Linguistic Fuzzy-Logic Game Theory

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The author develops a new game-theoretic approach, anchored not in Boolean two-valued logic but instead in linguistic fuzzy logic. The latter is characterized by two key features. First, the truth values of logical propositions span a set of linguistic terms such as *true*, *very true*, *almost false*, *very false*, and *false*. Second, the logic allows logical categories to overlap in contrast to Boolean logic, where the two possible logical categories, "true" and "false," are sharply distinct. A game becomes a linguistic fuzzy logic game by turning strategies into linguistic fuzzy strategies, players' preferences into linguistic fuzzy preferences, and the rules of reasoning and inferences of the game into linguistic fuzzy reasoning operating according to linguistic fuzzy logic. This leads to the introduction of a new notion of linguistic fuzzy domination and linguistic Nash equilibrium.

Keywords: *game theory; fuzzy logic; Boolean logic; Nash equilibrium; prisoner's dilemma*

> Vagueness is no more to be done away with in the world of logic than friction in mechanics.
>
> —Charles Sanders Peirce

> Everything is vague to a degree you do not realize till you have tried to make it precise.
>
> —Bertrand Russell

> As complexity rises, precise statements lose meaning and meaningful statements lose precision.
>
> —Lotfi Zadeh

Conventional game theory faces a dilemma rooted in an incompatibility between the Boolean logical foundation of the game and the linguistic nature of strategic communication—linguistic fuzzy logic that "computes with words" all the way down offers a way out of this dilemma. On one hand, game-theoretic methodologies are inescapably based on a posited underlying logic. The latter is what guides us in judging whether a game-theoretic methodology (and theory) possesses logical coherence and consistency. Boolean two-valued logic underpins conventional game theory. All game-theoretic arguments and the very notion of consistency/inconsistency are based on a sharp Boolean logic true-false dichotomy. On the other hand, vagueness and

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equivocation are constitutive features of human communication. Human beings cannot live and communicate without a "language," with the latter being inherently vague. Game theory is essentially the study of strategic communication of information through language in a rigorous and stylized way. However, stylization and rigor cannot totally eradicate vagueness: there is always a remainder of linguistic vagueness at the very heart of the conceptual tools—communicative devices—used in game theory. This creates a foundational dilemma for conventional game theory. Is it possible to resolve this incompatibility between, on one hand, taking *the dichotomous Boolean two-valued logic* as the logical foundation of game theory and, on the other hand, *the vagueness that inheres in strategic communication* due to the very nature of language? This difficulty is often glossed over or not even sensed at all in conventional game-theoretic works. I propose a remedy to this dilemma by anchoring game theory in linguistic fuzzy logic.

I lay the grounds for and formulate a game-theoretic approach founded not on a Boolean logic but rather on linguistic fuzzy logic. I thence inscribe vagueness into the logical foundations of game theory. This is a fundamental departure from conventional game theory on two fronts at least. First, positing a fuzzy logic at the foundations of strategic communication and action means that truth values are not only TRUE and FALSE (as in Boolean logic) but can also assume in-between values, such as *very true*, *less true*, *extremely false*, and *approximately false*. A number of works have recently been published, mostly in the economics literature, presenting a fuzzy version of game theory by adopting a fuzzy-set framework, which is conventionally developed using the notion of gradual membership to sets (Butnariu 1978; Borges et al. 1997; Billot 1992; Maeda 2000, 2003; Kim and Lee 2001; Li et al. 2001; De Wilde 2004).

This work differs from these various efforts on an important aspect, which is the second fundamental departure from conventional game theoretic. I use a linguistic formalism that is anchored in so-called linguistic fuzzy logic, not fuzzy-set theory. Hence, I do not use the notion of a gradual membership to express the fuzziness of concepts and objects of study. *I "compute with words" all the way down into the inner logic of game theory.*

To do this, I go back to the logical foundations of game theory and work my way up to strategic interaction. This paves the way for introducing new notions of fuzzy strategy, fuzzy dominant strategy, and fuzzy Nash equilibrium, which are in some sense (i.e., fuzzy-logic sense) extensions of those used in conventional game theory with ordinal preferences. A key concept of this reconceptualization is the notion of truth value of a choice, termed in this article as the *feasibility value*. Everything becomes fuzzy, everything becomes more or less this or that—everything acquires a linguistic value that is essentially *more or less nuanced and more or less feasible*! We can, for example, talk of nuanced cooperation such as *high cooperation* with a *very high feasibility* or nuanced cooperation such as *very high cooperation* with a *very low feasibility*, not just of cooperation or no cooperation. If the linguistic fuzzy relations and all other components of the game structure are de-fuzzified by anchoring them in a crisp two-valued logic, the linguistic fuzzy game simply reduces to the conventional game. I apply the new approach to 2×2 prisoner's dilemma (PD) game. I find that there is always a strong Nash equilibrium that is Pareto optimal, thereby lifting the dilemma that emerges in the crisp PD game. I begin by introducing key notions and elements of linguistic fuzzy logic.[1] Second, I analyze a conventional 2×2 PD game, highlighting the elements of Boolean logic that anchor the logical structure of the game. I then develop the different elements of a linguistic fuzzy-logic game theory—termed *LFL game theory*. This article is obviously intended to be an introduction to a new game-theoretic method and, as such, does not seek to apply the approach to concrete situations of empirical analyses.

THE VALUE OF LINGUISTIC FUZZY LOGIC

Most of today's social sciences are built on a Boolean logical framework. This assumption, however, runs counter to many of our observations and conceptualizations about social/political phenomena. For example, we know that it is both theoretically and empirically impossible to unequivocally and precisely define what a developing state is and where the boundary between the sets of developing and developed states lies. However, we often ignore this problem simply by assuming Boolean logic as the underlying logic of our conceptual apparatus and analysis—that is, by saying that a state is either developed or not developed. One can argue that we may instead characterize the state as developing with a higher or lesser level of development. However, this does not alleviate all forms of equivocation since to carry out the empirical analysis, we definitely need to classify the set of so-characterized developing states into categories that do not admit overlapping cases; that is, boundaries between categories would need to be sharply delineated. The process of classifying involves verifying that the cases can be unambiguously assigned to the appropriate categories. Thus, should we be unable to categorize a case, we would not include it in the analysis. Put differently, a case can belong or not belong to any single category—in-between positions described as belonging more or less to two or more different categories are not allowed to exist in Boolean logic thinking. One way to address this problem of overlapping categories and fuzzy boundaries is to reconsider the assumption of Boolean logic underlying much of what counts today as social sciences. Doing so has a number of benefits.

First, a non-Boolean approach would radically redefine our conception of what is a consistent and coherent theory. There is widespread belief that we can ascertain the consistency of our analyses and formal theories only through an exploration of their underlying logic. This belief is rooted in one unspoken assumption—that is, to take the principle of Boolean contradiction as a fundamental pillar in defining logical consistency. The interesting exchange between Stephen Walt and his critics on the criteria with which we should judge formal theories in international security studies is illustrative (Brown et al. 2000). All participants agree that a good theory should be logically consistent. Formal and mathematical modeling is praised by its advocates precisely because it is believed to help guard against inconsistencies due to logical failure of the theory or resulting from ambiguities inherent in natural language. As put by Zagare,

1. See Arfi (2005) on the linguistic fuzzy-set approach in the study of decision making in politics.

"There can be no compromise here. Without a logically consistent theoretical structure to explain them, empirical observations are impossible to evaluate; without a *logically consistent* theoretical structure to constrain them, original and creative theories are of limited utility; and without a *logically consistent* argument to support them, even entirely laudable conclusions . . . lose much of their intellectual force" (Brown et al. 2000, 103, emphasis added). A resolution of logical inconsistencies through the exploration of the logic of a theory is thus a central part of the scientific endeavor. These notions of logical coherence and consistency are assumed to be Boolean (true/false) in nature. This is well reflected in the rhetoric of "paradox." To highlight the value added of their work, many political/social scientists phrase their research questions as a paradox. However, what goes unnoticed in this deployment of rhetorical power is that *this paradox is only a paradox given the posited underlying logic*. For a different logic—and there are so many other logics besides Boolean on-off logic—this paradox might not be a paradox at all but rather a consequence of the axioms of the logic. For example, paraconsistent logic is built around precisely negating this principle (Heyting 1971; Priest, Routley, and Norman 1989).[2] A theory built on the premise of a paraconsistent logic, which inherently takes contradiction as an axiom, will undoubtedly have no room for such an argumentation procedure! Likewise, starting from so-called intuitionistic logic, which is a classical logic without the Aristotelian law of excluded middle: ($A \vee \neg A$), would logically validate Boolean logic–type inconsistencies in a theory. This hence raises a serious question on the conventional wisdom behind the requirement of logical consistency—it becomes contextualized (i.e., dependent on the underlying logic).

Conventional wisdom has it that "an inconsistent theory creates a false picture of the world. Inconsistent theories are also more difficult to test because it is harder to know if the available evidence supports the theory" (Brown et al. 2000, 8). I suggest that, instead of taking it for granted that "logical inconsistencies" are the first culprits to be eliminated from a theory as a way of improving its empirical validation, we need first to realize that this "logical inconsistency" is contextual (i.e., there is an inconsistency only within the purview of the Boolean logic that is being assumed). To avoid any misunderstanding here, let me be clear that I am not calling for sloppiness and anything-goes attitude. To the contrary, I am suggesting that we need to put our research endeavor on firmer logical grounds by being precisely clear on the logic that underpins our theorizing and empirical testing works. Being clear at the forefront about the limitations of one's theory and method is highly praised in social science. However, we often stop short of fulfilling this commitment to the full extent, right down to the logic underlying our theories and methods. This is problematic because understanding the inner working of the underlying logic and how it is axiomatically constructed opens up the possibility for considering other types of underlying logics of relationships among variables—and there is no shortage of other logics (e.g., fuzzy logic, modal logic, intuitionistic logic, paraconsistent logic, quantum logic, modal logic, etc.). Boolean

2. The liar paradox "This sentence is not true" is illustrative. We have two options—either the sentence is true or it is not. (1) If we suppose it to be true, then what it says (i.e., this sentence is not true) is the case. Therefore, the sentence is not true. (2) If we suppose that it is not true, which is what it says, then the sentence is true. In either case (1) or (2), *it is both true and not true*.

logic still largely underpins the overwhelming majority of works in social sciences for the most part due to genealogical reasons and resistance due to path dependency. Yet, there are no ontologically, epistemologically, theoretically, empirically, or methodologically insurmountable impediments or raison d'être why we cannot explore the impact of the vast realm of other logics on the theorization, design, and execution of social science inquiry. As the burgeoning literature on the fuzzy-set theoretic approach to social, economic, and political inquiry testifies, we have much to gain from doing so.[3] This article seeks to contribute to this burgeoning literature.

