We compute mu, for a path of length l (ell) with endvertices A and B, with rooted trees T_A and N_B in these two vertices
Here we denote \overline N_A as Na2 etc

> mu := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)→((Ra·Nb + Na·Nb·(l − 1) + Rb·Na + l·Ra + l·Rb + binomial(l, 2)·(Na + Nb) + Ra2 + Rb2) + binomial(l + 1, 3))/(Na·Nb + l·Na + l·Nb + Na2 + Nb2 + binomial(l, 2))

$$\mu := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto \frac{1}{Na \cdot Nb + l \cdot Na + l \cdot Nb + Na2 + Nb2 + \binom{l}{2}} \left(Nb \cdot Ra + Na \cdot Nb \cdot (l-1) + Rb \cdot Na + l \cdot Ra + l \cdot Rb + \binom{l}{2} \cdot (Na + Nb) + Ra2 + Rb2 + \binom{l+1}{3} \right) \quad \textbf{(1)}$$

We consider the numerator and denominator separately and compute it for the difference between mu for the the tree T and T\e (path of length l-1)

> Nu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := (Ra·Nb + Na·Nb·(l − 1) + Rb·Na + l·Ra + l·Rb + binomial(l, 2)·(Na + Nb) + Ra2 + Rb2) + binomial(l + 1, 3)

$$N := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto Nb \cdot Ra + Na \cdot Nb \cdot (l-1) + Rb \cdot Na + l \cdot Ra + l \cdot Rb + \binom{l}{2} \cdot (Na + Nb) + Ra2 + Rb2 + \binom{l+1}{3} \quad \textbf{(2)}$$

> De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := Na·Nb + l·Na + l·Nb + Na2 + Nb2 + binomial(l, 2)

$$De := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto Na \cdot Nb + l \cdot Na + l \cdot Nb + Na2 + Nb2 + \binom{l}{2} \quad \textbf{(3)}$$

> d := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)→μ(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) − μ(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l − 1)

$$d := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto \mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) \quad \textbf{(4)}$$

> Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := Nu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)·De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l − 1)−Nu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l − 1)·De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)

$$Nd := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto N(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) - N(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \quad \textbf{(5)}$$

> Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)·De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l − 1)

$$Dd := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) \quad \textbf{(6)}$$

Check

> simplify(mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)

$$- \frac{\mathrm{Nu}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}{De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}\Bigg)$$

$$0 \qquad (7)$$

> $simplify\Big(d(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \frac{Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}{Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}\Big)$

$$0 \qquad (8)$$

We first consider the 3 cases where T_A and T_B are equal to rooted P_3 or P_4 and note that the expression 3*Nd-Dd is strictly positive,
i.e. Nd/Dd>1/3.

> $Pos(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := 3\cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)$

$$Pos := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto 3\cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \qquad (9)$$

> $simplify(Pos(8, 4, 2, 2, 8, 4, 2, 2, l))$

$$2\, l^2 + 22\, l + 88 \qquad (10)$$

> $simplify(Pos(15, 6, 5, 4, 8, 4, 2, 2, l))$

$$243 + \frac{105}{2}\, l + \frac{9}{2}\, l^2 \qquad (11)$$

> $simplify(Pos(15, 6, 5, 4, 15, 6, 5, 4, l))$

$$7\, l^2 + 103\, l + 594 \qquad (12)$$

>

Next we consider the expansion of 3*Nd-Dd

> $collect(simplify(expand(simplify(3\cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)))), l)$

$$\left(\frac{1}{2}\, Na^2 + \frac{1}{2}\, Nb^2 + Na + \frac{1}{2}\, Na2 + Nb + \frac{1}{2}\, Nb2 - \frac{3}{2}\, Ra - \frac{3}{2}\, Rb\right) l^2 + \left(\frac{(2\, Nb - 1)\, Na^2}{2} + (Nb^2 + Na2 + 4\, Nb + Nb2 - 3\, Rb - 1)\, Na - \frac{Nb^2}{2} + (-1 - 3\, Ra + Na2 + Nb2)\, Nb + \frac{3\, Ra}{2} + \frac{3\, Rb}{2} + \frac{Na2}{2} + \frac{Nb2}{2} - 3\, Ra2 - 3\, Rb2\right) l + (2\, Nb^2 + Nb - 3\, Rb)\, Na^2 + (Nb^2 + (-4 + Na2 + Nb2)\, Nb + 3\, Rb - 2\, Na2 - 2\, Nb2 - 3\, Ra2 - 3\, Rb2)\, Na - 3\, Nb^2\, Ra + (3\, Ra - 2\, Na2 - 2\, Nb2 - 3\, Ra2 - 3\, Rb2)\, Nb - Na2^2 + (-1 + 3\, Ra + 3\, Rb - 2\, Nb2)\, Na2 - Nb2^2 + (-1 + 3\, Ra + 3\, Rb)\, Nb2 + 3\, Ra2 + 3\, Rb2 \qquad (13)$$

