We compute mu, for a path of length l (ell) with endvertices A and B, with rooted trees T_A and N_B in these two vertices
Here we denote \overline N_A as Na2 etc

> mu := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)→((Ra·Nb + Na·Nb·(l − 1) + Rb·Na + l·Ra + l·Rb + binomial(l, 2)·(Na + Nb) + Ra2 + Rb2) + binomial(l + 1, 3))/(Na·Nb + l·Na + l·Nb + Na2 + Nb2 + binomial(l, 2))

$$\mu := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto \frac{1}{Na\cdot Nb + l\cdot Na + l\cdot Nb + Na2 + Nb2 + \binom{l}{2}} \left(Nb\cdot Ra + Na\cdot Nb\cdot (l-1) + Rb\cdot Na + l\cdot Ra + l\cdot Rb + \binom{l}{2}\cdot (Na + Nb) + Ra2 + Rb2 + \binom{l+1}{3} \right) \quad \textbf{(1)}$$

We consider the numerator (Nu) and denominator (De) separately and compute it for the difference (d) between mu for the the tree T and T\e (path of length l-1)

> Nu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := (Ra·Nb + Na·Nb·(l − 1) + Rb·Na + l·Ra + l·Rb + binomial(l, 2)·(Na + Nb) + Ra2 + Rb2) + binomial(l + 1, 3)

$$\mathrm{N} := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto Nb\cdot Ra + Na\cdot Nb\cdot (l-1) + Rb\cdot Na + l\cdot Ra + l\cdot Rb + \binom{l}{2}\cdot (Na + Nb) + Ra2 + Rb2 + \binom{l+1}{3} \quad \textbf{(2)}$$

> De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := Na·Nb + l·Na + l·Nb + Na2 + Nb2 + binomial(l, 2)

$$De := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto Na\cdot Nb + l\cdot Na + l\cdot Nb + Na2 + Nb2 + \binom{l}{2} \quad \textbf{(3)}$$

> d := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)→μ(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) − μ(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l − 1)

$$d := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto \mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) \quad \textbf{(4)}$$

We compute the numerator (Nd) and denominator (Dd) of d

> Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := Nu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)·De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l − 1) − Nu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l − 1)·De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)

$$Nd := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto \mathrm{N}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)\cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) - \mathrm{N}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1)\cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \quad \textbf{(5)}$$

> Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)·De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l − 1)

$$Dd := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)\cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) \quad \textbf{(6)}$$

We add a check

> simplify(mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)

$$- \frac{\mathrm{Nu}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}{De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)} \Bigg)$$

$$0 \tag{7}$$

> $simplify\Big(d(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \frac{Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}{Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}\Big)$

$$0 \tag{8}$$

We first consider the 3 cases where T_A and T_B are equal to rooted P_3 or P_4 and note that the expression 3*Nd-Dd is strictly positive,
i.e. Nd/Dd>1/3.

> $Pos(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := 3 \cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)$

$$Pos := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto 3 \cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \tag{9}$$

> $simplify(Pos(8, 4, 2, 2, 8, 4, 2, 2, l))$

$$2\, l^2 + 22\, l + 88 \tag{10}$$

> $simplify(Pos(15, 6, 5, 4, 8, 4, 2, 2, l))$

$$243 + \frac{105}{2}\, l + \frac{9}{2}\, l^2 \tag{11}$$

> $simplify(Pos(15, 6, 5, 4, 15, 6, 5, 4, l))$

$$7\, l^2 + 103\, l + 594 \tag{12}$$

>

Next we consider the expansion of 3*Nd-Dd for the general case.

> $collect(simplify(expand(simplify(3 \cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)))), l)$

$$\begin{aligned}
&\left(\frac{1}{2}\, Na^2 + \frac{1}{2}\, Nb^2 + Na + \frac{1}{2}\, Na2 + Nb + \frac{1}{2}\, Nb2 - \frac{3}{2}\, Ra - \frac{3}{2}\, Rb\right) l^2 \\
&\quad + \left(\frac{(2\, Nb - 1)\, Na^2}{2} + \left(Nb^2 + Na2 + 4\, Nb + Nb2 - 3\, Rb - 1\right) Na - \frac{Nb^2}{2} + (-1 \right. \\
&\quad \left. - 3\, Ra + Na2 + Nb2)\, Nb + \frac{3\, Ra}{2} + \frac{3\, Rb}{2} + \frac{Na2}{2} + \frac{Nb2}{2} - 3\, Ra2 - 3\, Rb2\right) l \\
&\quad + \left(2\, Nb^2 + Nb - 3\, Rb\right) Na^2 + \left(Nb^2 + (-4 + Na2 + Nb2)\, Nb + 3\, Rb - 2\, Na2 - 2\, Nb2 \right. \\
&\quad \left. - 3\, Ra2 - 3\, Rb2\right) Na - 3\, Nb^2\, Ra + (3\, Ra - 2\, Na2 - 2\, Nb2 - 3\, Ra2 - 3\, Rb2)\, Nb - Na2^2 \\
&\quad + (-1 + 3\, Ra + 3\, Rb - 2\, Nb2)\, Na2 - Nb2^2 + (-1 + 3\, Ra + 3\, Rb)\, Nb2 + 3\, Ra2 \\
&\quad + 3\, Rb2
\end{aligned} \tag{13}$$