As an illustration of the great potential of considering different underlying logics, let us briefly see how fuzzy logic might reshape the seemingly never-ending debates in the literature on democratic peace. In a very innovative work, Zinnes (2004) uses propositional calculus (based on Boolean logic) to explore the logic underpinning the various arguments of the democratic peace. As she puts it, "Just as we moved from initial democratic-peace statistical results to explanations for them and then to critiques of those explanations, we now need to take the next step and expand and revise the logics to better incorporate such observations as the war initiation behavior and war victory rate by democracies" (p. 453). Yet, the "logics" that Zinnes is referring to are all underpinned by Boolean logic. The fact that Zinnes was able to logically validate the normative and institutional arguments of the democratic peace using Boolean propositional calculus raises the following question: what if we were to use a different foundational logic, such as fuzzy logic? Would this help us make more sense of the democratic arguments and the diversity of its empirical validation? A fuzzy-logic approach to the problem would, for instance, rephrase the basic propositions that Zinnes starts from very differently with a strong potential of not only reconfirming Zinnes's conclusions but also going beyond them, thereby raising new questions that are outside the purview of the Boolean world. Analyzing the democratic peace through a fuzzy-logic approach will definitely lead to a different propositional setup and broader conclusions.

Mentioning a few examples suffices to make the point (LFL counterparts are in italics). When Zinnes (2004) writes, "If state is a democracy, then all decisions involve the participation of the population and its representative institutions," or "If state is a nondemocracy, then all decisions are made by a small group of elite leaders," the fuzzy-logic counterpart will be "*If state is a more or less democracy, then all decisions more or less involve the participation of the population and its representative institutions*." Likewise, when Zinnes writes, "If a state uses bargaining to settle internal societal conflicts and its security is not threatened, then it will use bargaining to settle internation conflicts," we would have "*If a state more or less uses bargaining to settle internal societal conflicts and its security is more or less threatened, then it will more or less use bargaining to settle internation conflicts*." Similarly, "If states X and Y are in conflict and bargaining is used to settle internation conflicts, then states X and Y do not use force to settle internation conflicts" would turn into "*If states X and Y are more or less in conflict and bargaining is more or less used to settle internation conflicts, then states X and Y more or less use force to settle internation conflicts*." We can see that this produces much vagueness in the various propositions. Thus, in fuzzy logic, we can

3. See Arfi (2005) and Ragin (2000). See also http://www.compasss.org.

conclude that two states can go to war or not go to war—just like in Boolean logic. We also can conclude that two states can more or less go to war—that is, the two states are in a no-peace situation but not in a fully-fledged state of war. Although it is premature to advance any concrete conclusions about a fuzzy-logic approach to the democratic peace, it nonetheless opens up a new way to addressing the still-unresolved issue. In sum, explicating the logic underpinning the analysis opens up the horizon for new questions, new methods of thinking about and investigating social and political phenomena, and new results. This would help us relinquish many of the homogenizing assumptions that underpin much of quantitative analysis such as populations, cases, and causality (as pointed out by Ragin 2000, 5).

Second, we can do quantitative analysis without assuming a Boolean logic, but of different kinds based on different logics and approaches. In this article, I advocate "computing with words" as another way of doing formal game-theoretic analysis. It is quite amazing that while scholars and researchers in computer science, information sciences, and engineering are seeking to compute with words (Zadeh 1999), most social scientists seek to "crispen" our methods even more in seeking rigorous and scientific ways to understand social and political phenomena. To be clear, conventional numericalization of knowledge in social sciences has achieved much by seeking more sophistication and rigor. However, we can also achieve rigor and mathematical precision using words expressed in a natural language, in such a way as to strengthen much more the connection between empirical data analysis and theory, without losing the complexity that natural languages possess. This approach is termed the *linguistic fuzzy logic* (LFL) approach in this article.

Third, ironically, what makes Boolean logic a simple, rigorous basis of analysis is also its weakest point—dichotomization of the truth values of the antecedent and consequent variables. Boolean logic forces us to think only in terms of true and false statements. Yet, as the wealth of statistical and qualitative studies show, vagueness is inescapable in all our theoretical and empirical analyses of social and political phenomena. Statements that such and such variable causes more or less such and such outcome are pervasive in all social sciences. What is important to note here is that this vagueness is not just a type of epistemic uncertainty that might be resolved with better tools of analysis. Rather, linguistic vagueness goes deep down into the logic underpinning our analyses. This vagueness eludes Boolean dichotomization and hence drops out in the stylization process of formal theories. A key idea in fuzzy logics is to inscribe this vagueness in the logic of our theories, positing that the truth value of a proposition can assume other values than just "true" or "false." Thus, it is incorrect to think that to know the world, it is sufficient to know the "truths"; we also need to know the "falsities," as well as the degrees of both. This is very different from Boolean logic, which upholds that {*if a proposition p is not true, then p is false. Should we thus know that p is not true, there is no need to inquire whether p is false? it is taken for granted to be false*}. There are many situations in daily life when this logic is simply not applicable. A simple counterexample from ordinary language usage is when one answers by stating, *more or less yes and more or less no*. This is beyond Boolean logic!

Fourth, in Boolean logic, it does not matter much whether one uses either {0, 1} or {True, False} to express the truth values of the logical propositions. In fuzzy logic, the choice matters. Indeed, we can have a multivalued fuzzy logic, the truth values of which are expressed using numerical values belonging to the set [0, 1]. However, this version of fuzzy logic still relies on the concept of membership function (i.e., the degree of belonging to a set) and runs into many problems, especially in using membership functions as a tool to analyze the sociopolitical world (Ragin 2000). The linguistic fuzzy-logic approach uses natural language to express a fuzzy logic in which the truth values of propositions are expressed in natural language terms such as *true*, *very true*, *less true*, and *false*. To grasp the novelty of doing so, we need to introduce some elements of linguistic fuzzy logic such as the concepts of a linguistic variable, a linguistic truth value, and the method of aggregating linguistic variables.

ELEMENTS OF LINGUISTIC FUZZY LOGIC

This section presents a rather brief introduction to the linguistic fuzzy-logic framework needed to formulate LFL-Game (Herrera and Herrera-Viedman 2000). The starting point is to choose a context-dependent linguistic terms set to describe vague variables. Each value of a linguistic variable is characterized by a *syntactic label* and a *semantic value* or *meaning*. The label is a word or a sentence belonging to the chosen linguistic terms set. For example, a linguistic terms set for the linguistic variable *Feasible* is as follows: *F*(*Feasible*) = {*fully feasible*, *very feasible*, *approximately feasible*, *possibly feasible*, *not feasible*}. Formally,

> A linguistic variable is a quintuplet (*L*, *S(L)*, *U*, *G*, *M*), where *L* is the name of the linguistic variable. *S(L)* denotes the term set of *L* (i.e., the set of names that provide the linguistic values of *L*), with each value being a fuzzy variable ranging across a universe of discourse *U* (a finite set of words and phrases). *G* is a syntactic rule (which usually takes the form of a grammar) for generating the names of the values of *L*. *M* is a semantic rule for associating meanings *M(L)* to *L*, with *M(L)* being a subset of *U*.

This definition is implemented by means of an ordered structure of linguistic terms. The cardinality of the terms set must be small enough so as not to introduce useless precision in the analysis since vagueness is of the essence in the approach. Yet, it ought to be rich enough to allow for enough discrimination among the values of the linguistic variables. A set of seven terms *S* used to describe various levels of cooperation would be given as *S-cooperation* = {s_0 = *Null*, s_1 = *Very Low*, s_2 = *Low*, s_3 = *Moderate*, s_4 = *High*, s_5 = *Very High*, s_6 = *Full*}*-cooperation*. This set is intuitively ordered in the sense that $s_a \prec s_b$ if, and only if, $a < b$. Note that the strict ordering $s_a \prec s_b$ would become partial ordering $s_a f s_b$ for any two adjacent ranks *a* and *b* since in fuzzy logic, there always is an unavoidable (in fact, desirable!) residual overlap between two adjacent hedges, such as between *high cooperation* and *very high cooperation*.

In Boolean logic, stating that a variable acquires a certain value implies that the truth value of this assignment is “true.” If the variable does not acquire such a value, then the truth value of this assignment is “false.” The truth assignment is always one of two choices—true or false. The situation is radically different in linguistic fuzzy logic.

In addition to possessing a linguistic value (such as *big*, *small*, *extremely low*, or *approximately high*), each value assignment to a linguistic variable has a degree of truth. For example, a variable with a linguistic value of *low cooperation* could have a *very high*, *moderate*, *low*, or *very low* degree of truth. Manipulating linguistic variables using linguistic fuzzy logic thus introduces two "degrees of freedom": one dealing with the linguistic value of the variable, termed in this article as the *nuance* degree (or level), and one dealing with the truth value of this assigned nuance, termed in this article as the *feasibility* degree (or level). The meanings of these two concepts can be intuitively grasped through a simple example of cooperation and defection as two linguistic variables. As Robert Jervis (1988, 329) retorted, although "the concepts of cooperation and defection are crucial.... These terms work well for a laboratory Prisoner's Dilemma, but most situations are more complex.... Perhaps we should think not of a dichotomy, but of a *continuum*.... Some policies express a high degree of both cooperation and defection simultaneously" (emphasis added).

In other words, instead of speaking of cooperation and defection in a dichotomous way, we speak of *nuanced cooperation* and *nuanced defection*, that is, *more or less cooperation* and *more or less defection*. We then pose the following question: how feasible is each of the value assignments or nuances? Is *low cooperation* more feasible than *high cooperation*? In other words, each of these nuances of cooperation is *more or less feasible*. The manipulation of these linguistic values of nuance and feasibility calls for an algebra of linguistic variables, which is introduced next.

ALGEBRA OF LINGUISTIC VARIABLES

Consider the following set of possible linguistic values of a linguistic variable *C* (*cooperation*) = {*Null*, *Very Low*, *Low*, *Moderate*, *High*, *Very High*, *Full*}-*cooperation*. Suppose that we have two linguistic variables, *A* and *B*, with the value of *A* being *low cooperation* and that of *B* being *very high cooperation*. What would be the linguistic value of the aggregation *A AND B*? The axiomatic system of an algebraic structure would enable us to make such evaluations. Such a system is based on the idea of using linguistic hedges (or modifiers) to change the meaning and hence the linguistic values of basic elements such as *cooperation*. For example, using the hedge *high* on the variable *cooperation*, we would obtain a new value, *high cooperation*. The resulting algebra is called hedge algebra; it defines and determines how the hedges are combined with the basic elements (e.g., cooperation) to form a system of evaluating logical connectives and inferences. This algebra is based on natural semantic properties of the linguistic hedges, the set of which is equipped with an ordering relation. Linguistic hedges are useful because they possess two semantic properties, making their algebraic manipulation intuitively meaningful. First, the hedges are semantic modifiers with different degrees of modification. This makes it possible to compare the effects of two hedges on the same primary term. Hence, we would have *little* < *highly* (i.e., the effect of *little* is smaller than that of *highly*), which makes *little cooperative* < *highly cooperative*, in accordance with our intuitive knowledge that *highly cooperative* is much closer to *cooperative* than *little cooperative*, with the latter being rather closer to *not cooperative*. Second, the hedges change the meaning of the primary term but nevertheless do preserve a trace of the original meaning of the primary term—this is called semantic heredity. For example, from *little cooperative* < *highly cooperative*, we have *possibly little cooperative* < *possibly highly cooperative*.

