

Double-Clamped Beam Analysis (5LMA0)

1749471: Stijn van den Dungen **1739468:** Thom Nelissen
Group: 18

Boundary Conditions – Double-Clamped Case

$$\begin{aligned} \text{Left side: } w(0, t) &= 0, & \frac{\partial w}{\partial x}(0, t) &= 0 \\ \text{Right side: } w(L, t) &= 0, & \frac{\partial w}{\partial x}(L, t) &= 0 \end{aligned}$$

1 Natural Frequencies and Vibration Modes

1.1 Question 1: Linearity and Time-Invariance

Prove that the model (1) with the physical specifications of Table 2 and with the double-clamped boundary conditions is linear and time-invariant.

Recall: The Euler-Bernoulli beam equation is

$$\frac{\partial^2}{\partial x^2} \left(EI \frac{\partial^2 w}{\partial x^2} \right) + \rho A \frac{\partial^2 w}{\partial t^2} = q \quad (1)$$

Answer:

For a homogeneous material with constant properties E , I , ρ and A , (1) becomes:

$$EI \frac{\partial^4 w}{\partial x^4} + \rho A \frac{\partial^2 w}{\partial t^2} = q(x, t), \quad (2)$$

which is linear because:

1. All terms are linear combinations of w and its partial derivatives (first-order in terms of the operator applied to w).
2. There are no products of w with itself or its derivatives (no $w \cdot \frac{\partial w}{\partial t}$ terms).
3. There are no nonlinear functions of w or its derivatives.
4. The coefficients EI and ρA are constants that do not depend on w or its derivatives.

Hence, to prove linearity, we only have to prove that superposition holds. Suppose $w_1(x, t)$ is a solution to the equation with input $q_1(x, t)$, and $w_2(x, t)$ is a solution with input $q_2(x, t)$. Then:

$$EI \frac{\partial^4 w_1}{\partial x^4} + \rho A \frac{\partial^2 w_1}{\partial t^2} = q_1(x, t), \quad EI \frac{\partial^4 w_2}{\partial x^4} + \rho A \frac{\partial^2 w_2}{\partial t^2} = q_2(x, t) \quad (3)$$

Now consider the linear combination $w(x, t) = \alpha w_1(x, t) + \beta w_2(x, t)$ for arbitrary constants $\alpha, \beta \in \mathbb{R}$.

Substituting into the left-hand side of the (2):

$$EI \frac{\partial^4(\alpha w_1 + \beta w_2)}{\partial x^4} + \rho A \frac{\partial^2(\alpha w_1 + \beta w_2)}{\partial t^2} \quad (4)$$

$$= \alpha \left(EI \frac{\partial^4 w_1}{\partial x^4} + \rho A \frac{\partial^2 w_1}{\partial t^2} \right) + \beta \left(EI \frac{\partial^4 w_2}{\partial x^4} + \rho A \frac{\partial^2 w_2}{\partial t^2} \right) \quad (5)$$

$$= \alpha q_1(x, t) + \beta q_2(x, t) \quad (6)$$

Therefore, $w(x, t) = \alpha w_1(x, t) + \beta w_2(x, t)$ satisfies the equation with input $q(x, t) = \alpha q_1(x, t) + \beta q_2(x, t)$. This proves the superposition principle for linear systems.

Next to that, (1) is time-invariant because:

1. The coefficients EI and ρA are constants independent of time.
2. The differential equation structure does not change with time.

Then, suppose $w(x, t)$ is a solution to (2) with input $q(x, t)$, and suppose the time-shifted solution $\tilde{w}(x, t) := w(x, t - \tau)$ for any constant $\tau \in \mathbb{R}$.

We compute the partial derivatives of \tilde{w} with respect to x and t :

$$\frac{\partial \tilde{w}}{\partial x} = \frac{\partial w}{\partial x}(x, t - \tau), \quad \frac{\partial^4 \tilde{w}}{\partial x^4} = \frac{\partial^4 w}{\partial x^4}(x, t - \tau) \quad (7)$$

For the time derivatives, using the chain rule:

$$\frac{\partial \tilde{w}}{\partial t} = \frac{\partial}{\partial t}[w(x, t - \tau)] = \frac{\partial w}{\partial t}(x, t - \tau) \cdot 1 = \frac{\partial w}{\partial t}(x, t - \tau) \quad (8)$$

$$\frac{\partial^2 \tilde{w}}{\partial t^2} = \frac{\partial^2 w}{\partial t^2}(x, t - \tau) \quad (9)$$

Substituting \tilde{w} into the left-hand side of (2):

$$EI \frac{\partial^4 \tilde{w}}{\partial x^4} + \rho A \frac{\partial^2 \tilde{w}}{\partial t^2} = EI \frac{\partial^4 w}{\partial x^4}(x, t - \tau) + \rho A \frac{\partial^2 w}{\partial t^2}(x, t - \tau) \quad (10)$$

$$= q(x, t - \tau) \quad (11)$$

where the last equality follows from the fact that $w(x, t)$ satisfies the original equation (2). Therefore, $\tilde{w}(x, t) = w(x, t - \tau)$ is a solution to the equation with input $\tilde{q}(x, t) = q(x, t - \tau)$. This proves that the system is time-invariant.

1.2 Question 2: Free Harmonic Vibration

Prove that, for arbitrary real constants $C_1, C_2 \in \mathbb{R}$, the function $w(x, t) = \hat{w}(x)[C_1 \sin(\omega t) + C_2 \cos(\omega t)]$ is a free harmonic vibration of the beam if and only if $\hat{w} : [0, L] \rightarrow \mathbb{R}$ satisfies the differential equation

$$EI \frac{d^4 \hat{w}}{dx^4} - \rho A \omega^2 \hat{w} = 0 \quad (12)$$

In addition, show that any solution \hat{w} of (12) can be written as

$$\hat{w}(x) = A_1 \sin(\beta x) + A_2 \cos(\beta x) + A_3 \sinh(\beta x) + A_4 \cosh(\beta x) \quad (13)$$

where $\beta = \left(\frac{\rho A \omega^2}{EI}\right)^{1/4}$ and where (A_1, A_2, A_3, A_4) is a quadruple of real constants.

