In this document, we verify the essential inequality necessary to prove 2M_T>mu(T;v) by considering the critical values of mu^bullet, when there are (essentially) 5 sub-k-trees

We first write the formulas in general

> $k := 5$

$$k := 5 \tag{1}$$

> $LHS := (N1, N2, N3, N4, N5, mu1, mu2, mu3, mu4, mu5) \to (N1 \cdot N2 \cdot N3 \cdot N4 \cdot N5 + 1) \cdot (1 + mu1 + mu2 + mu3 + mu4 + mu5 + k) + k + mu1 \cdot (1 + k + mu1) \cdot N1 + mu2 \cdot (1 + k + mu2) \cdot N2 + mu3 \cdot (1 + k + mu3) \cdot N3 + mu4 \cdot (1 + k + mu4) \cdot N4 + mu5 \cdot (1 + k + mu5) \cdot N5$

$$LHS := (N1, N2, N3, N4, N5, \mu 1, \mu 2, \mu 3, \mu 4, \mu 5) \mapsto (N1 \cdot N2 \cdot N3 \cdot N4 \cdot N5 + 1) \cdot (1 + \mu 1 + \mu 2 + \mu 3 + \mu 4 + \mu 5 + k) + k + \mu 1 \cdot (1 + k + \mu 1) \cdot N1 + \mu 2 \cdot (1 + k + \mu 2) \cdot N2 + \mu 3 \cdot (1 + k + \mu 3) \cdot N3 + \mu 4 \cdot (1 + k + \mu 4) \cdot N4 + \mu 5 \cdot (1 + k + \mu 5) \cdot N5 \tag{2}$$

> $RHS := (N1, N2, N3, N4, N5, mu1, mu2, mu3, mu4, mu5) \to (1 + mu1 + mu2 + mu3 + mu4 + mu5) \cdot ((1 + k + mu1) \cdot N1 + (1 + k + mu2) \cdot N2 + (1 + k + mu3) \cdot N3 + (1 + k + mu4) \cdot N4 + (1 + k + mu5) \cdot N5)$

$$RHS := (N1, N2, N3, N4, N5, \mu 1, \mu 2, \mu 3, \mu 4, \mu 5) \mapsto (1 + \mu 1 + \mu 2 + \mu 3 + \mu 4 + \mu 5) \cdot ((1 + k + \mu 1) \cdot N1 + (1 + k + \mu 2) \cdot N2 + (1 + k + \mu 3) \cdot N3 + (1 + k + \mu 4) \cdot N4 + (1 + k + \mu 5) \cdot N5) \tag{3}$$

> $Dif := (N1, N2, N3, N4, N5, mu1, mu2, mu3, mu4, mu5) \to LHS(N1, N2, N3, N4, N5, mu1, mu2, mu3, mu4, mu5) - RHS(N1, N2, N3, N4, N5, mu1, mu2, mu3, mu4, mu5)$

$$Dif := (N1, N2, N3, N4, N5, \mu 1, \mu 2, \mu 3, \mu 4, \mu 5) \mapsto LHS(N1, N2, N3, N4, N5, \mu 1, \mu 2, \mu 3, \mu 4, \mu 5) - RHS(N1, N2, N3, N4, N5, \mu 1, \mu 2, \mu 3, \mu 4, \mu 5) \tag{4}$$

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> $f1 := (N1, N2, N3, N4, N5) \to Dif(N1, N2, N3, N4, N5, 0, 0, 0, 0, 0)$

$$f1 := (N1, N2, N3, N4, N5) \mapsto Dif(N1, N2, N3, N4, N5, 0, 0, 0, 0, 0) \tag{5}$$

> $f2 := (N1, N2, N3, N4, N5) \to Dif\left(N1, N2, N3, N4, N5, \frac{N1 - 1}{2}, 0, 0, 0, 0\right)$

$$f2 := (N1, N2, N3, N4, N5) \mapsto Dif\left(N1, N2, N3, N4, N5, \frac{N1}{2} - \frac{1}{2}, 0, 0, 0, 0\right) \tag{6}$$

> $f3 := (N1, N2, N3, N4, N5) \to Dif\left(N1, N2, N3, N4, N5, \frac{N1 - 1}{2}, \frac{N2 - 1}{2}, 0, 0, 0\right)$

$$f3 := (N1, N2, N3, N4, N5) \mapsto Dif\left(N1, N2, N3, N4, N5, \frac{N1}{2} - \frac{1}{2}, \frac{N2}{2} - \frac{1}{2}, 0, 0, 0\right) \tag{7}$$

