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In this document, we verify the essential inequality necessary to prove 2M_T>mu(T;v) by considering the critical values of mu^bullet, when there are 2 sub-k-trees

We first write the formulas in general

> $k := 2$

$$k := 2 \tag{1}$$

> $LHS := (N1, N2, mu1, mu2) \rightarrow (N1 \cdot N2 + 1) \cdot (1 + mu1 + mu2 + k) + k + mu1 \cdot (1 + k + mu1) \cdot N1 + mu2 \cdot (1 + k + mu2) \cdot N2$

$$LHS := (N1, N2, \mu 1, \mu 2) \mapsto (N2 \cdot N1 + 1) \cdot (1 + \mu 1 + \mu 2 + k) + k + \mu 1 \cdot (1 + k + \mu 1) \cdot N1 + \mu 2 \cdot (1 + k + \mu 2) \cdot N2 \tag{2}$$

> $RHS := (N1, N2, mu1, mu2) \rightarrow (1 + mu1 + mu2) \cdot ((1 + k + mu1) \cdot N1 + (1 + k + mu2) \cdot N2)$

$$RHS := (N1, N2, \mu 1, \mu 2) \mapsto (1 + \mu 1 + \mu 2) \cdot ((1 + k + \mu 1) \cdot N1 + (1 + k + \mu 2) \cdot N2) \tag{3}$$

> $Dif := (N1, N2, mu1, mu2) \rightarrow LHS(N1, N2, mu1, mu2) - RHS(N1, N2, mu1, mu2)$

$$Dif := (N1, N2, \mu 1, \mu 2) \mapsto LHS(N1, N2, \mu 1, \mu 2) - RHS(N1, N2, \mu 1, \mu 2) \tag{4}$$

Next, we consider the three extremes and note that these are positive

> $simplify(Dif(N1, N2, 0, 0))$

$$(3\, N2 - 3)\, N1 - 3\, N2 + 5 \tag{5}$$

3(N1-1)(N2-1)+2 \ge 5

> $simplify\left(Dif\left(N1, N2, \frac{N1}{2}, 0\right)\right)$

$$\frac{(N2 - 1)\, N1^2}{2} + \frac{(3\, N2 - 5)\, N1}{2} - 3\, N2 + 5 \tag{6}$$

This is increasing in N1, so it is sufficient to check N1=2.

> $Dif(2, x, 1, 0)$

$$2\, x - 2 \tag{7}$$

> $simplify\left(4 \cdot Dif\left(N1, N2, \frac{N1 - 1}{2}, \frac{N2 - 1}{2}\right) - ((N1 + N2) \cdot ((N1 - 1) \cdot (N2 - 1) - 4) + 16)\right)$

$$0 \tag{8}$$

When min(N1,N2) \ge 3,
$((N1 + N2) \cdot ((N1 - 1) \cdot (N2 - 1) - 4) + 16)$ *is clearly positive. When* $N2 = 2$, $N1 = x$, *we have* $x^2 - 3\, x + 6 > 0$