $g \in \mathscr{L}_1$ such that $g \geq z_n$ for all n. Then by dominated convergence, we obtain

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \int_X z_n \,\mathrm{d}\mu = \int_X \lim_{n\to\infty} z_n \,\mathrm{d}\mu = \int_X f \,\mathrm{d}\mu.$$

If no such g exists, then $f \notin \mathscr{L}_1$, so $\int_X f \,\mathrm{d}\mu = \infty$. By Fatou:

$$\infty = \int_X f \,\mathrm{d}\mu = \int_X \lim_{n\to\infty} z_n \,\mathrm{d}\mu \leq \liminf_{n\to\infty} \int_X z_n \,\mathrm{d}\mu = \liminf_{n\to\infty} s_n,$$

so $s_n \to \infty, n \to \infty$. $\square$