Answer:

Assuming $w(x, t) = \hat{w}(x)[C_1 \sin(\omega t) + C_2 \cos(\omega t)]$ is a solution to the free vibration problem (with $q = 0$), we can compute the time derivatives:

$$\frac{\partial w}{\partial t} = \hat{w}(x)[C_1 \omega \cos(\omega t) - C_2 \omega \sin(\omega t)] \quad (14)$$

$$\frac{\partial^2 w}{\partial t^2} = \hat{w}(x)[C_1(-\omega^2) \sin(\omega t) + C_2(-\omega^2) \cos(\omega t)] = -\omega^2 w(x, t) \quad (15)$$

Computing the spatial derivatives:

$$\frac{\partial^4 w}{\partial x^4} = \frac{d^4 \hat{w}}{dx^4}[C_1 \sin(\omega t) + C_2 \cos(\omega t)] \quad (16)$$

Substituting into the free vibration equation $EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} = 0$:

$$EI\frac{\partial^4 \hat{w}}{\partial x^4}[C_1 \sin(\omega t) + C_2 \cos(\omega t)] + \rho A(-\omega^2)\hat{w}(x)[C_1 \sin(\omega t) + C_2 \cos(\omega t)] = 0 \quad (17)$$

$$\left[EI\frac{\partial^4 \hat{w}}{\partial x^4} - \rho A\omega^2 \hat{w}(x) \right] [C_1 \sin(\omega t) + C_2 \cos(\omega t)] = 0 \quad (18)$$

For this to hold for all t , and for arbitrary constants C_1, C_2 (not both zero), we must have:

$$EI\frac{\partial^4 \hat{w}}{\partial x^4} - \rho A\omega^2 \hat{w} = 0 \quad (19)$$

If $\hat{w}(x)$ satisfies $EI\frac{\partial^4 \hat{w}}{\partial x^4} - \rho A\omega^2 \hat{w} = 0$, then direct substitution into the free vibration equation confirms that $w(x, t) = \hat{w}(x)[C_1 \sin(\omega t) + C_2 \cos(\omega t)]$ is indeed a solution.

Now, we seek exponential solutions of the form $\hat{w}(x) = e^{rx}$. We start by substituting $\hat{w} = e^{rx}$ into the ODE:

$$EIr^4 e^{rx} - \rho A\omega^2 e^{rx} = 0 \quad (20)$$

Since $e^{rx} \neq 0$, we obtain the characteristic equation:

$$EIr^4 - \rho A\omega^2 = 0 \Rightarrow r^4 = \frac{\rho A\omega^2}{EI} \quad (21)$$

Let $\beta^4 := \frac{\rho A\omega^2}{EI}$, so $\beta = \left(\frac{\rho A\omega^2}{EI}\right)^{1/4}$, then the four roots of $r^4 = \beta^4$ are:

$$r_1 = \beta, \quad r_2 = -\beta, \quad r_3 = i\beta, \quad r_4 = -i\beta \quad (22)$$

The corresponding exponential solutions are:

$$e^{\beta x}, \quad e^{-\beta x}, \quad e^{i\beta x}, \quad e^{-i\beta x} \quad (23)$$

which can be converted to real-valued functions. Then, the solution to the ODE is:

$$\hat{w}(x) = A_1 \sin(\beta x) + A_2 \cos(\beta x) + A_3 \sinh(\beta x) + A_4 \cosh(\beta x) \quad (24)$$

where $A_1, A_2, A_3, A_4 \in \mathbb{R}$ are arbitrary constants determined by boundary conditions.

1.3 Question 3: Natural Vibration Modes for Double-Clamped Beam

Prove that the natural vibration modes of the beam subject to double-clamped boundary conditions are given by

$$\hat{w}_n(x) = A_{n,1} \sin(\beta_n x) + A_{n,2} \cos(\beta_n x) + A_{n,3} \sinh(\beta_n x) + A_{n,4} \cosh(\beta_n x), \quad n = 1, 2, \dots \quad (25)$$

for a specific relation among the amplitudes $(A_{n,1}, A_{n,2}, A_{n,3}, A_{n,4})$. Derive that relation, and give an expression for the frequencies β_n as the roots of a transcendental equation of the form

$$f(\beta) = 0 \quad (26)$$

for a suitable function f . Show that this implies that the natural frequencies of the beam are given by

$$\omega_n = \beta_n^2 \left(\frac{EI}{\rho A} \right)^{1/2} \quad (27)$$

and that the free harmonic motion of the n th vibration mode assumes the form

$$w_n(x, t) = \hat{w}_n(x)[C_{1,n} \sin(\omega_n t) + C_{2,n} \cos(\omega_n t)] \quad (28)$$

for real constants $C_{1,n}$ and $C_{2,n}$ and for natural frequencies ω_n .

Answer:

From Question 2, the general spatial solution for the n -th mode is:

$$\hat{w}_n(x) = A_{n,1} \sin(\beta_n x) + A_{n,2} \cos(\beta_n x) + A_{n,3} \sinh(\beta_n x) + A_{n,4} \cosh(\beta_n x). \quad (29)$$

The boundary conditions are zero displacement and zero slope at both ends: $\hat{w}_n(0) = \hat{w}'_n(0) = 0$ and $\hat{w}_n(L) = \hat{w}'_n(L) = 0$.

1. Conditions at $x = 0$: Substituting $x = 0$ into the general solution:

$$\hat{w}_n(0) = A_{n,2} + A_{n,4} = 0 \implies A_{n,4} = -A_{n,2} \quad (30)$$

$$\hat{w}'_n(0) = \beta_n(A_{n,1} + A_{n,3}) = 0 \implies A_{n,3} = -A_{n,1} \quad (\text{since } \beta_n \neq 0) \quad (31)$$

Substituting these relations back simplifies the expression to:

$$\hat{w}_n(x) = A_{n,1}(\sin(\beta_n x) - \sinh(\beta_n x)) + A_{n,2}(\cos(\beta_n x) - \cosh(\beta_n x)). \quad (32)$$

2. Conditions at $x = L$: Applying $\hat{w}_n(L) = 0$ and $\hat{w}'_n(L) = 0$ results in a system for $A_{n,1}$ and $A_{n,2}$ of the form $M\mathbf{v} = 0$:

$$\begin{pmatrix} \sin(\beta_n L) - \sinh(\beta_n L) & \cos(\beta_n L) - \cosh(\beta_n L) \\ \cos(\beta_n L) - \cosh(\beta_n L) & -\sin(\beta_n L) - \sinh(\beta_n L) \end{pmatrix} \begin{pmatrix} A_{n,1} \\ A_{n,2} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (33)$$

For a non-trivial solution, so $A_{n,1}, A_{n,2}$ not both zero, the determinant of the matrix M must be zero:

$$\det(M) = -(\sin(\beta_n L) - \sinh(\beta_n L))(\sin(\beta_n L) + \sinh(\beta_n L)) - (\cos(\beta_n L) - \cosh(\beta_n L))^2 \quad (34)$$

$$= -(\sin^2(\beta_n L) - \sinh^2(\beta_n L)) - (\cos^2(\beta_n L) - 2 \cos(\beta_n L) \cosh(\beta_n L) + \cosh^2(\beta_n L)) \quad (35)$$

$$= -(\sin^2(\beta_n L) + \cos^2(\beta_n L)) + (\sinh^2(\beta_n L) - \cosh^2(\beta_n L)) + 2 \cos(\beta_n L) \cosh(\beta_n L). \quad (36)$$

Using the identities $\sin^2 \theta + \cos^2 \theta = 1$ and $\cosh^2 \theta - \sinh^2 \theta = 1$, this simplifies to:

$$-2 + 2 \cos(\beta_n L) \cosh(\beta_n L) = 0. \quad (37)$$

Thus, the natural frequencies are determined by the transcendental equation:

$$f(\beta) = \cos(\beta L) \cosh(\beta L) - 1 = 0. \quad (38)$$

of which the roots are the values β_n ($n = 1, 2, \dots$).

