In this file, we prove the following proposition

Let \$T\$ be a tree of order **for** *which no vertex has degree \$2\$, except possibly the vertex \$v\$.*
Then \$\overline N_v R_v − N_v \overline R_v \le N_v(N_v + 2\overline N_v).\$

when deg v \ge 2.

First, we compute $Nvb \cdot Rv - Nv \cdot Rvb$ **in** *terms of quantities of T_1* **and** *T_2.*
Here Nvb is \overline N_v **and** *analogous.*

> $Nv := N1 \cdot N2;$

$$Nv := N1\,N2 \tag{1}$$

> $Nvb := N1b + N2b;$

$$Nvb := N1b + N2b \tag{2}$$

> $Rv := N1 \cdot R2 + N2 \cdot R1 - N1 \cdot N2;$

$$Rv := -N1\,N2 + N1\,R2 + N2\,R1 \tag{3}$$

> $Rvb := R1b + R2b$

$$Rvb := R1b + R2b \tag{4}$$

> $simplify(Nvb \cdot Rv - Nv \cdot Rvb\ )$

$$((-N1b - N2b - R1b - R2b)\,N2 + R2\,(N1b + N2b))\,N1 + N2\,R1\,(N1b + N2b) \tag{5}$$

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Next, we **use** *estimates* **to** *prove most cases.*
For this, we rewrite $Nvb \cdot Rv - Nv \cdot Rvb$ *as* $N1(N2b \cdot R2 - N2 \cdot R2b) + N2(N1b \cdot R1 - N1 \cdot R1b) + N2b \cdot N2 \cdot R1 + N1b \cdot N1 \cdot R2 - N1N2(N1b + N2b)$ **and use** *the induction hypothesis* **to** *get :*
$Nvb \cdot Rv - Nv \cdot Rvb \le N1N2(N2 + 2\,N2b) + N1 \cdot N2(N1 + 2 \cdot N1b) + N2b \cdot N2 \cdot R1 + N1b \cdot N1 \cdot R2 - N1N2(N1b + N2b) = N2b \cdot N2 \cdot R1 + N1b \cdot N1 \cdot R2 + N1 \cdot N2 \cdot (N1b + N1 + N2b + N2)$

Using the latter, we derive the following lower bound f **for** $Nv(Nv + 2 \cdot Nvb) - (Nvb \cdot Rv - Nv \cdot Rvb\ )$ *which we need* **to** *prove* **to** *be positive.*

> $f(N1b, N1, R1, N2b, N2, R2) := simplify(N1 \cdot N2 \cdot (N1 \cdot N2 + 2 \cdot (N1b + N2b)) - (N2b \cdot N2 \cdot R1 + N1b \cdot N1 \cdot R2 + N1 \cdot N2 \cdot (N1b + N1 + N2b + N2))\ )$

$$f := (N1b, N1, R1, N2b, N2, R2) \mapsto simplify(N1 \cdot N2 \cdot (N1 \cdot N2 + 2 \cdot N1b + 2 \cdot N2b) - N2b \cdot N2 \cdot R1 - N1 \cdot N1b \cdot R2 - N1 \cdot N2 \cdot (N1b + N1 + N2b + N2)) \tag{6}$$

> $f(N1b, N1, R1, N2b, N2, R2)$

$$(N2^2 - N2)\,N1^2 + \left(-N2^2 + (N1b + N2b)\,N2 - R2\,N1b\right)N1 - N2\,N2b\,R1 \tag{7}$$

The latter is positive **if** $R\_i \le \frac{N\_i^2}{4}$ **for** *i* **in** $\{1, 2\}$ *since f is decreasing* **in** *R2* **and** *R1,* **and**

> $simplify\left(f\left(N1b, N1, \frac{N1^2}{4}, N2b, N2, \frac{N2^2}{4}\right)\right)$

$$\left(\left(N2 - \frac{N2b}{4} - 1\right)N1 + \left(-\frac{N1b}{4} - 1\right)N2 + N1b + N2b\right)N2\,N1 \tag{8}$$

Note **for** *this that* $\frac{2 \cdot N2 - N2b - 4}{4}$ **and** $\frac{N1}{2} - \frac{N1b}{4} - 1$ *are both nonnegative* (*a lemma says that* $Nvb \le 2\,Nv - 4$)

There are 7 *choices* **for** *subtrees T_i* **to not** *satisfy* $R\_i \le \frac{N\_i^2}{4}$

(**for** *this, we checked trees with at most* 10 *vertices* **in** *the document Relations_N&R_smalltrees, concluding as* **for** *larger trees N_i \ge* 32 **and then** *\mu_i* $< \frac{3}{2} log\_2(N\_i) < \frac{N\_i}{4}$)

We first exclude the 5 *larger ones.*

If T_2 is one of these 5 *larger trees,* **and** *T1 is a larger subtree* $\left(\textit{satisfying } R1 \le \frac{N1^2}{4}\right)$, *we can conclude using that N1b \le* 2 *N1* **and** $R\_1 \le \frac{N\_1^2}{4} \cdot$ (**and** *f being decreasing* **in** *R1*)

