

1 **Implementation of Type Theory based on**
2 **dependent Inductive and Coinductive**
3 **Types**

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9 Masterarbeit

10 **Implementation of Type Theory based on**
11 **dependent inductive and coinductive types**

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25 Florian Engel (Matrikelnummer 3860700), November 20, 2020

26 **Abstract**

27 Dependent types are an useful tool to restrict types even further then types of
28 strongly typed languages like Haskell. This gives us further type safety. With them
29 we can also proof theorems. Coinductive types allow us to define types by their
30 observations rather then by their constructors. This is useful for infinite types like
31 streams. In many common dependently typed languages , like coq and agda, we
32 can define inductive types which depend on values and coinductive types but not
33 coinductive types, which depend on values.

34 In this work I will first give a survey of coinductive types in these languages and
35 then implement the type theory from [BG16]. This type theory has both dependent
36 inductive types and dependent coinductive types. In this type theory the dependent
37 function space becomes definable. This leads to a more symmetrical approach of
38 coinduction in dependently typed languages.

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1. Introduction

In functional programming we have functions which get an input and produce an output. These functions don't depend on multiple values i.e. if there is no IO involved, they produce for the same input always the same output. For example, if we call a function `or` on the values `true` and `false` we always get `true`. This makes the code more predictable.

The `or` function should only be working on Booleans. To call it on strings `'foo'` and `'bar'` wouldn't make sense i.e. there is no defined output for these inputs. To prevent calls like these, some functional programming languages introduced types. Types contain only certain values. For example the type for truth values contains only the values for true and false. In Haskell we can define it like the following:

```
data Bool = True | False
```

This says we can construct values of type `Bool` with the constructors `True` and `False`. These types which have constructors are called inductive types. We can then define `or` like this.

```
or :: Bool -> Bool -> Bool
or True _    = True
or _    True = True
or _    _    = False
```

Here, we just list equations which say what the output for a given input is. For example, in the first equation, we say if the first value is constructed with the constructor `True`, we give back `True`. We don't care about the second value, therefore we write `_`. We are matching on the construction of the input values. Therefore we call this method pattern matching. If we call this function somewhere in the code on values which aren't of type `Bool`, Haskell won't compile our code. Instead it gives back a type error.

If we now want to change `Bool` to a three-valued logic, we have to add a third constructor to `Bool`. After that we have to change every function which pattern matches on `Bool`. If there are a lot of those kinds of functions, this would be a lot of repetitive work. If Haskell would have coinductive type, this could be a lot less work. Coinductive types are types which are, in contrary to inductive types defined over their destruction. So we could define `Bool` over its destructors. These would be `or`, `and`, etc.

Through this work we will explain coinductive types at the examples of streams and functions. They will be generalized to partial streams and the `Pi` type in dependently

typed languages. Streams are lists which are infinitely long. They are useful for modeling many IO interactions. For example, a chat of a text messenger might be infinitely long. We can never know if the chat is finished. This is of course limited by the hardware, but we are interested in abstract models. Functions are used everywhere in functional programming. In most of these languages they are first-class objects. But in languages with coinductive types we can define them. If we only have inductive and coinductive types, we get a symmetrical language. This is useful, because then we can change an inductive type to a coinductive one and vice versa. It is straight forward to add functions which destruct an inductive type by pattern matching on the constructor. But it is hard to add a new constructor. We then have to add this constructor to every pattern matching on that type. For coinductive types its the other way around. For more on this see [BJSO19]. In the implemented syntax we can define streams like the following:

```

119 codata Stream(A : Set) : Set where Hd : Stream (succ @ k)
120   Hd : Stream → A
121   Tl : Stream → Stream

```

And functions like follows:

```

123 codata Fun(A : Set, B : Set) : Set where
124   Inst : (x : A) → Fun → B

```

We can generalize streams to partial streams as the following:

```

126 codata PStr(A : Set) : (n : Conat) → Set where
127   Hd : (k : Conat) → PStr (succ @ k) → A
128   Tl : (k : Conat) → PStr (succ @ k) → PStr @ k

```

And functions to the Pi type.

```

130 codata Pi(A : Set, B : (x : A) → Set) : Set where
131   Inst : (x : A) → Pi → B @ x

```

The rest of this thesis will be structured as follows:

- In chapter 2 we will see how coinductive types can be defined. Here we will be defining the stream and function type. We will also define some functions on the stream.
- We will see in chapter 3 how coinductive types are defined in the dependently typed languages coq and agda. We will see that we can define them positive or negative. We will show why defining them positive leads to problems.
- In chapter 4 we see how they are defined in [BG16]. With this theory we can than define coinductive types which depend on values. But we can not define types which depend on types.
- We will then in chapter 5 explain how this theory is implemented. Therefore we need to rewrite the typing rules. It will also be possible to define type schemata.

- 145 • At last we look at the examples from this paper in the implemented syntax.
146 Here we will see the reduction steps for recursion and corecursion. We will con-
147 clude this section with the example of partial streams, which is a coinductive
148 type which depends on a value.

149 2. Coinductive Types

150 Inductive types are defined via their constructors. Coinductive types on the other
 151 hand are defined via their destructors. In the paper [APTS13] functions, which have
 152 coinductive types as their output, are implemented via copattern matching. In this
 153 paper streams are defined like the following:

```
154 record Stream A = { head : A,  
155                    tail : Stream A }
```

156 The **A** in the definition should be a concrete type. The type system in the paper
 157 don't has dependent types. What differentiate this from regular record types (for
 158 example in Haskell), is the recursive field **tail**. So they call it a recursive record. In
 159 a strict language without coinductive types we could never instantiate such a type,
 160 because to do this we already need something of type **Stream A** to fill in the field **tail**.
 161 To remedy this the paper defines copattern matching. With the help of copattern
 162 matching we can define functions which outputs expressions of type **Stream A**. As an
 163 example we look at the definition of **repeat**. This function takes in a value of type
 164 **Nat** and generates a stream which just infinitely repeats it.

```
165 repeat : Nat → Stream Nat  
166 head (repeat x) = x  
167 tail (repeat x) = repeat x
```

168 As you can see copattern matching works via observations i.e. we define what should
 169 be the output of the fields applied to the result of the function. These fields are
 170 also called observers, because we observe parts of the type. Because inhabitants
 171 of **Stream** are infinitely long we can't print out a stream. Because of this we also
 172 consider each expression with has a type, which is coinductive, as a value. To get
 173 a subpart of this value we have to use observers. For example we can look at the
 174 third value of **repeat 2** via **head (tail (tail (repeat 2)))** which should evaluate to
 175 2. We can also implement a function which looks at the *n*th. value. Here it is:

```
176 nth : Nat → Stream A → A  
177 nth 0 x = head x  
178 nth (S n) x = nth n (tail x)
```

179 As you can see we use ordinary pattern matching on the left hand side and observers
 180 on the right hand side. **nth 3 (repeat 2)** will output 2 as expected. Functions can also
 181 be defined via a recursive record. It is defined like the following:

```
182 record A → B = { apply : A → B }
```

183 Here we differentiate between our defined function $\mathbf{A} \rightarrow \mathbf{B}$ and \leadsto in the destructor.
 184 Constructor application or, as is the case here, destructor application is not the same
 185 as function application, like in Haskell. In the paper $\mathbf{f} \mathbf{x}$ means `apply f x`. We will
 186 also use this convention in the following. In fact we already used it in the definitions
 187 of the functions `repeat` and `nth`. `nth 0 x = head x` is just a nested copattern. We
 188 can also write it with ‘`apply`’ like so: `apply (apply nth 0) x = head x`. Here we use
 189 currying. So the first `apply` is the sole observer of type $\mathbf{Stream} \mathbf{A} \rightarrow \mathbf{A}$ and the second
 190 of type $\mathbf{Nat} \rightarrow (\mathbf{Stream} \mathbf{A} \rightarrow \mathbf{A})$.

3. Coinductive Types in dependent languages

In this section we will look how coinductive types are implemented in dependently typed languages. In dependently typed languages types can depend on values. The classical example for such a type is the vector. Vectors are like list, except their length is contained in their type. For example a vector of natural numbers of length 2 has type `Vec Nat 2`. This type depends on two things. Namely the type `Nat` and the value 2, which is itself of type `Nat`. We can define vectors in coq like follows:

```
Inductive Vec (A : Set) : nat -> Set :=
| Nil : Vec A 0
| Cons : forall {k : nat}, A -> Vec A k -> Vec A (S k).
```

Contrary to a list the type constructor `Vec` has a second argument `nat`. This is the already mentioned length of the vector. A Vector has two constructors. One for the empty vector called `Nil` and one to append a element at the front of a vector called `Cons`. `Nil` just returns a vector of length 0. And `Cons` gets an `A` and a vector of length `k`. It returns a vector of length `S k` (`S` is just the successor of `k`). This type can also be defined in agda like follows:

```
data Vec (A : Set) : ℕ → Set where
  Nil : Vec A 0
  Cons : {k : ℕ} → A → Vec A k → Vec A (suc k)
```

One advantage of vectors over list is that we can define a total function (a function which is defined for every input) which takes the head of a vector. This function can't be total for lists, because we can't know if the input list is empty. an empty list has no head. For vectors we can enforce this in coq like follow.

```
Definition hd {A : Set} {k : nat} (v : Vec A (S k)) : A :=
  match v with
  | Cons _ x _ => x
end.
```

We just pattern match on `v`. The only patten is for the `Cons` constructor. The `Nil` constructor is a vector of length 0. But `v` has type `Vec A (S k)`. So it can't be a vector of length 0. In agda the function looks like follow.

```
hd : {A : Set} {k : ℕ} → Vec A (suc k) → A
hd (cons x _) = x
```

That terms can occur in types makes it necessary to ensure that function terminate. Otherwise type checking wouldn't be decidable. If we have a function `f : Nat → Nat` and we want to check a value `a` against a type `Vec (f 1)` we have to

215 know what `f 1` evaluates to. So `f` has to terminate. We check termination in `coq`
 216 via a structural decreasing argument. An argument is structural decreasing, if it
 217 is structural smaller in a recursive call. Structural smaller means it is a recursive
 218 occurrence in a constructor. As an example we look at the definition of the natural
 219 numbers and the function for addition on them. We define the natural numbers in
 220 `coq` like follows:

```
Inductive nat : Set :=
| 0 : nat
| S : nat -> nat.
```

221 `0` is the constructor for 0 and `S` is the successor of its argument. Here the recursive
 222 argument to `S` is structural smaller than `S` applied to it i.e. `n` is structural smaller
 223 than `S n`. Then we can define addition like follows:

```
Fixpoint add (n m : nat) : nat :=
match n with
| 0 => m
| S p => S (add p m)
end.
```

224 In the recursive call the first argument is structural decreasing. `p` is smaller than
 225 `S p`. So `coq` accepts this definition. The classical example for a function where an
 226 argument is decreasing, but not structural decreasing is Quicksort. A naive imple-
 227 mentation would be the following:

```
Fixpoint quicksort (l : list nat) : list nat :=
match l with
| nil => nil
| cons x xs => match split x xs with
| (lower, upper) => app (quicksort lower) (cons x (quicksort upper))
end
end.
```

228 Here `split` is just a function which gets a number and a list of numbers. It gives back
 229 a pair of two lists where the left list are all elements of the input list which are smaller
 230 than the input number and the right these which are bigger. It is clear that these
 231 lists can't be longer than the input list. So `lower` and `upper` can't be longer than
 232 `xs`. Here `xs` is structural smaller than the input `cons x xs`. So `lower` and `upper` are
 233 smaller than the input. Therefore we know that `quicksort` is terminating. But `coq`
 234 won't accept our code, because no argument is structural decreasing.