3. Mode Shapes and Frequencies: Next to that, from the first row of the matrix equation (33) (the boundary conditions at $x = L$, we can derive the relation between $A_{n,1}$ and $A_{n,2}$:

$$A_{n,2} = -A_{n,1} \frac{\sin(\beta_n L)}{\cos(\beta_n L) - \cosh(\beta_n L)}. \quad (39)$$

Then, using $A_{n,3} = -A_{n,1}$ and $A_{n,4} = -A_{n,2}$, the specific relations are:

$$A_{n,3} = -A_{n,1}, \quad (40)$$

$$A_{n,4} = -A_{n,2} = A_{n,1} \frac{\sin(\beta_n L)}{\cos(\beta_n L) - \cosh(\beta_n L)}. \quad (41)$$

Hence, we can write the natural vibration modes of the beam as

$$\hat{w}_n(x) = C_n \left[(\sin(\beta_n x) - \sinh(\beta_n x)) - \frac{\sin(\beta_n L) - \sinh(\beta_n L)}{\cos(\beta_n L) - \cosh(\beta_n L)} (\cos(\beta_n x) - \cosh(\beta_n x)) \right] \quad (42)$$

Combined with (40) and (41), this defines the specific relations among all amplitudes up to a scalar factor $A_{n,1}$ (or C_n). Finally, recalling $\beta_n = (\rho A \omega_n^2 / EI)^{1/4}$, the natural frequencies are:

$$\omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho A}}. \quad (43)$$

The free harmonic motion for the n -th mode is then:

$$w_n(x, t) = \hat{w}_n(x)[C_{1,n} \sin(\omega_n t) + C_{2,n} \cos(\omega_n t)] \quad (44)$$

1.4 Question 4: Numerical Solution

Numerically solve the transcendental equation (38) and find, for $N \geq 10$, the frequencies β_n , $n = 1, \dots, N$, the corresponding natural frequencies ω_n and the natural vibration modes \hat{w}_n . Plot the vibration modes $\hat{w}_n(x)$ for the wave numbers $n = 1, \dots, N$.

Answer:

The transcendental equation $\cos(\beta L) \cosh(\beta L) - 1 = 0$ was solved numerically using MATLAB's symbolic solver (`vpasolve`). For $N = 12$ modes, we systematically searched for roots near the initial guesses $\beta \approx i\pi/L$ for $i = 2, 3, \dots, N + 1$, which correspond to the approximate wavenumbers of the vibration modes.

The natural vibration modes are computed using (42) derived in Question 3. The constant was chosen as $C_n = 1$. The numerical results for the first 12 modes are summarized in Table 1.

Table 1: Numerical results for natural frequencies

Mode	1	2	3	4	5	6	7	8	9	10	11	12
β_n [1/m]	0.95	1.57	2.20	2.83	3.46	4.08	4.71	5.34	5.97	6.60	7.23	7.85
ω_n [rad/s]	26.1	71.9	141.0	233.0	348.1	486.2	647.3	831.4	1038.6	1268.8	1521.9	1798.1

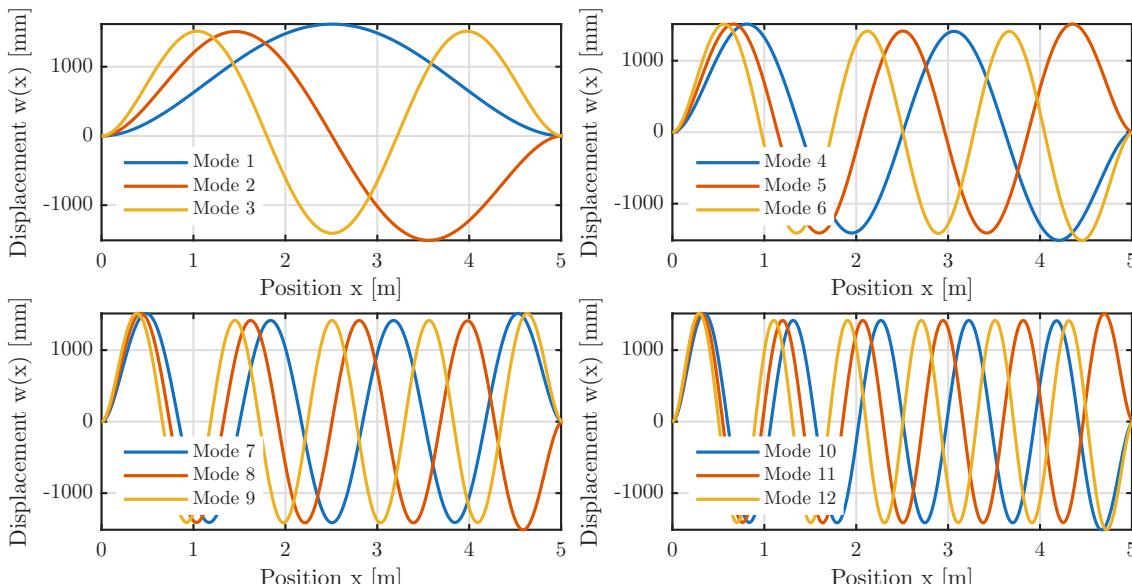


Figure 1: Natural vibration modes $\hat{w}_n(x)$ for the double-clamped beam ($n = 1, \dots, 12$)

The vibration modes $\hat{w}_n(x)$ for $n = 1, \dots, 12$ are visualized in Figure 1. The modes are spread over four figures, to keep the plots readable. The modes exhibit the characteristic behavior of double-clamped beams, with zero displacement and zero slope at both boundaries ($x = 0$ and $x = L$). Higher modes display an increasing number of oscillations across the beam length, consistent with the increasing wavenumbers β_n .

