We compute mu, for a path of length l (ell) with endvertices A and B, with rooted trees T_A and T_B in these two vertices
Here we denote \overline N_A as Na2 etc

> mu := $(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \to ((Ra \cdot Nb + Na \cdot Nb \cdot (l-1) + Rb \cdot Na + l \cdot Ra + l \cdot Rb + \mathrm{binomial}(l, 2) \cdot (Na + Nb) + Ra2 + Rb2) + \mathrm{binomial}(l+1, 3)) / (Na \cdot Nb + l \cdot Na + l \cdot Nb + Na2 + Nb2 + \mathrm{binomial}(l, 2))$

$$\mu := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto \frac{1}{Na \cdot Nb + l \cdot Na + l \cdot Nb + Na2 + Nb2 + \binom{l}{2}} \left(Nb \cdot Ra + Na \cdot Nb \cdot (l-1) + Rb \cdot Na + l \cdot Ra + l \cdot Rb + \binom{l}{2} \cdot (Na + Nb) + Ra2 + Rb2 + \binom{l+1}{3} \right) \quad (1)$$

We consider the numerator (Nu) and denominator (De) separately and compute it for the difference (d) between mu for the the tree T and T\e (path of length l-1)

> $\mathrm{Nu}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := (Ra \cdot Nb + Na \cdot Nb \cdot (l-1) + Rb \cdot Na + l \cdot Ra + l \cdot Rb + \mathrm{binomial}(l, 2) \cdot (Na + Nb) + Ra2 + Rb2) + \mathrm{binomial}(l+1, 3)$

$$\mathrm{N} := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto Nb \cdot Ra + Na \cdot Nb \cdot (l-1) + Rb \cdot Na + l \cdot Ra + l \cdot Rb + \binom{l}{2} \cdot (Na + Nb) + Ra2 + Rb2 + \binom{l+1}{3} \quad (2)$$

> $De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := Na \cdot Nb + l \cdot Na + l \cdot Nb + Na2 + Nb2 + \mathrm{binomial}(l, 2)$

$$De := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto Na \cdot Nb + l \cdot Na + l \cdot Nb + Na2 + Nb2 + \binom{l}{2} \quad (3)$$

> $d := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \to \mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1)$

$$d := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto \mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \mu(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) \quad (4)$$

We compute the numerator (Nd) and denominator (Dd) of d

> $Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := \mathrm{Nu}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) - \mathrm{Nu}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)$

$$Nd := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto \mathrm{N}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) - \mathrm{N}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \quad (5)$$

> $Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1)$

$$Dd := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \cdot De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l-1) \quad (6)$$

We add a check

> $simplify\Big(\mathrm{mu}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)$

$$- \frac{\mathrm{Nu}(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}{De(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)} \Big)$$

$$0 \tag{7}$$

> $simplify\Big(d(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - \frac{Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)}{Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)} \Big)$

$$0 \tag{8}$$

We first consider the 3 cases where T_A and T_B are equal to rooted P_3 or P_4 and note that the expression 3*Nd-Dd is strictly positive,
i.e. Nd/Dd>1/3.

> $Pos(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) := 3 \cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)$

$$Pos := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \mapsto 3 \cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) \tag{9}$$

> $simplify(Pos(8, 4, 2, 2, 8, 4, 2, 2, l))$

$$2 l^2 + 22 l + 88 \tag{10}$$

> $simplify(Pos(15, 6, 5, 4, 8, 4, 2, 2, l))$

$$243 + \frac{105}{2} l + \frac{9}{2} l^2 \tag{11}$$

> $simplify(Pos(15, 6, 5, 4, 15, 6, 5, 4, l))$

$$7 l^2 + 103 l + 594 \tag{12}$$

>

Next we consider the expansion of 3*Nd-Dd for the general case.

> $collect(simplify(expand(simplify(3 \cdot Nd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l) - Dd(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2, l)))), l)$

$$\begin{aligned}
&\left(\frac{1}{2} Na^2 + \frac{1}{2} Nb^2 + Na + \frac{1}{2} Na2 + Nb + \frac{1}{2} Nb2 - \frac{3}{2} Ra - \frac{3}{2} Rb \right) l^2 \\
&+ \left(\frac{(2\, Nb - 1)\, Na^2}{2} + (Nb^2 + Na2 + 4\, Nb + Nb2 - 3\, Rb - 1)\, Na - \frac{Nb^2}{2} + (-1 \right. \\
&\left. - 3\, Ra + Na2 + Nb2)\, Nb + \frac{3\, Ra}{2} + \frac{3\, Rb}{2} + \frac{Na2}{2} + \frac{Nb2}{2} - 3\, Ra2 - 3\, Rb2 \right) l \\
&+ (2\, Nb^2 + Nb - 3\, Rb)\, Na^2 + (Nb^2 + (-4 + Na2 + Nb2)\, Nb + 3\, Rb - 2\, Na2 - 2\, Nb2 \\
&- 3\, Ra2 - 3\, Rb2)\, Na - 3\, Nb^2\, Ra + (3\, Ra - 2\, Na2 - 2\, Nb2 - 3\, Ra2 - 3\, Rb2)\, Nb - Na2^2 \\
&+ (-1 + 3\, Ra + 3\, Rb - 2\, Nb2)\, Na2 - Nb2^2 + (-1 + 3\, Ra + 3\, Rb)\, Nb2 + 3\, Ra2 \\
&+ 3\, Rb2
\end{aligned} \tag{13}$$