To make the system of linguistic hedges a closed one for evaluating the combination and manipulation of linguistic variables (i.e., to ensure that our symbolic operations would not produce elements that cannot be represented using the initial set of hedges), we need to equip the latter with a joint ("and") operation, a meet ("or") operation, a negation operation, and an implication operation (Ho and Khang 1999; Ho and Nam 2002). The properties of these operations enable us to make logical evaluations of various combinations of linguistic variables and propositions. Let us denote the set of linguistic hedges by $\Theta_M = \{\tau_\alpha, \alpha \in [1, M]\}$, where τ_α is a linguistic hedge. We would, for example, have the following set of seven hedges Θ_7 = {*Null*, *Very Low*, *Low*, *Moderate*, *High*, *Very High*, *Full*}, which, when applied to the primary term *cooperation*, produce the following nuances: {*Null Cooperation*, *Very Low Cooperation*, *Low Cooperation*, *Moderate Cooperation*, *High Cooperation*, *Very High Cooperation*, *Full Cooperation*}. The algebraic structure of the set of hedges is denoted by $H = \{\Theta_M, \vee, \wedge, <, \rightarrow, \neg\}$, where M is the number of hedges, $\vee$ and $\wedge$ are respectively the disjunction and conjunction, $\rightarrow$ is the implication, $\neg$ is the negation, and $<$ is an ordering of the hedges (defined earlier).[4] The conjunction, disjunction, and implication operations, $\wedge$, $\vee$, and $\rightarrow$, of linguistic fuzzy logic are defined using three operators. These are LWC (linguistic weighted conjunction), LWD (linguistic disjunction), and LI (linguistic implication) (Herrera and Herrera-Viedman 1997). To define the operations LWC, LWD, and LI, we need the following intermediary operations. Let $S = \{s_0, s_1, \ldots, s_T\}$ denote a set of linguistic, ordered terms.

Negation operator: $Neg(s_i) = s_{T-i}$.
Maximization operator: $MAX(s_i, s_j) = \{s_i \text{ if } i \geq j;\ s_j \text{ otherwise}\}$.
Minimization operator: $MIN(s_i, s_j) = \{s_i \text{ if } i \leq j;\ s_j \text{ otherwise}\}$.

As an illustration, consider the set Θ_7 = {*Null*, *Very Low*, *Low*, *Moderate*, *High*, *Very High*, *Full*}. This set is ordered since we can intuitively see that *Null* ≤ *Very Low* ≤ *Low* ≤ *Moderate* ≤ *High* ≤ *Very High* ≤ *Full*. The negation operator gives *Neg*(*Null*) = *Full* and *Neg*(*Low*) = *High*, etc. The maximization operator gives *MAX*(*Null*, *Moderate*) = *Moderate* and *MAX*(*Very High*, *Moderate*) = *Very High*, etc. The minimization operator gives *MIN*(*Low*, *High*) = *Low* and *MIN*(*Full*, *High*) = *High*, etc. Because in linguistic fuzzy logic, each variable is defined by two degrees of freedom—nuance and feasibility—the symbolic manipulation of linguistic variables calls for two sorts of aggregating operations to evaluate the operations of conjunction, disjunction, and implication, as well as various combinations thereof: (1) *an aggregation of the nuance values of the variables* (each variable has its own linguistic nuance value, and hence

4. I take H to be a De Morgan lattice, satisfying the following definition. A lattice $L = \{M, \vee, \wedge, <, \neg\}$ is called a De Morgan lattice if (1) $\{M, \vee, \wedge, <\}$ is a distributive lattice, and (2) the operation $\neg: M \rightarrow M$ is such that (a) for all $u \in M$, we have $\neg u = u$, and (b) for all $u, v \in M$, we have $\neg(u \wedge v) = \neg u \vee \neg v$ and $\neg(u \vee v) = \neg u \vee \neg v$. De Morgan lattices are distributive lattices with a complement operation that satisfies the law of double complement, the De Morgan rules, and the law of contraposition for partial ordering <.

we need to aggregate these linguistic nuance values to obtain the collective nuance value for the aggregate) and (2) *an aggregation of linguistic degrees of feasibility* (we need to combine the feasibility degrees of the variables). Let us denote by ϕ_m the feasibility value of a linguistic variable L_m and ν_m its nuance degree: $L_m = (\nu_m, \phi_m)$. The set of linguistic feasibility degrees and the set of nuance degrees corresponding to the variations of the same variable have the same cardinality—each value of the linguistic variable has one degree of nuance and one corresponding feasibility value. Without any loss of generality, we assume that both ν and ϕ are expressed using the same terms set S.

LWC AGGREGATION

The LWC aggregation of m linguistic variables $\{(\nu_1, \phi_1, \ldots, (\nu_m, \phi_m)\}$ is defined by $(\nu_{LWC}, \phi_{LWC}) = LWC\{(\nu_1, \phi_1), \ldots, (\nu_m, \phi_m)\}$, where the aggregated nuance value is $\nu_{LWC} = MIN_{i=1,\ldots,m}\{MAX(Neg(\nu_i), \phi_i)\}$, and the aggregated feasibility degree is $\phi_{LWC} = LOWA(\phi_1, \ldots, \phi_m)$. The linguistic ordered weighted aggregation (LOWA) is defined and illustrated through an example in Appendix A.[5]

As an example of using LWC, let us consider $(\nu_{LWC}, \phi_{LWC}) = LWC[(LLL, VHH), (HHH, LLL)]$; that is, we want to aggregate two linguistic variables $A = (LLL, VHH)$ and $B = (HHH, LLL)$ with, respectively, a degree of nuance LLL for A and HHH for B, as well as a feasibility degree VHH for A and LLL for B. The degree of nuance of the aggregate is obtained as $\nu_{LWC} = MIN\{MAX(Neg(LLL), VHH); MAX\ (Neg\ (HHH)\ LLL)\}$. Using the hedge algebra $\{NNN, VLL, LLL, MMM, HHH, VHH, FFF\}$, we have $Neg(LLL) = HHH$ and $Neg(HHH) = LLL$. Thus, $\nu_{LWC} = MIN\{MAX(HHH, VHH); MAX(LLL, LLL)\} = MIN\{VHH; LLL\} = LLL$.

The degree of feasibility is obtained as an aggregation of the respective degrees of feasibility for A and B; that is, $\phi_{LWC} = LOWA(VHH, LLL)$. Using LOWA as given in Appendix A, we obtain $\phi_{LWC} = LOWA(VHH, LLL) = MMM$. Hence, $(\nu_{LWC}, \phi_{LWC}) = (LLL, MMM)$; the aggregated variable has a *low degree of nuance* with a *moderate degree of feasibility.*

LWD AGGREGATION

The LWD aggregation of m linguistic variables $\{(\nu_1, \phi_1, \ldots, (\nu_m, \phi_m)\}$ is defined by $(\nu_{LWC}, \phi_{LWC}) = LWD[(\nu_1, \phi_1), \ldots, (\nu_m, \phi_m)]$, where the aggregated nuance value is $\nu_{LWD} = MAX_{i=1,\ldots,m}\{MIN(\nu_i, \phi_i)\}$, and the feasibility degree of the aggregated variable is $\phi_{LWD} = LOWA(\phi_1, \ldots, \phi_m)$. Note that *LWC* and *LWD* are formally obtained from one another by exchanging the positions of the *MAX* and *MIN* operators.

5. The appendix is posted at http://jcr.sagepub.com/cgi/content/full/50/1/28/DC1/.

LI OPERATION[6]

The linguistic fuzzy logic implication is defined through the LWD operator by (ν_{LI}, $\phi_{LI} = LI[(\nu_i, \phi_i), (\nu_j, \phi_j)] = LWD[Neg\{(\nu_i, \phi_i)\};(\nu_j, \phi_j)]$; that is, $\nu_{LI} = MAX\{MIN[Neg(\nu_i, \phi_i)]; MIN[(\nu_j, \phi_j)]\}$ and $\phi_{LI} = LOWA(Neg(\phi_i), \phi_j)$.

As an illustration, let us consider the following operation: $(\nu_{LI}; \phi_{LI}) = LI\{(LLL, MMM);(MMM, HHH)\}$; that is, we want to evaluate the nuance degree and feasibility degree of the following linguistic fuzzy logic implication: $(LLL, MMM) \Rightarrow (MMM, HHH)$ with the antecedent having a low degree of nuance and moderate degree of feasibility and the consequent having a moderate degree of nuance and a high degree of feasibility. We effectively are posing the following questions: how nuanced would this implication be—that is, how is the consequent more or less implied by the antecedent? And how feasible would such a nuanced implication be? This means evaluating $(\nu_{LI}; \phi_{LI}) = LWD[Neg\{(LLL, MMM)\};(MMM, HHH)]$. Using the above definition, $\nu_{LI} = MAX\{MIN[Neg(LLL, MMM)];MIN[(MMM, HHH)]\} = MAX\{MMM; MMM\} = MMM$ and $\phi_{LI} = LOWA\{MMM, HHH\}$. Using LOWA, we obtain $(\nu_{LI}; \phi_{LI}) = (MMM; HHH)$. In words, the antecedent moderately implies the consequent with a high degree of feasibility. I now use this formalism to define and analyze a linguistic fuzzy-logic version of 2×2 PD game.

LINGUISTIC FUZZY LOGIC AND GAME THEORY

Before introducing key notions of LFL game theory, let us first reformulate a conventional PD game, explicitly using Boolean logic formalism. In the crisp 2×2 PD game, the payoffs are defined as follows: $(C, C) = (\alpha, \alpha)$, $(C, D) = (\delta, \gamma)$, $(D, C) = (\gamma, \delta)$, $(D, D) = (\beta, \beta)$, with the condition $\gamma > \alpha > \beta > \delta$. Let s_A and s_B be the crisp choices of A and B, respectively. Let u_A (u_B) be the crisp payoff of A (B). To clearly define the notion of strategy in linguistic fuzzy logic, from now on I differentiate between initial choices and strategies or strategic arrangements. A set of initial choices is the set of individual moves that an actor might potentially turn into strategies of the game since strategies are strictly defined in games. This differentiation, which is usually redundant in game-theoretic literature, is important because it facilitates the formulation of logical formulas for strategic interaction. Using the concept of inference rules highlights the importance of making such a differentiation, as explained in the following. The rules of inference $\Re_{ij}$ in the game for A (and similar expression for B) in the crisp PD game are as follows:

$\Re_{11}$ If $\{s_A = C$ and $s_B = C\}$ then $\{u_A = \alpha\} \Leftrightarrow ((s_A = C) \wedge (s_B = C)) \rightarrow (u_A = \alpha)$.

$\Re_{21}$ If $\{s_A = D$ and $s_B = C\}$ then $\{u_A = \gamma\} \Leftrightarrow ((s_A = D) \wedge (s_B = C)) \rightarrow (u_A = \gamma)$.

$\Re_{12}$ If $\{s_A = C$ and $s_B = D\}$ then $\{u_A = \delta\} \Leftrightarrow ((s_A = C) \wedge (s_B = D)) \rightarrow (u_A = \delta)$.

$\Re_{22}$ If $\{s_A = D$ and $s_B = D\}$ then $\{u_A = \beta\} \Leftrightarrow ((s_A = D) \wedge (s_B = D)) \rightarrow (u_A = \beta)$.

