Implementation of Type Theory based on Dependent Inductive and Coinductive Types

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Inductive Types in Coq

- ▶ defined over their constructors
- ▶ each constructor has to give back this type

Examples

- ▶ Booleans
  ```
  Inductive bool : Set :=
  | True : bool
  | False : bool.
  ```
- ▶ Natural numbers
  ```
  Inductive nat : Set :=
  | O : nat
  | S : nat -> nat.
  ```

Destructing Inductive Types in Coq

- ▶ also called recursion
- ▶ pattern matches on the constructor
- ▶ gives back values of same type in each match

Examples

- ▶ negation
  ```
  Definition neg (b : bool) :=
    match b with
    | True => False
    | False => True
    end.
  ```
- ▶ isZero
  ```
  Definition isZero (n : nat) :=
    match n with
    | O => True
    | S _ => False
    end.
  ```

Coinductive Types

- positive coinductive types
- negative coinductive types

Positive Coinductive Types

- ▶ treats recursive occurrence like a value
- ▶ otherwise like inductive types
- ▶ functions which produce such types have to be productive

Example

- ▶ Stream in coq
  ```
  CoInductive Stream (A : Set) : Set :=
    Cons : A -> Stream A -> Stream A.
  ```

- ▶ repeat function
  ```
  CoFixpoint repeat (A : Set) (x : A) : Stream A :=
    Cons A x (repeat A x).
  ```

What is wrong about Positive Coinductive Types

- Symmetry with inductive types not clear
- Breaks subject reduction
  - Subject reduction: types are preserved after reduction

Ourys Counterexample

```
CoInductive U : Set := In : U -> U.

CoFixpoint u : U := In u.

Definition force (x: U) : U :=
  match x with
    In y => In y
  end.

Compute u.
> cofix Fcofix : U := In Fcofix : U

Compute (force u).
> In (cofix Fcofix : U := In Fcofix) : U

Definition eq (x : U) : x = force x :=
  match x with
    In y => eq_refl
  end.

Definition eq_u : u = In u := eq u
```

Negative Coinductive Types

- defined over their destructors
- functions use copattern matching

Examples

- Stream
  ```
  CoFixpoint repeat (A : Set) (x : A) : Stream A :=
    {| hd := x; tl := repeat A x|}.
  ```

- repeat function
  ```
  CoFixpoint repeat (A : Set) (x : A) : Stream A :=
    {| hd := x; tl := repeat A x|}.
  ```

Type Theory Based on Dependent Inductive and Coinductive Types

- inductive types: $\mu(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A})$
- coinductive types $\nu(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A})$
- constructors: α_i^μ
- destructors: ξ_i^μ
- recursion: $\text{rec } \overrightarrow{(\Gamma_k, y_k).g_k}$
- corecursion: $\text{corec } \overrightarrow{(\Gamma_k, y_k).g_k}$

Symmetry betweeen Inductive and Coinductive Types

$$\text{Product } A\ B = \mu(X : * ; (()); \top)$$

$$\Gamma_1 = (x : A, y : B)$$

```
data Product⟨A,B⟩ : Set where
   MkProduct : (x : A, x : B) ⇸ Unit ⇸ Product
fst⟨A : Set, B : Set⟩ =
  rec Product<A,B> to A where
    { MkProduct x y u = x }
snd⟨A : Set, B : Set⟩ =
  rec Product<A,B> to B where
    { MkProduct x y u = y }
```

$$\text{Product } A\ B = \nu(X : * ; ((), ()); (A, B))$$

$$\Gamma_1 = \Gamma_2 = \emptyset$$

```
codata Product⟨A : Set, B : Set⟩ : Set where
   Fst : Product ⇸ A
   Snd : Product ⇸ B
mkProduct⟨A : Set, B : Set⟩ (x:A, y:B) =
  corec Unit to Product<A,B> where
    { Fst u ⇸ x
    ; Snd u ⇸ y } @ ◊
```

Dependent Coinductive Types

▶ Partial streams which depend on their definition depth

$$\text{PStr } A = \nu(X : (k : Conat) \rightarrow *; ((succ@k), (succ@k)); (A, X@k)$$

$$\Gamma_1 = \Gamma_2 = (k : Conat)$$

```
codata PStr⟨A : Set⟩ : (n : Conat) → Set where
  Hd : (k : Conat) → PStr (succ @ k) → A
  Tl : (k : Conat) → PStr (succ @ k) → PStr @ k
```

▶ Dependent functions

$$\text{Pi } A \ B = \nu(X : *; (()); (B@x)$$

$$\Gamma_1 = (x : A)$$

```
codata Pi⟨A : Set, B : (x : A) → Set⟩ : Set where
  Inst : (x : A) → Pi → B @ x
```

Demo

Other Topics in the Thesis

- Comparison with Agda
- Termination and productivity checking with sized types
- Difference between paper and implementation
  - Rules rewritten to syntax directed one
  - Added type "polymorphism"
  - De-Brujin indexes