2 Simulation of Low Order Approximations

2.1 Question 5: Initial Deflection Simulation

Define an arbitrary, but physically realistic initial deflection $w(x, 0) = w_0(x)$ of the beam that complies to your boundary conditions. Assume that the initial velocity $\frac{\partial w}{\partial t}(x, 0) = 0$ and simulate for various values of N the deflection

$$w(x, t) = \sum_{n=1}^N \hat{w}_n(x) [C_{1,n} \sin(\omega_n t) + C_{2,n} \cos(\omega_n t)] \quad (45)$$

over a suitably long time period (without external load).

Answer:

The initial deflection must have zero slope and zero displacement at both ends of the beam. It must also be continuous and smooth, with enough oscillations to excite multiple modes for the purpose of this demonstration. A suitable choice is:

$$w_0(x) = \sin\left(\frac{3\pi x}{L}\right) \left(1 - \frac{x}{L}\right)^3 \left(\frac{x}{L}\right)^3 \quad (46)$$

This function has been plotted in Figure 2 along with the approximations for $N = 1, 3, 5$.

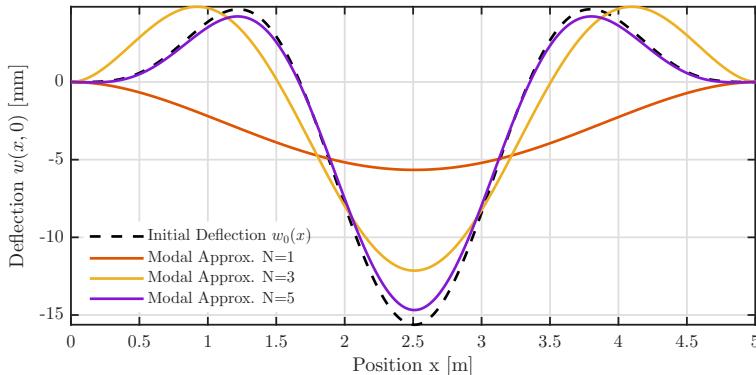
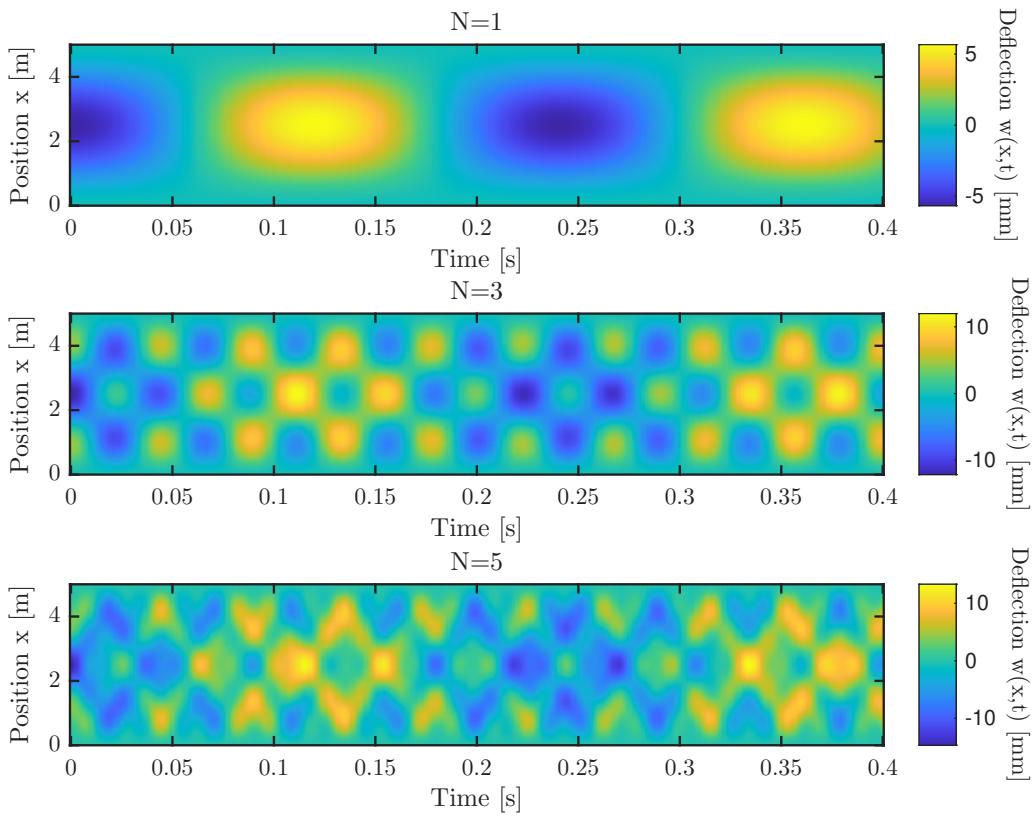


Figure 2: Initial deflection $w_0(x)$ and its approximations for $N = 1, 3, 5$

The coefficients $C_{2,n}$ are determined by projecting the initial deflection onto the normalized mode shapes:

$$C_{2,n} = \int_0^L w_0(x) \varphi_n(x) dx \quad (47)$$

The coefficients $C_{1,n}$ are set to zero due to the initial velocity condition, which the sine function does not satisfy. The deflection $w(x, t)$ is then simulated over a time period of 0.4 seconds for various values of N . The results show that as N increases, the approximation of the initial deflection improves significantly, capturing more of the beam's dynamic behavior.

Figure 3: Beam deflection $w(x, t)$ over time for various values of N

Define the normalized mode shape functions

$$\varphi_n(x) := \frac{\hat{w}_n(x)}{\|\hat{w}_n\|}, \quad n = 1, 2, \dots \quad (48)$$

where $\|\hat{w}\|^2 := \int_0^L \hat{w}(x)^2 dx$.

2.2 Question 6: Orthonormality

Prove that $\{\varphi_n\}_{n=1}^N$ is an orthonormal set of functions in the standard inner product space $L_2([0, L])$ of square integrable functions on $[0, L]$.