This is a polynomial of the form c2*l^2+c1*l+c0.
To prove that this is always positive, we prove that each of the coefficients is positive

> $c2(Ra, Na, Na2, Rb, Nb, Nb2) := \Big(\frac{1}{2} Na^2 + \frac{1}{2} Nb^2 + Na + \frac{1}{2} Na2 + Nb + \frac{1}{2} Nb2$

$$-\frac{3}{2}\,Ra-\frac{3}{2}\,Rb\Big)$$

$$c2 := (Ra, Na, Na2, Rb, Nb, Nb2) \mapsto \frac{1}{2}\cdot Na^2+\frac{1}{2}\cdot Nb^2+Na+\frac{1}{2}\cdot Na2+Nb+\frac{1}{2}\cdot Nb2-\frac{3}{2}\cdot Ra-\frac{3}{2}\cdot Rb \quad (14)$$

> $$c1(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) := \frac{(2\,Nb-1)\,Na^2}{2}+\left(Nb^2+Na2+4\,Nb+Nb2-3\,Rb-1\right)Na-\frac{Nb^2}{2}+(-1-3\,Ra+Na2+Nb2)\,Nb+\frac{3\,Ra}{2}+\frac{3\,Rb}{2}+\frac{Na2}{2}+\frac{Nb2}{2}-3\,Ra2-3\,Rb2$$

$$c1 := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) \mapsto \frac{(2\cdot Nb-1)\cdot Na^2}{2}+\left(Nb^2+Na2+4\cdot Nb+Nb2-3\cdot Rb-1\right)\cdot Na-\frac{Nb^2}{2}+(-1-3\cdot Ra+Na2+Nb2)\cdot Nb+\frac{3\cdot Ra}{2}+\frac{3\cdot Rb}{2}+\frac{Na2}{2}+\frac{Nb2}{2}-3\cdot Ra2-3\cdot Rb2 \quad (15)$$

> $$c0(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) := \left(2\,Nb^2+Nb-3\,Rb\right)Na^2+\left(Nb^2+(-4+Na2+Nb2)\,Nb+3\,Rb-2\,Na2-2\,Nb2-3\,Ra2-3\,Rb2\right)Na-3\,Nb^2\,Ra+(3\,Ra-2\,Na2-2\,Nb2-3\,Ra2-3\,Rb2)\,Nb-Na2^2+(-1+3\,Ra+3\,Rb-2\,Nb2)\,Na2-Nb2^2+(-1+3\,Ra+3\,Rb)\,Nb2+3\,Ra2+3\,Rb2$$

$$c0 := (Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) \mapsto \left(2\cdot Nb^2+Nb-3\cdot Rb\right)\cdot Na^2+\left(Nb^2+(-4+Na2+Nb2)\cdot Nb+3\cdot Rb-2\cdot Na2-2\cdot Nb2-3\cdot Ra2-3\cdot Rb2\right)\cdot Na-3\cdot Nb^2\cdot Ra+(3\cdot Ra-2\cdot Na2-2\cdot Nb2-3\cdot Ra2-3\cdot Rb2)\cdot Nb-Na2^2+(-1+3\cdot Ra+3\cdot Rb-2\cdot Nb2)\cdot Na2-Nb2^2+(-1+3\cdot Ra+3\cdot Rb)\cdot Nb2+3\cdot Ra2+3\cdot Rb2 \quad (16)$$

>

We have to verify that all 3 are positive.
We first do so for the case where Na and Nb are both at least equal to 6, and so all preliminary inequalities can be used.