6. This is a generalization of the usual definition of S-implication as $a \rightarrow b \equiv \neg a \vee b$.

These rules of inference are based on two-valued Boolean logic. For example, rule 11 for A translates in English as the following: if we know that $s_A = C$ is feasible AND we know that $s_B = C$ is feasible, then this implies that $u_A = \alpha$ is feasible. A strategy for a player is then a disjunction of two rules. For example, strategy C for A means that A uses either $\mathfrak{R}_{11}$ or $\mathfrak{R}_{12}$. We thus have for the strategic game, as perceived by A (and similar expressions for B), the following:

$$C\text{: } \mathfrak{R}_{11} \vee \mathfrak{R}_{12}\ \{((s_A = C) \wedge (s_B = C)) \rightarrow (u_A = \alpha)\} \vee \{((s_A = C) \wedge (s_B = D)) \rightarrow (u_A = \delta)\}.$$

$$D\text{: } \mathfrak{R}_{21} \vee \mathfrak{R}_{22}\ \{((s_A = D) \wedge (s_B = C)) \rightarrow (u_A = \gamma)\} \vee \{((s_A = D) \wedge (s_B = D)) \rightarrow (u_A = \beta)\}.$$

We can say that A (B) has four initial choices that correspond to the four entries of the normal form of the PD game. These four choices correspond to four inference rules. Therefore, in this notation, a strategy is effectively a disjunction of a number of inference rules. We see, for example, that A (B) has a strictly dominant strategy: D expressed as $\{\mathfrak{R}_{21} \vee \mathfrak{R}_{22}\}_{A(B)}$ and a strictly dominated strategy: C expressed as $\{\mathfrak{R}_{11} \vee \mathfrak{R}_{12}\}_{A(B)}$. The solution of the game, which in this case is DD, is the conjunction of the two dominant strategies of the two players:

$$DD \Leftrightarrow \{\mathfrak{R}_{21} \vee \mathfrak{R}_{22}\}_A \wedge \{\mathfrak{R}_{21} \vee \mathfrak{R}_{22}\}_B.$$

This method of explicating the underlying Boolean logic of the crisp game paves the way for explicating the linguistic fuzzy-logic approach to game theory. To obtain a linguistic fuzzy game from a crisp game, we hence need to fuzzify the following:

1. The initial set of crisp choices of each player into linguistic fuzzy choices: the choices become linguistic variables assuming linguistic values—termed as *nuanced choices*, with each having a feasibility degree.
2. The crisp orderings of the alternative strategies into linguistic orderings: the ranking preferences become linguistic variables assuming linguistic values—termed as *nuanced preferences*, each with a feasibility degree.
3. The two-valued Boolean logic conjunction AND into linguistic fuzzy-logic conjunction, LWC.
4. The two-valued Boolean logic disjunction OR into linguistic fuzzy-logic disjunction, LWD.
5. The two-valued Boolean logic implication $\rightarrow$ into linguistic fuzzy-logic implication, LI.
6. The rules of the game $\mathfrak{R}_{ij}$: changing them from inference based on two-valued logic to inference based on linguistic fuzzy logic.

Note that to each choice, we hence associate a primary term, a linguistic degree of nuance expressed using a linguistic hedge ν applied to the primary term of the variable, as well as a feasibility value expressed using a linguistic hedge ϕ applied to the primary term of a designated feasibility value such as *feasible*. For example, for a feasible nuanced choice of cooperation, we would have the term set of hedged ν-cooperation. We could, for example, have *low cooperation*, *moderate cooperation*, *high cooperation*, and so on. The hedges (or *nuancers*) in this case are $\nu \in \{$*low*, *moderate*, *high*$\}$. Each of these linguistic values of nuanced cooperation would have a linguistic degree of feasibility, such as *low to moderate feasibility*, *high feasibility*, and *moderate to high feasibility*, respectively; hence, we have the set of feasibility hedges $\varphi \in$ {*low to moderate*, *high*, *moderate to high*}. The player's choice set would be represented as $s = \{(\nu$-*cooperation*, φ-*feasible*), etc.}. Formally, we would write $s_i = (\nu_{[s_i = \ldots]}, \varphi_{[s_i = \ldots]})$. The degree of feasibility can be interpreted in a PD game as a player's predisposition toward being more nice-spirited or more mean-spirited (as illustrated down below in a full solution of PD). *Full feasibility* would stand for either full nice-spiritedness or full mean-spiritedness. *Null feasibility* would stand for either null nice-spiritedness or null mean-spiritedness. An intermediate value, such as *moderate feasibility*, would stand for being undecided or agnostic toward either nice- or mean-spiritedness. An LFL approach hence allows us to explore the impact of players' predispositions as part of the game structure—*players' predispositions are endogenous in the LFL game*, rather than being assumed as given by nature, as is usually done in conventional game theory. This opens up the possibility for richer notions of strategic dominance and Nash equilibrium, as discussed shortly. Similarly, we represent the preference rankings as $R_i = (\nu_{[R_i = \ldots]}, \varphi_{[R_i = \ldots]})$ and the inference rules as $\Re_{ij} = (\nu_{[\Re_{ij} = \ldots]}; \varphi_{[\Re_{ij} = \ldots]})$.

The next step is to express the rules of the game in a linguistic fuzzy-logic form. This would enable us to define the notions of a linguistic fuzzy-logic dominant strategy and linguistic fuzzy-logic Nash equilibrium. In doing so, we must keep in mind that all variables are linguistically expressed and symbolically manipulated, as well as all convex combinations thereof formed by using the four connectives ($\wedge$ = AND, $\vee$ = *OR*, $\rightarrow$ = Implication, $\neg$ = Negation). To this end, I use the algebra introduced earlier in the article. To fuzzify a PD game, we start with the ordinal preferences of an ordinal PD game. Each player has a matrix of preference relations for a particular strategy. In the case of the 2×2 PD game, A's preference relation would hence be $R_A(C; C)$ for both A and B cooperating and $R_A(C; D)$ for A cooperating but B defecting. A's and B's respective matrices of preference relations are given as

$$A\colon\ R_A = \begin{pmatrix} R_A(C;C) & R_A(C;D) \\ R_A(D;C) & R_A(D;D) \end{pmatrix}; \qquad B\colon\ R_B = \begin{pmatrix} R_B(C;C) & R_B(C;D) \\ R_B(D;C) & R_B(D;D) \end{pmatrix}.$$

In the case of the PD game, we have $R_A(D; C) > R_A(C; C) > R_A(D; D) > R_A(C; D)$ and $R_B(D; C) > R_B(C; C) > R_B(D; D) > R_B(C; D)$. The various choices are ranked from best to worst as 1, 2, 3, 4. The dilemma in the PD game is that both A and B end up rationally getting the outcome (3,3) by seeking to obtain either (1,4) for A or (4,1) for B—the outcome (3,3) corresponding to mutual defection is worse than that corresponding to mutual cooperation (2,2).

We can express the crisp game using the rules of inference of the underlying Boolean logic for player A as

$\Re_{11}$ If $\{s_A = C$ and $s_B = C\}$ then $\{R_A(C; C) = 2\} \Leftrightarrow ((s_A = C) \wedge (s_B = C)) \rightarrow (R_A(C; C) = 2)$

$\Re_{21}$ If $\{s_A = D$ and $s_B = C\}$ then $\{R_A(D; C) = 1\} \Leftrightarrow ((s_A = D) \wedge (s_B = C)) \rightarrow (R_A(D; C) = 1)$

$\Re_{12}$ If $\{s_A = C$ and $s_B = D\}$ then $\{R_A(C; D) = 4\} \Leftrightarrow ((s_A = C) \wedge (s_B = D)) \rightarrow (R_A(C; D) = 4)$

$\Re_{22}$ If $\{s_A = D$ and $s_B = D\}$ then $\{R_A(D; D) = 3\} \Leftrightarrow ((s_A = D) \wedge (s_B = D)) \rightarrow (R_A(D; D) = 3)$

(and similar expressions for B). We can rewrite these by using the notion of strategy as defined earlier in terms of inference rules.

C: $\Re_{11} \vee \Re_{12}$: $\{((s_A = C) \wedge (s_B = C)) \rightarrow (R_A(C; C) = 2)\} \vee \{((s_A = C) \wedge (s_B = D)) \rightarrow (R_A(C; D) = 4)\}$.

D: $\Re_{21} \vee \Re_{22}$: $\{((s_A = D) \wedge (s_B = C)) \rightarrow (R_A(D; C) = 1)\} \vee \{((s_A = D) \wedge (s_B = D)) \rightarrow (R_A(D; D) = 3)\}$.

It is clear that in both cases, D is a strictly dominant strategy for both players. The Nash equilibrium is thus $DD \Leftrightarrow \{\Re_{21} \vee \Re_{22}\}_A \wedge \{\Re_{21} \vee \Re_{22}\}_B$.

LINGUISTIC FUZZY-LOGIC GAME (LFL GAME)

To move from a crisp game to an LFL game, we transform the above crisp relations into linguistic fuzzy preference relations. Before doing so, let us consider as an example the linguistic fuzzy-logic equivalent of rule 11 for A: $\Re_{11} = s_A^1 \wedge s_B^1 \rightarrow R_A(s_A^1; s_B^1)$; that is,

> To what degrees of nuance $\nu^A_{\Re_{11}}$ and feasibility $\varphi^A_{\Re_{11}}$ can we infer the following: IF [{strategy s_A^1 has nuance degree $\nu_{[s_A^1]}$} with a feasibility degree $\varphi_{[s_A^1]}$] AND [{strategy s_B^1 has nuance degree $\nu_{[s_B^1]}$} with a feasibility degree $\varphi_{[s_B^1]}$ THEN [{preference relation $R_A(s_A^1; s_B^1)$ has nuance degree $\nu_{R_A(s_A^1;s_B^1)}$} with a feasibility degree $\varphi_{R_A(s_A^1;s_B^1)}$]?

Using the LWC and LI operations, we write the feasibility and nuance degrees of rule 11 for A as

$$(\nu^A_{\Re_{11}}; \varphi^A_{\Re_{11}}) = LI\left\{LWC[(\nu_{[s_A]}, \varphi_{[s_A]}), (\nu_{[s_B]}, \varphi_{[s_B]})], (\nu_{R_A(s_A;s_B)}, \varphi_{R_A(s_A;s_B)})\right\}.$$

The linguistic fuzzy-logic equivalents of the remaining rules (12, 21, 22) are written in similar fashions. A strategic choice s_A^1 for player A then consists of choosing a linguistic fuzzy-logic strategy as a disjunction of linguistic fuzzy-logic inference rules (formally, just like in a crisp game):

$$\{\Re_{11} \vee \Re_{12}\}_A = \left\{s_A^1 \wedge s_B^1 \rightarrow R_A(s_A^1; s_B^1)\right\} \vee \left\{s_A^1 \wedge s_B^2 \rightarrow R_A(s_A^1; s_B^2)\right\}.$$

The degrees of nuance and feasibility for this strategy are given by

$$(\nu^A_{11,12}, \varphi^A_{11,12}) = Disjunction\left\{\left(\nu^A_{\Re_{11}}, \varphi^A_{\Re_{11}}\right); \left(\nu^A_{\Re_{12}}; \varphi^A_{\Re_{12}}\right)\right\} = LWD\{\Re_{11}; \Re_{12}\}_A.$$

Rationally choosing a strategy then would consist in optimizing the degrees of nuance and feasibility—that is, choosing a strategy with the highest degree of nuance and/or the highest degree of feasibility of the disjunction of rules ($\Re_{11}$ or $\Re_{12}$). An outcome (profile) of the game would then be a linguistic conjunction of two strategies of the two players, for example, $\{\Re_{11} \vee \Re_{12}\}_A \wedge \{\Re_{11} \vee \Re_{12}\}_B$. The degrees of nuance and feasibility of this outcome will be obtained as

$$(\nu_{GAME}, \varphi_{GAME}) = LCW\{LWD\{\Re_{11};\Re_{12}\}_A; LWD\{\Re_{11};\Re_{12}\}_B\}.$$