Answer:

To prove orthonormality, we must show that for the inner product $\langle f, g \rangle := \int_0^L f(x)g(x) dx$ in $L_2([0, L])$:

$$\langle \varphi_m, \varphi_n \rangle = \delta_{mn} = \begin{cases} 1 & \text{if } m = n \\ 0 & \text{if } m \neq n \end{cases} \quad (49)$$

We first establish orthogonality of the unnormalized modes \hat{w}_n . Recall from (19) and (43) that each mode \hat{w}_n satisfies:

$$EI \frac{\partial^4 \hat{w}_n}{\partial x^4} = \rho A \omega_n^2 \hat{w}_n, \quad \text{where } \omega_n = \beta_n^2 \sqrt{\frac{EI}{\rho A}} \quad (50)$$

For two distinct modes $m \neq n$, multiply the equation for mode m by \hat{w}_n and integrate:

$$\int_0^L \hat{w}_n(x) \cdot EI \frac{\partial^4 \hat{w}_m}{\partial x^4} dx = \rho A \omega_m^2 \int_0^L \hat{w}_n(x) \hat{w}_m(x) dx \quad (51)$$

Apply integration by parts twice to the left-hand side. First integration:

$$\int_0^L \hat{w}_n \frac{\partial^4 \hat{w}_m}{\partial x^4} dx = \left[\hat{w}_n \frac{\partial^3 \hat{w}_m}{\partial x^3} \right]_0^L - \int_0^L \frac{\partial \hat{w}_n}{\partial x} \frac{\partial^3 \hat{w}_m}{\partial x^3} dx \quad (52)$$

The boundary term vanishes because $\hat{w}_n(0) = \hat{w}_n(L) = 0$ (double-clamped conditions). Second integration:

$$-\int_0^L \frac{\partial \hat{w}_n}{\partial x} \frac{\partial^3 \hat{w}_m}{\partial x^3} dx = -\left[\frac{\partial \hat{w}_n}{\partial x} \frac{\partial^2 \hat{w}_m}{\partial x^2} \right]_0^L + \int_0^L \frac{\partial^2 \hat{w}_n}{\partial x^2} \frac{\partial^2 \hat{w}_m}{\partial x^2} dx \quad (53)$$

Again, the boundary term vanishes because $\hat{w}'_n(0) = \hat{w}'_n(L) = 0$. Thus:

$$EI \int_0^L \frac{\partial^2 \hat{w}_n}{\partial x^2} \frac{\partial^2 \hat{w}_m}{\partial x^2} dx = \rho A \omega_m^2 \int_0^L \hat{w}_n \hat{w}_m dx \quad (54)$$

By symmetry (interchanging m and n), we also have:

$$EI \int_0^L \frac{\partial^2 \hat{w}_m}{\partial x^2} \frac{\partial^2 \hat{w}_n}{\partial x^2} dx = \rho A \omega_n^2 \int_0^L \hat{w}_m \hat{w}_n dx \quad (55)$$

Since the left-hand sides are identical, subtracting these equations yields:

$$0 = \rho A (\omega_m^2 - \omega_n^2) \int_0^L \hat{w}_m \hat{w}_n dx \quad (56)$$

For distinct modes ($m \neq n$), we have $\omega_m \neq \omega_n$ (distinct eigenvalues), so $\omega_m^2 - \omega_n^2 \neq 0$. Therefore:

$$\int_0^L \hat{w}_m(x) \hat{w}_n(x) dx = 0 \quad \text{for } m \neq n \quad (57)$$

This proves orthogonality of the unnormalized modes. Then, for the normalized modes:

$$\langle \varphi_m, \varphi_n \rangle = \int_0^L \frac{\hat{w}_m(x)}{\|\hat{w}_m\|} \cdot \frac{\hat{w}_n(x)}{\|\hat{w}_n\|} dx = \frac{1}{\|\hat{w}_m\| \|\hat{w}_n\|} \int_0^L \hat{w}_m(x) \hat{w}_n(x) dx \quad (58)$$

For $m \neq n$, the integral is zero (orthogonality), so $\langle \varphi_m, \varphi_n \rangle = 0$.

For $m = n$:

$$\langle \varphi_n, \varphi_n \rangle = \frac{1}{\|\hat{w}_n\|^2} \int_0^L \hat{w}_n(x)^2 dx = \frac{1}{\|\hat{w}_n\|^2} \cdot \|\hat{w}_n\|^2 = 1 \quad (59)$$

Therefore, $\{\varphi_n\}_{n=1}^N$ is an orthonormal set in $L_2([0, L])$.

3 Response to External Forcing

Consider the beam under excitation of the distributed force $q(x, t) = \ell(x)u(t)$ where ℓ is defined in (2) and where the input $u(t)$ is a time-varying force that is (homogeneously) applied to the beam at the actuator position L_0 . Postulate that the response to this force assumes the form

$$w(x, t) = \sum_{n=1}^N a_n(t) \varphi_n(x) \quad (60)$$

for suitable coefficients $a_n(t)$ and for N sufficiently large.

3.1 Question 7: Galerkin Projection and State Space Model

Use a Galerkin projection to derive an approximate state space model of the beam that takes $u(t)$ as input and coefficients $a(t) = \text{col}(a_1(t), \dots, a_N(t))$ as output.

Answer: We start the Galerkin projection by postulating the solution to the Euler-Bernoulli beam equation as a modal expansion (60), where $\varphi_n(x)$ are the normalized natural vibration modes derived in Questions 3-4, by substituting this expansion into the PDE:

$$EI \sum_{n=1}^N a_n(t) \frac{d^4 \varphi_n}{dx^4} + \rho A \sum_{n=1}^N \ddot{a}_n(t) \varphi_n(x) = q(x, t) = \ell(x)u(t). \quad (61)$$

Next to that, from the vibration problem (Question 2-3), each mode should satisfy:

$$EI \frac{d^4 \varphi_n}{dx^4} = \rho A \omega_n^2 \varphi_n. \quad (62)$$

If we substitute this relation into the PDE, we get:

$$\rho A \sum_{n=1}^N \omega_n^2 a_n(t) \varphi_n(x) + \rho A \sum_{n=1}^N \ddot{a}_n(t) \varphi_n(x) = \ell(x)u(t). \quad (63)$$

Now, we can apply the Galerkin projection procedure. We multiply both sides by the function $\varphi_m(x)$, and integrate over $[0, L]$:

$$\rho A \sum_{n=1}^N \omega_n^2 a_n(t) \int_0^L \langle \varphi_m(x), \varphi_n(x) \rangle dx + \rho A \sum_{n=1}^N \ddot{a}_n(t) \int_0^L \langle \varphi_m(x), \varphi_n(x) \rangle dx = u(t) \int_0^L \langle \ell(x), \varphi_m(x) \rangle dx \quad (64)$$

Using the orthonormality property from Question 6: $\int_0^L \langle \varphi_m(x), \varphi_n(x) \rangle dx = \delta_{mn}$:

$$\rho A \omega_m^2 a_m(t) + \rho A \ddot{a}_m(t) = u(t) \int_0^L \langle \ell(x), \varphi_m(x) \rangle dx. \quad (65)$$

Dividing by ρA and defining the modal input coefficient:

$$b_m := \frac{1}{\rho A} \int_0^L \langle \ell(x), \varphi_m(x) \rangle, dx. \quad (66)$$

