In this document, we verify the essential inequality necessary to prove 2M_T>mu(T;v) by considering the critical values of mu^bullet, when there are (essentially) 3 sub-k-trees

We first write the formulas in general

> $k := 3$

$$k := 3 \tag{1}$$

> $LHS := (N1, N2, N3, mu1, mu2, mu3) \rightarrow (N1 \cdot N2 \cdot N3 + 1) \cdot (1 + mu1 + mu2 + mu3 + k) + k + mu1 \cdot (1 + k + mu1) \cdot N1 + mu2 \cdot (1 + k + mu2) \cdot N2 + mu3 \cdot (1 + k + mu3) \cdot N3$

$$LHS := (N1, N2, N3, \mu1, \mu2, \mu3) \mapsto (N1 \cdot N2 \cdot N3 + 1) \cdot (1 + \mu1 + \mu2 + \mu3 + k) + k + \mu1 \cdot (1 + k + \mu1) \cdot N1 + \mu2 \cdot (1 + k + \mu2) \cdot N2 + \mu3 \cdot (1 + k + \mu3) \cdot N3 \tag{2}$$

> $RHS := (N1, N2, N3, mu1, mu2, mu3) \rightarrow (1 + mu1 + mu2 + mu3) \cdot ((1 + k + mu1) \cdot N1 + (1 + k + mu2) \cdot N2 + (1 + k + mu3) \cdot N3)$

$$RHS := (N1, N2, N3, \mu1, \mu2, \mu3) \mapsto (1 + \mu1 + \mu2 + \mu3) \cdot ((1 + k + \mu1) \cdot N1 + (1 + k + \mu2) \cdot N2 + (1 + k + \mu3) \cdot N3) \tag{3}$$

> $Dif := (N1, N2, N3, mu1, mu2, mu3) \rightarrow LHS(N1, N2, N3, mu1, mu2, mu3) - RHS(N1, N2, N3, mu1, mu2, mu3)$

$$Dif := (N1, N2, N3, \mu1, \mu2, \mu3) \mapsto LHS(N1, N2, N3, \mu1, \mu2, \mu3) - RHS(N1, N2, N3, \mu1, \mu2, \mu3) \tag{4}$$

>

> $f1(N1, N2, N3) := Dif(N1, N2, N3, 0, 0, 0)$

$$f1 := (N1, N2, N3) \mapsto Dif(N1, N2, N3, 0, 0, 0) \tag{5}$$

> $f2(N1, N2, N3) := Dif\left(N1, N2, N3, \frac{N1}{2}, 0, 0\right)$

$$f2 := (N1, N2, N3) \mapsto Dif\left(N1, N2, N3, \frac{N1}{2}, 0, 0\right) \tag{6}$$

> $f3 := (N1, N2, N3) \rightarrow Dif\left(N1, N2, N3, \frac{N1 - 1}{2}, \frac{N2 - 1}{2}, 0\right)$

$$f3 := (N1, N2, N3) \mapsto Dif\left(N1, N2, N3, \frac{N1}{2} - \frac{1}{2}, \frac{N2}{2} - \frac{1}{2}, 0\right) \tag{7}$$

> $f4 := (N1, N2, N3) \rightarrow Dif\left(N1, N2, N3, \frac{N1 - 1}{2}, \frac{N2 - 1}{2}, \frac{(N3 - 1)}{2}\right)$

$$f4 := (N1, N2, N3) \mapsto Dif\left(N1, N2, N3, \frac{N1}{2} - \frac{1}{2}, \frac{N2}{2} - \frac{1}{2}, \frac{N3}{2} - \frac{1}{2}\right) \tag{8}$$

>

> $minimize(f1(N1, N2, N3), N1 = 2 ..infinity, N2 = 2 .. infinity, N3 = 2 .. infinity)$

$$15 \tag{9}$$

> $minimize(f2(N1, N2, N3), N1 = 2 ..infinity, N2 = 2 .. infinity, N3 = 2 .. infinity)$

$$6 \tag{10}$$

> $minimize(f3(N1, N2, N3), N1 = 2 ..infinity, N2 = 2 .. infinity, N3 = 2 .. infinity)$

$$5 \tag{11}$$

> $minimize(f4(N1, N2, N3), N1 = 3 ..infinity, N2 = 2 .. infinity, N3 = 2 .. infinity)$

$$6 \tag{12}$$

```
> f4(2, 2, 2)
```

$$-\frac{3}{2} \tag{13}$$

There is one case which does not work. But N1=N2=N3=2 corresponds with a case where there are k+4 vertices.
Since mu(T) \ge k, when k\ge 4, mu(T; C) \le 2mu(T) is clear. When k=3, mu(T;C)\le 4+3/2<2k, it is again true.

```
>
```