235 For coinductive types termination means that functions which produce them should
 236 be productive. If a function is productive it produces in each step a new part of the
 237 infinitely large coinductive type.

238 In section 3.1 we will look at the implementation in `coq`. There are two ways to
 239 define them. The older way uses positive coinductive types. This is known to violate
 240 subject reduction. Therefore it is highly discouraged to use them. To fix this the new
 241 way uses negative coinductive types. In section 3.2 we look at the implementation in
 242 `agda`. `Agda` also has the two ways of defining such types. One special thing about it,
 243 is that it implements copattern matching. To help `agda` with termination checking

we can use sized types. We will explain them in section 3.2.3.

3.1. Coinductive Types in Coq

There are two approaches to define coinductive types in coq. The older one is described in 3.1.1. It works over constructors. Therefore they are called positive coinductive types. The newer and recommended one is described in section 3.1.2. They are defined over primitive records (a relatively new feature of coq). Therefore they are called negative coinductive Types.

3.1.1. Positive Coinductive Types

Positive coinductive types are defined over constructors in coq. The keyword **CoInductive** is used to indicate that we about to define a coinductive type. This is the only syntactical difference from the definition of inductive types. For example streams are defined like the following:

```
CoInductive Stream (A : Set) : Set :=
  Cons : A -> Stream A -> Stream A.
```

If this was an inductive type we couldn't generate a value of this type. To generate values of coinductive types coq uses guarded recursion. This checks if the recursive call to the function occurs as an argument to a coinductive constructor. In addition to the guard condition the constructor can only be nested in other constructors, fun or match expressions. With all of this in mind we can define **repeat** like the following:

```
CoFixpoint repeat (A : Set) (x : A) : Stream A := Cons A x (repeat A x).
```

Then we can produce the constant zero stream with **repeat nat 0**. If we used a normal coq function i.e. write **Fixpoint** instead of **CoFixpoint** coq wouldn't except our code. It rejects it, because there is no argument which is structural decreasing. **x** stays always the same. **CoFixpoint** on the other hand only checks the previously mentioned conditions. It sees the recursive call **repeat A x** occurs as an argument to constructor **Cons** of the coinductive type **Stream**. This constructor is also not nested. So our definition is accepted.

We can use the normal pattern matching of coq to destruct a coinductive type. We define **nth** like the following:

```
Fixpoint nth (A : Set) (n : nat) (s : Stream A) {struct n} : A :=
  match s with
  | Cons _ a s' =>
    match n with 0 => a | S p => nth A p s' end
  end.
```

The guard condition is necessary to ensure every expression is terminating. If we didn't have the guard condition we could define the following:

CoFixpoint loop (A : Set) : Stream A = loop A.

273 Here the recursive call doesn't occur in a constructor. So the guard condition is vio-
 274 lated. With this definition the expression **nth 0 loop** wouldn't terminate. **nth** would
 275 try to pattern match on **loop**. But to succeed in that **loop** has to unfold to some-
 276 thing of the form **Cons a ?** which it never does. So **nth 0 loop** will never evaluate to
 277 a value. This would lead to undecidable type checking.

278 We illustrate the purpose of the other conditions on an example taken from [Ch13].
 279 First we implement the function **tl** like so.

Definition tl A (s : Stream A) : Stream A :=
 match s with
 | Cons _ s' => s'
 end.

280 This is just one normal pattern match on **Stream**. If we didn't had the other condition
 281 we could define the following:

CoFixpoint bad : Stream nat := tl nat (Cons nat 0 bad).

282 This doesn't violate the guard condition. The recursive call **bad** is a argument to
 283 the constructor **Cons**. But the constructor is nested in a function. If we would allow
 284 this, **nth 0 bad** would loop forever. To understand why, we first unfold **tl** in **bad**. So
 285 we get

nth 0 (cofix bad : Stream nat :=
 match (Cons 0 bad) with
 | Cons _ s' => s'
 end)

286 We can now simplify this to just

nth 0 (cofix bad : Stream nat := bad)

287 After that **bad** isn't anymore an argument to a constructor. Here we can also see
 288 easily that the expression **cofix bad : Stream nat := bad** loops for ever. So we never
 289 get the value at position 0.

290 An important property of typed languages is subject reduction. Subject reduction
 291 says if we evaluate an expression e_1 of type t to an expression e_2 , e_2 should also be
 292 of type t . With positive coinductive types subject reduction is no longer valid. We
 293 illustrate this by Oury's counterexample [Our08]. First we define the codata type
 294 **U** as follows

CoInductive U : Set := In : U -> U.

295 We can now define a value of **u** with the following **Cofixpoint** like so

CoFixpoint u : U := In u.

296 This generates an infinite succession of **In**. We use the function **force** to force **U** to
 297 evaluate one step i.e. **x** becomes **In y**.

Definition force (x: U) : U :=
 match x with

```

    In y => In y
end.

```

298 The same trick will be used to define **eq** which states that **x** is definitional equal to
 299 **force x**.

```

Definition eq (x : U) : x = force x :=
  match x with
  | In y => eq_refl
  end.

```

300 This first matches on **x** to force it, to reduce to **In y**. Then the new goal becomes
 301 **In y = force (In y)**. **force (In y)** evaluates to just **In y**, as it is just pattern match-
 302 ing on **In y**. So the final goal is **In y = In y** which can be shown by **eq_refl**.
 303 **eq_refl** is a constructor for **=**, where both sides of **=** are exactly the same. If we
 304 now instantiate **eq** with **u** we become **eq u**.

```

Definition eq_u : u = In u := eq u

```

305 But **u** is not definitional equal to **In u**. As mentioned above expression with a coin-
 306 ductive type are always values to prevent infinite evaluation. So **In u** is a value and
 307 **u** is also a value. But values are only definitional equal, if they are exactly the same.
 308 The next section will solve this problem through negative coinductive types.

309 3.1.2. Negative Coinductive Types

310 In coq 8.5. primitive records were introduced. With this it is now possible to define
 311 types over there destructors. So we can have negative , especially negative coinduc-
 312 tive, types in coq. With primitive records we can define streams like the following:

```

CoInductive Stream (A : Set) : Set :=
  Seq { hd : A; tl : Stream A }.

```

313 Now we can define **repeat** over the fields of **Stream**.

```

CoFixpoint repeat (A : Set) (x : A) : Stream A :=
  {| hd := x; tl := repeat A x|}.

```

314 To define **repeat** we must define what is the head of the constructed stream and
 315 what it is tail. The guard condition says now that corecursive occurrences must be
 316 guarded by a record field. We can see that the corecursive call **repeat** is a direct
 317 argument to the field **tl** of the corecursive type **Stream A**. This means coq accepts
 318 the above definition. If we want to access parts of a stream we use the destructors
 319 **hd** and **tl**. With them we can define **nth** again for the negative stream.

```

Fixpoint nth (A : Set) (n : nat) (s : Stream A) : list A :=
  match n with
  | 0 => s.(hd A)
  | S n' => nth A n' s.(tl A)
  end.

```

320 With negative coinductive types we can't form the above mentioned counterexample
 321 to subject reduction anymore, because we can't pattern match on negative types.

322 Oury's example becomes.

```
CoInductive U := { out : U }.
```

323 U is now defined over its destructor **out**, instead of its constructor **in**. Then **in**
324 becomes just a function. In Fact its just a definition, because we don't recurse or
325 corecure on it.

```
Definition In (y : U) : U := { | out := y | }.
```

326 We define it over the only field **out**. When we put a **y** in then we get the same **y** out.
327 We can also again define **u**.

```
CoFixpoint u : U := { | out := u | }.
```

328 With coinductive types it is know possible to define the pi type (the depend function
329 type).

```
CoInductive Pi (A : Set) (B : A -> Set) := { Apply (x : A) : B x }.
```

330 The pi type is defined over its destructor **Apply**. If we evaluate **Apply** on a value of
331 **Pi** (which is a function) and an argument, we get the result i.e. we apply the value to
332 the function. It looks like the pi type becomes definable in coq. But we are cheating.
333 The type of **Apply** is already a pi type. This is because we identify constructors
334 and destructors with functions. We will see that the theory of the paper avoids this
335 identification. To define a function we use **CoFixpoint**. As a simple non recursive,
336 non dependent example we use the function **plus2**.

```
CoFixpoint plus2 : Pi nat (fun _ => nat) :=  
  { | Apply x := S (S x) | }.
```

337 If we apply (i.e. call the destructor **Apply**) a **x** to **plus2** it gives back **S (S x)**. Which
338 is twice the successor on **x**. So we add 2 to **x**. We use **_** here because **plus2** is not
339 a dependent function i.e. the result type **nat** doesn't depend on the input value. To
340 define functions with more than one argument we just use currying i.e. we use the
341 type **Pi** as the second argument to **Pi**. For example a 2-ary non-dependent function
342 from **A** and **B** to **C** would have type **Pi A (fun _ => Pi B (fun _ => C))**. It would be
343 fortunate if we could define **plus** like the following:

```
CoFixpoint plus : Pi nat (fun _ => Pi nat (fun _ => nat)) :=  
  { | Apply := fun (n : nat) =>  
    match n with  
    | 0 => { | Apply (m : nat) := m | }  
    | S n' => { | Apply m := S (Apply _ _ (Apply _ _ plus n') m) | }  
    end  
  | }.
```

344 But coq doesn't accept this definition. The guard condition is violated. **plus n'** is
345 not a direct argument of the field **Apply**. The definition should terminate because
346 we are decreasing **n** and the case for **0** is accepted. In the case for **0**, there is no
347 recursive call.

348 We can also define a dependent function. We define **append2Units** like follows

```

CoFixpoint append2Units : Pi nat
  (fun n => Pi (Vec unit n)
    (fun _ => Vec unit (S (S n)))) :=
  {| Apply n := {| Apply v := Cons _ tt (Cons _ tt v) |} |}.

```

349 This just appends 2 units at a vector of length n . Here the second argument and the
 350 result depend on the first argument i.e. the first argument is the length of the input
 351 vector and the output vector is this length plus two.

352 3.2. Coinductive Types in Agda

353 In agda coinductive types were first also introduced as positive types. In the section
 354 3.2.1 we will look at them in detail. In section 3.2.2 we describe the correct way to
 355 implement coinductive types in agda. There are functions which terminate but are
 356 rejected by the type checker. In fact in any total language there have to be such
 357 functions. We can show that by trying to list all total functions. The following table
 358 lists functions per row. The columns say what the output of the functions for the
 359 given input is.