We obtain the modal equation for mode m :

$$\ddot{a}_m(t) + \omega_m^2 a_m(t) = b_m u(t), \quad m = 1, \dots, N. \quad (67)$$

To convert to state-space form, we define the state vector as: $\mathbf{x}(t) = (\mathbf{a}(t) \quad \dot{\mathbf{a}}(t))^T \in \mathbb{R}^{2N}$ where $\mathbf{a}(t) = \text{col}(a_1(t), \dots, a_N(t))$. Then, the state space model is:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + B\mathbf{u}(t), \quad (68)$$

$$\mathbf{y}(t) = C\mathbf{x}(t), \quad (69)$$

where the system matrices are:

$$A = \begin{pmatrix} \mathbf{0}_{N \times N} & \mathbf{I}_{N \times N} \\ -\Omega^2 & \mathbf{0}_{N \times N} \end{pmatrix} \in \mathbb{R}^{2N \times 2N} \quad (70)$$

$$B = \begin{pmatrix} \mathbf{0}_{N \times 1} \\ \mathbf{b} \end{pmatrix} \in \mathbb{R}^{2N \times 1} \quad (71)$$

$$C = (\mathbf{I}_{N \times N} \quad \mathbf{0}_{N \times N}) \in \mathbb{R}^{N \times 2N}. \quad (72)$$

Here, $\Omega^2 = \text{diag}(\omega_1^2, \omega_2^2, \dots, \omega_N^2)$ is the diagonal matrix of squared natural frequencies, and $\mathbf{b} = \text{col}(b_1, b_2, \dots, b_N)$ is the input distribution vector similarly defined as (66).

3.2 Question 8: Simulation Experiments

Use (60) and your state space model of item 7 to simulate beam deflections of your approximate model. Experiment with

- your favourite time-varying inputs $u(t)$,
- different values of the approximation order N in (60),
- different initial conditions $w_0(x)$.

Try to obtain conclusions on the accuracy of your model.

Answer:

The state-space model from Question 7 is simulated using `ode45`. Initial conditions are computed by projecting $w_0(x)$ onto normalized modes: $a_n(0) = \int_0^L w_0(x) \varphi_n(x) dx$ with $\dot{a}_n(0) = 0$. Three experiments validate model accuracy:

Experiment 1: Step Input A step input $u(t) = 100$ N is applied with zero initial deflection. Figure 4 shows convergence for $N \geq 3$, indicating the first 3 modes capture dominant dynamics.

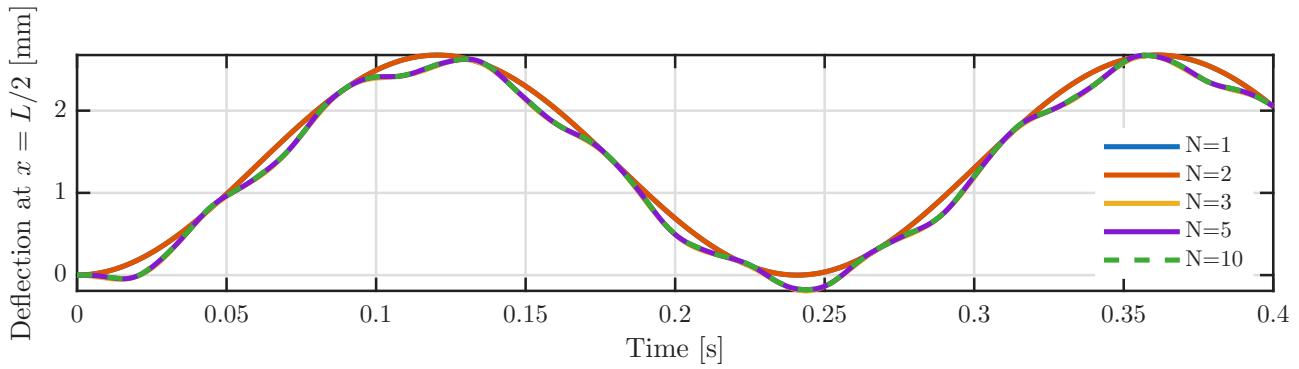


Figure 4: Beam deflection at midpoint under step input for approximation orders N .

Experiment 2: Sinusoidal Input at Resonance A sinusoidal input $u(t) = 50 \sin(\omega_3 t)$ at the third natural frequency excites resonance behavior. Figure 5 shows the beam oscillating with amplitudes dominated by the third mode shape, validating the decomposition.

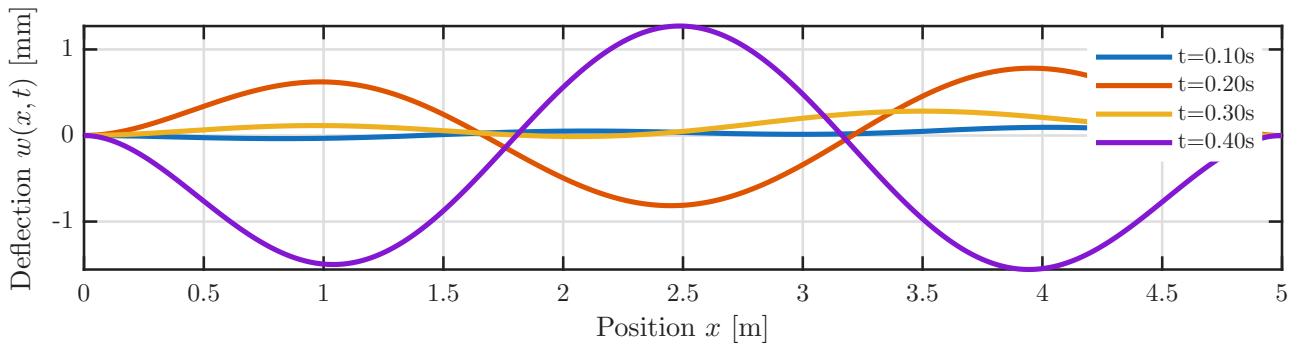


Figure 5: Spatial deflection at resonance ($\omega = \omega_3$) for $N = 10$ at different time instances.

Experiment 3: Free Vibration from Initial Deflection With initial deflection $w_0(x) = \sin(3\pi x/L)(x/L)^3(1 - x/L)^3$ and zero input, Figure 6 demonstrates improved approximation accuracy as N increases. As with Figure 4, Figure 6 shows convergence for $N \geq 5$, indicating the first 5 modes capture dominant dynamics.

