In this document, we verify the essential inequality necessary to prove 2M_T>mu(T;v) by considering the critical values of mu^bullet, when there are (essentially) 5 sub-k-trees

We first write the formulas in general

> $j := 5$

$$j := 5 \tag{1}$$

> $LHS := (N1, N2, N3, N4, N5, k, mu1, mu2, mu3, mu4, mu5) \to (N1 \cdot N2 \cdot N3 \cdot N4 \cdot N5 + 1) \cdot (1 + mu1 + mu2 + mu3 + mu4 + mu5 + k) + k \cdot (1 + k - j) + mu1 \cdot (1 + k + mu1) \cdot N1 + mu2 \cdot (1 + k + mu2) \cdot N2 + mu3 \cdot (1 + k + mu3) \cdot N3 + mu4 \cdot (1 + k + mu4) \cdot N4 + mu5 \cdot (1 + k + mu5) \cdot N5$

$$LHS := (N1, N2, N3, N4, N5, k, \mu1, \mu2, \mu3, \mu4, \mu5) \to (N1\, N2\, N3\, N4\, N5 + 1)\, (1 + \mu1 + \mu2 + \mu3 + \mu4 + \mu5 + k) + k\,(1 + k - j) + \mu1\,(1 + k + \mu1)\, N1 + \mu2\,(1 + k + \mu2)\, N2 + \mu3\,(1 + k + \mu3)\, N3 + \mu4\,(1 + k + \mu4)\, N4 + \mu5\,(1 + k + \mu5)\, N5 \tag{2}$$

> $RHS := (N1, N2, N3, N4, N5, k, mu1, mu2, mu3, mu4, mu5) \to (1 + mu1 + mu2 + mu3 + mu4 + mu5) \cdot ((1 + k + mu1) \cdot N1 + (1 + k + mu2) \cdot N2 + (1 + k + mu3) \cdot N3 + (1 + k + mu4) \cdot N4 + (1 + k + mu5) \cdot N5 + k - j)$

$$RHS := (N1, N2, N3, N4, N5, k, \mu1, \mu2, \mu3, \mu4, \mu5) \to (1 + \mu1 + \mu2 + \mu3 + \mu4 + \mu5)\, ((1 + k + \mu1)\, N1 + (1 + k + \mu2)\, N2 + (1 + k + \mu3)\, N3 + (1 + k + \mu4)\, N4 + (1 + k + \mu5)\, N5 + k - j) \tag{3}$$

> $Dif := (N1, N2, N3, N4, N5, k, mu1, mu2, mu3, mu4, mu5) \to LHS(N1, N2, N3, N4, N5, k, mu1, mu2, mu3, mu4, mu5) - RHS(N1, N2, N3, N4, N5, k, mu1, mu2, mu3, mu4, mu5)$

$$Dif := (N1, N2, N3, N4, N5, k, \mu1, \mu2, \mu3, \mu4, \mu5) \to LHS(N1, N2, N3, N4, N5, k, \mu1, \mu2, \mu3, \mu4, \mu5) - RHS(N1, N2, N3, N4, N5, k, \mu1, \mu2, \mu3, \mu4, \mu5) \tag{4}$$

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> $f1 := (N1, N2, N3, N4, N5, k) \to Dif(N1, N2, N3, N4, N5, k, 0, 0, 0, 0, 0)$

$$f1 := (N1, N2, N3, N4, N5, k) \to Dif(N1, N2, N3, N4, N5, k, 0, 0, 0, 0, 0) \tag{5}$$

> $f2 := (N1, N2, N3, N4, N5, k) \to Dif\left(N1, N2, N3, N4, N5, k, \frac{N1 - 1}{2}, 0, 0, 0, 0\right)$

$$f2 := (N1, N2, N3, N4, N5, k) \to Dif\left(N1, N2, N3, N4, N5, k, \frac{1}{2}\, N1 - \frac{1}{2}, 0, 0, 0, 0\right) \tag{6}$$

