In this document, we verify one of case of the essential inequality necessary to prove 2M_T>mu(T;v) for large k
We do the case where all mu_i^bullet are equal to Ni/2.

We start with the case where k>6.
We will write x=N1 and z=sum(N_i, 2 \le i \le k).
Note that product of (N_i, 2 \le i \le k) is at least f times the sum of them (z), where f=2^(k-2)/(k-1)> k-2.
As such, we have that the left hand side is at least equal to

> $L := (x, z, k, f) \to (x \cdot f \cdot z + 1) \cdot \left(1 + k + \frac{(x+z)}{2}\right) + \frac{x}{2} \cdot \left(x + k \cdot x + \frac{x^2}{2}\right)$

$$L := (x, z, k, f) \mapsto (x \cdot f \cdot z + 1) \cdot \left(1 + k + \frac{x}{2} + \frac{z}{2}\right) + \frac{x \cdot \left(x + k \cdot x + \frac{1}{2} \cdot x^2\right)}{2} \quad \textbf{(1)}$$

For the right hand side, we **use** *that overline* $Ni = (k+1)Ni + \frac{Ni^2}{2}$ *en* Ni^2*\le N1 Ni* **for** *every i \ge* 2.

> $R := (x, z, k) \to \left(1 + \frac{(x+z)}{2}\right) \cdot \left(\left(x + k \cdot x + \frac{x^2}{2}\right) + z \cdot (k+1) + \frac{x \cdot z}{2}\right)$

$$R := (x, z, k) \mapsto \left(1 + \frac{x}{2} + \frac{z}{2}\right) \cdot \left(x + k \cdot x + \frac{x^2}{2} + z \cdot (k+1) + \frac{z \cdot x}{2}\right) \quad \textbf{(2)}$$

Next we consider the difference of the two sides.

(3)

> $Dif(x, z, k, f) := simplify(L(x, z, k, f) - R(x, z, k))$

$$Dif := (x, z, k, f) \mapsto simplify(L(x, z, k, f) - R(x, z, k)) \quad \textbf{(4)}$$

> $collect(Dif(x, z, k, f), k)$

$$\left(-\frac{z^2}{2} + \frac{((4f-4)x-4)z}{4} - x + 1\right)k + \frac{((2f-1)x-2)z^2}{4} + \frac{((2f-2)x^2 + (4f-6)x-2)z}{4} - (x-1)\left(\frac{x}{2} + 1\right) \quad \textbf{(5)}$$

Using that z =< (k-1)x, k\ge 7, f> k-2 and x+z < xz (since x >=2, z>=12), we note that
$x f z - x z - x - z - \frac{1}{2} z^2 + 1$ > (k-7)/2 xz>=0.
So the first term is positive.
Next we consider the remaining terms.

> $Dif2(x, z, f) := \frac{((2f-1)x-2)z^2}{4} + \frac{((2f-2)x^2 + (4f-6)x-2)z}{4} - (x-1)\left(\frac{x}{2} + 1\right)$

$$Dif2 := (x, z, f) \mapsto \frac{((2 \cdot f-1) \cdot x-2) \cdot z^2}{4} + \frac{((2 \cdot f-2) \cdot x^2 + (4 \cdot f-6) \cdot x-2) \cdot z}{4} - (x-1) \quad \textbf{(6)}$$

$$\cdot\left(\frac{x}{2}+1\right)$$

The remaining is increasing in both f and z. Since f >5 and z >=2(k-1), it is sufficient to consider those two values.
After that we note that the expression is increasing in x, so it is sufficient to verify the case with x=2.