We split this expression in terms I1*l*(l-1)+II*(l-1/2)+III

> $I1(Ra, Na, Na2, Rb, Nb, Nb2) := \Big(\frac{1}{2}\, Na^2 + \frac{1}{2}\, Nb^2 + Na + \frac{1}{2}\, Na2 + Nb + \frac{1}{2}\, Nb2$

$$-\frac{3}{2}\,Ra-\frac{3}{2}\,Rb\Big)$$

$$I1 := (Ra, Na, Na2, Rb, Nb, Nb2) \mapsto \frac{1}{2}\cdot Na^2+\frac{1}{2}\cdot Nb^2+Na+\frac{1}{2}\cdot Na2+Nb+\frac{1}{2}\cdot Nb2-\frac{3}{2}\cdot Ra-\frac{3}{2}\cdot Rb \tag{14}$$

> $$II(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) := \frac{(2\,Nb-1)\,Na^2}{2}+\big(Nb^2+Na2+4\,Nb+Nb2-3\,Rb-1\big)\,Na-\frac{Nb^2}{2}+(-1-3\,Ra+Na2+Nb2)\,Nb+\frac{3\,Ra}{2}+\frac{3\,Rb}{2}+\frac{Na2}{2}+\frac{Nb2}{2}-3\,Ra2-3\,Rb2+I1(Ra, Na, Na2, Rb, Nb, Nb2)$$

$$II := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) \mapsto \frac{(2\cdot Nb-1)\cdot Na^2}{2}+\big(Nb^2+Na2+4\cdot Nb+Nb2-3\cdot Rb-1\big)\cdot Na-\frac{Nb^2}{2}+(-1-3\cdot Ra+Na2+Nb2)\cdot Nb+\frac{3\cdot Ra}{2}+\frac{3\cdot Rb}{2}+\frac{Na2}{2}+\frac{Nb2}{2}-3\cdot Ra2-3\cdot Rb2+I1(Ra, Na, Na2, Rb, Nb, Nb2) \tag{15}$$

> $$III(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) := \big(2\,Nb^2+Nb-3\,Rb\big)\,Na^2+\big(Nb^2+(-4+Na2+Nb2)\,Nb+3\,Rb-2\,Na2-2\,Nb2-3\,Ra2-3\,Rb2\big)\,Na-3\,Nb^2\,Ra+(3\,Ra-2\,Na2-2\,Nb2-3\,Ra2-3\,Rb2)\,Nb-Na2^2+(-1+3\,Ra+3\,Rb-2\,Nb2)\,Na2-Nb2^2+(-1+3\,Ra+3\,Rb)\,Nb2+3\,Ra2+3\,Rb2+\frac{1}{2}\cdot II(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2)$$

$$III := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) \mapsto \big(2\cdot Nb^2+Nb-3\cdot Rb\big)\cdot Na^2+\big(Nb^2+(-4+Na2+Nb2)\cdot Nb+3\cdot Rb-2\cdot Na2-2\cdot Nb2-3\cdot Ra2-3\cdot Rb2\big)\cdot Na-3\cdot Nb^2\cdot Ra+(3\cdot Ra-2\cdot Na2-2\cdot Nb2-3\cdot Ra2-3\cdot Rb2)\cdot Nb-Na2^2+(-1+3\cdot Ra+3\cdot Rb-2\cdot Nb2)\cdot Na2-Nb2^2+(-1+3\cdot Ra+3\cdot Rb)\cdot Nb2+3\cdot Ra2+3\cdot Rb2+\frac{II(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2)}{2} \tag{16}$$

We have to verify that all 3 are positive.
We first do so for the case where Na and Nb are both at least equal to 6, and so all preliminary identities can be used.