> $f4 := (N1, N2, N3, N4, N5) \to Dif\left(N1, N2, N3, N4, N5, \frac{N1 - 1}{2}, \frac{N2 - 1}{2}, \frac{(N3 - 1)}{2}, 0, 0\right)$

$$f4 := (N1, N2, N3, N4, N5) \mapsto Dif\left(N1, N2, N3, N4, N5, \frac{N1}{2} - \frac{1}{2}, \frac{N2}{2} - \frac{1}{2}, \frac{N3}{2} - \frac{1}{2}, 0, 0\right) \tag{8}$$

> $f5 := (N1, N2, N3, N4, N5) \to Dif\left(N1, N2, N3, N4, N5, \frac{N1 - 1}{2}, \frac{N2 - 1}{2}, \frac{(N3 - 1)}{2}, \frac{(N4 - 1)}{2}, 0\right)$

$$f5 := (N1, N2, N3, N4, N5) \mapsto Dif\left(N1, N2, N3, N4, N5, \frac{N1}{2} - \frac{1}{2}, \frac{N2}{2} - \frac{1}{2}, \frac{N3}{2} - \frac{1}{2}, \frac{N4}{2} - \frac{1}{2}, 0\right) \quad (9)$$

> $f6 := (N1, N2, N3, N4, N5, k) \to Dif\left(N1, N2, N3, N4, N5, \frac{N1-1}{2}, \frac{N2-1}{2}, \frac{(N3-1)}{2}, \frac{(N4-1)}{2}, \frac{N5-1}{2}\right)$

$$f6 := (N1, N2, N3, N4, N5, k) \mapsto Dif\left(N1, N2, N3, N4, N5, \frac{N1}{2} - \frac{1}{2}, \frac{N2}{2} - \frac{1}{2}, \frac{N3}{2} - \frac{1}{2}, \frac{N4}{2} - \frac{1}{2}, \frac{N5}{2} - \frac{1}{2}\right) \quad (10)$$

> $minimize(f1(N1, N2, N3, N4, N5), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity, N5 = 2..infinity)$

$$143 \quad (11)$$

> $minimize(f2(N1, N2, N3, N4, N5), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity, N5 = 2..infinity)$

$$\frac{269}{2} \quad (12)$$

> $minimize(f3(N1, N2, N3, N4, N5), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity, N5 = 2..infinity)$

$$125 \quad (13)$$

> $minimize(f4(N1, N2, N3, N4, N5), N1 = 2..infinity, N2 = 2..infinity, N3 = 2..infinity, N4 = 2..infinity, N5 = 2..infinity)$

$$\frac{229}{2} \quad (14)$$

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By checking the derivatives and using symmetry, we note that it is sufficient to consider the cases where most Ni are equal to the minimum (2)
By symmetry, for f6, all terms can be taken to be minimal
For f5, there is symmetry between Ni for 1 \le i \le4.
The derivatives are positive, as the higher order order terms are positive.
(proven exactly by grouping)

> $expand(simplify(4 \cdot diff(f6(N1, N2, N3, N4, N5), N1)))$

$$4\,N1\,N2\,N3\,N4\,N5 + 2\,N2^2\,N3\,N4\,N5 + 2\,N2\,N3^2\,N4\,N5 + 2\,N2\,N3\,N4^2\,N5 + 2\,N2\,N3\,N4\,N5^2 + 14\,N2\,N3\,N4\,N5 - 2\,N2\,N1 - 2\,N3\,N1 - 2\,N4\,N1 - 2\,N5\,N1 - N2^2 - N3^2 - N4^2 - N5^2 + 4\,N1 - 22\,N2 - 22\,N3 - 22\,N4 - 22\,N5 + 24 \quad (15)$$

> $expand(simplify(4 \cdot diff(f5(N1, N2, N3, N4, N5), N1)))$

$$4\,N1\,N2\,N3\,N4\,N5 + 2\,N2^2\,N3\,N4\,N5 + 2\,N2\,N3^2\,N4\,N5 + 2\,N2\,N3\,N4^2\,N5 + 16\,N2\,N3\,N4\,N5 - 2\,N2\,N1 - 2\,N3\,N1 - 2\,N4\,N1 - N2^2 - N3^2 - N4^2 + 2\,N1 - 22\,N2 - 22\,N3 - 22\,N4 - 12\,N5 + 13 \quad (16)$$

>

> $f5(2, 2, 2, 2, n, 5)$

$$110\,n - 117 \tag{17}$$

> $f6(2, 2, 2, 2, 2, 5)$

$$\frac{181}{2} \tag{18}$$

>

(19)