At this juncture, let us introduce some notations that would pave the way for a generalized definition of a dominant linguistic fuzzy-logic strategy and an LFL game profile. Let P be the set of q players. Let h be the total number of hedges (forming a set H) available to the players. If, for example, in the crisp ordinal game each player possesses one initial crisp choice, the player can apply a maximum of h hedges to this initial choice and hence would end up having to choose among h different linguistic fuzzy-logic choices, each having a linguistic degree of nuance and a linguistic degree of feasibility. Hence, for a player i, we have a set of possible linguistic fuzzy-logic choices defined as $S_i = \left\{ s_i^{l_i}(\nu_{l_i}, \varphi_{l_i}) | l_i = 1, \ldots, q_i; \nu_{l_i} \in H; \varphi_{l_i} \in H \right\}$. That is, each player i has LFL choices, and each of these choices has its own feasibility value ϕ and nuance value ν. For example, in the case of the crisp PD game, A has two crisp choices, C and D. From these two crisp choices, we would have two sets of LFL choices obtained by applying the elements of the hedge algebra set H to these crisp choices. Suppose, for example, that the hedge set is given by H = {*NNN*, *VLL*, *LLL*, *MMM*, *HHH*, *VHH*, *FFF*}, where *NNN* = null, *VLL* = very low, *LLL* = low, *MMM* = moderate, *HHH* = high, *VHH* = very high, and *FFF* = full. A would have the following two sets of LFL choices $\{\nu\text{-}C | \nu \in H\}$ = {*NNN*, *VLL*, *LLL*, *MMM*, *HHH*, *VHH*, *FFF*}-*cooperation* and $\{\nu\text{-}D | \nu \in H\}$ = {*NNN*, *VLL*, *LLL*, *MMM*, *HHH*, *VHH*, *FFF*}-*defection*. Each of these LFL choices would have a feasibility degree $\phi \in$ {*NNN*, *VLL*, *LLL*, *MMM*, *HHH*, *VHH*, *FFF*}-*feasibility*. A could, for example, choose *MMM-C* (*moderate cooperation*) with *VHH* (*very high degree of*) *feasibility*. The sets $\{\nu\text{-}C | \nu \in H\}$ and $\{\nu\text{-}D | \nu \in H\}$ are, semantically speaking, the mirror images of one another; that is, for example, *null cooperation* is the same as *full defection*. It is therefore enough to consider the set of LFL choices defined as $\{\nu\text{-}C | \nu \in H\}$. This is not always the case, especially starting with crisp games where the choice set is not two-dimensional or, more generally, when the crisp choices are not necessarily symmetrically related to one another in terms of their semantics.

If we write the linguistic feasibility values ϕ and the linguistic nuance degrees ν for the q_i choices for player i as $\left\{ \phi_i(s_i^k) | k = 1, \cdots, q_i \right\}$; $\left\{ \nu_i(s_i^k) | k = 1, \cdots, q_i \right\}$, the linguistic feasibility and nuance values for an inference rule $\Re_{kl}$ for player i would be given by

$$(\nu^i_{\Re_{kl}}; \varphi^i_{\Re_{kl}}) =$$
$$LI\left\{ LWC\left\{ \left(\nu_i(s_i^k), \varphi_i(s_i^k)\right); \left(\nu_i(s_{-i}^l), \varphi_i(s_{-i}^l)\right) \right\}; \left(\nu_i\left[R_i(s_i^k, s_{-i}^l)\right], \varphi_i\left[R_i(s_i^k, s_{-i}^l)\right]\right) \right\},$$

where, following a standard practice, I denote by a_{-i} the $n-1$-tuple that results from removing the ith element from the n-tuple $a = (a_1, \ldots, a_n)$. The set of LFL choices that player i's opponents may play is then denoted by $S_{-i} = S_1 \times \cdots \times S_{i-1} \times S_{i+1} \times \cdots S_q$. The linguistic fuzzy-logic choices of all other players $j \neq i$ are

$$S_{-i} = \left\{ s_j^{l_j}\left(\nu_{l_j}, \varphi_{l_j}\right) | j = 1, 2, \ldots, i-1, i+1, \ldots, q;\ l_j = 1, \ldots, q_j;\ \nu_{l_j} \in H; \varphi_{l_j} \in H \right\}.$$

The feasibility and nuance values for a player's choice are functions of three prior feasibility values and three prior degrees of nuance as the following example from the PD game shows:

$$(\nu^A_{\Re_{11}};\varphi^A_{\Re_{11}}) = LI\{LWC[(\nu_{[MMM]},\varphi_{[MMM]}),(\nu_{[HHH]},\varphi_{[HHH]})],\\ (\nu_{R_A(MMM;HHH)},\varphi_{R_A(MMM;HHH)})\}.$$

For example, for player A, the feasibility value $\varphi^A_{\Re_{11}}$ for playing $\Re_{11}$ is a function of

- The degrees of nuance and corresponding feasibility values $(\nu_{[s_A=MMM]},\varphi_{[s_A=MMM]})$ for A playing *MMM-C* (*moderate cooperation*) with a degree of feasibility $\varphi_{[s_A=MMM]}$ and $(\nu_{[s_B=HHH]},\varphi_{[s_B=HHH]})$ for B playing *HHH-C* (*high cooperation*) with a degree of feasibility $\varphi_{[s_B=HHH]}$.
- The degree of nuance and corresponding feasibility value of the preference relation of player A should this rule be chosen, $(\nu_{R_A(MMM;HHH)},\varphi_{R_A(MMM;HHH)})$.

The set of inference rules available to i playing against j are thus generally written as

$$\Re_{l_i l_j} = \left\{ s_i^{l_i} \wedge s_j^{l_j} \rightarrow R_i(s_i^{l_i};s_j^{l_j}) \mid l_i \in \{1,\ldots,q_i\}; l_j \in \{1,\ldots,q_j\} \right\}.$$

The strategy of player i consists of a disjunction of all j's choices, which in effect is nothing but player i's strategy when player i's initial choice is $s_i^{l_i}$. Hence, we can write the strategy $\zeta(.)$ for player i against player j as

$$\zeta(s_i^{l_i}) = \Re_{l_i\,(l_j)} = \Re_{l_i\,k_1} \vee \Re_{l_i\,k_2} \vee \ldots \vee \Re_{l_i k_{qj}} = LWD\{\Re_{l_i\,k_1};\Re_{l_i\,k_2};\ldots;\Re_{l_i\,k_{qj}}\}.$$

The linguistic nuance and feasibility values of i's strategy for when player i's choice is $s_i^{l_i}$ are

$$\zeta(\nu_i^{l_i};\varphi_i^{l_i}) = LWD\left\{(\nu_{\Re_{l_ik_1}};\varphi_{\Re_{l_ik_1}});(\nu_{\Re_{l_ik_1}};\varphi_{\Re_{l_ik_2}});\ldots;(\nu_{\Re_{l_ik_1}};\varphi_{\Re_{l_ik_{qj}}})\right\}.$$

We now are in a position to define three types of strictly dominant strategies.

Strictly F-Dominant Strategy

A strictly F-dominant linguistic fuzzy-logic strategy $\zeta(s_i^k)$ with a feasibility value φ_i^k for player i is such that $\varphi_i^k \succ \varphi_i^{[-k]}$, where [–$k$] stands for all linguistic fuzzy-logic strategies for player i other than strategy k. The relation of domination is partial if the strict relation > is replaced by the weaker relation, ≥.

Strictly N-Dominant Strategy

A strictly N-dominant linguistic fuzzy-logic strategy $\zeta(s_i^k)$ with a nuance value ν_i^k for player i is such that $\nu_i^k \succ \nu_i^{[-k]}$, where [–$k$] stands for all linguistic fuzzy-logic strategies for player i other than strategy k. The relation of domination is partial if the strict relation > is replaced by the weaker relation, ≥.

Strictly F-N-Dominant Strategy

A strictly *F-N*-dominant linguistic fuzzy-logic strategy $\zeta(s_i^k)$ with a nuance value ν_i^k and a feasibility value ϕ_i^k for player i is such that $\nu_i^k \succ \nu_i^{[-k]}$ and $\varphi_i^k \succ \varphi_i^{[-k]}$, where $[-k]$ stands for all linguistic fuzzy-logic strategies for player i other than strategy k. The relation of domination is partial if the strict relation > is replaced by the weaker relation, ≥.

Since the hedges are always (at least partially) ordered, the degrees of nuance (and the degrees of feasibility) for any two strategies of a player stand (at least partially) ordered toward one another. This implies that a player has always at least one partially, if not strictly, dominant strategy in one of the three meanings defined above. We now can provide the following complete definition.

Definition: Two-Player Linguistic Fuzzy-Logic (LFL) Game

An LFL game G is a couple (P, H), where P is a finite set of two players and H a finite set of h linguistic hedges, with the following elements:

1. For each player $i \in P$, there is a finite set of q_i initial LFL choices:

$$S_i = \left\{ s_i^{l_i}(\nu_{l_i}, \varphi_{l_i}) \,|\, l_i = 1, \ldots, q_i; \quad \nu_{l_i} \in H; \varphi_{l_i} \in H \right\},$$

   $\phi_{l_i} : S_i \to H$ is called the feasibility value of the LFL choice $s_i^{l_i}(\nu_{l_i}, \phi_{l_i})$.

   $\nu_{l_i} : S_i \to H$ is called the nuance degree of the LFL choice $s_i^{l_i}(\nu_{l_i}, \phi_{l_i})$.
2. For each player $i \in P$, there is an $R_i(s_i^{l_i}; s_{-i})$ matrix of preference relations with $q_1 \times q_2$ elements showing i's subjective ranking of simultaneous choices for both players in the game. Each matrix element has a nuance degree spanning a spectrum from fully preferred (i.e., best) to not preferred (i.e., worst) and a feasibility degree spanning a spectrum from fully feasible to not feasible.
3. The LFL game G has $q_1 \times q_2$ rules: rule $\Re_{kl}$ for $i \in P$ playing s_i^k against $j \in P$ playing s_j^l is

$$\left\{ \Re_{kl} = \left[(s_i^k \wedge s_j^l) \to R_i(s_i^k; s_j^l) \right] \,|\, k \in \{1, \ldots, q_1\};\ l \in \{1, \ldots, q_2\};\ i \in \{1,2\} \neq j \in \{1,2\} \right\}.$$

The degrees of nuance ν and feasibility ϕ of $\Re_{kl}$ are given by:

$$(\nu^i_{\Re_{kl}}; \varphi^i_{\Re_{kl}}) = LI\left\{ LWC\{(\nu_i(s_i^k), \varphi_i(s_i^k)); (\nu_i(s_j^l), \varphi_i(s_j^l))\}; (\nu_i[R_i(s_i^k, s_j^l)], \varphi_i[R_i(s_i^k, s_j^l)]) \right\}.$$

4. The LFL strategic arrangement for player i against player j when player i's initial choice is

$$s_i^{l_i} \text{ is: } \zeta(s_i^{l_i}) = \Re_{l_i(k_j)} = \Re_{l_i k_1} \vee \Re_{l_i k_2} \vee \ldots \vee \Re_{l_i k_{qj}} = LWD\{\Re_{l_i k_1}; \Re_{l_i k_2}; \ldots; \Re_{l_i k_{qj}}\}.$$