> $f3 := (N1, N2, N3, N4, N5, k) \to Dif\left(N1, N2, N3, N4, N5, k, \frac{N1 - 1}{2}, \frac{N2 - 1}{2}, 0, 0, 0\right)$

$$f3 := (N1, N2, N3, N4, N5, k) \to Dif\left(N1, N2, N3, N4, N5, k, \frac{1}{2}\, N1 - \frac{1}{2}, \frac{1}{2}\, N2 - \frac{1}{2}, 0, 0, 0\right) \tag{7}$$

> $f4 := (N1, N2, N3, N4, N5, k) \to Dif\left(N1, N2, N3, N4, N5, k, \frac{N1 - 1}{2}, \frac{N2 - 1}{2},\right.$

$$\frac{(N3-1)}{2}, 0, 0\Big)$$

$$f4 := (N1, N2, N3, N4, N5, k) \rightarrow Dif\left(N1, N2, N3, N4, N5, k, \frac{1}{2}\, N1 - \frac{1}{2}, \frac{1}{2}\, N2 - \frac{1}{2}, \frac{1}{2}\, N3 - \frac{1}{2}, 0, 0\right) \quad \textbf{(8)}$$

> $f5 := (N1, N2, N3, N4, N5, k) \rightarrow Dif\left(N1, N2, N3, N4, N5, k, \frac{N1-1}{2}, \frac{N2-1}{2}, \frac{(N3-1)}{2}, \frac{(N4-1)}{2}, 0\right)$

$$f5 := (N1, N2, N3, N4, N5, k) \rightarrow Dif\left(N1, N2, N3, N4, N5, k, \frac{1}{2}\, N1 - \frac{1}{2}, \frac{1}{2}\, N2 - \frac{1}{2}, \frac{1}{2}\, N3 - \frac{1}{2}, \frac{1}{2}\, N4 - \frac{1}{2}, 0\right) \quad \textbf{(9)}$$

> $f6 := (N1, N2, N3, N4, N5, k) \rightarrow Dif\left(N1, N2, N3, N4, N5, k, \frac{N1-1}{2}, \frac{N2-1}{2}, \frac{(N3-1)}{2}, \frac{(N4-1)}{2}, \frac{N5-1}{2}\right)$

$$f6 := (N1, N2, N3, N4, N5, k) \rightarrow Dif\left(N1, N2, N3, N4, N5, k, \frac{1}{2}\, N1 - \frac{1}{2}, \frac{1}{2}\, N2 - \frac{1}{2}, \frac{1}{2}\, N3 - \frac{1}{2}, \frac{1}{2}\, N4 - \frac{1}{2}, \frac{1}{2}\, N5 - \frac{1}{2}\right) \quad \textbf{(10)}$$

> $minimize(f1(N1, N2, N3, N4, N5, k), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity, N5 = 2..infinity, k = 5..infinity)$

$$143 \quad \textbf{(11)}$$

> $minimize(f2(N1, N2, N3, N4, N5, k), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity, N5 = 2..infinity, k = 5..infinity)$

$$\frac{269}{2} \quad \textbf{(12)}$$

Since the derivatives in k are positive, it is sufficient to check k=5

> $simplify(diff(f6(N1, N2, N3, N4, N5, k), k))$

$$-N2\,N1 - N3\,N1 - N2\,N3 - N2\,N4 - N2\,N5 - N3\,N4 - N3\,N5 - N4\,N5 - N4\,N1 - N5\,N1 + N1\,N2\,N3\,N4\,N5 - \frac{3}{2} + \frac{1}{2}\,N4 + \frac{1}{2}\,N5 + \frac{1}{2}\,N2 + \frac{1}{2}\,N3 + \frac{1}{2}\,N1 + 2\,k \quad \textbf{(13)}$$

> $simplify(diff(f5(N1, N2, N3, N4, N5, k), k))$

$$-2 - N2\,N1 - N3\,N1 - N2\,N3 - N2\,N4 - \frac{1}{2}\,N2\,N5 - N3\,N4 - \frac{1}{2}\,N3\,N5 - \frac{1}{2}\,N4\,N5 - N4\,N1 - \frac{1}{2}\,N5\,N1 + N1\,N2\,N3\,N4\,N5 + N5 + 2\,k \quad \textbf{(14)}$$