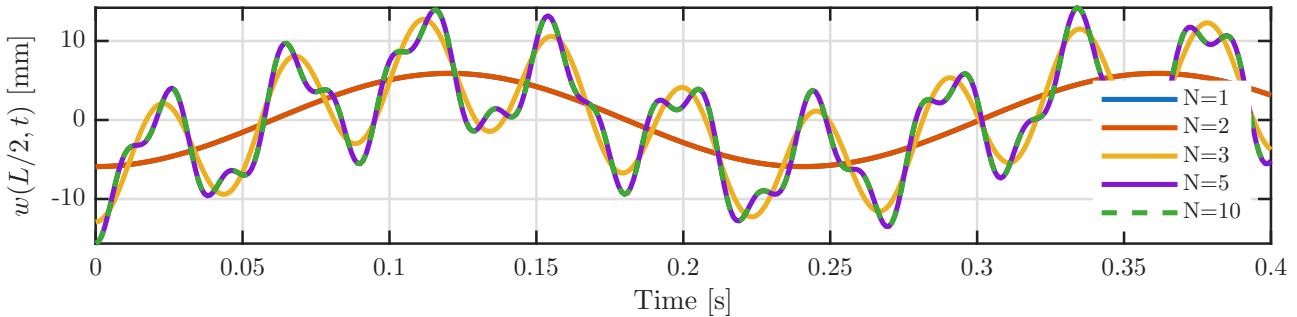


Figure 6: Comparison of different approximation orders N with an initial deflection.

To conclude, the model accuracy improves with N ; $N = 3$ is sufficient for most cases, beyond $N = 5$ improvements are marginal. Next to that, the resonance being captured accurately validates the eigenvalue solution (Questions 2–4). Finally, the first 3 modes dominate the response for smooth inputs; higher modes capture sharp spatial features.

3.3 Question 9: POD Basis

Generate, from one representative experiment of item 8 that you performed with a large value of N , the deflections $w(x, t)$. Use this data to compute a POD basis $\{\phi_n\}_{n=1,\dots,R}$ of order R with the method of Proper Orthogonal Decompositions. Implement a state space model that represents the evolution of the coefficients $a_n(t)$ in the expansion (6) for this POD basis.

Answer:

Starting again from the modal expansion substituted into the PDE, as in (61), we now do not fill in the relation for the natural modes. Instead, we immediately apply the Galerkin projection by multiplying by the POD basis functions $\phi_m(x)$ and integrating over $[0, L]$:

$$EI \sum_{n=1}^N a_n(t) \int_0^L \langle \phi_m(x), \frac{d^4 \phi_n}{dx^4} \rangle dx + \rho A \sum_{n=1}^N \ddot{a}_n(t) \int_0^L \langle \phi_m(x), \phi_n(x) \rangle dx = u(t) \int_0^L \langle \ell(x), \phi_m(x) \rangle dx \quad (73)$$

Using the orthonormality of the POD basis: $\int_0^L \langle \phi_m(x), \phi_n(x) \rangle dx = \delta_{mn}$:

$$EI \sum_{n=1}^R a_n(t) \int_0^L \langle \phi_m(x), \frac{d^4 \phi_n}{dx^4} \rangle dx + \rho A \ddot{a}_m(t) = u(t) \int_0^L \langle \ell(x), \phi_m(x) \rangle dx \quad (74)$$

Dividing by ρA and defining the modal stiffness matrix and input coefficients:

$$K_{mn} := \frac{EI}{\rho A} \int_0^L \langle \phi_m(x), \frac{d^4 \phi_n}{dx^4} \rangle dx, \quad (75)$$

$$b_m := \frac{1}{\rho A} \int_0^L \langle \ell(x), \phi_m(x) \rangle dx. \quad (76)$$

We obtain the modal equation for mode m :

$$\ddot{a}_m(t) + \sum_{n=1}^N K_{mn} a_n(t) = b_m u(t), \quad m = 1, \dots, R. \quad (77)$$

The state-space form is constructed similarly to Question 7, where the system matrices are:

$$A = \begin{pmatrix} \mathbf{0}_{R \times R} & \mathbf{I}_{R \times R} \\ -\mathbf{K} & \mathbf{0}_{R \times R} \end{pmatrix} \in \mathbb{R}^{2R \times 2R} \quad (78)$$

$$B = \begin{pmatrix} \mathbf{0}_{R \times 1} \\ \mathbf{b} \end{pmatrix} \in \mathbb{R}^{2R \times 1} \quad (79)$$

$$C = (\mathbf{I}_{R \times R} \quad \mathbf{0}_{R \times R}) \in \mathbb{R}^{R \times 2R}. \quad (80)$$

Here, $\mathbf{K} \in \mathbb{R}^{R \times R}$ is the modal stiffness matrix, and $\mathbf{b} = \text{col}(b_1, b_2, \dots, b_R)$ is the input distribution vector.

3.4 Question 10: POD Validation

Now repeat item 8 for the state model of item 9. Validate the POD basis by looking at different orders of R , different initial conditions $w_0(x)$ and different inputs $u(t)$. What are your conclusions? Can you make a statement on the quality of the POD basis when compared to the basis of normalized natural vibration modes?

Answer:

The state-space model derived from the POD basis is simulated using `ode45` for various orders R , initial conditions, and inputs. Two experiments model accuracy:

Experiment 1: Step Input A step input $u(t) = 100$ N is applied with zero initial deflection. Figure 7 shows convergence for $R \geq 2$, indicating the first 2 basis functions capture dominant dynamics.

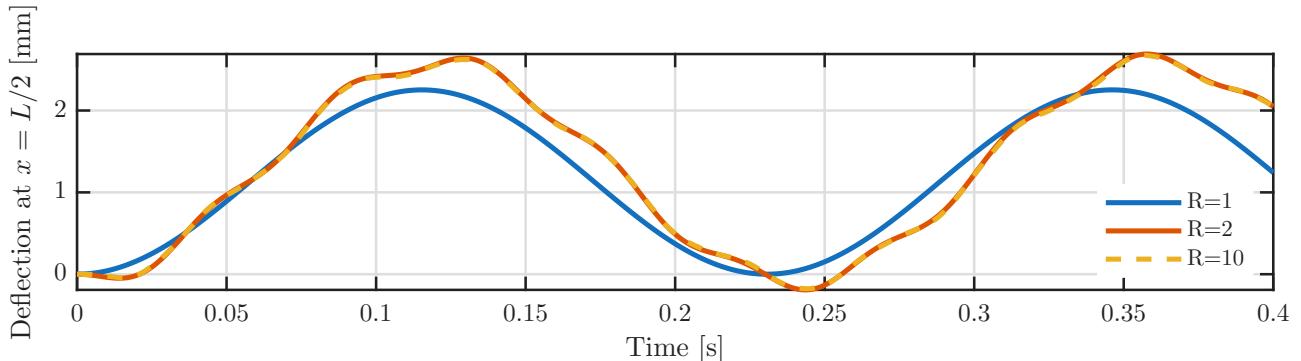


Figure 7: Beam deflection at midpoint under step input for POD approximation orders R .

