We compute mu, for a path of length l (ell) with endvertices A and B, with rooted trees T_A and T_B in these two vertices
Here we denote \overline N_A as Na2 etc

> mu := $(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \rightarrow ((Ra \cdot Nb + Na \cdot Nb \cdot (l-1) + Rb \cdot Na + l \cdot Ra + l \cdot Rb + \mathrm{binomial}(l, 2) \cdot (Na + Nb) + Ra2 + Rb2) + \mathrm{binomial}(l+1, 3)) / (Na \cdot Nb + l \cdot Na + l \cdot Nb + Na2 + Nb2 + \mathrm{binomial}(l, 2))$

We consider the numerator (Nu) and denominator (De) separately and compute it for the difference (d) between mu for the the tree T and T\e (path of length l-1)

> $\mathrm{Nu}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := (Ra \cdot Nb + Na \cdot Nb \cdot (l-1) + Rb \cdot Na + l \cdot Ra + l \cdot Rb + \mathrm{binomial}(l, 2) \cdot (Na + Nb) + Ra2 + Rb2) + \mathrm{binomial}(l+1, 3)$

> $De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := Na \cdot Nb + l \cdot Na + l \cdot Nb + Na2 + Nb2 + \mathrm{binomial}(l, 2)$

> $d := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \rightarrow \mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1)$

We compute the numerator (Nd) and denominator (Dd) of d

> $Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := \mathrm{Nu}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) - \mathrm{Nu}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)$

> $Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1)$

We add a check

> $simplify\left(\mathrm{mu}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \frac{\mathrm{Nu}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}{De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}\right)$

> $simplify\left(d(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \frac{Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}{Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}\right)$

We first consider the 3 cases where T_A and T_B are equal to rooted P_3 or P_4 and note that the expression 3*Nd-Dd is strictly positive,
i.e. Nd/Dd>1/3.

> $Pos(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := 3 \cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)$

> $simplify(Pos(8, 4, 2, 2, 8, 4, 2, 2, l))$

> $simplify(Pos(15, 6, 5, 4, 8, 4, 2, 2, l))$

> $simplify(Pos(15, 6, 5, 4, 15, 6, 5, 4, l))$

>

Next we consider the expansion of 3*Nd-Dd for the general case.

> $collect(simplify(expand(simplify(3 \cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)))), l)$

This is a polynomial of the form c2*l^2+c1*l+c0.
To prove that this is always positive, we prove that each of the coefficients is positive

> $c2(Ra, Na, Na2, Rb, Nb, Nb2) := \left(\frac{1}{2} Na^2 + \frac{1}{2} Nb^2 + Na + \frac{1}{2} Na2 + Nb + \frac{1}{2} Nb2 - \frac{3}{2} Ra - \frac{3}{2} Rb\right)$

> $c1(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) := \frac{(2\,Nb - 1)\,Na^2}{2} + \left(Nb^2 + Na2 + 4\,Nb + Nb2 - 3\,Rb - 1\right) Na - \frac{Nb^2}{2} + (-1 - 3\,Ra + Na2 + Nb2)\,Nb + \frac{3\,Ra}{2} + \frac{3\,Rb}{2} + \frac{Na2}{2} + \frac{Nb2}{2} - 3\,Ra2 - 3\,Rb2$

> $c0(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) := \left(2\,Nb^2 + Nb - 3\,Rb\right) Na^2 + \left(Nb^2 + (-4 + Na2 + Nb2)\,Nb + 3\,Rb - 2\,Na2 - 2\,Nb2 - 3\,Ra2 - 3\,Rb2\right) Na - 3\,Nb^2\,Ra + (3\,Ra - 2\,Na2 - 2\,Nb2 - 3\,Ra2 - 3\,Rb2)\,Nb - Na2^2 + (-1 + 3\,Ra + 3\,Rb - 2\,Nb2)\,Na2 - Nb2^2 + (-1 + 3\,Ra + 3\,Rb)\,Nb2 + 3\,Ra2 + 3\,Rb2$

>

We have to verify that all 3 are positive.
We first do so for the case where Na and Nb are both at least equal to 6, and so all preliminary inequalities can be used.