The nuance and feasibility values for this strategic arrangement are

$$(\nu_i^{l_i};\varphi_i^{l_i}) = LWD\{(\nu_{\Re_{l_ik_1}};\varphi_{\Re_{l_ik_1}});(\nu_{\Re_{l_ik_1}};\varphi_{\Re_{l_ik_2}});\ldots;(\nu_{\Re_{l_ik_1}};\varphi_{\Re_{l_ik_{qj}}})\}.$$

5. The Cartesian product $S = S_1 \times S_2$ is called the LFL strategy profile for the LFL game G. A profile GP for players i and j in the LFL game is given by

$$GP(i, l_i; j, n_j) = \Re_{l_i(k_j)} \wedge \Re_{(m_i)n_j} = LWC\{\Re_{l_i(k_j)}; \Re_{(m_i)n_j}\}.$$

The nuance and feasibility degrees of this profile are obtained as

$$(\nu_{l_il_j}^{pr};\varphi_{l_il_j}^{pr}) = LWC\ \{\zeta(\nu_i^{l_i};\varphi_i^{l_i});\zeta(\nu_j^{l_j};\varphi_j^{l_j})\}.$$

We can recover the case of a crisp 2×2 Boolean logic PD game by setting $P = \{A, B\}$; $H = \{NNN, FFF\}$ or, equivalently, $H = \{\text{false, true}\}$; $S = \{C; D\} \times \{C; D\}$; $R_A(D; C) \succ R_A(C; C) > R_A(D; D) \succ R_A(C; D)$; $R_B(D; C) \succ R_B(C; C) \succ R_B(D; D) \succ R_B(C; D)$. To define as a notion of the Nash equilibrium, let us recall that the notion of the Nash equilibrium for a crisp ordinal game is usually introduced through an ordering operation that ranks the preference relations of the players. In the LFL approach, the notion of dominant strategy is defined by using the degrees of nuance and feasibility, which are by construction always ordered. Because an LFL strategy has both a degree of nuance and a degree of feasibility, we can have three sorts of Nash equilibria.

Definition: Nash Equilibrium

Each player $i \in \{1, 2\}$ has a finite set of linguistic feasibility values ϕ_i^l and a finite set of linguistic nuance values ν_i^l defining its LFL strategies: $\{\zeta(s_i^{l_i}) = \zeta(\nu_i^{l_i};\varphi_i^{l_i}) | i \in \{1,2\}; l_i \in \{1,\ldots,q_i\}\}$. The game profile is given by $(\nu_{l_1l_2}^{pr};\varphi_{l_1l_2}^{pr}) = LWC\ \{\zeta(\nu_1^{l_1};\varphi_1^{l_1});\zeta(\nu_2^{l_2};\varphi_2^{l_2})\}$.

NNE: N-Nash equilibrium:

$(\nu_{l_1l_2}^{*pr};\varphi_{l_1l_2}^{pr})$ is called an N-Nash equilibrium (NNE) if:

$$(\nu_{l_1l_2}^{pr} \prec \nu_{l_1^*l_2^*}^{*pr}), \forall l_1 \in \{1,\ldots,1-l_1^*,1+l_1^*,\ldots,q_1\};\ \forall l_2 \in \{1,\ldots,1-l_2^*,1+l_2^*,\ldots q_2\}.$$

FNE: F-Nash equilibrium:

$(\nu_{l_1l_2}^{pr};\ \varphi_{l_1l_2}^{*pr})$ is called an F-Nash equilibrium (FNE) if

$$(\varphi_{l_1l_2}^{pr} \prec \varphi_{l_1^*l_2^*}^{*pr}),\ \forall l_1 \in \{1,\ldots,1-l_1^*,1+l_1^*,\ldots,q_1\};\ \forall l_2 \in \{1,\ldots,1-l_2^*,1+l_2^*,\ldots,q_2\}.$$

FNNE: F-N-Nash equilibrium:

$(\nu_{l_1l_2}^{*pr};\ \varphi_{l_1l_2}^{*pr})$ is called an F-N-Nash equilibrium (FNNE) if

$$(\nu^{pr}_{l_1 l_2} \prec \nu^{*pr}_{l_1^* l_2^*}) \text{ and } (\varphi^{pr}_{l_1 l_2} \prec \varphi^{*pr}_{l_1^* l_2^*}),$$
$$\forall l_1 \in \{1,\ldots,1-l_1^*, 1+l_1^*,\ldots,q_1\};\ \forall l_2 \in \{1,\ldots,1-l_2^*, 1+l_2^*,\ldots,q_2\}.$$

Because the set of hedges is an ordered algebra and the operations of LOWA, LWC, LWD, and LI are convex operations, we can see that the processes of linguistic aggregation produce linguistic terms that are elements of the same set of partially ordered hedges. This implies that there will always be a partial ordering of the profiles of a game. That is, the degrees of nuance and feasibility of the various profiles are always partially ordered as elements of the hedge algebra. This, in turn, implies that that there will at least be an *N* or an *F* Nash equilibrium as defined above. The simultaneous occurrence of *F* and *N* types of equilibrium produces an *F-N* Nash equilibrium. We thus can state the following theorem of existence.

Theorem: Existence of Nash Equilibrium

There always exists at least one Nash equilibrium in an LFL game. The Nash equilibrium can be of *F* type, *N* type, or both simultaneously, that is, *F-N* type.

To illustrate these notions of LFL-Nash equilibrium, let us consider the following game structure: the basic crisp strategy is cooperation.

- The hedge algebra with seven degrees of nuance and feasibility is {*NNN*, *VLL*, *LLL*, *MMM*, *HHH*, *VHH*, *FFF*}.[7]
- Assume that each player has seven LFL choices $\{s_i^j | i = 1, 2; j = 1, 2, 3, 4, 5, 6, 7\}$ with the following nuance degrees: {*NNN*, *VLL*, *LLL*, *MMM*, *HHH*, *VHH*, *FFF*}. This means that the choices are no cooperation, very low cooperation, low cooperation, moderate cooperation, high cooperation, very high cooperation, and full cooperation.
- Assume that player 1 has the following feasibility degrees for its LFL choices: {*LLL*, *HHH*, *VHH*, *VLL*, *FFF*, *NNN*, *LLL*}. This means player 1 perceives its choice as being no cooperation with a low degree of feasibility, very low cooperation with a high degree of feasibility, low cooperation with a very high degree of feasibility, moderate cooperation with a very low degree of feasibility, high cooperation with a full degree of feasibility, very high cooperation with a null degree of feasibility, and full cooperation with a low degree of feasibility.
- Assume that player 2 has the following feasibility degrees for its LFL choices: {*HHH*, *MMM*, *VHH*, *FFF*, *LLL*, *NNN*, *VLL*}. This means player 2 perceives its choices as being no cooperation with a high degree of feasibility, very low cooperation with a moderate degree of feasibility, low cooperation with a very high degree of feasibility, moderate cooperation with a full degree of feasibility, high cooperation with a low degree of feasibility, very high cooperation with a null degree of feasibility, and full cooperation with a very low degree of feasibility.

7. *NNN* = null, *VLL* = very low, *LLL* = low, *MMM* = moderate, *HHH* = high, *VHH* = very high, and *FFF* = full.

The elements of the matrix of strategic preference relations for player 1 (and similar expression for player 2) are $\{R_1(s_1^i; s_2^j)|i = 1, 2, 3, 4, 5, 6, 7; j = 1, 2, 3, 4, 5, 6, 7\}$. Each of these elements has a nuance degree spanning a spectrum from fully preferred (i.e., best) to not preferred (i.e., worst) and a feasibility degree spanning a spectrum from fully feasible to not feasible. We thus have as input for each player two 7×7 matrices, one for the degrees of nuance and one for the degrees of feasibility. These elements are defining features of the LFL game, much like the utility entries define crisp cardinal and ordinal games. The corresponding matrix for a crisp PD game would be read using the following ordinal utilities (with 1 = *best*, 4 = *worst*):

$$R_1^{ordinal} = \begin{array}{c c} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} 2 & 4 \\ 1 & 3 \end{bmatrix} \end{array}; \quad R_2^{ordinal} = \begin{array}{c c} & \begin{array}{cc} C & D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix} \end{array}.$$

For an LFL game, we have the matrix of preference relations for player 1:

$$R_1 = \{R_1(s_1^i; s_2^j)|\ i = 1, \ldots, 7;\ j = 1, \ldots, 7\}.$$

Each element of this matrix has a nuance degree and a feasibility degree. What matters for the assessment of a rule and the possibility of having a Nash equilibrium are these degrees of nuance and feasibility. To proceed further in the illustration of an LFL game, we hence need to postulate these values as a further specification of the structure of the LFL game. To keep a transparent connection with the Boolean logic ordinal PD game, we need to remember that for the latter, we have all degrees of feasibility equal to TRUE (or, in LFL denotation, *FFF*). As to the degrees of nuance, we have for player 1 the following rankings: 1 for strategy *DC*, 2 for strategy *CC*, 3 for strategy *DD*, and 4 for strategy *CD*. These rankings translate into the following degrees of nuance in LFL terminology: *FFF* for *DC*, *HHH* for *CC*, *LLL* for *DD*, and *NNN* for *CD*. We thus have the following two matrices for both players:

Matrix of preference feasabilities in crisp ordinal PD game:

$$\textit{Player 1: } \varphi[R_1^{ordinal}] = \begin{bmatrix} FFF & FFF \\ FFF & FFF \end{bmatrix}; \quad \textit{Player 2: } \varphi[R_2^{ordinal}] = \begin{bmatrix} FFF & FFF \\ FFF & FFF \end{bmatrix}.$$

Matrix of preference nuances in crisp ordinal PD game:

$$\textit{Player 1: } \nu[R_1^{ordinal}] = \begin{bmatrix} HHH & NNN \\ FFF & LLL \end{bmatrix}; \quad \textit{Player 2: } \nu[R_2^{ordinal}] = \begin{bmatrix} HHH & FFF \\ NNN & LLL \end{bmatrix}.$$

In the LFL-PD game, we would, for example, write the matrix of nuanced preferences as

Matrix of nuanced preferences:

$$\nu[R_1] = \begin{bmatrix} HHH & MMM & LLL & VLL & NNN & NNN & NNN \\ HHH & MMM & LLL & VLL & NNN & NNN & NNN \\ VHH & HHH & MMM & LLL & VLL & NNN & NNN \\ VHH & HHH & MMM & LLL & VLL & VLL & VLL \\ VHH & HHH & MMM & LLL & VLL & VLL & VLL \\ FFF & VHH & HHH & MMM & LLL & VLL & VLL \\ FFF & VHH & HHH & MMM & LLL & VLL & VLL \end{bmatrix},$$

Matrix of degrees of feasibility:

$$\varphi[R_1] = \begin{bmatrix} FFF & VHH & HHH & MMM & HHH & VHH & FFF \\ VHH & HHH & MMM & LLL & MMM & HHH & VHH \\ HHH & MMM & LLL & VLL & LLL & MMM & HHH \\ MMM & LLL & VLL & NNN & VLL & LLL & MMM \\ HHH & MMM & LLL & VLL & LLL & MMM & HHH \\ VHH & HHH & MMM & LLL & MMM & HHH & VHH \\ FFF & VHH & HHH & MMM & HHH & VHH & FFF \end{bmatrix}.$$

This means, for example, that for player 1, $\nu[R_1(s_1^3; s_2^5)] = VLL$; that is, player 1 ranks a strategy with (very high cooperation, no cooperation) (third row, fifth column) as having a very low degree of nuance of preference ranking. We also need a matrix of feasibility degrees corresponding to the matrix of nuanced preferences (shown above). This means, for example, that for player 1, $\phi[R_1(s_1^3; s_2^5)] = LLL$; that is, player 1 sees the strategy (moderate cooperation, no cooperation) as having a low degree of feasibility. From these, we can evaluate the degree of feasibility of, say, rule for player 1. Let us go on to the complete LFL-PD game.