Experiment 2: Free Vibration from Initial Deflection With identical initial deflection as in Question 8, Figure 8 demonstrates the approximations with increasing R . As with Figure 7, Figure 8 shows convergence for $R \geq 3$, indicating the first 3 basis functions capture dominant dynamics.

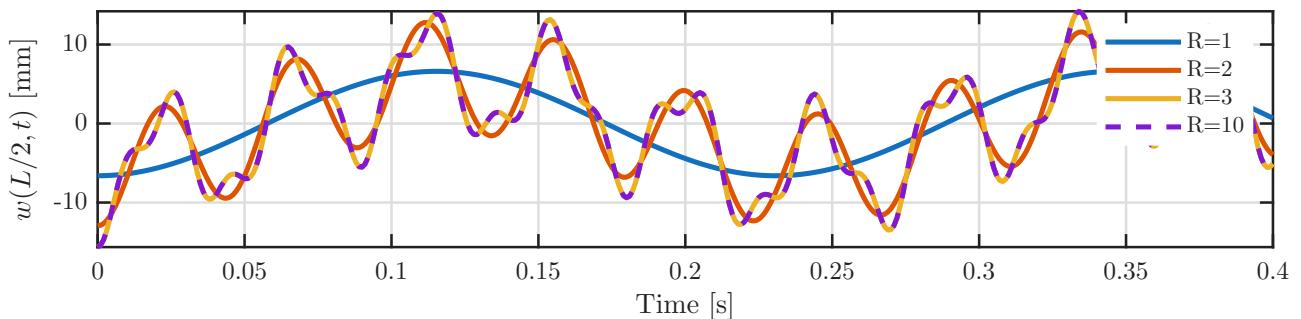


Figure 8: Comparison of different POD approximation orders R with an initial deflection.

Comparison with Natural Vibration Modes For this same experiment, a comparison is made between the POD basis and the natural vibration modes. Figure 9 shows that the POD basis captures the initial deflection more accurately than the natural modes while using a lower-order approximation. With $R = 3$, the POD model is indistinguishable from the high-order model, while the natural mode approximation with $N = 4$ still significantly deviates from it.

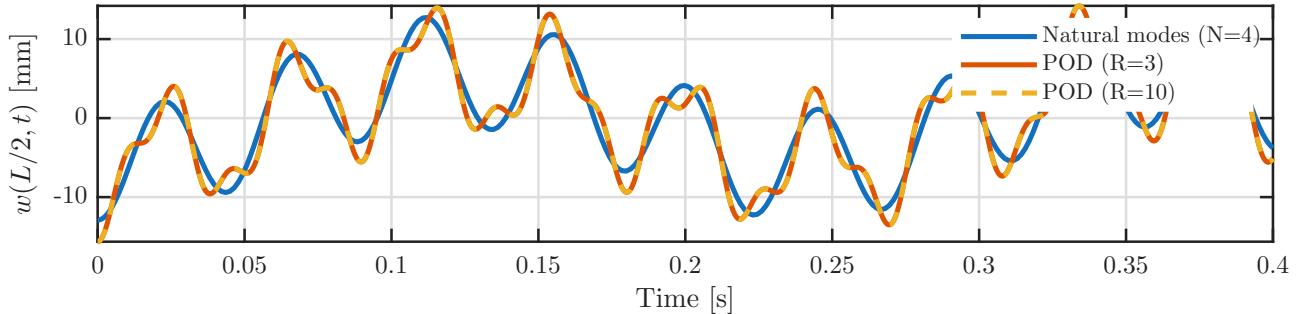


Figure 9: Comparison of POD approximation with natural vibration mode approximation.

The results show that the POD basis can capture the dominant dynamics of the beam with significantly fewer modes than the natural vibration modes. For instance, with $R = 3$, the POD model captures over 99% of the energy in the system, as indicated by the singular value spectrum in Figure 10. This is compared to the natural vibration modes, where $N \geq 5$ modes are required to capture a visually similar level of accuracy.

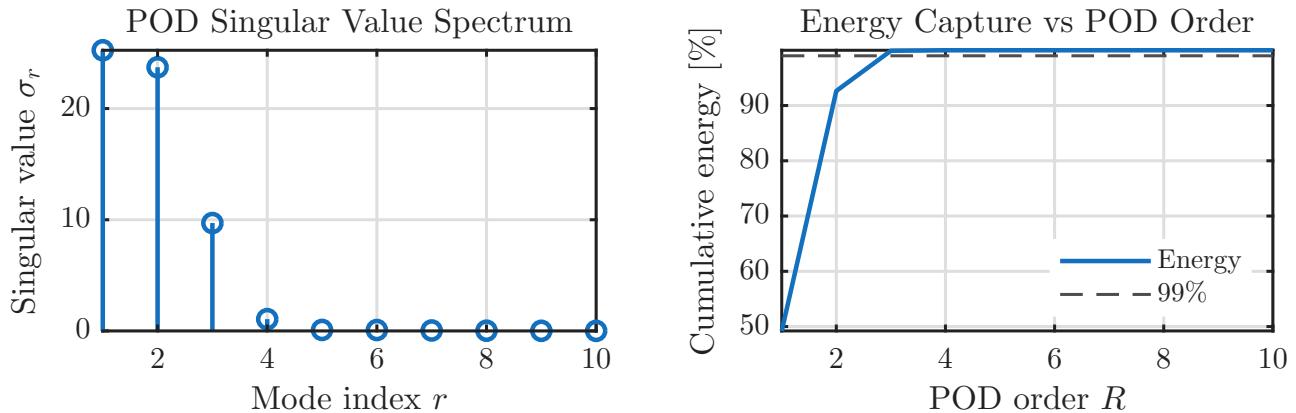


Figure 10: POD singular value spectrum and cumulative energy capture for different POD orders.

Appendix: Physical Specifications

Table 2: Physical Specifications

Variable	Unit	Value
L	m	5
A	m^2	4×10^{-4}
E	N/m^2	69.0×10^9
I	m^4	1.33×10^{-8}
ρ	kg/m^3	2700
L_0	m	4
W	m	0.05

External Force Distribution

The external force is a distributed function $q(x, t) = \ell(x)u(t)$ where $\ell(x)$ is an indicator function:

$$\ell(x) = \begin{cases} 1 & \text{if } x \in [L_0 - \frac{W}{2}, L_0 + \frac{W}{2}] \\ 0 & \text{otherwise} \end{cases} \quad (81)$$