> $$simplify\left(c2(Ra, Na, Na2, Rb, Nb, Nb2)-\frac{(Na\cdot(Na+1)-3\cdot Ra)}{2}-\frac{(Nb\cdot(Nb+1)-3\cdot Rb)}{2}\right)$$

$$\frac{Na}{2}+\frac{Na2}{2}+\frac{Nb}{2}+\frac{Nb2}{2} \quad (17)$$

> $$simplify\left(c1(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2)-\left(Nb^2+Nb-3\cdot Rb\right)\cdot\left(Na-\frac{1}{2}\right)-\left(Na^2+Na-3\cdot Ra\right)\cdot\left(Nb-\frac{1}{2}\right)-((Na+1)\cdot Na2-3\,Ra2)-((Nb+1)\cdot Nb2-3\,Rb2)\right)$$

$$(18)$$

$$\frac{(4\,Nb+2\,Nb2-1)\,Na}{2}+\frac{(2\,Na2-1)\,Nb}{2}-\frac{Na2}{2}-\frac{Nb2}{2} \quad \textbf{(18)}$$

(Note that this is positive, since all terms are at least one)

$$> simplify\Big(c0(Ra, Na, Ra2, Na2, Rb, Nb, Rb2, Nb2) - (Nb^2+Nb-3\cdot Rb)\cdot Na\cdot\Big(Na-\frac{1}{2}\Big) - (Na^2+Na-3\cdot Ra)\cdot Nb\cdot\Big(Nb-\frac{1}{2}\Big) - (3\cdot(Nb2\cdot Rb-Nb\cdot Rb2)-Nb2\cdot(Nb+Nb2)) - (3\cdot(Na2\cdot Ra-Na\cdot Ra2)-Na2\cdot(Na+Na2)) - (Na-1)\cdot((Nb+1)\cdot Nb2-3\cdot Rb2) - (Nb-1)\cdot((Na+1)\cdot Na2-3\cdot Ra2) - \frac{3}{2}\cdot(Na+Na2)\cdot(Rb-(Nb+Nb2)) - \frac{3}{2}\cdot(Nb+Nb2)\cdot(Ra-(Na+Na2))\Big)$$

$$\frac{(3\,Rb+2\,Nb2)\,Na2}{2}+\frac{3\,Ra\,Nb2}{2}+\frac{Na\,Nb\,(Na+Nb)}{2} \quad \textbf{(19)}$$

>

The case where T_A is P_3 and N_B>6

$$> simplify\Big(c2(8, 4, 2, Rb, Nb, Nb2) - \frac{(Nb\cdot(Nb+1)-3\cdot Rb)}{2}\Big)$$

$$1+\frac{Nb}{2}+\frac{Nb2}{2} \quad \textbf{(20)}$$

$$> simplify\Big(c1(8, 4, 2, 2, Rb, Nb, Rb2, Nb2) - \frac{7}{2}\cdot(Nb\cdot(Nb+1)-3\cdot Rb) - ((Nb+1)\cdot Nb2 - 3\cdot Rb2)\Big)$$

$$\frac{11\,Nb}{2}+3+\frac{7\,Nb2}{2} \quad \textbf{(21)}$$

$$> simplify(c0(8, 4, 2, 2, Rb, Nb, Rb2, Nb2) - (3\cdot(Nb2\cdot Rb-Nb\cdot Rb2)-Nb2\cdot(Nb+Nb2)) - 11\cdot(Nb^2+2\cdot Nb+Nb2-Rb\cdot 3) - 3\cdot(Nb\cdot Nb2+Nb2-3\cdot Rb2) - (Nb\cdot(Nb+1)-3\cdot Nb2))$$

$$-Nb+3\,Rb+8 \quad \textbf{(22)}$$

The case with T_A being P_4

$$> simplify\Big(c2(15, 6, 4, Rb, Nb, Nb2) - \frac{(Nb\cdot(Nb+1)-3\cdot Rb)}{2}\Big)$$

$$\frac{7}{2}+\frac{Nb}{2}+\frac{Nb2}{2} \quad \textbf{(23)}$$

$$> simplify\Big(c1(15, 6, 5, 4, Rb, Nb, Rb2, Nb2) - \frac{11}{2}\cdot(Nb\cdot(Nb+1)-3\cdot Rb) - ((Nb+1)\cdot Nb2 - 3\cdot Rb2)\Big)$$

(24)

$$\frac{25\,Nb}{2} + \frac{19}{2} + \frac{11\,Nb2}{2} \tag{24}$$

> $simplify\big(c0(15, 6, 5, 4, Rb, Nb, Rb2, Nb2) - (3\cdot(Nb2\cdot Rb - Nb\cdot Rb2) - Nb2\cdot(Nb + Nb2)) - 29\cdot\big(Nb^2 + 2\cdot Nb + Nb2 - Rb\cdot 3\big) - 5\cdot(Nb\cdot Nb2 + Nb2 - 3\cdot Rb2) - 4\cdot(Nb\cdot(Nb + 1) - 3\cdot Nb2)\big)$

$$-4\,Nb + 2\,Nb2 + 9\,Rb + 37 \tag{25}$$

>