> $collect(simplify(Dif(x, 2\cdot(k-1), 5)), x)$

$$\left(4k-\frac{9}{2}\right)x^2+\left(9k^2-11k+\frac{3}{2}\right)x-2k^2+3k \quad \textbf{(7)}$$

> $Dif(2, 2\cdot(k-1), 5)$

$$4(2k-2)^2+29k-31 \quad \textbf{(8)}$$

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Finally, we consider the case k=6 separately. In this case, if z>=11, we have that f \ge 48/11>4,
We now also use that sum_{2 \le i\le 6}(Ni^2) \ge 1/5*z^2 (QM-AM)

> $L := (x, z, k)\rightarrow(x\cdot4\cdot z+1)\cdot\left(1+k+\frac{(x+z)}{2}\right)+\frac{x}{2}\cdot\left(x+k\cdot x+\frac{x^2}{2}\right)+\frac{1}{10}k\cdot z^2$

$$L := (x, z, k)\mapsto(4\cdot z\cdot x+1)\cdot\left(1+k+\frac{x}{2}+\frac{z}{2}\right)+\frac{x\cdot\left(x+k\cdot x+\frac{1}{2}\cdot x^2\right)}{2}+\frac{k\cdot z^2}{10} \quad \textbf{(9)}$$

> $R := (x, z, k)\rightarrow\left(1+\frac{(x+z)}{2}\right)\cdot\left(\left(x+k\cdot x+\frac{x^2}{2}\right)+z\cdot(k+1)+\frac{x\cdot z}{2}\right)$

$$R := (x, z, k)\mapsto\left(1+\frac{x}{2}+\frac{z}{2}\right)\cdot\left(x+k\cdot x+\frac{x^2}{2}+z\cdot(k+1)+\frac{z\cdot x}{2}\right) \quad \textbf{(10)}$$

> $collect(simplify(L(x, z, k)-R(x, z, k)), k)$

$$\left(-\frac{2z^2}{5}+\frac{(6x-2)z}{2}-x+1\right)k+\frac{(35x-10)z^2}{20}+\frac{(3x^2+5x-1)z}{2}-(x-1)\left(\frac{x}{2}+1\right) \quad \textbf{(11)}$$

Since z<=5x, x>=3 and z>=11, we have that
$-\frac{2}{5}z^2+3xz-z-x\geq xz-z-x>0$ **and** *thus the linear factor* **in** k *is positive.*
The remaining factor is increasing **in** z, **and** *filling* **in** $z=2$ *even gives the result*

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> $simplify\left(\frac{(35x-10)\cdot2^2}{20}+\frac{(3x^2+5x-1)\cdot2}{2}-(x-1)\left(\frac{x}{2}+1\right)\right)$

$$\frac{23}{2}x-2+\frac{5}{2}x^2 \quad \textbf{(12)}$$

In the final case, **if** $z=10$ **and** *thus* all Ni are 2, except from possibly x. We can use that mu_i^bullet is bounded by 1/2 and do the computations more exact

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> $L := (x, k) \rightarrow (\mathrm{x}\cdot 32 + 1)\cdot\left(1 + k + \frac{(x+5)}{2}\right) + \frac{x}{2}\cdot\left(x + k\cdot x + \frac{x^2}{2}\right) + 5\cdot\left(1 + k + \frac{1}{2}\right)\cdot 2$

$$L := (x, k) \mapsto (32\cdot x + 1)\cdot\left(\frac{7}{2} + k + \frac{x}{2}\right) + \frac{x\cdot\left(x + k\cdot x + \frac{1}{2}\cdot x^2\right)}{2} + 15 + 10\cdot k \qquad \textbf{(13)}$$

> $R := (x, k) \rightarrow \left(1 + \frac{(x+5)}{2}\right)\cdot\left(\left(x + k\cdot x + \frac{x^2}{2}\right) + 5\cdot\left(1 + k + \frac{1}{2}\right)\cdot 2\right)$

$$R := (x, k) \mapsto \left(\frac{7}{2} + \frac{x}{2}\right)\cdot\left(x + k\cdot x + \frac{1}{2}\cdot x^2 + 15 + 10\cdot k\right) \qquad \textbf{(14)}$$

> $simplify(L(x, k) - R(x, k))$

$$\frac{57\, x^2}{4} + \frac{(94\, k + 406)\, x}{4} - 24\, k - 34 \qquad \textbf{(15)}$$

Since this is increasing in x, it is sufficient to check x=2

> $L(2, k) - R(2, k)$

$$226 + 23\, k \qquad \textbf{(16)}$$

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