LFL-PD GAME

We may, for example, consider three different situations depending on the type of player, that is, whether the player is mean-spirited or nice-spirited. Each of these two types is in fact a set of linguistic subtypes assuming a linguistic value—there is graduality in meanness and niceness of the actors. I present the solutions to the game in terms of six sets of couplets. ν_1^k and ϕ_1^k stand for the nuance and feasibility degrees of an initial choice k for player 1. ν_2^l and ϕ_2^l stand for the nuance and feasibility degrees of an initial choice l for player 2. $\zeta(\nu_1^k)$ and $\zeta(\phi_1^k)$ stand for the nuance and feasibility degrees of a strategic arrangement k for player 1. $\zeta(\nu_2^k)$ and $\zeta(\phi_2^k)$ stand for the nuance and feasibility degrees of a strategic arrangement l for player 2. The nuance and feasibility degrees of Nash solutions are denoted by $\nu^{*pr}_{k_1^* l_2^*}$ and $\varphi^{*pr}_{k_1^* l_2^*}$. I present the different solutions obtained for various values of orness ω for each actor (Appendix A explicates how ω intervenes in the aggregation process).

The various values for ω (orness) stand for different ways of combining the feasibility degrees of the possible rules to form a strategic arrangement. ω is a measure of how disjunctively like (OR-like) or conjunctively like (AND-like) the various feasibility degrees are combined (Arfi 2005). LOWA is an "orand" operator located somewhere between the "AND" and "OR" logical operations—LOWA has simultaneously a finite degree of "orness" and a finite degree of "andness." Moreover, an or-like LOWA operator with $\omega > 0.5$ describes the case of a player who prefers nuances (i.e., levels) of cooperation with high levels of feasibility. This means that this player will be more trusting and less cautious when deciding on the strategic arrangement. Conversely, an and-like LOWA operator with $\omega > 0.5$ describes the case of a player who prefers nuances of cooperation with low levels of feasibility. This means that this player will be less trusting and more cautious when deciding on the strategic arrangement.

Although this looks similar to the notion of mixed strategies of conventional game theory, it is fundamentally different from it. In a mixed strategy, a player would have a probability for using any one of the possible rules, with the understanding that the player might only use ONE such rule at any point in time. In the LFL approach, the player will combine more or less the feasibility degrees of all rules to design a strategy. The LOWA operator allows us to incorporate vagueness (or fuzzy overlap) in the degree of feasibility of the strategic arrangement. Moreover, the type of Nash equilibrium and its degrees of nuance and feasibility depend on the value of orness. To facilitate a comparison with the crisp PD game, I make the *C* strategy correspond to a nuance value $\nu = FFF$ (and feasibility degree $\phi = FFF$) and *D* correspond to a nuance value $\nu = NNN$ (and feasibility degree $\phi = FFF$). The Nash solution of the crisp PD game is thus *DD*, with a nuance value $\nu = NNN$ and a feasibility value $\phi = FFF$. A fully mean-spirited (nice-spirited) player is one who has an *FFF* (*FFF*) degree of feasibility for *NNN*- (*FFF*-) cooperation and an *NNN* (*NNN*) degree of feasibility for *FFF*- (*NNN*) cooperation. That is, a mean-spirited (nice-spirited) player is more prone to see null (full) cooperation as fully (fully) feasible and, by the same token, is more prone to see full (null) cooperation with a null (null) degree of feasibility. Thus, we have the nuance degrees for a mean-spirited player as ν-cooperation = {*FFF*, *VHH*, *HHH*, *MMM*, *LLL*, *VLL*, *NNN*} with the corresponding feasibility degrees ϕ-feasibility = {*NNN*, *VLL*, *LLL*, *MMM*, *HHH*, *VHH*, *FFF*}. For a nice-spirited player, we have ν-cooperation = {*FFF*, *VHH*, *HHH*, *MMM*, *LLL*, *VLL*, *NNN*} and the corresponding feasibility degrees ϕ-feasibility = {*FFF*, *VHH*, *HHH*, *MMM*, *LLL*, *VLL*, *NNN*}.

Due to a lack of space, I only present the results for a game where we have two more or less mean-spirited players, with varying degrees of meanness, for two values of orness. We can see from Table 1 that there is a large number (14) of *F-N* Nash equilibria for a small value of orness $\omega = 0.1$, that is, in situations where the players, when evaluating the feasibility of strategic arrangements of the game, favor strongly those with low degrees of feasibility (*MMM* = moderate or *LLL* = low). This is displayed in columns 7 and 9, which show the degrees of feasibility of the two players for strategic arrangements that result from initial choices of the players (columns 2 and 3 for player 1; columns 4 and 5 for player 2). In Table 1, all equilibria have the same degree of nuance and feasibility for $\omega = 0.1$; that is, *MMM*—the *F-N* Nash equilibrium is charac-

TABLE 1

Nash Equilibria for Linguistic Fuzzy-Logic/Prisoner's Dilemma Game

	Row	ν_1^k	ϕ_1^k	ν_2^l	ϕ_2^l	$\zeta(\nu_1^k)$	$\zeta(\phi_1^k)$	$\zeta(\nu_2^l)$	$\zeta(\phi_2^l)$	$v^{*pr}_{s_1^* l_2^*}$	$\varphi^{*pr}_{k_1^* l_1^*}$	*Nash*
$\omega = 0.1$												
	1	*FFF*	*NNN*	*FFF*	*NNN*	*FFF*	*MMM*	*FFF*	*MMM*	*MMM*	*MMM*	*F-N*
	2	*FFF*	*NNN*	*VHH*	*VLL*	*FFF*	*MMM*	*VHH*	*MMM*	*MMM*	*MMM*	*F-N*
	3	*FFF*	*NNN*	*NNN*	*FFF*	*FFF*	*MMM*	*FFF*	*MMM*	*MMM*	*MMM*	*F-N*
	4	*VHH*	*VLL*	*FFF*	*NNN*	*VHH*	*MMM*	*FFF*	*MMM*	*MMM*	*MMM*	*F-N*
	5	*VHH*	*VLL*	*VHH*	*VLL*	*VHH*	*MMM*	*VHH*	*MMM*	*MMM*	*MMM*	*F-N*
	6	*VHH*	*VLL*	*MMM*	*MMM*	*VHH*	*MMM*	*MMM*	*LLL*	*MMM*	*MMM*	*F-N*
	7	*VHH*	*VLL*	*NNN*	*FFF*	*VHH*	*MMM*	*FFF*	*MMM*	*MMM*	*MMM*	*F-N*
	8	*MMM*	*MMM*	*FFF*	*NNN*	*MMM*	*LLL*	*FFF*	*MMM*	*MMM*	*MMM*	*F-N*
	9	*MMM*	*MMM*	*VHH*	*VLL*	*MMM*	*LLL*	*VHH*	*MMM*	*MMM*	*MMM*	*F-N*
	10	*MMM*	*MMM*	*NNN*	*FFF*	*MMM*	*LLL*	*FFF*	*MMM*	*MMM*	*MMM*	*F-N*
	11	*NNN*	*FFF*	*FFF*	*NNN*	*FFF*	*MMM*	*FFF*	*MMM*	*MMM*	*MMM*	*F-N*
	12	*NNN*	*FFF*	*VHH*	*VLL*	*FFF*	*MMM*	*VHH*	*MMM*	*MMM*	*MMM*	*F-N*
	13	*NNN*	*FFF*	*MMM*	*MMM*	*FFF*	*MMM*	*MMM*	*LLL*	*MMM*	*MMM*	*F-N*
	14	*NNN*	*FFF*	*NNN*	*FFF*	*FFF*	*MMM*	*FFF*	*MMM*	*MMM*	*MMM*	*F-N*
$\omega = 0.8$												
	15	*FFF*	*NNN*	*FFF*	*NNN*	*FFF*	*HHH*	*FFF*	*HHH*	*HHH*	*HHH*	*F-N*
	16	*FFF*	*NNN*	*VHH*	*VLL*	*FFF*	*HHH*	*VHH*	*HHH*	*HHH*	*HHH*	*F-N*
	17	*FFF*	*NNN*	*NNN*	*FFF*	*FFF*	*HHH*	*VHH*	*HHH*	*HHH*	*HHH*	*F-N*
	18	*VHH*	*VLL*	*FFF*	*NNN*	*VHH*	*HHH*	*FFF*	*HHH*	*HHH*	*HHH*	*F-N*
	19	*VHH*	*VLL*	*VHH*	*VLL*	*VHH*	*HHH*	*VHH*	*HHH*	*HHH*	*HHH*	*F-N*
	20	*VHH*	*VLL*	*NNN*	*FFF*	*VHH*	*HHH*	*VHH*	*HHH*	*HHH*	*HHH*	*F-N*
	21	*NNN*	*FFF*	*FFF*	*NNN*	*VHH*	*HHH*	*FFF*	*HHH*	*HHH*	*HHH*	*F-N*
	22	*NNN*	*FFF*	*VHH*	*VLL*	*VHH*	*HHH*	*VHH*	*HHH*	*HHH*	*HHH*	*F-N*
	23	*NNN*	*FFF*	*NNN*	*FFF*	*VHH*	*HHH*	*VHH*	*HHH*	*HHH*	*HHH*	*F-N*

terized by moderate cooperation and is perceived to be moderately feasible. This equilibrium can be reached from a variety of combinations of initial choices of the two players. A remarkable result is shown in row 14 of the table, where the *F-N* Nash equilibrium is obtained from an initial situation where both players perceive noncooperation (*NNN*) as fully feasible (*FFF*). The converse of this situation is shown in row 1, where the two players perceive full cooperation as not feasible.

Increasing the degree of orness to $\omega = 0.8$ (i.e., the players put much more emphasis on those strategic arrangements with high degrees of feasibility, as shown in columns 7 and 9), we obtain 9 Nash equilibria, all of which are of the *F-N* type. From the table, we can see that high cooperation with a high degree of feasibility can be arrived at starting from different initial choices of the individual players, that is, before strategic interaction is taken into account. Rows 15, 17, 21, and 23 show that although one or both players might perceive full cooperation as not feasible (or equivalently, no cooperation as fully feasible), strategic interaction under linguistic fuzzy logic leads to high cooperation with a high degree of feasibility. In sum, when $\omega = 0.1$ or 0.8, even when both players initially prefer full defection, playing the LFL game leads to a Nash equilibrium of the strongest type for moderate cooperation. The dilemma of the crisp Boolean logic PD game is lifted in the sense that moderate (high) cooperation for $\omega = 0.1$ (0.8) is rationally preferred to full defection. Similar results obtain for other combinations of players' predispositions such as more or less nice-spirited versus more or less mean-spirited, or when both players are more or less nice-spirited.