> $$simplify\left(I1(Ra, Na, Na2, Rb, Nb, Nb2)-\frac{(Na\cdot(Na+1)-3\cdot Ra)}{2}-\frac{(Nb\cdot(Nb+1)-3\cdot Rb)}{2}\right)$$

$$\frac{Na}{2}+\frac{Na2}{2}+\frac{Nb}{2}+\frac{Nb2}{2} \tag{17}$$

> $$simplify\big(II(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2)-\big(Nb^2+Nb-3\cdot Rb\big)\cdot Na-\big(Na^2+Na-3\cdot Ra\big)\cdot Nb-((Na+1)\cdot Na2-3\,Ra2)-((Nb+1)\cdot Nb2-3\,Rb2)\big)$$

$$(2\,Nb + Nb2)\,Na + Nb\,Na2 \quad \textbf{(18)}$$

> $simplify\Big(III(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) - (Nb^2 + Nb - 3\cdot Rb)\cdot Na^2 - (Na^2 + Na - 3\cdot Ra)\cdot Nb^2 - (3\cdot(Nb2\cdot Rb - Nb\cdot Rb2) - Nb2\cdot(Nb + Nb2)) - (3\cdot(Na2\cdot Ra - Na\cdot Ra2) - Na2\cdot(Na + Na2)) - \frac{1}{2}\cdot(Na - 1)\cdot((Nb + 1)\cdot Nb2 - 3\cdot Rb2) - \frac{1}{2}\cdot(Nb - 1)\cdot((Na + 1)\cdot Na2 - 3\cdot Ra2) - \frac{Na}{2}\cdot((Nb + 1)\cdot(Nb + Nb2) - 3\cdot Rb - 3\cdot Rb2) - \frac{Nb}{2}\cdot((Na + 1)\cdot(Na + Na2) - 3\cdot Ra - 3\cdot Ra2) - 3\cdot(Na + Na2)\cdot(Rb - (Nb + Nb2)) - 3\cdot(Nb + Nb2)\cdot(Ra - (Na + Na2))\Big)$

$$\frac{(7\,Nb + 8\,Nb2)\,Na2}{2} + 3\left(Nb + \frac{7\,Nb2}{6}\right)Na \quad \textbf{(19)}$$

>

>

The case where T_A is P_3 and N_B>6

> $simplify\left(I1(8, 4, 2, Rb, Nb, Nb2) - \frac{(Nb\cdot(Nb + 1) - 3\cdot Rb)}{2}\right)$

$$1 + \frac{Nb}{2} + \frac{Nb2}{2} \quad \textbf{(20)}$$

> $simplify(II(8, 4, 2, 2, Rb, Nb, Rb2, Nb2) - 4\cdot(Nb\cdot(Nb + 1) - 3\cdot Rb) - ((Nb + 1)\cdot Nb2 - 3\cdot Rb2))$

$$6\,Nb + 4\,Nb2 + 4 \quad \textbf{(21)}$$

>

> $simplify(2\cdot III(8, 4, 2, 2, Rb, Nb, Rb2, Nb2) - 2\cdot(3\cdot(Nb2\cdot Rb - Nb\cdot Rb2) - Nb2\cdot(Nb + Nb2)) - 24\cdot(Nb^2 + 2\cdot Nb + Nb2 - Rb\cdot 3) - 7\cdot(Nb\cdot Nb2 + Nb2 - 3\cdot Rb2) - 2\cdot(Nb\cdot(Nb + 1) - 3\cdot Nb2))$

$$2\,Nb^2 + 4\,Nb + 2\,Nb2 + 20 \quad \textbf{(22)}$$

>

(23)

>

> $simplify\left(I1(15, 6, 4, Rb, Nb, Nb2) - \frac{(Nb\cdot(Nb + 1) - 3\cdot Rb)}{2}\right)$

$$\frac{7}{2} + \frac{Nb}{2} + \frac{Nb2}{2} \quad \textbf{(24)}$$

> $simplify(II(15, 6, 5, 4, Rb, Nb, Rb2, Nb2) - 6\cdot(Nb\cdot(Nb + 1) - 3\cdot Rb) - ((Nb + 1)\cdot Nb2 - 3\cdot Rb2))$

$$13\,Nb + 6\,Nb2 + 13 \quad \textbf{(25)}$$

> $simplify(2\cdot III(15, 6, 5, 4, Rb, Nb, Rb2, Nb2) - 2\cdot(3\cdot(Nb2\cdot Rb - Nb\cdot Rb2) - Nb2\cdot(Nb + Nb2)) - 58\cdot(Nb^2 + 2\cdot Nb + Nb2 - Rb\cdot 3) - 11\cdot(Nb\cdot Nb2 + Nb2 - 3\cdot Rb2) - 5\cdot(Nb\cdot(Nb + 1) - 3\cdot Nb2))$

$$9\,Nb^2 + 14\,Nb + Nb2 + 87 \tag{26}$$

>