> $simplify\left(c2(Ra, Na, Na2, Rb, Nb, Nb2) - \frac{(Na\cdot(Na+1) - 3\cdot Ra)}{2} - \frac{(Nb\cdot(Nb+1) - 3\cdot Rb)}{2}\right)$

> $simplify\left(c1(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) - \left(Nb^2 + Nb - 3\cdot Rb\right)\cdot\left(Na - \frac{1}{2}\right) - \left(Na^2 + Na - 3\cdot Ra\right)\cdot\left(Nb - \frac{1}{2}\right) - ((Na+1)\cdot Na2 - 3\,Ra2) - ((Nb+1)\cdot Nb2 - 3\,Rb2)\right)$

> $simplify\left(c0(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) - \left(Nb^2 + Nb - 3\cdot Rb\right)\cdot Na\cdot\left(Na - \frac{1}{2}\right) - \left(Na^2 + Na - 3\cdot Ra\right)\cdot Nb\cdot\left(Nb - \frac{1}{2}\right) - (3\cdot(Nb2\cdot Rb - Nb\cdot Rb2) - Nb2\cdot(Nb + Nb2)) - (3\cdot(Na2\cdot Ra - Na\cdot Ra2) - Na2\cdot(Na + Na2)) - (Na - 1)\cdot((Nb+1)\cdot Nb2 - 3\cdot Rb2) - (Nb - 1)\cdot((Na+1)\cdot Na2 - 3\cdot Ra2) - \frac{3}{2}\cdot(Na + Na2)\cdot(Rb - (Nb + Nb2)) - \frac{3}{2}\cdot(Nb + Nb2)\cdot(Ra - (Na + Na2))\right)$

>

The case where T_A is P_3 and N_B>6

> $simplify\left(c2(8, 4, 2, Rb, Nb, Nb2) - \frac{(Nb\cdot(Nb+1) - 3\cdot Rb)}{2}\right)$

> $simplify\Big(c1(8, 4, 2, 2, Rb, Nb, Rb2, Nb2) - \frac{7}{2}\cdot(Nb\cdot(Nb+1) - 3\cdot Rb) - ((Nb+1)\cdot Nb2$

$- 3 \cdot Rb2) \Big)$

> $simplify\big(\ c0(8, 4, 2, 2, Rb, Nb, Rb2, Nb2) - (3 \cdot (Nb2 \cdot Rb - Nb \cdot Rb2) - Nb2 \cdot (Nb + Nb2))$
> $- 11 \cdot \big(Nb^2 + 2 \cdot Nb + Nb2 - Rb \cdot 3\big) - 3 \cdot (Nb \cdot Nb2 + Nb2 - 3\ \cdot Rb2) - (Nb \cdot (Nb + 1)$
> $- 3 \cdot Nb2)\big)$

The case with T_A being P_4

> $simplify\left(c2(15, 6, 4, Rb, Nb, Nb2) - \frac{(Nb \cdot (Nb + 1) - 3 \cdot Rb)}{2}\right)$

> $simplify\Big(c1(15, 6, 5, 4, Rb, Nb, Rb2, Nb2) - \frac{11}{2} \cdot (Nb \cdot (Nb + 1) - 3 \cdot Rb) - ((Nb + 1) \cdot Nb2$
> $- 3 \cdot Rb2) \Big)$

> $simplify\big(c0(15, 6, 5, 4, Rb, Nb, Rb2, Nb2) - (3 \cdot (Nb2 \cdot Rb - Nb \cdot Rb2) - Nb2 \cdot (Nb + Nb2))$
> $- 29 \cdot \big(Nb^2 + 2 \cdot Nb + Nb2 - Rb \cdot 3\big) - 5 \cdot (Nb \cdot Nb2 + Nb2 - 3\ \cdot Rb2) - 4 \cdot (Nb \cdot (Nb + 1)$
> $- 3 \cdot Nb2)\big)$

>