> $f\left(N1b, N1, \frac{N1^2}{4}, 17, 11, 42\right)$

$$\frac{253}{4} N1^2 - 31\, N1\, N1b + 66\, N1 \quad \textbf{(9)}$$

> $f\left(N1b, N1, \frac{N1^2}{4}, 3, 8, 20\right)$

$$2\, N1\, (25\, N1 - 6\, N1b - 20) \quad \textbf{(10)}$$

> $f\left(N1b, N1, \frac{N1^2}{4}, 11, 9, 29\right)$

$$\frac{N1\, (189\, N1 - 80\, N1b + 72)}{4} \quad \textbf{(11)}$$

> $simplify\left(f\left(N1b, N1, \frac{N1^2}{4}, 2, 4, 8\right)\right)$

$$2\, N1\, (5\, N1 - 2\, N1b - 4) \quad \textbf{(12)}$$

> $simplify\left(f\left(N1b, N1, \frac{N1^2}{4}, 7, 10, 31\right)\right)$

$$\frac{N1\, (145\, N1 - 42\, N1b - 60)}{2} \quad \textbf{(13)}$$

If both T1 and T2 are one of the five, we just *checking the remaining* 15 *combinations of them.*

> $f(3, 8, 20, 2, 4, 8)$

$$448 \quad \textbf{(14)}$$

> $f(3, 8, 20, 7, 10, 31)$

$$3616 \quad \textbf{(15)}$$

> $f(3, 8, 20, 17, 11, 42)$

(16)

$$3084 \quad (16)$$

> $f(3, 8, 20, 11, 9, 29)$

$$2292 \quad (17)$$

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> $f(2, 4, 8, 2, 4, 8)$

$$64 \quad (18)$$

> $f(7, 10, 31, 7, 10, 31)$

$$5060 \quad (19)$$

> $f(17, 11, 42, 17, 11, 42)$

$$385 \quad (20)$$

> $f(11, 9, 29, 11, 9, 29)$

$$1143 \quad (21)$$

> $f(2, 4, 8, 7, 10, 31)$

$$592 \quad (22)$$

> $f(2, 4, 8, 17, 11, 42)$

$$280 \quad (23)$$

> $f(2, 4, 8, 11, 9, 29)$

$$272 \quad (24)$$

> $f(7, 10, 31, 17, 11, 42)$

$$3693 \quad (25)$$

> $f(7, 10, 31, 11, 9, 29)$

$$2911 \quad (26)$$

> $f(17, 11, 42, 11, 9, 29)$

$$1012 \quad (27)$$

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For the case where T_i is P_2 or S_4, we do not estimate from the start and use the exact formula (h). Hereby, we consider the expression in terms of the quantities of T_w=T_1 \v.

> $h(N1b, R1b, N1, R1, N2b, R2b, N2, R2) := N1 \cdot N2 \cdot (N1 \cdot N2 + 2 \cdot (N1b + N2b)) - (((-N1b - N2b - R1b - R2b) N2 + R2 (N1b + N2b)) N1 + N2 R1 (N1b + N2b))$

$$h := (N1b, R1b, N1, R1, N2b, R2b, N2, R2) \mapsto N1 \cdot N2 \cdot (N1 \cdot N2 + 2 \cdot N1b + 2 \cdot N2b) - ((-N1b - N2b - R1b - R2b) \cdot N2 + R2 \cdot (N1b + N2b)) \cdot N1 - N2 \cdot R1 \cdot (N1b + N2b) \quad (28)$$

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case T2=P2

> $simplify(\ h(Nw + Nwb, Rw + Rwb, Nw + 1, Rw + Nw + 1, 1, 1, 2, 3))$

$$5\, Nw^2 + (Nwb + 2\, Rwb + 12)\, Nw + (-2\, Rw + 1)\, Nwb + 2\, Rwb + 7 \quad (29)$$

> $simplify(2 \cdot (Nw \cdot (Nw + Nwb) - (Rw * Nwb - Rwb * Nw)) + Nw \cdot (Nw - Nwb) + 12 \cdot Nw + 2 \cdot Nw^2 + Nwb + 2 \cdot Rwb + 7)$

(30)

$$5\,Nw^2 + (Nwb + 2\,Rwb + 12)\,Nw + (-2\,Rw + 1)\,Nwb + 2\,Rwb + 7 \qquad \textbf{(30)}$$

The first expression is always positive, as Rw*Nwb-Rwb*Nw <= Nw(Nw+Nwb) and Nw(Nw-Nwb) >= 0.

case T2=S4

> $simplify(\ h(Nw + Nwb, Rw + Rwb, Nw + 1, Rw + Nw + 1, 6, 10, 5, 13))$

$$22\,Nw^2 + (-3\,Nwb + 5\,Rwb + 79)\,Nw + (-5\,Rw - 3)\,Nwb - 25\,Rw + 5\,Rwb + 57 \qquad \textbf{(31)}$$

> $simplify(17\,Nw\text{^}2 - 13\,Nw * Nwb - 25\,Rw + 5 \cdot (Nw \cdot (Nw + Nwb) - (Rw * Nwb - Rwb * Nw)) - h(Nw + Nwb, Rw + Rwb, Nw + 1, Rw + Nw + 1, 6, 10, 5, 13))$

$$(-5\,Nw + 3)\,Nwb - 79\,Nw - 5\,Rwb - 57 \qquad \textbf{(32)}$$

The second expression h(...,6,10,5,13) is larger than 17Nw^2-13Nw*Nwb-25Rw, which is positive when Rw \le 4/25Nw^2 (since Nwb <Nw).
Since Rw \le Nw* (3/2 log_2(N_w) -1) , the last is satisfied whenever Nw \ge 64, which is the case when T_2 \backslash v has at least order 11.
For the remaining trees with less than 10 vertices, the expression can be checked by ranging in a brute force way over all of them (again in the document Relations_N&R_smalltrees)