	1	2	3	4	...
f_1	2	7	8	6	...
f_2	4	4	6	19	...
360 f_3	6	257	1	2	...
f_4	7	121	23188	2313	...
\vdots	\vdots	\vdots	\vdots	\vdots	\ddots

361 We can now define a function $g(n) = f_n(n) + 1$ this function is total and not in the list,
 362 because it is different to any function in the list for at least one input. To allow more
 363 functions we can use an unique feature of agda, sized types. They are described in
 364 section 3.2.3.

365 3.2.1. Positive Coinductive Types in Agda

366 Agda doesn't have a special keyword to define coinductive types like `coq`. It uses
 367 the symbol ∞ to mark arguments to constructors as coinductive. This symbol says
 368 that the computation of arguments of this type are suspended. ∞ is just a type
 369 constructor. So agda ensures productivity over type checking. We define streams
 370 like so.

```

data Stream (A : Set) : Set where
  cons : A → ∞ (Stream A) → Stream A

```

371 Here the second argument to `cons` is marked with ∞ . This is the tail of the stream.
 372 Because it is infinitely long (we don't have a constructor of an empty stream) we can't

373 compute it completely, so we suspend the computation. We can delay a computation
 374 with the constructor \sharp and force it with the function \flat . Their types are given below.

```
 $\sharp\_ : \forall \{a\} \{A : \mathbf{Set} \ a\} \rightarrow A \rightarrow \infty \ A$   

 $\flat : \forall \{a\} \{A : \mathbf{Set} \ a\} \rightarrow \infty \ A \rightarrow A$ 
```

375 We can now again define our usual functions. We begin with **repeat**.

```
repeat : {A : Set}  $\rightarrow A \rightarrow \mathbf{Stream} \ A$   

repeat x = cons x ( $\sharp$  (repeat x))
```

376 We first apply **cons** to **x**. So the head of the stream is **x**. We then apply it to the
 377 corecursive call **repeat**. So the tail will be a repetition of **xs**. We have to call the
 378 **repeat** with \sharp to suspend the computation. Otherwise the code doesn't type check.
 379 If we would write this function without \sharp on a stream which has no ∞ on the second
 380 argument of **cons**, the function would run forever. In fact the termination checker
 381 won't allow us to write such a function. We can also write **nth** again, which consumes
 382 a stream.

```
nth : {A : Set}  $\rightarrow \mathbb{N} \rightarrow \mathbf{Stream} \ A \rightarrow A$   

nth 0 (cons x _) = x  

nth (suc n) (cons _ xs) = nth n ( $\flat$  xs)
```

383 Here we have to use \flat on the right hand side of the second case, to force the computa-
 384 tion of the tail of the input stream. We have to do that because **nth** wants a stream.
 385 It doesn't want a suspended stream. Productivity on coinductive types like stream
 386 is checked by only allowing non decreasing recursive calls behind the \sharp constructor.

387 3.2.2. Negative Coinductive Types in Agda

388 In agda we can also define negative coinductive types. This is the recommended
 389 way. Agda implements the previously mentioned copattern matching. We can define
 390 a record with the keyword **record**. We use the keyword **coinductive** to make it
 391 possible to define recursive fields. Stream is defined like the following:

```
record Stream (A : Set) : Set where  

  coinductive  

  field  

  hd : A  

  tl : Stream A
```

392 A Stream has 2 fields. **hd** is the head of the stream. It has type **A**. **tl** is the tail
 393 of the stream. It is another stream, so it has type **Stream A**. **tl** is a recursive field.
 394 So agda wouldn't accept the definition without **coinductive**. Stream can never be
 395 empty. Every stream has a head (a field **hd**) and an empty stream wouldn't have a
 396 head. So the tail of a stream can never be empty. Therefore every stream is infinitely
 397 long. We can now define **repeat** with copattern matching.

```
repeat :  $\forall \{A : \mathbf{Set}\} \rightarrow A \rightarrow \mathbf{Stream} \ A$   

hd (repeat x) = x  

tl (repeat x) = repeat x
```


We have to copattern match on every field of **Stream**, namely **hd** and **tl**. Because agda is total it won't accept non-exhaustive (co)pattern matches like Haskell. First we define what the head of **repeat x** is. We just repeat **x** infinitely often. So every element of the stream is **x**, including the head. Therefore we just write **x**. In the second and last copattern we define what the tail of the stream is. The tail is just **repeat x**. Infinitely often repeated **x** is the same as **x** and then infinitely repeated **x**. We can use normal pattern matching and the destructors for functions which consume streams. We define **nth** like the following:

```
nth : ∀ {A : Set} → ℕ → Stream A → A
nth zero s = hd s
nth (suc n) s = nth n (tl s)
```

Here we just pattern match on the first argument (excluding the implicit argument of the type). If it is zero the result is just the head of the stream. If it is $n + 1$ the result is the recursive call of **nth** on **n** and **tl s**. Agda accepts this code, because it is structural decreasing on the first (or second if we count the implicit) argument.

We can also define the pi type. We use **__\$** as the apply operator. This operator is taken from Haskell.

```
record Pi (A : Set) (B : A → Set) : Set where
  field _$_ : (x : A) → B x
  infixl 20 _$_
  open Pi
```

Like in coq we are using the first-class pi type to define the pi type. We can also define a function which adds 2 to a number **plus2** in agda.

```
plus2 : ℕ → 'ℕ
plus2 $ x = suc (suc x)
```

We just use copattern matching to define it. If we apply a **x** to **plus2** we get **suc (suc x)**. \rightarrow' is just the non-dependent function it is defined using our pi type. Here it is:

```
→' : Set → Set → Set
A →' B = Pi A (λ _ → B)
infixr 20 →'_
```

In agda it becomes possible to define plus. We just use nested copattern matching.

```
plus : ℕ → 'ℕ →' ℕ
plus $ 0 $ m = m
plus $ (suc n) $ m = suc (plus $ n $ m)
```

If we change \rightarrow' to \rightarrow and remove **\$** we get the standard definition for plus in agda.

We can also define a dependent function **repeatUnit** like follow

```
repeatUnit : Pi ℕ (λ n → Vec τ n)
repeatUnit $ 0 = nil
repeatUnit $ suc n = tt :: (repeatUnit $ n)
```

This function gives back a vector with the length of the input, where every element is unit.

422 3.2.3. Termination Checking with Sized Types

423 They are many functions, which are total but are not accepted by Agda's termination
 424 checker. For example we could try to define division with rest on natural numbers
 425 like the following:

```

_/_ : ℕ → ℕ → ℕ
zero / y = zero
suc x / y = suc ( (x - y) / y)

```

426 The problem with this definition is that agda doesn't know that $x - y$ is smaller
 427 than $x + 1$, which is clearly the case (x and y are positive). This definition would
 428 work perfectly fine in a language without termination checking (like Haskell). Agda
 429 only checks if an argument is structurally decreasing. Here it is neither the case for
 430 x nor for y .

431 To remedy this problem sized types were introduced first to mini-agda (a language
 432 specifically developed to explore them) by [Abe10]. Later they got introduced to
 433 agda itself. Sized types allow us to annotate data with their size. Functions can use
 434 this sizes to check termination and productivity.

435 We can now define the natural numbers depending on a size argument.

```

data ℕ (i : Size) : Set where
  zero : ℕ i
  suc : ∀ {j : Size < i} → ℕ j → ℕ i

```

436 The natural number now depends on a size i . The constructor **zero** is of arbitrary
 437 size i . **suc** gets a size j which is smaller than i , a natural number of size j and gives
 438 back a natural number of size i . This means the size of the input is smaller than the
 439 size of the output. For inductive types, a size is an upper bound on the number of
 440 constructors. With **suc** we add a constructor so the size has to increase i . We can
 441 now define subtraction on these sized natural numbers.

```

_/_ : {i : Size} → ℕ i → ℕ ∞ → ℕ i
zero - _ = zero
n - zero = n
(suc n) - (suc m) = n - m

```

442 Through the sized annotations, we know now that the result isn't larger than the
 443 first input. ∞ means that the size isn't bound. If the first argument is zero the result
 444 is also zero, which has the same type. If the second argument is zero we return
 445 just the first. In the last case both arguments are non-zero. We call subtraction
 446 recursively on the predecessors of the inputs. Here the size and both arguments are
 447 smaller. So the function terminates. Though the type is smaller than i , the result type
 448 checks because sizes are upper bounds. We can now define division.

```

_/_ : {i : Size} → ℕ i → ℕ ∞ → ℕ i
zero / _ = zero
suc x / y = suc ( (x - y) / y)

```

449 From the definition of **suc** we know that the size of x is smaller than i . Because the

450 result of $-$ has the same size as its first input (here x), we also know that $(x - y)$ has
 451 the same size as x . Therefore $(x - y)$ is smaller than $\text{succ } x$ and the function is decreasing
 452 on the first argument. Also, agda accepts this definition.

453 We can also use sized types for coinductive types. To show this we will define the
 454 hamming function. This produces a stream of all composites of two and three in
 455 order. First we will define the sized stream type.

```
record Stream (i : Size) (A : Set) : Set where
  coinductive
  field
    hd : A
    tl :  $\forall \{j : \text{Size} < i\} \rightarrow \text{Stream } j \text{ } A$ 
  open Stream
```

456 This stream has a new parameter of type **Size**. This size gives the minimal definition
 457 depth of the stream. The definition depth says how often we can destruct the stream
 458 without diverging. If we take the tail of an stream, the output streams depth would
 459 be one smaller. Because in agda coinductive types can't have indexes, we can only say
 460 that its depth is smaller. We will now define some helper functions for the hamming
 461 function. First we need a cons function.

```
cons : {i : Size} {A : Set}  $\rightarrow A \rightarrow \text{Stream } i \text{ } A \rightarrow \text{Stream } i \text{ } A$ 
hd (cons x _) = x
tl (cons _ xs) = xs
```

462 This just appends an element at the front of the stream. Because the output streams
 463 depth is larger than the input and the size is a minimum, we can give the output
 464 the same size parameter as the input. Now we will define map over stream.

```
map : {A B : Set} {i : Size}  $\rightarrow (A \rightarrow B) \rightarrow \text{Stream } i \text{ } A \rightarrow \text{Stream } i \text{ } B$ 
hd (map f xs) = f (hd xs)
tl (map f xs) = map f (tl xs)
```

465 This function just changes the content of the stream so the size stays the same. The
 466 last helper function we need is the merge function.

```
merge : {i : Size}  $\rightarrow \text{Stream } i \text{ } \mathbb{N} \rightarrow \text{Stream } i \text{ } \mathbb{N} \rightarrow \text{Stream } i \text{ } \mathbb{N}$ 
hd (merge xs ys) = hd xs  $\sqcap$  hd ys
tl (merge xs ys) = if [ hd xs  $\leq?$  hd ys ]
  then cons (hd ys) (merge (tl xs) (tl ys))
  else cons (hd xs) (merge (tl xs) (tl ys))
```

