

We compute μ , for a path of length l (ell) with endvertices A and B, with rooted trees T_A and N_B in these two vertices

Here we denote $\overline{N_A}$ as $Na2$ etc

$$\mu := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \rightarrow ((Ra \cdot Nb + Na \cdot Nb \cdot (l - 1) + Rb \cdot Na + l \cdot Ra + l \cdot Rb + \text{binomial}(l, 2) \cdot (Na + Nb) + Ra2 + Rb2) + \text{binomial}(l + 1, 3)) / (Na \cdot Nb + l \cdot Na + l \cdot Nb + Na2 + Nb2 + \text{binomial}(l, 2))$$

We consider the numerator (Nu) and denominator (De) separately and compute it for the difference (d) between μ for the tree T and T_e (path of length $l-1$)

$$Nu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := (Ra \cdot Nb + Na \cdot Nb \cdot (l - 1) + Rb \cdot Na + l \cdot Ra + l \cdot Rb + \text{binomial}(l, 2) \cdot (Na + Nb) + Ra2 + Rb2) + \text{binomial}(l + 1, 3)$$

$$De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := Na \cdot Nb + l \cdot Na + l \cdot Nb + Na2 + Nb2 + \text{binomial}(l, 2)$$

$$d := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \rightarrow \mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l - 1)$$

We compute the numerator (Nd) and denominator (Dd) of d

$$Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := Nu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l - 1) - Nu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l - 1) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)$$

$$Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l - 1)$$

We add a check

$$\text{simplify} \left(\mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \frac{Nu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}{De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)} \right)$$

$$\text{simplify} \left(d(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \frac{Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}{Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)} \right)$$

We first consider the 3 cases where T_A and T_B are equal to rooted P_3 or P_4 and note that the expression $3 \cdot Nd - Dd$ is strictly positive, i.e. $Nd/Dd > 1/3$.

$$Pos(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := 3 \cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)$$

$$\text{simplify}(Pos(8, 4, 2, 2, 8, 4, 2, 2, l))$$

$$\text{simplify}(Pos(15, 6, 5, 4, 8, 4, 2, 2, l))$$

$$\text{simplify}(Pos(15, 6, 5, 4, 15, 6, 5, 4, l))$$

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Next we consider the expansion of $3 \cdot Nd - Dd$ for the general case.

$$\text{collect}(\text{simplify}(\text{expand}(\text{simplify}(3 \cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l))), l))$$

This is a polynomial of the form $c2 \cdot l^2 + c1 \cdot l + c0$.

To prove that this is always positive, we prove that each of the coefficients is positive

$$> c2(Ra, Na, Na2, Rb, Nb, Nb2) := \left(\frac{1}{2} Na^2 + \frac{1}{2} Nb^2 + Na + \frac{1}{2} Na2 + Nb + \frac{1}{2} Nb2 - \frac{3}{2} Ra - \frac{3}{2} Rb \right)$$

$$> c1(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) := \frac{(2 Nb - 1) Na^2}{2} + (Nb^2 + Na2 + 4 Nb + Nb2 - 3 Rb - 1) Na - \frac{Nb^2}{2} + (-1 - 3 Ra + Na2 + Nb2) Nb + \frac{3 Ra}{2} + \frac{3 Rb}{2} + \frac{Na2}{2} + \frac{Nb2}{2} - 3 Ra2 - 3 Rb2$$

$$> c0(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) := (2 Nb^2 + Nb - 3 Rb) Na^2 + (Nb^2 + (-4 + Na2 + Nb2) Nb + 3 Rb - 2 Na2 - 2 Nb2 - 3 Ra2 - 3 Rb2) Na - 3 Nb^2 Ra + (3 Ra - 2 Na2 - 2 Nb2 - 3 Ra2 - 3 Rb2) Nb - Na2^2 + (-1 + 3 Ra + 3 Rb - 2 Nb2) Na2 - Nb2^2 + (-1 + 3 Ra + 3 Rb) Nb2 + 3 Ra2 + 3 Rb2$$

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We have to verify that all 3 are positive.