ROLE OF REASSURANCE IN INTERNATIONAL RELATIONS: AN ILLUSTRATION

Andrew Kydd (2000) used game theory to develop a costly signaling theory of reassurance, which focuses on the sending and interpretation of costly signals, as a way of resolving the problem of mistrust in international relations (IR). Signals are made costly enough to persuade the receiver that the sender is trustworthy (Kydd 2000, 326). In this pursuit, Kydd modified the usual trust game modeled using a PD game by dividing the game into two rounds: a lesser initial round, followed by a final, more important, round. The actors can hence predicate their choices in the second round on what happened in the first one. Kydd introduces a parameter $0 < \alpha < 1$ as a measure of signal costliness. Round 1 of the game is worth α while round 2 is worth $1 - \alpha$. The most promising equilibrium—termed *separating*—occurs over an interval of α^* values. Kydd shows that in this parameter space, cooperation would occur at much lower levels of trust than in a one-round trust game. "The signals must be costly, but not too costly. Make them too easy and they become cheap talk. . . . Such claims are unpersuasive because there is nothing to prevent untrustworthy types from making them. However, the signals cannot be made too costly either, or the nice types will be too fearful to send them . . . cooperation may be possible for levels of trust that are quite low—low enough to preclude cooperation in the simple one-round trust game" (Kydd 2000, 340). As an illustration of his game-theoretic argument, Kydd looks at the negotiations that led to the end of the cold war, arguing that Gorbachev engaged U.S. leaders into a game of reassurance. The latter, however, succeeded in fostering trust only when the signals (i.e., unilateral concessions made by Gorbachev) became truly costly for the Soviet Union (Kydd 2000, 341).

I address the same issue but use an approach based on LFL game theory. Following Kydd's (2000) approach, I allow the players to play two consecutive PD games; that is, the second game takes as initial choices of the players the outcome of the first game. One key difference from Kydd's game is that I do not use signaling costs as a way of conveying the actors' levels of readiness to cooperate with one another. I instead use the notion of feasibility in combination with the notion of nuanced cooperation (or defection) to describe how the actors communicate with one another. For example, Kydd argues that "the secret to making the separating equilibrium work is finding a signal that is adequately costly to deter the mean types from sending it but not so costly that the nice type is afraid to send it" (p. 338). The LFL approach counterpart would be the following: *the secret to making reassurance work is that the actors choose strategies that have higher degrees of feasibility, although the corresponding degrees of nuanced cooperation might not be the highest one available. Conversely, the equivalent of a low-cost signal will be to choose a strategy that has a higher level of nuanced cooperation but with a low level of feasibility.* This differs from Kydd's Boolean logic approach on two main points. First, the actors are not just looking for cooperation or defection. They are instead looking for more or less cooperation. They want to know the following: how nuanced will their mutual cooperation be? Is it very weak or rather strong or somewhere in between? Second, how feasible are such levels of cooperation? Undoubtedly, these degrees of feasibility are shaped by cost analysis and cost signaling. However, the notion of feasibility is not restricted to these conventional measures for it includes the actors' perceptions of the past, present, and the future, not only in terms of costs but also in terms of intangibles such as identity (and emotions). As explained in the previous section, in the LFL game, actors have a propensity toward seeking more or less feasible strategies, an aspect of the game that is modeled by the level of orness ω. The actors thus inquire the following: at what levels of orness would an actor have a propensity of favoring low (high) degrees of feasibility? In the two-round version of the reassurance game, the actors would then have two different levels of orness: ω_1 and ω_2, one for each round.

Following Kydd's (2000) model, one would assume that $\omega_1 + \omega_2 = 1$. That is, the propensity toward choosing more or less feasible strategies in the two rounds of the game is mutually compensatory—a propensity to favor high levels of feasibility during round 1 would be compensated with a propensity to favor low levels of feasibility during round 2 and vice versa. In this interpretation, the degree of feasibility becomes the equivalent of cost in Kydd's formulation. However, this is only one possible combination of the degrees of orness of the two rounds. Indeed, in the LFL game, we have two degrees of freedom: the level or nuance of strategies and the degree of feasibility of these nuanced strategies. We do not just have cooperation and defection, which are either chosen or rejected due to their cost/benefit. This opens up the possibility of many more ways of combining the propensities (i.e., levels of orness) of the two rounds. For example, at the end of round 1, depending on the value of orness ω_1, a player possesses a number of choices, all of which are equilibria (either of *N*, *F*, or *F-N* types, as explained in the previous section). These equilibria produce at least moderate (*MMM*) cooperation, which is at least highly (*HHH*) feasible. As shown in Table 2, I find at the end of round 1 moderate cooperation with a high degree of feasibility for $\omega_1 \leq 0.2$, high cooperation with a high degree of feasibility for $0.2 \leq \omega_1 \leq 0.8$, and full cooperation with a full degree of feasibility for $0.8 \leq \omega_1$.

Translated in policy terms, these results mean that negotiators do not have just to cooperate or defect; they can engage one another at various levels of cooperation, which they believe as more or less feasible! Therefore, the negotiators can move on to the second round of the reassurance game along many different equilibrium paths, each characterized with a level of cooperation and a corresponding degree of feasibility. The degrees of nuance and feasibility for the game profile at the end of the second round are shown in Table 2, where ω_1 and ω_2 stand for orness levels for rounds 1 and 2. One notable result is that very high ($\nu_{GP} = VHH$) cooperation is fully ($\phi_{GP} = FFF$) feasible at the end of the second round at any orness level of the first round (note that we get *FFF-feasible*, *FFF-cooperation* for $\omega_2 > 0.90$). In other words, even starting with (moderately or highly feasible) moderate levels of cooperation at the end of round 1 can produce fully feasible, very high, or full cooperation at the end of round 2.

In terms of practical application, this opens up the possibility for continuing the negotiations even when the players are not still fully committed to full cooperation or even when they do not see full cooperation as fully feasible at the end of the first round of negotiations. We can also interpret this in terms of path-dependency reasoning; that is, moderately feasible, moderate cooperation at the end of the first round puts the players on a path-dependent equilibrium path, which self-feeds to preserve the momentum of cooperation going on. This can also be phrased in terms of "cheap-talk" jargon used in conventional game theory. LFL game theory shows that even "cheap talks" can build enough momentum for full cooperation in later stages. This runs in contradiction to Kydd's (2000, 343) argument that, when Gorbachev began to make some arms control advances toward the United States, such as suggesting a moratorium on nuclear testing and one on SS20 intermediate-range ballistic missile (IRBM) deployments, "these gestures can be usefully conceived of as signals that fall short of the crucial level . . . these signals were 'cheap talk' rather than 'costly signals.' These moves were ineffective in changing opinions in the West." In light of LFL game theory, I disagree with Kydd's assessment of this historical event and instead concur with Collins (1998, cited by Kydd 2000) and Joshua Goldstein and John Freeman (1990, cited by Kydd 2000), who reason that although there was no direct response from the United States, Gorbachev's gestures did initiate a path-dependent process in a more subtle way, thereby setting the stage for upcoming important gestures and explicit cooperation. Instead of arguing as Kydd did that "these signals failed, largely because they were regarded as moves that did not really hurt the Soviets; that is, they were signals with little or no cost" (p. 343), I think that these events showed that cooperation, if only of a moderate level, can be more or less feasible.

In this respect, we can see how a fuzzy-logic approach leads to another important difference with Boolean logic–based analysis (including game theory). For instance, Kydd's (2000) conceptualization of the notion of a threshold at which point signals become costly is a "yes/no" one. That is, signals remain "cheap talks" until they reach

TABLE 2

Profile of Reassurance Game

ROUND 1 OF THE GAME

	ω_1								
	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Degrees of Nuance: ν_{GP}	*MMM*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*FFF*
Degrees of Feasibility: ϕ_{GP}	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*FFF*

ROUND 2 OF THE GAME

Degrees of Nuance: ν_{GP}

ω_1	ω_2: 0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0.1	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*HHH*	*VHH*
0.2	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*HHH*	*VHH*
0.3	*MMM*	*MMM*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*VHH*
0.4	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*HHH*	*VHH*
0.5	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*HHH*	*VHH*
0.6	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*HHH*	*VHH*
0.7	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*VHH*
0.8	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*VHH*
0.9	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*VHH*

Degrees of Feasibility: ϕ_{GP}

ω_2

ω_1		0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
	0.1	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*HHH*	*FFF*
	0.2	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*HHH*	*FFF*
	0.3	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*FFF*
	0.4	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*FFF*
	0.5	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*HHH*	*FFF*
	0.6	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*HHH*	*FFF*
	0.7	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*HHH*	*FFF*
	0.8	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*FFF*
	0.9	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*MMM*	*HHH*	*HHH*	*FFF*

the threshold after which they become effectively credible. Thresholds in fuzzy logic are (of course!) fuzzy—they are continuous boundaries between nuanced levels of signaling and cooperation instead of being step-like changes. Even in the cursory analysis that Kydd (understandably) offers from the rise of Gorbachev to the final agreements, we can discern a continuous, fuzzy momentum of negotiations from one round to the next, with every new round producing higher levels of cooperation than previous ones. This fits very well with the above briefly discussed LFL game of reassurance. In fact, taking both sides of the negotiations together, it is very hard to see (except through post hoc reassessment) when was the precise threshold that Kydd argues had occurred. At any point in time, the actors not only kept coming back to the table of negotiations but also kept cooperating even more. More important, they truly believed increasingly more that *higher levels of cooperation* were *more feasible*. One can even venture to say that the Soviet "New Thinking" was precisely about believing and being able to convince the West to believe that consecutively higher levels of cooperation between the East and the West were possible with increasingly higher degrees of feasibility. I hence somewhat agree with Kydd's rendering of the end of the cold war when he says that "one can observe a series of costly signals leading to mutual trust between former adversaries. The attitudes of Western leaders, press, and publics toward the Soviet Union all underwent a substantial transformation" (p. 350). However, his Boolean logic game-theoretic analysis stops short in seeing that there was a continuous progression in the degrees of feasibility and levels of cooperation that underpinned the evolution of the New Thinking and consequent transformation of Western attitudes and strategies from very low levels of feasible cooperation to fully feasible, full cooperation. To sum up this very brief fuzzy-logic rendering of the end of the cold war, Kydd's analysis is not wrong but misses much of the *fuzzy* dynamics that underpinned the progression from (fully feasible) no cooperation to fully feasible, full cooperation. I understand that much more empirical work is needed to really make this case than what the small space allotted for this discussion allows for. Yet, I believe that LFL has much to offer in terms of explaining the fuzzy-logical nature of major historical events of our times.

CONCLUSION

This article develops a new kind of game theory—termed *LFL game theory*—which departs from conventional (crisp) game theory by being based on linguistic fuzzy logic rather than Boolean two-valued logic. Going from a Boolean logic to LFL lifts the PD dilemma and allows the emergence of a strong Nash equilibrium. Moreover, there always exists at least one Pareto-optimum Nash equilibrium. A second major departure of this work is the use of "words to compute" the solutions of the game. The vagueness that inherently exists in real strategic situations is preserved in LFL games. It is not considered an epistemic issue but rather as an ontological issue and is hence incorporated in the very logic underpinning the reasoning process. A third departure has to do with the extent of stylization in analyzing real situations. Although one cannot avoid a certain level of stylization in formalizing strategic interactions between actors, the level of stylization in LFL games is much less than in conventional game theory. This makes LFL game theory much more straightforwardly relevant to analyzing real-world situations of strategic interaction.

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