467 This functions just merges two streams. It always compares one element of each
 468 stream with each other and puts the bigger after the smaller. This is clear in the
 469 case for **hd** (\sqcap is just the binary minimum function in agda). in the **tl** case we just
 470 compare the heads of the stream and construct the tail with **cons** accordingly. Both
 471 input streams have a minimal definition depth of i . Because **cons** isn't destructing
 472 the stream (the minimal depth doesn't get smaller) we can say that the minimal
 473 depth of the output also won't get smaller. With all this function we can now define
 474 the ham function. Here it is:

```
ham : {i : Size}  $\rightarrow \text{Stream } i \text{ } \mathbb{N}$ 
hd ham = 1
tl ham = (merge (map ( $\_$ * 2) ham) (map ( $\_$ * 3) ham))
```

Chapter 3. Coinductive Types in dependent languages

475 None of the used function is destructing the stream, so this definition gets accepted.

4. Type Theory based on dependent Inductive and Coinductive Types

In the paper [BG16] a type theory, where inductive types and coinductive types can depend on values, is developed. For example we can, in contrast to the coinductive types of coq and agda, define streams which depend on their definition length. The theory differentiates types from terms. We don't have infinite universes, where a term in universe n has a type in universe $n+1$ (This is how it is done in coq [ST14] and agda [agd]). Therefore types can only depend on values, not on other types. We only have functions on the type level. These functions abstract over terms. For example $\lambda x.A$ is a type where all occurrences of the term variable x in A are bound. We will see that functions are definable on the term level. We can apply types to terms. For example $A@t$ means we apply the term A to x . Every type has a kind. A kind is either $*$ or $\Gamma \rightarrow *$. Here Γ is a context, which states to what terms we can apply the type. For example we can apply A of kind $(x : B) \rightarrow *$ only to a term of type B . If we apply it to t of type B , we get a type of kind $*$. We write \rightarrow instead of \rightarrow to indicate, that these are not functions. We can also apply a term to another term. For example $t@s$ means we apply the term t to the term s . Terms also can depend on contexts. For example if we have a term t of type $(x : A) \rightarrow B$ and apply it to a term s of type A we get a term of type B . We can also define our own types. $\mu(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A})$ is an inductive type and $\nu(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A})$ is a coinductive type. X is a variable which stands for the recursive occurrence of the type. It has the same kind $\Gamma \rightarrow *$ as the defined type. The \vec{A} can contain this variable. There are also contexts $\vec{\Gamma}$, which are implicit in the paper. σ_k and A_k can contain variables from Γ_k . σ_k is a context morphism from Γ_k to Γ . A context morphism is a sequence of terms, which depend on Γ_k and instantiate Γ . $\vec{\sigma}$, \vec{A} and $\vec{\Gamma}$ are of the same length.

In this theory we can define partial streams on some type A like the following:

$$\begin{aligned} \text{PStr } A &:= \nu(X : (n : \text{Conat}) \rightarrow *; (\text{succ}@n, \text{succ}@n); (A, X@n)) \\ &\text{with } \Gamma_1 = (n : \text{Conat}) \text{ and } \Gamma_2 = (n : \text{Conat}) \end{aligned}$$

Here **succ** is the successor on co-natural numbers. Co-natural numbers are natural numbers with one additional element, infinity. See 6.2 for their definition. Here the first destructor is the head. It becomes a stream with length $\text{succ}@N$ and returns an A . The second destructor is the tail. It becomes also a stream of length $\text{succ}@N$. It gives back an $X@n$, which is a stream of length n . We can also define the Pi type

from A to B , where B can depend on A .

$$\begin{aligned} \Pi x : A. B &:= \nu(_ : *; \epsilon_1; B) \\ \text{with } \Gamma_1 &= (x : A) \end{aligned}$$

501 By $_$ we mean, we are ignoring this variable. ϵ_1 is one empty context morphism.
 502 So the only destructor gives back a B which can depend on x of type A . It is the
 503 function application.

504 To construct an inductive types we use constructors (written $\alpha_k^{\mu(X:\Gamma \rightarrow *; \vec{\sigma}; \vec{A})}$ in the
 505 paper, which is the k -th constructor of the given type). We can destruct it with
 506 recursion (written $\mathbf{rec} \overrightarrow{(\Gamma_k.y_k).g_k}$). Coinductive type work the other way around. We
 507 destruct them with destructors (written $\xi_k^{\nu(X:\Gamma \rightarrow *; \vec{\sigma}; \vec{A})}$) and construct them with core-
 508 cursion (written $\mathbf{corec} \overrightarrow{(\Gamma_k.y_k).g_k}$).

509 We will give the rules for the theory in section 5.3 and a detailed explanation of the
 510 reduction in 5.4.

511 5. Implementation

512 In this section we look at the implementation details. We use the functional pro-
513 gramming language Haskell for implementing the theory. Haskell is a pure language.
514 This means functions which aren't in the IO monad have no side effects. The only
515 IO we are doing is reading a file and as the last step printing it. Because everything
516 between is pure, we can test it without bordering on side effects. Another feature
517 of Haskell, which will be get useful in our implementation is pattern matching. We
518 will see its usefulness in section 5.3.

519 In section 5.1 we will develop the abstract syntax of our language from the raw
520 syntax in the paper. Then we rewrite the typing rules in 5.3. At last we look at the
521 implementation of the reduction in 5.4

522 5.1. Abstract Syntax

523 In the following we will scratch out the abstract syntax. In contrast to [BG16] we
524 can't write anonymous inductive and coinductive types. We will give every inductive
525 and coinductive type a name. They will be defined via declarations. In these declara-
526 tions we will give, their constructors/destructors. They will also be given names. In
527 [BG16] they are anonymous. We can then refer to the previously defined types. We
528 will described declarations in section 5.1.1. We will also be able to bind expressions
529 to names. In section 5.1.2 we will define the syntax of expressions. This will mostly
530 be in one to one correspondence with the syntax of [BG16]. Note however that we
531 use the names of the constructors instead of anonymous constructors together with
532 their type and number. Also the order of the matches in **rec** and **corec** is irrelevant.
533 We use the names of the Con/Destructors to identify them. In the following section
534 6 we will see how the examples from the paper look in our concrete syntax.

535 5.1.1. Declarations

536 The abstract syntax is given in figure 5.1. With the keywords **data** and **codata**
537 we define inductive and coinductive types respectively. After that we will write
538 the name. We can only use names which aren't used already. Behind that we can
539 give a parameter context. This is a type context. These types are not polymorphic.
540 They are merely macros to make the code more readable and allow the definition of

N	$:= [A - Z][a - zA - Z0 - 9]^*$	Names for types, constructors and destructors
n	$:= [a - z][a - zA - Z0 - 9]^*$	Names for expressions
EV	$:= x, y, z, \dots$	Expression variables
TV	$:= X, Y, Z, \dots$	Type expression variables
PV	$:= A, B, C, \dots$	Parameter variables
EC	$:= \diamond$ $ (EV : TV, EV : TV)^*$	Expression Context
PC	$:= \langle \rangle$ $ \langle (PV : EC \rightarrow \text{Set})^* \rangle$	Parameter Context
$Decl$	$:= \text{data } N \text{ } PC : (EC \rightarrow)? \text{ Set where}$ $(N : (EC \rightarrow)? \text{TypeExpr} \rightarrow N \text{Expr})^*$ $ \text{codata } N \text{ } PC : (EC \rightarrow)? \text{ Set where}$ $(N : (EC \rightarrow)? N \text{Expr}^* \rightarrow \text{TypeExpr})^*$ $ n \text{ } PC \text{ } EC = \text{Expr}$	Declarations

Figure 5.1.: Syntax for declarations

541 nested types. If we want to use these types we have to fully instantiate this context.
 542 These types can occur everywhere in the definition where a type is expected. A
 543 (co)inductive type can have a context, which is written before an arrow. **Set** stands
 544 for type (or $*$ in the paper). If a type don't has a context we omit the arrow. We
 545 will also give names to every constructor and destructor. These names have to be
 546 unique. Constructors and destructors also have contexts. Additionally they have
 547 one argument which can has a recursive occurrence of the type we are defining. A
 548 constructor gives back a value of the type, where its context is instantiated. This
 549 instantiation corresponds to the sigmas in the paper. If we write a name before an
 550 equal sign we can bind the following expression to the name. Every such defined
 551 name can depend on a parameter context and an argument context. We write the
 552 parameter context like in the case for data types behind the name. After that we
 553 can give a term context between round parenthesis.

554 The declarations in Figure 5.1 correspond to $\rho(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A}) : \Gamma \rightarrow *$ as follows:

- 555 • The first N is X
- 556 • The other N will be used later for $\alpha_1^{\mu(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A})}, \alpha_2^{\mu(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A})}, \dots$ in the case of
 557 inductive types and $\xi_1^{\nu(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A})}, \xi_2^{\nu(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A})}, \dots$ in the coinductive case
- 558 • The TypeExpr are the \vec{A}

- 559 • The $Expr^*$ are the $\vec{\sigma}$
- 560 • The first EC is Γ
- 561 • The other EC stand for $\Gamma_1, \dots, \Gamma_m$

562 To parse the abstract syntax we use Megaparsec. The parser generates an abstract
 563 syntax tree, which is given for declarations in Listing 1. The field **ty** in **ExprDef** is
 564 used later in type checking. The parser just fills them in with **Nothing**. data and co-
 565 data definitions are both saved in **TypeDef**. The Haskell type **OpenDuctive** contains
 566 all the information for inductive and coinductive types. It corresponds to μ and
 567 ν in the paper. We use an **OpenDuctive** where the field **inOrCoin** is **IsIn** for μ
 568 and an **OpenDuctive** where the field **inOrCoin** is **IsCoin** for ν . The Haskell type
 569 **StrDef** ensures that the sigmas, as and gammals have the same length. We omit
 570 the implementation details for the parser, because we are manly focused on type
 571 checking.

```

data Decl = ExprDef { name :: Text
                      , tyParameterCtx :: TyCtx
                      , exprParameterCtx :: Ctx
                      , expr :: Expr
                      , ty :: Maybe Type
                      }
  | TypeDef OpenDuctive
  | Expression Expr

data OpenDuctive = OpenDuctive { nameDuc :: Text
                                , inOrCoin :: InOrCoin
                                , parameterCtx :: TyCtx
                                , gamma :: Ctx
                                , strDefs :: [StrDef]
                                }

data StrDef = StrDef { sigma :: [Expr]
                     , a :: TypeExpr
                     , gammal :: Ctx
                     , strName :: Text
                     }

```

Listing 1: Implementation of the abstract syntax of fig. 5.1

572 5.1.2. Expressions

573 The abstract syntax for expression is given in figure 5.2. We will separate expres-
 574 sion in expressions for terms and expressions for types. There are given as regular
 575 expressions in **Expr** and **TypeExpr** respectively.