We first do so for the case where Na and Nb are both at least equal to 6, and so all preliminary identities can be used.

$$> simplify\left(c2(Ra, Na, Na2, Rb, Nb, Nb2) - \frac{(Na \cdot (Na + 1) - 3 \cdot Ra)}{2} - \frac{(Nb \cdot (Nb + 1) - 3 \cdot Rb)}{2}\right)$$

$$> simplify\left(c1(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) - (Nb^2 + Nb - 3 \cdot Rb) \cdot \left(Na - \frac{1}{2}\right) - (Na^2 + Na - 3 \cdot Ra) \cdot \left(Nb - \frac{1}{2}\right) - ((Na + 1) \cdot Na2 - 3 Ra2) - ((Nb + 1) \cdot Nb2 - 3 Rb2)\right)$$

$$> simplify\left(c0(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) - (Nb^2 + Nb - 3 \cdot Rb) \cdot Na \cdot \left(Na - \frac{1}{2}\right) - (Na^2 + Na - 3 \cdot Ra) \cdot Nb \cdot \left(Nb - \frac{1}{2}\right) - (3 \cdot (Nb2 \cdot Rb - Nb \cdot Rb2) - Nb2 \cdot (Nb + Nb2)) - (3 \cdot (Na2 \cdot Ra - Na \cdot Ra2) - Na2 \cdot (Na + Na2)) - (Na - 1) \cdot ((Nb + 1) \cdot Nb2 - 3 \cdot Rb2) - (Nb - 1) \cdot ((Na + 1) \cdot Na2 - 3 \cdot Ra2) - \frac{3}{2} \cdot (Na + Na2) \cdot (Rb - (Nb + Nb2)) - \frac{3}{2} \cdot (Nb + Nb2) \cdot (Ra - (Na + Na2))\right)$$

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The case where T_A is P_3 and N_B>6

$$> simplify\left(c2(8, 4, 2, Rb, Nb, Nb2) - \frac{(Nb \cdot (Nb + 1) - 3 \cdot Rb)}{2}\right)$$

$$> simplify\left(c1(8, 4, 2, 2, Rb, Nb, Rb2, Nb2) - \frac{7}{2} \cdot (Nb \cdot (Nb + 1) - 3 \cdot Rb) - ((Nb + 1) \cdot Nb2\right)$$

$$- 3 \cdot Rb2) \Big)$$

$$\begin{aligned} &> \text{simplify} \Big(c0(8, 4, 2, 2, Rb, Nb, Rb2, Nb2) - (3 \cdot (Nb2 \cdot Rb - Nb \cdot Rb2) - Nb2 \cdot (Nb + Nb2)) \\ &\quad - 11 \cdot (Nb^2 + 2 \cdot Nb + Nb2 - Rb \cdot 3) - 3 \cdot (Nb \cdot Nb2 + Nb2 - 3 \cdot Rb2) - (Nb \cdot (Nb + 1) \\ &\quad - 3 \cdot Nb2) \Big) \end{aligned}$$

The case with T_A being P_4

$$> \text{simplify} \Big(c2(15, 6, 4, Rb, Nb, Nb2) - \frac{(Nb \cdot (Nb + 1) - 3 \cdot Rb)}{2} \Big)$$

$$\begin{aligned} &> \text{simplify} \Big(c1(15, 6, 5, 4, Rb, Nb, Rb2, Nb2) - \frac{11}{2} \cdot (Nb \cdot (Nb + 1) - 3 \cdot Rb) - ((Nb + 1) \cdot Nb2 \\ &\quad - 3 \cdot Rb2) \Big) \end{aligned}$$

$$\begin{aligned} &> \text{simplify} \Big(c0(15, 6, 5, 4, Rb, Nb, Rb2, Nb2) - (3 \cdot (Nb2 \cdot Rb - Nb \cdot Rb2) - Nb2 \cdot (Nb + Nb2)) \\ &\quad - 29 \cdot (Nb^2 + 2 \cdot Nb + Nb2 - Rb \cdot 3) - 5 \cdot (Nb \cdot Nb2 + Nb2 - 3 \cdot Rb2) - 4 \cdot (Nb \cdot (Nb + 1) \\ &\quad - 3 \cdot Nb2) \Big) \end{aligned}$$

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