In this document, we verify the essential inequality necessary to prove 2M_T>mu(T;v) by considering the critical values of mu^bullet, when there are (essentially) 4 sub-k-trees

We first write the formulas in general

> $j := 4$

$$j := 4 \quad \textbf{(1)}$$

> $LHS := (N1, N2, N3, N4, k, mu1, mu2, mu3, mu4) \rightarrow (N1 \cdot N2 \cdot N3 \cdot N4 + 1) \cdot (1 + mu1 + mu2 + mu3 + mu4 + k) + k \cdot (k + 1 - j) + mu1 \cdot (1 + k + mu1) \cdot N1 + mu2 \cdot (1 + k + mu2) \cdot N2 + mu3 \cdot (1 + k + mu3) \cdot N3 + mu4 \cdot (1 + k + mu4) \cdot N4$

$$LHS := (N1, N2, N3, N4, k, \mu1, \mu2, \mu3, \mu4) \rightarrow (N1\, N2\, N3\, N4 + 1)\, (1 + \mu1 + \mu2 + \mu3 + \mu4 + k) + k\, (k + 1 - j) + \mu1\, (1 + k + \mu1)\, N1 + \mu2\, (1 + k + \mu2)\, N2 + \mu3\, (1 + k + \mu3)\, N3 + \mu4\, (1 + k + \mu4)\, N4 \quad \textbf{(2)}$$

> $RHS := (N1, N2, N3, N4, k, mu1, mu2, mu3, mu4) \rightarrow (1 + mu1 + mu2 + mu3 + mu4) \cdot ((1 + k + mu1) \cdot N1 + (1 + k + mu2) \cdot N2 + (1 + k + mu3) \cdot N3 + (1 + k + mu4) \cdot N4 + k - j)$

$$RHS := (N1, N2, N3, N4, k, \mu1, \mu2, \mu3, \mu4) \rightarrow (1 + \mu1 + \mu2 + \mu3 + \mu4)\, ((1 + k + \mu1)\, N1 + (1 + k + \mu2)\, N2 + (1 + k + \mu3)\, N3 + (1 + k + \mu4)\, N4 + k - j) \quad \textbf{(3)}$$

> $Dif := (N1, N2, N3, N4, k, mu1, mu2, mu3, mu4) \rightarrow LHS(N1, N2, N3, N4, k, mu1, mu2, mu3, mu4) - RHS(N1, N2, N3, N4, k, mu1, mu2, mu3, mu4)$

$$Dif := (N1, N2, N3, N4, k, \mu1, \mu2, \mu3, \mu4) \rightarrow LHS(N1, N2, N3, N4, k, \mu1, \mu2, \mu3, \mu4) - RHS(N1, N2, N3, N4, k, \mu1, \mu2, \mu3, \mu4) \quad \textbf{(4)}$$

>

> $f1 := (N1, N2, N3, N4, k) \rightarrow Dif(N1, N2, N3, N4, k, 0, 0, 0, 0)$

$$f1 := (N1, N2, N3, N4, k) \rightarrow Dif(N1, N2, N3, N4, k, 0, 0, 0, 0) \quad \textbf{(5)}$$

> $f2 := (N1, N2, N3, N4, k) \rightarrow Dif\left(N1, N2, N3, N4, k, \frac{N1 - 1}{2}, 0, 0, 0\right)$

$$f2 := (N1, N2, N3, N4, k) \rightarrow Dif\left(N1, N2, N3, N4, k, \frac{1}{2}\, N1 - \frac{1}{2}, 0, 0, 0\right) \quad \textbf{(6)}$$

> $f3 := (N1, N2, N3, N4, k) \rightarrow Dif\left(N1, N2, N3, N4, k, \frac{N1 - 1}{2}, \frac{N2 - 1}{2}, 0, 0\right)$

$$f3 := (N1, N2, N3, N4, k) \rightarrow Dif\left(N1, N2, N3, N4, k, \frac{1}{2}\, N1 - \frac{1}{2}, \frac{1}{2}\, N2 - \frac{1}{2}, 0, 0\right) \quad \textbf{(7)}$$

> $f4 := (N1, N2, N3, N4, k) \rightarrow Dif\left(N1, N2, N3, N4, k, \frac{N1 - 1}{2}, \frac{N2 - 1}{2}, \frac{(N3 - 1)}{2}, 0\right)$

$$f4 := (N1, N2, N3, N4, k) \rightarrow Dif\left(N1, N2, N3, N4, k, \frac{1}{2}\, N1 - \frac{1}{2}, \frac{1}{2}\, N2 - \frac{1}{2}, \frac{1}{2}\, N3 - \frac{1}{2}, 0\right) \quad \textbf{(8)}$$

> $f5 := (N1, N2, N3, N4, k) \rightarrow Dif\Big(N1, N2, N3, N4, k, \frac{N1 - 1}{2}, \frac{N2 - 1}{2}, \frac{(N3 - 1)}{2},$

$$\frac{(N4-1)}{2}\Big)$$

$$f5 := (N1, N2, N3, N4, k) \to Dif\left(N1, N2, N3, N4, k, \frac{1}{2}\,N1 - \frac{1}{2}, \frac{1}{2}\,N2 - \frac{1}{2}, \frac{1}{2}\,N3 - \frac{1}{2}, \frac{1}{2}\,N4 - \frac{1}{2}\right) \tag{9}$$

\>

\> $minimize(f1(N1, N2, N3, N4, k), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity, k = 4..infinity)$

$$49 \tag{10}$$

\> $minimize(f2(N1, N2, N3, N4, k), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity, k = 4..infinity)$

$$\frac{83}{2} \tag{11}$$

For larger values, we can check that the derivatives of the function in k are positive, except for f5 when all Ni are 2 (which is checked separately,

\> $simplify(diff(f3(N1, N2, N3, N4, k), k))$

$$N1\,N2\,N3\,N4 - 2 + 2\,k - N1 - N2 - N1\,N2 - \frac{1}{2}\,N1\,N3 - \frac{1}{2}\,N1\,N4 - \frac{1}{2}\,N2\,N3 - \frac{1}{2}\,N2\,N4 \tag{12}$$

\> $simplify(diff(f4(N1, N2, N3, N4, k), k))$

$$N1\,N2\,N3\,N4 - \frac{3}{2} + 2\,k - \frac{1}{2}\,N1 - \frac{1}{2}\,N2 - \frac{1}{2}\,N3 + \frac{1}{2}\,N4 - N1\,N2 - N1\,N3 - \frac{1}{2}\,N1\,N4 - N2\,N3 - \frac{1}{2}\,N2\,N4 - \frac{1}{2}\,N3\,N4 \tag{13}$$

\> $simplify(diff(f5(N1, N2, N3, N4, k), k))$

$$N1\,N2\,N3\,N4 - N1\,N2 - N1\,N3 - N1\,N4 - N2\,N3 - N2\,N4 - N3\,N4 + 2\,k - 1 \tag{14}$$

\> $f5(2, 2, 2, 2, k)$

$$33 - 6\,k + k\,(k-3) \tag{15}$$

\> $minimize(f3(N1, N2, N3, N4, 4), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity)$

$$33 \tag{16}$$

\> $minimize(f4(N1, N2, N3, N4, 4), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity)$

$$\frac{47}{2} \tag{17}$$

\> $minimize(f5(N1, N2, N3, N4, 4), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity)$

$$13 \tag{18}$$

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