>

Here we denote \overline N_A as Na2 etc

> $\mathrm{mu} := (Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) \rightarrow \frac{(Ra \cdot Nb + Rb \cdot Na + Ra + Ra2 + Rb + Rb2)}{Na \cdot Nb + Na + Nb + Na2 + Nb2}$

$$\mu := (Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) \mapsto \frac{Ra \cdot Nb + Rb \cdot Na + Ra + Ra2 + Rb + Rb2}{Na \cdot Nb + Na + Nb + Na2 + Nb2} \quad (1)$$

> $mue := (Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) \rightarrow \frac{(Ra \cdot Nb + Rb \cdot Na - Na \cdot Nb + Ra2 + Rb2)}{Na \cdot Nb + Na2 + Nb2}$

$$mue := (Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) \mapsto \frac{Ra \cdot Nb + Rb \cdot Na - Na \cdot Nb + Ra2 + Rb2}{Na \cdot Nb + Na2 + Nb2} \quad (2)$$

> $Pos := (Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) \rightarrow simplify\left(2 \cdot (Na \cdot Nb + Na2 + Nb2) \cdot (Na \cdot Nb + Na + Nb + Na2 + Nb2) \cdot \left(\mu(Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) - mue(Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) - \frac{1}{2}\right)\right)$

$$Pos := (Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) \mapsto simplify\left((2 \cdot Na \cdot Nb + 2 \cdot Na2 + 2 \cdot Nb2) \cdot (Na \cdot Nb + Na + Nb + Na2 + Nb2) \cdot \left(\mu(Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) - mue(Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) - \frac{1}{2}\right)\right) \quad (3)$$

We verify that the function is increasing in Rb.

> $collect(simplify(expand(Pos(1, 1, Rb, Nb, Rb2, 0, 0, Nb2))), Rb)$

$$(2\,Nb2 - 2)\,Rb - Nb2^2 + (-Nb + 1)\,Nb2 + (-2\,Rb2 + 1)\,Nb - 2\,Rb2 \quad (4)$$

It is thus sufficient to consider a lowerbound for Rb.
We consider two cases.
ell <=2
In this case, we have an lower bound equal to $\frac{Rb2 \cdot Nb}{Nb2} + \frac{4}{3} \cdot Nb$

> $G(Nb, Rb2, Nb2) := simplify\left(expand\left(Pos\left(1, 1, \frac{Rb2 \cdot Nb}{Nb2} + \frac{4}{3} \cdot Nb, Nb, Rb2, 0, 0, Nb2\right)\right)\right)$

$$G := (Nb, Rb2, Nb2) \mapsto simplify\left(expand\left(Pos\left(1, 1, \frac{Rb2 \cdot Nb}{Nb2} + \frac{4 \cdot Nb}{3}, Nb, Rb2, 0, 0, Nb2\right)\right)\right) \quad (5)$$

> $diff(\ G(Nb, Rb2, Nb2), Rb2)$

$$-\frac{2\,Nb}{Nb2} - 2 \quad (6)$$

This function is decreasing in Rb2, so it is sufficient to consider an upper bound.
One upperbound is Rb2 <= Nb2*mu_b, where mu_B < 3/2*log_2(N_b)-1

> $F(Nb, Nb2) := G\left(Nb, \left(\frac{3}{2} \cdot log\_2(Nb) - 1\right) \cdot Nb2, Nb2\right)$

$$F := (Nb, Nb2) \mapsto G\left(Nb, \left(\frac{3 \cdot log\_2(Nb)}{2} - 1\right) \cdot Nb2, Nb2\right) \quad \textbf{(7)}$$

> collect(F(Nb, Nb2), Nb2)

$$-Nb2^2 + \left(-3\ log\_2(Nb) + \frac{5\ Nb}{3} + 3\right) Nb2 - 3\ Nb\ log\_2(Nb) + \frac{Nb}{3} \quad \textbf{(8)}$$

The extrema are attained when Nb2 is at one of its boundaries.
Now the conclusion follows as the following two expressions are always positive as Nb\ge 2^9.

>

> F(Nb, Nb)

$$\frac{2\ Nb\ (Nb + 5 - 9\ log\_2(Nb))}{3} \quad \textbf{(9)}$$

> $F\left(Nb, \frac{Nb}{4}\right)$

$$\frac{Nb\ (17\ Nb + 52 - 180\ log\_2(Nb))}{48} \quad \textbf{(10)}$$

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In the other case, we have **ell \ge 3.**

In this case, we have an lower bound equal to $\frac{Rb2 \cdot Nb}{Nb2} + \frac{7}{3} \cdot Nb$

> $G(Nb, Rb2, Nb2) := simplify\left(expand\left(Pos\left(1, 1, \frac{Rb2 \cdot Nb}{Nb2} + \frac{7}{3} \cdot Nb, Nb, Rb2, 0, 0, Nb2\right)\right)\right)$

$$G := (Nb, Rb2, Nb2) \mapsto simplify\left(expand\left(Pos\left(1, 1, \frac{Rb2 \cdot Nb}{Nb2} + \frac{7 \cdot Nb}{3}, Nb, Rb2, 0, 0, Nb2\right)\right)\right) \quad \textbf{(11)}$$

> diff(G(Nb, Rb2, Nb2), Rb2)

$$-\frac{2\ Nb}{Nb2} - 2 \quad \textbf{(12)}$$

This function is decreasing in Rb2, so it is sufficient to consider an upper bound.
One upperbound is Rb2 <= Nb2*mu_b, where mu_B < 3/2*log_2(N_b)-1

> $F(Nb, Nb2) := G\left(Nb, \left(\frac{3}{2} \cdot log\_2(Nb) - 1\right) \cdot Nb2, Nb2\right)$

$$F := (Nb, Nb2) \mapsto G\left(Nb, \left(\frac{3 \cdot log\_2(Nb)}{2} - 1\right) \cdot Nb2, Nb2\right) \quad \textbf{(13)}$$

> collect(F(Nb, Nb2), Nb2)

$$-Nb2^2 + \left(-3\ log\_2(Nb) + \frac{11\ Nb}{3} + 3\right) Nb2 - 3\ Nb\ log\_2(Nb) - \frac{5\ Nb}{3} \quad \textbf{(14)}$$

> diff(F(Nb, Nb2), Nb2)

$$-3\ log\_2(Nb) - 2\ Nb2 + \frac{11\ Nb}{3} + 3 \quad \textbf{(15)}$$

The extrema are attained when Nb2 is at one of its boundaries.
Now the conclusion follows as the following two expressions are always positive as Nb\ge 2^9.

>

> $F(Nb, Nb)$

$$\frac{8\,Nb^2}{3} - 6\,Nb\,log\_2(Nb) + \frac{4\,Nb}{3} \tag{16}$$

> $F(Nb, log\_2(Nb))$

$$\frac{2\,Nb\,log\_2(Nb)}{3} - 4\,log\_2(Nb)^2 - \frac{5\,Nb}{3} + 3\,log\_2(Nb) \tag{17}$$

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