576 An **Expr** is either a **rec**, a **corec**, a con/destructor, an application @, the only prim-
 577 itive unit expression \diamond or a variable. With the keyword **rec** we can destruct an
 578 inductive type. We write **N ParInst? to TypeExpr**, where **N** is a previously defined
 579 inductive type and **ParInst?** the instantiation of its parameter context, after **rec** to
 580 facilitate type checking. It says we want to destruct an inductive type to some other

$ParInst$	$:= \langle TypeExpr, TypeExpr \rangle^*$	Instantiations for parameter contexts
$ExprInst$	$:= (Expr, Expr)^*$	Instantiations for expression contexts
$Expr$	$:= \text{rec } N \text{ } ParInst? \text{ to } TypeExpr \text{ where}$ $\quad Match^*$ $\quad \text{ corec } TypeExpr \text{ to } N \text{ } ParInst? \text{ where}$ $\quad \quad Match^*$ $\quad Expr @ Expr$ $\quad \Diamond$ $\quad EV$ $\quad n \text{ } ParInst \text{ } ExprInst$	expression
$Match$	$:= N \text{ } EV^* = Expr$	match
$TypeExpr$	$:= (EV : TypeExpr).TypeExpr$ $\quad TypeExpr @ Expr$ $\quad Unit$ $\quad TV$ $\quad N \text{ } ParInst?$	Type expressions

Figure 5.2.: Syntax for expressions

type . We have to list all the constructors above one another. For each constructor we write an expression behind the equal sign, which should be of type **TypeExpr** which we have given above. In this expression we can use variables given in the match expression. The last one is the recursive occurrence. With the keyword **corec** we can do the same thing to construct a coinductive type. Here we have to swap the **NParInst?** and the **TypeExpr** and list the destructors. All con/destructors have to be instantiate with all variables in the parameter contexts of their types. This is done by giving types of the expected kinds separated by ',' enclosed in \langle and \rangle . The variables are separated in local variables and global variables. Global variables refer to previously defined expressions. We have to fully instantiate they parameter contexts and their expression contexts. We can also apply an expression to another with **@**. This application is left associative. So if we write **t @ s @ v** we mean **(t @ s) @ v**.

The **typeExpr** is either the unit type **Unit**, a lambda abstraction on types, an application or a variable. In the lambda expression we have to give the type of the variable. We apply a type to a term (types can only depend on terms) with **@**. As in the case of term application this is also left associative. The unit type is the only primitive type expression.

The generated abstract syntax tree is given in listing 2. The variables for expressions are separated in **LocalExprVar** and **GlobalExprVar**. **LocalExprVar** should refer to variables which are only locally defined i.e. in **Rec** and **Corec**. We use de Bruijn indexes for them. This facilitates substitution which we will describe in section 5.2. **GlobalExprVar** refers to variables from definitions. Here we just use names. We do the same thing for **LocalTypeVar** and **GlobalTypeVar**. In the abstract syntax tree we use anonymous constructors like in the paper. We combine them to the Haskell constructor **Structor**. We know from the field **ductive** if it is a constructor or a destructor. The types in field **parameters** are to fill in the parameter context of the field **ductive**. The field **nameStr** in **Constructor** and **Destructor** are just for printing. We combine **rec** and **corec** to **Iter**.

5.2. Substitution

In the following we will write $t[s/x]$ for "substitute every free occurrences of x in t by s ". Substitution is done in the module **Subst.hs**. We use de Bruijn indexes [DB72] for bound variables to facilitate substitution. With this method every bound variable is a number instead of a string. The number says where the variable is bound. To find the binder of a variable we go outwards from it and count every binder until we reach the number of the variable. For example $\lambda.\lambda.\lambda.1$ says that the variable is bound by the second binder (we start counting at zero). This would be the same as $\lambda x.\lambda y.\lambda z.y$. This means we never have to generate fresh names. We just shift the free variables in the term with which we substitute by one, every time we encounter a binder. This shifting is done in the module **ShiftFreeVars.hs**. We

```

data TypeExpr = UnitType
  | TypeExpr :@ Expr
  | LocalTypeVar Int Bool Text
  | Parameter Int Bool Text
  | GlobalTypeVar Text [TypeExpr]
  | Abstr Text TypeExpr TypeExpr
  | Ductive { openDuctive :: OpenDuctive
             , parametersTyExpr :: [TypeExpr] }

data Expr = UnitExpr
  | LocalExprVar Int Bool Text
  | GlobalExprVar Text [TypeExpr] [Expr]
  | Expr :@: Expr
  | Structor { ductive :: OpenDuctive
              , parameters :: [TypeExpr]
              , num :: Int
              }
  | Iter { ductive :: OpenDuctive
          , parameters :: [TypeExpr]
          , motive :: TypeExpr
          , matches :: [(Text, Expr)]
          }

```

Listing 2: Implementation of the abstract syntax of fig. 5.2

620 also want to be able to substitute multiple variables simultaneously. If we would
 621 just substitute one term after another we could substitute into a previous term. For
 622 example the substitution $x[y/x][z/y]$ would yield z if we substitute sequential and y if
 623 we substitute simultaneously. To make simultaneous substitution possible every local
 624 variable has a boolean flag. If this flag is set to true substitution won't substitute
 625 for that variable. So for simultaneous substitution we just set this flag to true for all
 626 terms with which we want to substitute. Then we substitute with them. In the last
 627 step we just have to set the flags to false in the result. This setting (marking of the
 628 variables) is done in the module **Mark.hs**.

629 5.3. Typing rules

630 A typing rule says that some expression or declaration is of some type, given some
 631 premises. If we can for every declaration or expression form a tree of such rules
 632 with no open premises, our program type checks. We have to rewrite the typing
 633 rules of the paper, to get rules which are syntax directed. Syntax directed means
 634 we can infer from the syntax alone what we have to check next i. e. which rule with
 635 which premises we have to apply. In the paper their are rules containing variables
 636 in the premises where their type isn't in the conclusion. So if we want to type-check
 637 something which is the conclusion of such a rule we have no way of knowing what
 638 these variables are.

639 We don't need the weakening rules because we can lookup a variable in a context.
 640 So we ignore them in our implementation.

The order in **TyCtx** isn't relevant so we can use a map for it. In the code we use a list, because the names of the variables are the index of their type in the context. The order of **Ctx** is relevant because types of later variables can refer to former variables and application instantiate the first variable in **Ctx**. We add a new context for data types. We also need a context for the parameters. **Ctx** can contain variables from this context, but not from **TyCtx**.

We also rewrite the rules which are already syntax-directed to rules which work on our syntax. We will mark semantic differences in the rewritten rules gray. We use variables $\Phi, \Phi', \Phi_1, \Phi_2, \dots$ for parameter contexts, $\Theta, \Theta', \Theta_1, \Theta_2, \dots$ for type variable contexts and $\Gamma, \Gamma', \Gamma_1, \Gamma_2, \dots$ for term variable contexts. The judgments in our rules are of one of the following form.

- $\Phi | \Theta | \Gamma \vdash \Theta'$ - The type variable context Θ' is well formed in the combined context $\Phi | \Theta | \Gamma$.
- $\Phi | \Theta | \Gamma \vdash \Gamma'$ - The term variable context Γ' is well formed in the combined context $\Phi | \Theta | \Gamma$.
- $\Phi | \Theta | \Gamma \vdash \Phi'$ - The parameter variable context Φ' is well formed in the combined context $\Phi | \Theta | \Gamma$.
- $A \longrightarrow_T^* B$ - The type A fully evaluates to type B .
- $A \equiv_\beta B$ - The type A is computational equivalent to type B .
- $\Phi | \Theta | \Gamma \vdash A : \Gamma_2 \rightarrow *$ - The type A is well formed in the combined context $\Phi | \Theta | \Gamma$ and can be instantiated with arguments according to context Γ_2 .
- $\Phi | \Theta | \Gamma \vdash t : \Gamma_2 \rightarrow A$ - The term t is well formed in the combined context $\Phi | \Theta | \Gamma$ and can be instantiated with arguments according to context Γ_2 . After this instantiation it is of type A , where the arguments are substituted in A .
- $\Phi \vdash \sigma : \Gamma_1 \triangleright \Gamma_2$ - The context morphism σ is a well-formed substitution for Γ_2 with terms in context Γ_1 in parameter context Φ .

We will write \vdash for $\Phi | \Theta | \Gamma \vdash$ where Φ, Θ and Γ are arbitrary and aren't referred to by the right hand side.

In the module **TypeChecker** we will implement the following rules. It defines a monad **TI** which can throw errors and has a reader on the contexts in which we are type checking. To add something to a context we use the function **local**. This function gets a function to change the current content of the reader monad and executes a reader on this changed context in the current monad.

5.3.1. Context rules

The rules for valid contexts are already syntax directed so we take just them.

$$\frac{}{\vdash \emptyset \text{ TyCtx}} \quad \frac{\vdash \Theta \text{ TyCtx} \quad \vdash \Gamma \text{ Ctx}}{\vdash \Theta, X : \Gamma \rightarrow * \text{ TyCtx}}$$

$$\frac{}{\vdash \emptyset \text{ Ctx}} \quad \frac{|\emptyset| \Gamma \vdash A : *}{\vdash \Gamma, x : A \text{ Ctx}}$$

In the rules for valid contexts we ensure that the types in the context can not depend on **TyCtx**. Note however that they can depend on **ParCtx**. This ensures that only strictly positive types are possible.

We also need new rules for checking if a parameter context is valid.

$$\frac{}{\vdash \emptyset \text{ ParCtx}} \quad \frac{\vdash \Phi \text{ ParCtx} \quad \vdash \Gamma \text{ Ctx}}{\vdash \Phi, X : \Gamma \rightarrow * \text{ ParCtx}}$$

This are structural the same rule as this for **TyCtx**. The difference is that **ParCtx** and **TyCtx** are used differently in the other rules, as we have already seen in the rule for **Ctx**.

We use the notation $\Theta(X) \rightsquigarrow \Gamma \rightarrow *$ for looking up the type variable X in type context Θ yields type $\Gamma \rightarrow *$. We add 2 rules for looking up something in a type context. They are:

$$\frac{\vdash \Theta \text{ TyCtx} \quad \vdash \Gamma \text{ Ctx}}{\Theta, X : \Gamma \rightarrow *(X) \rightsquigarrow \Gamma \rightarrow *} \quad \frac{\vdash \Gamma_1 \text{ Ctx} \quad \Theta(X) \rightsquigarrow \Gamma_2 \rightarrow *}{\Theta, Y : \Gamma_1 \rightarrow *(X) \rightsquigarrow \Gamma_2 \rightarrow *}$$

Here Y and X are different variables.

