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In this file, we verify that mu(T)-mu(T-e)>0.5 if e is a pendent edge of a series-reduced tree T of order at least 19.

Here we denote \overline N_A as Na2 etc

> mu := $(Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) \rightarrow \frac{(Ra \cdot Nb + Rb \cdot Na + Ra + Ra2 + Rb + Rb2)}{Na \cdot Nb + Na + Nb + Na2 + Nb2}$

$$\mu := (Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) \mapsto \frac{Ra \cdot Nb + Rb \cdot Na + Ra + Ra2 + Rb + Rb2}{Na \cdot Nb + Na + Nb + Na2 + Nb2} \quad (1)$$

> $mue := (Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) \rightarrow \frac{(Ra \cdot Nb + Rb \cdot Na - Na \cdot Nb + Ra2 + Rb2)}{Na \cdot Nb + Na2 + Nb2}$

$$mue := (Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) \mapsto \frac{Ra \cdot Nb + Rb \cdot Na - Na \cdot Nb + Ra2 + Rb2}{Na \cdot Nb + Na2 + Nb2} \quad (2)$$

> $Pos := (Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) \rightarrow simplify\left(2 \cdot (Na \cdot Nb + Na2 + Nb2) \cdot (Na \cdot Nb + Na + Nb + Na2 + Nb2) \cdot \left(\mu(Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) - mue(Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) - \frac{1}{2}\right)\right)$

$$Pos := (Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) \mapsto simplify\left((2 \cdot Na \cdot Nb + 2 \cdot Na2 + 2 \cdot Nb2) \cdot (Na \cdot Nb + Na + Nb + Na2 + Nb2) \cdot \left(\mu(Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) - mue(Ra, Na, Rb, Nb, Rb2, Ra2, Na2, Nb2) - \frac{1}{2}\right)\right) \quad (3)$$

We verify that the function is increasing in Rb.

> $collect(simplify(expand(Pos(1, 1, Rb, Nb, Rb2, 0, 0, Nb2))), Rb)$

$$(2\, Nb2 - 2)\, Rb - Nb2^2 + (-Nb + 1)\, Nb2 + (-2\, Rb2 + 1)\, Nb - 2\, Rb2 \quad (4)$$

It is thus sufficient to consider a lowerbound for Rb.
We consider two cases.
ell <=2

In this case, we have an lower bound equal to $\frac{Rb2 \cdot Nb}{Nb2} + \frac{4}{3} \cdot Nb$

> $G(Nb, Rb2, Nb2) := simplify\left(expand\left(Pos\left(1, 1, \frac{Rb2 \cdot Nb}{Nb2} + \frac{4}{3} \cdot Nb, Nb, Rb2, 0, 0, Nb2\right)\right)\right)$

$$G := (Nb, Rb2, Nb2) \mapsto simplify\left(expand\left(Pos\left(1, 1, \frac{Rb2 \cdot Nb}{Nb2} + \frac{4 \cdot Nb}{3}, Nb, Rb2, 0, 0, Nb2\right)\right)\right) \quad (5)$$

> $diff(\ G(Nb, Rb2, Nb2), Rb2)$

$$-\frac{2\, Nb}{Nb2} - 2 \quad (6)$$

This function G is decreasing in Rb2, so it is sufficient to consider an upper bound.
One upper bound is Rb2 <= Nb2*mu_b, where mu_B <= 3/2*log_2(N_b)-1

> $F(Nb, Nb2) := G\left(Nb, \left(\frac{3}{2}\cdot log\_2(Nb) - 1\right)\cdot Nb2, Nb2\right)$

$$F := (Nb, Nb2) \mapsto G\left(Nb, \left(\frac{3\cdot log\_2(Nb)}{2} - 1\right)\cdot Nb2, Nb2\right) \tag{7}$$

> $collect(F(Nb, Nb2), Nb2)$

$$-Nb2^2 + \left(-3\ log\_2(Nb) + \frac{5\ Nb}{3} + 3\right) Nb2 - 3\ Nb\ log\_2(Nb) + \frac{Nb}{3} \tag{8}$$

The minima are attained when Nb2 is at one of its boundaries.
Now the conclusion follows as the following two expressions are always positive as Nb\ge 2^9.

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> $F(Nb, Nb)$

$$\frac{2\ Nb\ (Nb + 5 - 9\ log\_2(Nb))}{3} \tag{9}$$

> $F\left(Nb, \frac{Nb}{4}\right)$

$$\frac{Nb\ (17\ Nb + 52 - 180\ log\_2(Nb))}{48} \tag{10}$$

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In the other case, we have **ell \ge 3.**

In this case, we have an lower bound equal to $\frac{Rb2\cdot Nb}{Nb2} + \frac{7}{3}\cdot Nb$

> $G(Nb, Rb2, Nb2) := simplify\left(expand\left(Pos\left(1, 1, \frac{Rb2\cdot Nb}{Nb2} + \frac{7}{3}\cdot Nb, Nb, Rb2, 0, 0, Nb2\right)\right)\right)$

$$G := (Nb, Rb2, Nb2) \mapsto simplify\left(expand\left(Pos\left(1, 1, \frac{Rb2\cdot Nb}{Nb2} + \frac{7\cdot Nb}{3}, Nb, Rb2, 0, 0, Nb2\right)\right)\right) \tag{11}$$

> $diff(\ G(Nb, Rb2, Nb2), Rb2)$

$$-\frac{2\ Nb}{Nb2} - 2 \tag{12}$$

This function G is decreasing in Rb2, so it is sufficient to consider an upper bound.
One upperbound is Rb2 <= Nb2*mu_b, where mu_B < 3/2*log_2(N_b)-1

> $F(Nb, Nb2) := G\left(Nb, \left(\frac{3}{2}\cdot log\_2(Nb) - 1\right)\cdot Nb2, Nb2\right)$

$$F := (Nb, Nb2) \mapsto G\left(Nb, \left(\frac{3\cdot log\_2(Nb)}{2} - 1\right)\cdot Nb2, Nb2\right) \tag{13}$$

> $collect(F(Nb, Nb2), Nb2)$

$$-Nb2^2 + \left(-3\ log\_2(Nb) + \frac{11\ Nb}{3} + 3\right) Nb2 - 3\ Nb\ log\_2(Nb) - \frac{5\ Nb}{3} \tag{14}$$

> $diff(F(Nb, Nb2), Nb2)$

(15)

$$-3\, log\_2(Nb) - 2\, Nb2 + \frac{11\, Nb}{3} + 3 \tag{15}$$

The extrema are attained when Nb2 is at one of its boundaries.
Now the conclusion follows as the following two expressions are always positive as Nb\ge 2^9.
(note that Nb2 \ge number of vertices different from B, and Nb is bounded by 2^{ number of vertices different from B})

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> $F(Nb, Nb)$

$$\frac{8\, Nb^2}{3} - 6\, Nb\, log\_2(Nb) + \frac{4\, Nb}{3} \tag{16}$$

> $F(Nb, log\_2(Nb))$

$$\frac{2\, Nb\, log\_2(Nb)}{3} - 4\, log\_2(Nb)^2 - \frac{5\, Nb}{3} + 3\, log\_2(Nb) \tag{17}$$

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