This is a polynomial of the form c2*l^2+c1*l+c0.
To prove that this is always positive, we prove that each of the coefficients is positive

> $c2(Ra, Na, Na2, Rb, Nb, Nb2) := \Big(\frac{1}{2}\, Na^2 + \frac{1}{2}\, Nb^2 + Na + \frac{1}{2}\, Na2 + Nb + \frac{1}{2}\, Nb2$

$$- \frac{3}{2}\, Ra - \frac{3}{2}\, Rb\Big)$$

$$c2 := (Ra, Na, Na2, Rb, Nb, Nb2) \mapsto \frac{1}{2}\cdot Na^2 + \frac{1}{2}\cdot Nb^2 + Na + \frac{1}{2}\cdot Na2 + Nb + \frac{1}{2}\cdot Nb2 - \frac{3}{2}\cdot Ra - \frac{3}{2}\cdot Rb \quad \textbf{(14)}$$

> $c1(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) := \frac{(2\, Nb - 1)\, Na^2}{2} + \big(Nb^2 + Na2 + 4\, Nb + Nb2 - 3\, Rb - 1\big)\, Na - \frac{Nb^2}{2} + (-1 - 3\, Ra + Na2 + Nb2)\, Nb + \frac{3\, Ra}{2} + \frac{3\, Rb}{2} + \frac{Na2}{2} + \frac{Nb2}{2} - 3\, Ra2 - 3\, Rb2$

$$c1 := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) \mapsto \frac{(2\cdot Nb - 1)\cdot Na^2}{2} + \big(Nb^2 + Na2 + 4\cdot Nb + Nb2 - 3\cdot Rb - 1\big)\cdot Na - \frac{Nb^2}{2} + (-1 - 3\cdot Ra + Na2 + Nb2)\cdot Nb + \frac{3\cdot Ra}{2} + \frac{3\cdot Rb}{2} + \frac{Na2}{2} + \frac{Nb2}{2} - 3\cdot Ra2 - 3\cdot Rb2 \quad \textbf{(15)}$$

> $c0(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) := \big(2\, Nb^2 + Nb - 3\, Rb\big)\, Na^2 + \big(Nb^2 + (-4 + Na2 + Nb2)\, Nb + 3\, Rb - 2\, Na2 - 2\, Nb2 - 3\, Ra2 - 3\, Rb2\big)\, Na - 3\, Nb^2\, Ra + (3\, Ra - 2\, Na2 - 2\, Nb2 - 3\, Ra2 - 3\, Rb2)\, Nb - Na2^2 + (-1 + 3\, Ra + 3\, Rb - 2\, Nb2)\, Na2 - Nb2^2 + (-1 + 3\, Ra + 3\, Rb)\, Nb2 + 3\, Ra2 + 3\, Rb2$

$$c0 := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) \mapsto \big(2\cdot Nb^2 + Nb - 3\cdot Rb\big)\cdot Na^2 + \big(Nb^2 + (-4 + Na2 + Nb2)\cdot Nb + 3\cdot Rb - 2\cdot Na2 - 2\cdot Nb2 - 3\cdot Ra2 - 3\cdot Rb2\big)\cdot Na - 3\cdot Nb^2\cdot Ra + (3\cdot Ra - 2\cdot Na2 - 2\cdot Nb2 - 3\cdot Ra2 - 3\cdot Rb2)\cdot Nb - Na2^2 + (-1 + 3\cdot Ra + 3\cdot Rb - 2\cdot Nb2)\cdot Na2 - Nb2^2 + (-1 + 3\cdot Ra + 3\cdot Rb)\cdot Nb2 + 3\cdot Ra2 + 3\cdot Rb2 \quad \textbf{(16)}$$

>

We have to verify that all 3 are positive.
We first do so for the case where Na and Nb are both at least equal to 6, and so all preliminary identities can be used.