> $simplify(diff(f4(N1, N2, N3, N4, N5, k), k))$

$$N1\,N2\,N3\,N4\,N5-\frac{5}{2}+2\,k-\frac{1}{2}\,N1-\frac{1}{2}\,N2-\frac{1}{2}\,N3+\frac{1}{2}\,N4+\frac{1}{2}\,N5-N2\,N1-N3\,N1-\frac{1}{2}\,N4\,N1-\frac{1}{2}\,N5\,N1-N2\,N3-\frac{1}{2}\,N2\,N4-\frac{1}{2}\,N2\,N5-\frac{1}{2}\,N3\,N4-\frac{1}{2}\,N3\,N5 \quad \textbf{(15)}$$

> $simplify(diff(f3(N1, N2, N3, N4, N5, k), k))$

$$N1\,N2\,N3\,N4\,N5-3+2\,k-N1-N2-N2\,N1-\frac{1}{2}\,N3\,N1-\frac{1}{2}\,N4\,N1-\frac{1}{2}\,N5\,N1-\frac{1}{2}\,N2\,N3-\frac{1}{2}\,N2\,N4-\frac{1}{2}\,N2\,N5 \quad \textbf{(16)}$$

> $minimize(f3(N1, N2, N3, N4, N5, 5), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity, N5 = 2..infinity)$

$$125 \quad \textbf{(17)}$$

> $minimize(f4(N1, N2, N3, N4, N5, 5), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity, N5 = 2..infinity)$

$$\frac{229}{2} \quad \textbf{(18)}$$

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By checking the derivatives and using symmetry, we note that it is sufficient to consider the cases where most Ni are equal to the minimum (2)

> $simplify(diff(f6(N1, N2, N3, N4, N5, 5), N1))$

$$6-\frac{1}{4}\,N2^2-\frac{1}{4}\,N3^2-\frac{1}{2}\,N2\,N1-\frac{1}{2}\,N3\,N1+\frac{1}{2}\,N2\,N3\,N4^2\,N5+\frac{1}{2}\,N2\,N3\,N4\,N5^2-\frac{1}{4}\,N4^2-\frac{1}{4}\,N5^2-\frac{1}{2}\,N4\,N1-\frac{1}{2}\,N5\,N1+\frac{7}{2}\,N2\,N3\,N4\,N5+\frac{1}{2}\,N2^2\,N3\,N4\,N5+\frac{1}{2}\,N2\,N3^2\,N4\,N5+N1\,N2\,N3\,N4\,N5-\frac{11}{2}\,N4-\frac{11}{2}\,N5-\frac{11}{2}\,N2-\frac{11}{2}\,N3+N1 \quad \textbf{(19)}$$

> $simplify(diff(f5(N1, N2, N3, N4, N5, 5), N1))$

$$-\frac{1}{4}\,N2^2-\frac{1}{4}\,N3^2-\frac{1}{2}\,N2\,N1-\frac{1}{2}\,N3\,N1+\frac{1}{2}\,N2\,N3\,N4^2\,N5-\frac{1}{4}\,N4^2-\frac{1}{2}\,N4\,N1+4\,N2\,N3\,N4\,N5+\frac{1}{2}\,N2^2\,N3\,N4\,N5+\frac{1}{2}\,N2\,N3^2\,N4\,N5+\frac{13}{4}+N1\,N2\,N3\,N4\,N5-\frac{11}{2}\,N4-3\,N5-\frac{11}{2}\,N2-\frac{11}{2}\,N3+\frac{1}{2}\,N1 \quad \textbf{(20)}$$

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> $f5(2, 2, 2, 2, n, 5)$

$$110\,n-117 \quad \textbf{(21)}$$

> $f6(2, 2, 2, 2, 2, 5)$

$$\frac{181}{2} \tag{22}$$

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(23)