The rules for looking up something in a parameter context are principally the same.

$$\frac{\vdash \Phi \text{ ParCtx} \quad \vdash \Gamma \text{ Ctx}}{\Phi, X : \Gamma \rightarrow *(X) \rightsquigarrow \Gamma \rightarrow *} \quad \frac{\vdash \Gamma_1 \text{ Ctx} \quad \Phi(X) \rightsquigarrow \Gamma_2 \rightarrow *}{\Phi, Y : \Gamma_1 \rightarrow *(X) \rightsquigarrow \Gamma_2 \rightarrow *}$$

Respectively the notation $\Gamma(x) \rightsquigarrow A$ means looking up the term variable x in term context Γ yields type A . The rules for term contexts are:

$$\frac{\vdash \Gamma \text{ Ctx} \quad \Gamma \vdash A : *}{\Gamma, x : A(x) \rightsquigarrow A} \quad \frac{\Gamma(x) \rightsquigarrow A \quad \Gamma \vdash B : *}{\Gamma, y : B(x) \rightsquigarrow A}$$

5.3.2. Beta-equivalence

Two types are beta equivalent if they evaluate to the same type. Because our language is deterministic this just means if we fully evaluate both of them they are alpha equivalent. Alpha equivalence means we can substitute some variables in both of them and get the same type. So we first need to define rules which say what full evaluation means. We write $A \longrightarrow_T^* B$ for evaluating A as long as it is possible yields B .

The rules are:

$$\frac{\neg \exists B : A \longrightarrow_T B}{A \longrightarrow_T^* A} \quad \frac{A \longrightarrow_T B \quad B \longrightarrow_T^* C}{A \longrightarrow_T^* C}$$

705 \longrightarrow_T is defined in section 5.4.

706 We can then introduce a new rule for beta-equivalence.

$$707 \frac{A \longrightarrow_T^* A' \quad B \longrightarrow_T^* B' \quad A' \equiv_\alpha B'}{A \equiv_\beta B}$$

708 This rule says if A evaluates to A' , B to B' and A' and B' are alpha equivalent, then
 709 A and B are beta equivalent. In the implementation \equiv_α is trivial, because we use *de*
 710 *Bruijn indices*.

711 We also add some rules to check if two contexts are the same.

$$712 \frac{}{\emptyset \equiv_\beta \emptyset} \quad \frac{\Gamma_1 \equiv_\beta \Gamma_2 \quad A \equiv_\beta B}{\Gamma_1, x : A \equiv_\beta \Gamma_2, y : B}$$

713 5.3.3. Unit type and expression introduction

714 The paper defines one rule for the unit type and one for the unit value. These are.

$$715 \frac{}{\vdash \top : *} \text{ (}\top\text{-I)} \quad \frac{}{\vdash \diamond : \top} \text{ (}\top\text{-I)}$$

716 The first rule says that the type \top has always an empty context. The second rule
 717 says its value \diamond is always of type \top . These rules get rewritten to.

$$718 \frac{}{\Phi \mid \Theta \mid \Gamma \vdash \text{Unit} : *} \text{ (Unit-I)} \quad \frac{}{\Phi \mid \Theta \mid \Gamma \vdash \diamond : \text{Unit}} \text{ (}\top\text{-I)}$$

719 We change the syntax " \top " to "Unit" and add the contexts Φ, Θ, Γ . We will do this for
 720 every rule which has empty contexts to subsume the weakening rules of the paper.
 721 The unit term always has the unit type as its type.

722 5.3.4. Variable lookup

723 We have three kinds of variables we can lookup. They are type variables, term vari-
 724 ables and parameters. The paper already has rules for the type and term variables.
 725 We need to rewrite them. We add a new rule for looking up a parameter.

726 The rule

$$727 \frac{\vdash \Theta \quad \text{TyCtx} \quad \vdash \Gamma \quad \text{Ctx}}{\Theta, X : \Gamma \rightarrow * \mid \emptyset \vdash X : \Gamma \rightarrow *} \text{TyVar-I}$$

728 gets rewritten to

$$729 \frac{\Theta(X) \rightsquigarrow \Gamma \rightarrow * \quad \vdash \Gamma_1 \quad \text{Ctx}}{\Phi \mid \Theta \mid \Gamma_1 \vdash X : \Gamma \rightarrow *} \text{TyVar-I}$$

730 The rule

$$731 \frac{\Gamma \vdash A : *}{\Gamma, x : A \vdash x : A} \text{ (Proj)}$$

732 gets rewritten to

$$733 \quad \frac{\Gamma(x) \rightsquigarrow A}{\Phi \mid \Theta \mid \Gamma \vdash x : A} \text{ (Proj)}$$

734 The rule for looking something up in the parameter context is:

$$735 \quad \frac{\Phi(X) \rightsquigarrow \Gamma \rightarrow * \quad \vdash \Gamma_1 \text{ Ctx}}{\Phi \mid \Theta \mid \Gamma_1 \vdash X : \Gamma \rightarrow *} \text{ TyVar-I}$$

736 In the rule from the paper we can only infer the type or kind of the last variable
 737 in the context. In our rules we just look up the variable in the context. These rules
 738 can check the same thing if we take the weakening rules into account. With them
 739 we can just weaken the context until we get to the desired variable.

740 5.3.5. Type and expression instantiation

741 We can instantiate types and terms. The rule

$$742 \quad \frac{\Theta \mid \Gamma_1 \vdash A : (x : B, \Gamma_2) \rightarrow * \quad \Gamma_1 \vdash t : B}{\Theta \mid \Gamma_1 \vdash A@t : \Gamma_2[t/x] \rightarrow *} \text{ (Ty-Inst)}$$

743 for instantiating types gets rewritten to

$$744 \quad \frac{\Phi \mid \Theta \mid \Gamma_1 \vdash A : (x : B, \Gamma_2) \rightarrow * \quad \Phi \mid \Theta \mid \Gamma_1 \vdash t : B' \quad B \equiv_\beta B'}{\Phi \mid \Theta \mid \Gamma_1 \vdash A@t : \Gamma_2[t/x] \rightarrow *} \text{ (Ty-Inst)}$$

745 For this rule we have to check if t has the expected type for the first variable in the
 746 context of A . In our version we just infer the type for A and t . Then we check if the
 747 first variable in the context is beta-equal to the type of t . If that isn't the case type
 748 checking fails. Otherwise we just substitute in the remaining context.

749 We also have a rule to instantiate terms. This rule

$$750 \quad \frac{\Gamma_1 \vdash t : (x : A, \Gamma_2) \rightarrow B \quad \Gamma_1 \vdash s : A}{\Gamma_1 \vdash t@s : \Gamma_2[s/x] \rightarrow B[s/x]} \text{ (Inst)}$$

751 gets rewritten to

$$752 \quad \frac{\Phi \mid \Theta \mid \Gamma_1 \vdash t : (x : A, \Gamma_2) \rightarrow B \quad \Phi \mid \Theta \mid \Gamma_1 \vdash s : A' \quad A \equiv_\beta A'}{\Phi \mid \Theta \mid \Gamma_1 \vdash t@s : \Gamma_2[s/x] \rightarrow B[s/x]} \text{ (Inst)}$$

753 These rules are similar to the rule for type instantiation. Here we have to check(or
 754 infer) a term instead of a type. We also have to substitute s in the result type of t (in
 755 the case of types its always $*$, which obviously has no free variables).

5.3.6. Parameter abstraction

The rule

$$\frac{\Theta \mid \Gamma_1, x : A \vdash B : \Gamma_2 \rightarrow *}{\Theta \mid \Gamma_1 \vdash (x).B : (x : A, \Gamma_2) \rightarrow *} \text{ (Param-Abstr)}$$

gets rewritten to

$$\frac{\Phi \mid \Theta \mid \Gamma_1, x : A \vdash B : \Gamma_2 \rightarrow *}{\Phi \mid \Theta \mid \Gamma_1 \vdash (x:\mathbf{A}).B : (x : A, \Gamma_2) \rightarrow *} \text{ (Param-Abstr)}$$

Here we just add the argument of the lambda to the expression context. Then we check the body of the lambda. In the syntax directed version we have to annotate the variable with its type, so we know which type we have to add to the context.

5.3.7. (co)inductive types

We have to separate the rule

$$\frac{\sigma_k : \Gamma_k \triangleright \Gamma \quad \Theta, X : \Gamma \rightarrow * \mid \Gamma_k \vdash A_k : *}{\Theta \mid \emptyset \vdash \rho(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A}) : \Gamma \rightarrow *} \text{ (FP-Ty)}$$

into multiple rules. First we need rules to check the definitions of (co)inductive types. These are

$$\frac{\sigma_k : \Gamma_k \triangleright \Gamma \quad \Phi \mid X : \Gamma \rightarrow * \mid \Gamma_k \vdash A_k : * \quad \vdash \phi \text{ ParCtx}}{\vdash \text{data } X\langle\Phi\rangle : \Gamma \rightarrow \text{Set where; } \overline{\text{Constr}_k : \Gamma_k \rightarrow A_k \rightarrow X\sigma_k}} \text{ (FP-Ty)}$$

and

$$\frac{\sigma_k : \Gamma_k \triangleright \Gamma \quad \Phi \mid X : \Gamma \rightarrow * \mid \Gamma_k \vdash A_k : * \quad \vdash \phi \text{ ParCtx}}{\vdash \text{codata } X\langle\Phi\rangle : \Gamma \rightarrow \text{Set where; } \overline{\text{Destr}_k : \Gamma_k \rightarrow X\sigma_k \rightarrow A_k}} \text{ (FP-Ty)}$$

Because we only allow top level definitions of (co)inductive types our rules have empty contexts. We first have to check if σ_k is a context morphism from Γ_k to Γ . This basically means that the terms in σ_k are of the types in Γ , if we check them in Γ_k . After that we have to check if the \vec{A} (the arguments where we can have a recursive occurrence) are of kind $*$. Because this is a top level definition the context ϕ is provided by the code. So we have to check if it is valid. We will now have to rewrite the rules for context morphism. Here we just add the parameter context to the rules of the paper.

$$\frac{}{\Phi \vdash () : \Gamma_1 \triangleright \emptyset} \quad \frac{\Phi \vdash \sigma : \Gamma_1 \triangleright \Gamma_2 \quad \Phi \mid \Gamma_1 \vdash t : A[\sigma]}{\Phi \vdash (\sigma, t) : \Gamma_1 \triangleright (\Gamma_2, x : A)}$$

We also need a rule for the cases in which we are using these defined variables. This is:

$$\frac{\Phi \mid \Theta \mid \Gamma' \vdash \vec{A} : \Gamma_i \rightarrow *}{\Phi \mid \Theta \mid \Gamma' \vdash X(\vec{A}) : \Gamma[\vec{A}] \rightarrow *} \quad (783)$$

784 Here X is a data or codata definition. The parser can decide if a variable is a such a
 785 definition or a local definition. Because we are type checking on the abstract syntax
 786 tree we also know Γ and Φ' . Γ is just the context from the definition and Φ is the
 787 parameter context. Because we already typed checked this definition we just have to
 788 check if the types given for the parameters have the right kind. Then we substitute
 789 these parameters in its type. We will now give the rules for checking if a list of
 790 parameters matches a parameter context.

$$\frac{}{\Phi \mid \Theta \mid \Gamma \vdash () : ()} \quad \frac{\Phi \mid \Theta \mid \Gamma \vdash A : \Gamma' \rightarrow * \quad \Phi \mid \Theta \mid \Gamma \vdash \vec{A} : \Phi'[A/X]}{\Phi \mid \Theta \mid \Gamma \vdash A, \vec{A} : (X : \Gamma' \rightarrow *, \Phi')} \quad (791)$$

792 We just check every variable for the kinds in Φ' one after the other. We also have
 793 to substitute the type into the context. Because kinds in a parameter context can
 794 depend on variables previously defined in this context.