> $simplify\left(c2(Ra, Na, Na2, Rb, Nb, Nb2) - \frac{(Na\cdot(Na+1) - 3\cdot Ra)}{2} - \frac{(Nb\cdot(Nb+1) - 3\cdot Rb)}{2}\right)$

$$\frac{Na}{2} + \frac{Na2}{2} + \frac{Nb}{2} + \frac{Nb2}{2} \quad \textbf{(17)}$$

> $simplify\left(c1(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) - \big(Nb^2 + Nb - 3\cdot Rb\big)\cdot\left(Na - \frac{1}{2}\right) - \big(Na^2 + Na - 3\cdot Ra\big)\cdot\left(Nb - \frac{1}{2}\right) - ((Na+1)\cdot Na2 - 3\, Ra2) - ((Nb+1)\cdot Nb2 - 3\, Rb2)\right)$

$$\textbf{(18)}$$

$$\frac{(4\,Nb + 2\,Nb2 - 1)\,Na}{2} + \frac{(2\,Na2 - 1)\,Nb}{2} - \frac{Na2}{2} - \frac{Nb2}{2} \quad \textbf{(18)}$$

> $simplify\Big(c0(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) - (Nb^2 + Nb - 3\cdot Rb)\cdot Na\cdot\Big(Na - \frac{1}{2}\Big) - (Na^2 + Na - 3\cdot Ra)\cdot Nb\cdot\Big(Nb - \frac{1}{2}\Big) - (3\cdot(Nb2\cdot Rb - Nb\cdot Rb2) - Nb2\cdot(Nb + Nb2)) - (3\cdot(Na2\cdot Ra - Na\cdot Ra2) - Na2\cdot(Na + Na2)) - (Na - 1)\cdot((Nb + 1)\cdot Nb2 - 3\cdot Rb2) - (Nb - 1)\cdot((Na + 1)\cdot Na2 - 3\cdot Ra2) - \frac{3}{2}\cdot(Na + Na2)\cdot(Rb - (Nb + Nb2)) - \frac{3}{2}\cdot(Nb + Nb2)\cdot(Ra - (Na + Na2))\Big)$

$$\frac{(3\,Rb + 2\,Nb2)\,Na2}{2} + \frac{3\,Ra\,Nb2}{2} + \frac{Na\,Nb\,(Na + Nb)}{2} \quad \textbf{(19)}$$

>

The case where T_A is P_3 and N_B>6

> $simplify\Big(c2(8, 4, 2, Rb, Nb, Nb2) - \frac{(Nb\cdot(Nb + 1) - 3\cdot Rb)}{2}\Big)$

$$1 + \frac{Nb}{2} + \frac{Nb2}{2} \quad \textbf{(20)}$$

> $simplify\Big(c1(8, 4, 2, 2, Rb, Nb, Rb2, Nb2) - \frac{7}{2}\cdot(Nb\cdot(Nb + 1) - 3\cdot Rb) - ((Nb + 1)\cdot Nb2 - 3\cdot Rb2)\Big)$

$$\frac{11\,Nb}{2} + 3 + \frac{7\,Nb2}{2} \quad \textbf{(21)}$$

> $simplify(\ c0(8, 4, 2, 2, Rb, Nb, Rb2, Nb2) - (3\cdot(Nb2\cdot Rb - Nb\cdot Rb2) - Nb2\cdot(Nb + Nb2)) - 11\cdot(Nb^2 + 2\cdot Nb + Nb2 - Rb\cdot 3) - 3\cdot(Nb\cdot Nb2 + Nb2 - 3\cdot Rb2) - (Nb\cdot(Nb + 1) - 3\cdot Nb2))$

$$-Nb + 3\,Rb + 8 \quad \textbf{(22)}$$

The case with T_A being P_4

> $simplify\Big(c2(15, 6, 4, Rb, Nb, Nb2) - \frac{(Nb\cdot(Nb + 1) - 3\cdot Rb)}{2}\Big)$

$$\frac{7}{2} + \frac{Nb}{2} + \frac{Nb2}{2} \quad \textbf{(23)}$$

> $simplify\Big(c1(15, 6, 5, 4, Rb, Nb, Rb2, Nb2) - \frac{11}{2}\cdot(Nb\cdot(Nb + 1) - 3\cdot Rb) - ((Nb + 1)\cdot Nb2 - 3\cdot Rb2)\Big)$

$$\frac{25\,Nb}{2} + \frac{19}{2} + \frac{11\,Nb2}{2} \quad \textbf{(24)}$$

> $simplify\left(c0(15, 6, 5, 4, Rb, Nb, Rb2, Nb2) - (3\cdot(Nb2\cdot Rb - Nb\cdot Rb2) - Nb2\cdot(Nb + Nb2)) - 29\cdot\left(Nb^2 + 2\cdot Nb + Nb2 - Rb\cdot 3\right) - 5\cdot(Nb\cdot Nb2 + Nb2 - 3\cdot Rb2) - 4\cdot(Nb\cdot(Nb + 1) - 3\cdot Nb2)\right)$

$$-4\,Nb + 2\,Nb2 + 9\,Rb + 37 \tag{25}$$

>