795 5.3.8. Constructor and Destructor

796 The rule for constructors

$$\frac{\mu(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A}) : \Gamma \rightarrow * \quad 1 \leq k \leq |\vec{A}|}{\vdash a_k^{\mu(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A})} : (\Gamma_k, y : A_k[\mu/X]) \rightarrow \mu @_{\sigma_k}} \quad (\text{Ind-I}) \quad (797)$$

798 gets rewritten to

$$\frac{\Phi \mid \Theta \mid \Gamma \vdash \vec{B} : \Phi'}{\Phi \mid \Theta \mid \Gamma \vdash \text{Constr}(\vec{B}) : (\Gamma_k[\vec{B}], y : A_k[\mu/X][\vec{B}]) \rightarrow \mu @_{\sigma_k}[\vec{B}]} \quad (\text{Ind-I}) \quad (799)$$

800 The rule for destructors

$$\frac{\nu(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A}) : \Gamma \rightarrow * \quad 1 \leq k \leq |\vec{A}|}{\vdash \xi_k^{\nu(X : \Gamma \rightarrow *; \vec{\sigma}; \vec{A})} : (\Gamma_k, y : \nu @_{\sigma_k}) \rightarrow A_k[\nu/X]} \quad (\text{Coind-E}) \quad (801)$$

802 gets rewritten to

$$\frac{\Phi \mid \Theta \mid \Gamma \vdash \vec{B} : \Phi'}{\Phi \mid \Theta \mid \Gamma \vdash \text{Destr}(\vec{B}) : (\Gamma_k[\vec{B}], y : \nu @_{\sigma_k}[\vec{B}]) \rightarrow A_k[\nu/X][\vec{B}]} \quad (\text{Ind-I}) \quad (803)$$

804 In the paper de/constructors are anonymous. They come together with their type.
 805 Therefor we have to check if this type is valid. Constructors construct their type. So
 806 their output value is their type μ applied to the context morphism σ_k , where k is the
 807 number of the constructor. They become as input the context Γ_k , which is implicit
 808 in the paper, and a value of type $A_k[\mu/X]$, which is the type, which can contain the
 809 recursive occurrence. Destructors are destructing their type so we get their type ν
 810 applied to σ_k as input and $A_k[\nu/X]$ as output.

811 In our rules, in contrast to the paper, the de/constructors refer to some type which
 812 we have already type checked. We just have to check the parameters. Every term
 813 we need is in the Haskell representation of the de/constructor. The de/constructor
 814 has the type which we have defined in the data definition. We just substitute the
 815 type itself for the free variable. At last we need to substitute the parameters for the
 816 respective variables.

817 5.3.9. Recursion and Corecursion

818 The rule

$$819 \frac{\vdash C : \Gamma \rightarrow * \quad \Delta, \Gamma_k, y_k : A_k[C/X] \vdash g_k : (C@_{\sigma_k}) \quad \forall k = 1, \dots, n}{\Delta \vdash \text{rec } (\overrightarrow{\Gamma_k, y_k}).g_k : (\Gamma, y : \mu@id_\Gamma) \rightarrow C@id_\Gamma} \text{ (Ind-E)}$$

820 gets rewritten to

$$821 \frac{\vdash C : \Gamma \rightarrow * \quad \frac{\vdash \Gamma \equiv_\beta \Gamma'[\vec{D}]}{\vdash B_k \equiv_\beta (C@_{\sigma_k}[\vec{D}])} \quad \frac{\Phi \mid \Theta \mid \Delta \vdash \vec{D} : \Phi'}{\Phi \parallel \Delta, \Gamma_k[\vec{D}], y_k : A_k[\vec{D}][C/X] \vdash g_k : B_k}}{\Phi \mid \Theta \mid \Delta \vdash \text{rec } \mu(\vec{D}) \text{ to } C; \text{Constr}_k \vec{x}_k y_k = g_k : (\Gamma, y : \mu[\vec{D}]@id_\Gamma) \rightarrow C@id_\Gamma} \text{ (Ind-E)}$$

822 We are recursing over some previously inductively defined type μ to some type C .
 823 This types must have the same context. Recursing is done by listing each constructor
 824 with the result, which the whole expression should have if we apply it to this con-
 825 structor. This result can refer to the arguments of the constructor via the variables
 826 \vec{x}_k, y_k . The type must be the result type C applied to the σ_k of this constructor. In
 827 the syntax directed version we also have to check the parameters. We check if the
 828 types match by inferring them and compare them on beta equality.

829 We have a similar rule for corecursion. It

$$830 \frac{\vdash C : \Gamma \rightarrow * \quad \Delta, \Gamma_k, y_k : (C@_{\sigma_k}) \vdash g_k : A_k[C/X] \quad \forall k = 1, \dots, n}{\Delta \vdash \text{corec } (\overrightarrow{\Gamma_k, y_k}).g_k : (\Gamma, y : C@id_\Gamma) \rightarrow v@id_\Gamma} \text{ (Coind-I)}$$

831 gets rewritten to

$$832 \frac{\vdash C : \Gamma \rightarrow * \quad \frac{\vdash \Gamma \equiv_\beta \Gamma'[\vec{D}]}{\vdash B_k \equiv_\beta A_k[\vec{D}][C/X]} \quad \frac{\Phi \mid \Theta \mid \Delta \vdash \vec{D} : \Phi'}{\Phi \parallel \Delta, \Gamma_k[\vec{D}], y_k : (C@_{\sigma_k}[\vec{D}]) \vdash g_k : B_k}}{\Phi \mid \Theta \mid \Delta \vdash \text{corec } C \text{ to } v(\vec{D}); \text{Destr}_k \vec{x}_k y_k = g_k : (\Gamma, y : C@id_\Gamma) \rightarrow v[\vec{D}]@id_\Gamma} \text{ (Coind-I)}$$

834 A corecursion produces a coinductive type v . We have to give it a type C and list
 835 the destructors together with the expression they should be destructed to. We get
 836 the syntax directed rule analog as in the case of recursion.

837 5.4. Evaluation

838 There are two kinds of reduction steps in this system. The implementation of this is
839 in `Eval.hs`. Will give the formal definition in the following.

The first is a reduction on the type level, \longrightarrow . It is defined like follows:

$$((x).A)@t \longrightarrow_p A[t/x]$$

840 It is standard beta reduction. If we apply a lambda $(x).A$ to a term t we substitute
841 this term for the binding variable x in the body. This body is then the result of the
842 reduction.

843 The other is the reduction on the term level, \succ . To define this reduction we need a
844 action on types (written $\widehat{C}(A)$) and terms (written $\widehat{C}(t)$), where the following holds.

$$845 \frac{X : \Gamma_1 \rightarrow * \mid \Gamma'_2 \vdash C : \Gamma_2 \rightarrow * \quad \Gamma_1, x : A \vdash t : B}{\Gamma'_2, \Gamma_2, x : \widehat{C}(A) \vdash \widehat{C}(t) : \widehat{C}(B)}$$

846 Here we have a type C with a free type variable X and a term t of type B with a free
847 term variable x of type A . If we use the action of this type on t we get a term with a
848 type of this action on B . This term contains a free term variable x of type, the action
849 applied to A . The type action is implemented in the module `TypeAction.hs`. Both
850 the type action and the evaluation are done in the `Eval` monad. This monad has
851 access to the previously defined declarations. We will now define the type action.

Definition 1. *Let $n \in \mathbb{N}$ and $1 \leq i \leq n$. Let:*

$$\begin{aligned} X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \Gamma' \vdash C : \Gamma \rightarrow * \\ \Gamma_i \vdash A_i : * \\ \Gamma_i \vdash B_i : * \\ \Gamma_i, x : A_i \vdash t_i : B_i \end{aligned}$$

Then we define the type action on terms inductively over C

$$\begin{aligned}
\widehat{C}(\vec{t}, t_{n+1}) &= \widehat{C}(\vec{t}) && \text{for } (\mathbf{TyVarWeak}) \\
\widehat{X}_i(\vec{t}) &= t_i \\
\widehat{C'@s}(\vec{t}) &= \widehat{C'}(\vec{t})[s/y], && \text{for } \Theta \mid \Gamma' \vdash C' : (y, \Gamma) \rightarrow * \\
(\widehat{y}).\widehat{C'}(\vec{t}) &= \widehat{C'}(\vec{t}), && \text{for } \Theta \mid (\Gamma', y) \vdash C' : \Gamma \rightarrow * \\
\mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D}) &= \overrightarrow{\text{rec}^{R_A}(\Delta_k, x).g_k@id_\Gamma @x} && \text{for } \Theta, Y : \Gamma \rightarrow * \mid \Delta_k \vdash D_k : * \\
&\text{with } g_k = \alpha_k^{R_B}@id_{\Delta_k}@\widehat{D_k}(\vec{t}, x) \\
&\text{and } R_A = \mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D}[(\Gamma_i).\vec{A}/\vec{X}]) \\
&\text{and } R_B = \mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D}[(\Gamma_i).\vec{B}/\vec{X}]) \\
\nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D}) &= \overrightarrow{\text{corec}^{R_B}(\Delta_k, x).g_k@id_\Gamma @x} && \text{for } \Theta, Y : \Gamma \rightarrow * \mid \Delta_k \vdash D_k : * \\
&\text{with } g_k = \widehat{D_k}(\vec{t}, x)[(\xi_k^{R_A}@id_{\Delta_k}@x)/x] \\
&\text{and } R_A = \mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D}[(\Gamma_i).\vec{A}/\vec{X}]) \\
&\text{and } R_B = \mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D}[(\Gamma_i).\vec{B}/\vec{X}])
\end{aligned}$$

And the type action on types as follow

$$\widehat{C}(\vec{A}) = C[(\Gamma_i).\vec{A}/\vec{X}]@id_\Gamma$$

852 The type action generates a term with a free variable x . In the type of this term
853 we have changed all the free variables to the types of \vec{t} . We will show the proof in
854 appendix A.

855 The reduction on terms is subdivided into an reduction on recursion and one on
856 corecursion. Here $\sigma_k \bullet \tau$ is a context morphism, where we first substitute with τ and
857 then with σ_k .

The reduction on recursion is defined as follows

$$\overrightarrow{\text{rec}(\Gamma_k, y_k).g_k@(\sigma_k \bullet \tau)@(\alpha_k@ \tau @u)} > g_k[\widehat{A_k}(\overrightarrow{\text{rec}(\Gamma_k, y_k).g_k@id_\Gamma @x})/y_k][\tau, u]$$

858 If we apply a recursion $\overrightarrow{\text{rec}(\Gamma_k, y_k).g_k}$ to this context morphism and a constructor
859 $\alpha_k@ \tau @u$, which is fully applied, we lookup the case for this constructor. In this case
860 we substitute τ for the variables from Γ_k and u , where we apply the recursion to
861 all recursive occurrences, for y_k . For this application we need the type action. So a
862 recursion is destructing an inductive type and all its recursive occurrences to another
863 type, while we use different cases for the different constructors of the type.

On the contrary corecursion is constructing a coinductive type. It is defined like follows:

$$\xi_k@ \tau @(\overrightarrow{\text{corec}(\Gamma_k, y_k).g_k} @(\sigma_k \bullet \tau) @u) > \widehat{A_k}(\overrightarrow{\text{corec}(\Gamma_k, y_k).g_k} @id_\Gamma @x)[g_k/x][\tau, u]$$

864 If we apply a destructor together with its arguments for it context $\xi_k @ \tau$, on such a
 865 construction $(\text{corec}(\overline{\Gamma_k}, y_k) \cdot \overrightarrow{g_k} @ (\sigma_k \bullet \tau) @ u)$, we are taking the case of this destructor.
 866 In this case we are applying the corecursion to all recursive occurrences. τ and u are
 867 substituted as in recursion.

868 6. Examples

869 In this section we reiterate the example types from the paper. We use our syntax,
870 which is defined in 5.1. We will also show some functions on these types. On some
871 of them we will show the reduction steps in detail.

872 6.1. Terminal and Initial Object

873 The terminal object is a type which has exactly one value. In category theory every
874 object in the category has an unique morphism to it. We define it as a coinductive
875 type **Terminal** with no destructors . It gets a terminal and returns a terminal. To
876 get a terminal value we use corecursion on the unit type, which is the first class
877 terminal object.

```
878 codata Terminal : Set where  
879 terminal = corec Unit to Terminal where @ ◇
```

880 Contrary to the definition in the paper there is no destructor **Terminal**. In the
881 paper definitions of coinductive or inductive types need at least one de/constructor.
882 Therefore our definition wouldn't work.

883 The initial object is a type which has no values. In category theory it is the object
884 which has an unique morphism to every other object in the category. We define
885 it inductively as **Intial** with no constructor. In the paper it is defined with one
886 constructor. This constructor want's one value of the same type. We can't have a
887 value of this type, because to get one we already need one. Our way of defining it
888 is shorter and more clear. We can't construct an value of this type because we have
889 no constructors. If we could get something of type **Intial**, we could generate with
890 **exfalsum** a value of arbitrary type **C**.

```
891 data Initial : Set where  
892 exfalsum(C : Set) = rec Initial to C where
```

893 6.2. Natural Numbers and Extended Naturals

894 We use the classical peano numbers to define natural numbers. Therefor we use
895 the inductive type **Nat** with the constructors **Zero** and **Suc**. **Zero** is just the number
896 zero. Every constructor has to have an argument, which can contain a recursive

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occurrence. Every Type **A** is isomorphic to the function type **Terminal** \rightarrow **A**. So we use **Terminal** for this occurrence. **Suc** is the successor. So the meaning of **Suc n** is $n + 1$.

```

900 data Nat : Set where
901     Zero : Terminal  $\rightarrow$  Nat
902     Suc  : Nat  $\rightarrow$  Nat
903 zero = Zero @  $\diamond$ 
904 one  = Suc @ zero

```

We can then define a identity recursion on it to see how reduction works. It's a recursion which goes from a **Nat** to **Nat** and gives back in every case its input.

```

907 id = rec Nat to Nat where
908     Zero u = Zero @ u
909     Succ n = Succ @ n

```

We use it on one to see all cases.

```

911 id @ one = id @ (Succ @ zero)
912           > Succ @ n[ $\widehat{X}(\text{id @ x})/n$ ] [zero]
913           = Succ @  $\widehat{X}(\text{id @ x})$  [zero]
914           = Succ @ (id @ x)[zero]
915           = Succ @ (id @ zero)
916           = Succ @ (id @ (Zero @  $\diamond$ ))
917           > Succ @ (Zero @ u[ $\widehat{\text{Unit}}(\text{id @ x})/u$ ][ $\diamond$ ])
918           = Succ @ (Zero @ u[ $\widehat{\text{Unit}}(\text{id @ x})/u$ ][ $\diamond$ ])
919           = Succ @ (Zero @  $\widehat{\text{Unit}}(\text{id @ x})$ )[ $\diamond$ ]
920           = Succ @ (Zero @ x)[ $\diamond$ ]
921           = Succ @ (Zero @ x) = Succ @ zero = one

```

As expected the identity recursion applied to one gives back one.

We will now define extended naturals. There are also called co-natural numbers. There are natural numbers with an additional value, infinity. We define it coinductively with the predecessor as its only destructor. The predecessor is either not defined or another natural number. We use the type **Maybe** to describe something which is either present (the constructor **Just**) or absent (the constructor **Nothing**). We can define the successor as a corecursion. The predecessor of the successor of **x** is just **x**. So the only case of **corec** returns a **Just x** (remember **Prec** returns a **Maybe<Conat>** not a **Conat**).

```

931 data Maybe(A : Set) : Set where
932     Nothing : Unit  $\rightarrow$  Maybe
933     Just    : A  $\rightarrow$  Maybe
934 nothing(A) = Nothing(A) @  $\diamond$ 
935 codata Conat : Set where
936     Prec : Conat  $\rightarrow$  Maybe(Conat)
937 succ = corec Conat to Conat where
938     Prec x = Just(Conat) @ x

```

We now define the values zero and infinity

```

940 zero = (corec Unit to Conat where
941     {Prev x = nothing(Unit)}) @  $\diamond$ 

```


942 `infinity = (corec Unit to Conat where`
 943 `{Prev x = Just<Conat> @ x}) @ \diamond`

For **zero** the predecessor is absent, there is no predecessor of 0 in the natural numbers, so we give back **Nothing**. We then have to apply the **corec** to \diamond to get the value. The predecessor of **infinity** should also be **infinity**. We apply the **corec** to another **Conat**, so the **x** is also a **Conat**. We will know see that the predecessor on this values give back the right value.

$$\begin{aligned}
 \text{Prev @zero} &> \widehat{\text{Maybe}(X)} \left(\underbrace{\text{corec Unit to Conat where } \{ \text{Prev } x = \text{nothing}(\text{Unit}) \}}_{t_1} @ x \right) [\text{nothing}(\text{Unit})/x][\diamond] \\
 &= \text{rec Maybe}(\text{Unit}) \text{ to Maybe}(\text{Conat}) \text{ where} \\
 &\quad \{ \text{Nothing } u = \text{Nothing}(\text{Conat}) @ \widehat{\text{Unit}}(t_1, u) \\
 &\quad \text{Just } c = \text{Just}(\text{Conat}) @ \widehat{X}(t_1, c) \} @ x [\text{nothing}(\text{Unit})/x][\diamond] \\
 &= \underbrace{\text{rec Maybe}(\text{Unit}) \text{ to Maybe}(\text{Conat}) \text{ where} \\
 &\quad \{ \text{Nothing } u = \text{Nothing}(\text{Conat}) @ u \\
 &\quad \text{Just } c = \text{Just}(\text{Conat}) @ t_1 \}}_{t_2} @ \text{nothing}(\text{Unit}) \\
 &> \text{Nothing}(\text{Conat}) @ u [\widehat{\text{Unit}}(t_2 @ x)/u][\diamond] \\
 &= \text{Nothing}(\text{Conat}) @ u [x/u][\diamond] \\
 &= \text{Nothing}(\text{Conat}) @ \diamond
 \end{aligned}$$

$$\begin{aligned}
 \text{Prev @infinity} &> \widehat{\text{Maybe}(X)} \left(\underbrace{\text{corec Unit to Conat where } \{ \text{Prev } x = \text{Just}(\text{Unit}) @ x \}}_{t_1} @ x \right) [\text{Just}(\text{Unit}) @ /x][\diamond] \\
 &= \text{rec Maybe}(\text{Unit}) \text{ to Maybe}(\text{Conat}) \text{ where} \\
 &\quad \{ \text{Nothing } u = \text{Nothing}(\text{Conat}) @ \widehat{\text{Unit}}(t_1, u) \\
 &\quad \text{Just } c = \text{Just}(\text{Conat}) @ \widehat{X}(t_1, c) \} @ x [\text{Just}(\text{Unit}) @ /x][\diamond] \\
 &= \underbrace{\text{rec Maybe}(\text{Unit}) \text{ to Maybe}(\text{Conat}) \text{ where} \\
 &\quad \{ \text{Nothing } u = \text{Nothing}(\text{Conat}) @ u \\
 &\quad \text{Just } c = \text{Just}(\text{Conat}) @ t_1 \}}_{t_2} @ \text{Just}(\text{Unit}) @ \\
 &> \text{Just}(\text{Conat}) @ t_1 [\widehat{\text{Unit}}(t_2 @ x)/x][\diamond] \\
 &= \text{Just}(\text{Conat}) @ t_1 [x/x][\diamond] \\
 &= \text{Just}(\text{Conat}) @ \text{infinity}
 \end{aligned}$$

944

945 6.3. Binary Product and Coproduct

946 The product is defined as a coinductive type. It has two destructors. The first gives
 947 back the first element. And the second the second. To use this type, the types A and
 948 B have to be instantiated to concrete types. We don't have type polymorphism in our
 949 language. We also define a pair expression which generates a pair over corecursion.

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```

950 codata Product(A : Set , B : Set) : Set where
951     Fst : Product → A
952     Snd : Product → B
953 pair(A : Set , B : Set) (x:A, y:B) = corec Unit where
954     { Fst u → x
955       ; Snd u → y } @ ◇

```

For types with other contexts we have to define different Products. For example if **B** depends on **Nat**, we define product like the following:

```

958 codata Pair(A : Set , B : (n : Nat) → Set) : (n : Nat) → Set where
959     First : (n : Nat) → Pair n → A
960     Second : (n : Nat) → Pair n → B @ n

```

Here the product also depends on **Nat**. If **A** or **B** depends on values the product must also depend on these values. This is the product, which is used for the definition of vectors in [BG16].

On **Product** we can define the swap function.

```

965 swap(A : Set , B : Set) =
966     corec Product(A,B) to Product(B,A) where
967         Fst x → Snd x
968         Snd x → Fst x

```

This is a well typed function as shown by the following proof

$$\begin{array}{c}
 (A : *, B : *) \parallel (x : A) \vdash \text{Snd} @ x : \text{Product}(A, B) \text{ (a)} \\
 (A : *, B : *) \parallel \vdash \text{Product}(A, B) : * \quad (A : *, B : *) \parallel (y : B) \vdash \text{Fst} @ y : \text{Product}(A, B) \text{ (b)} \\
 \hline
 (A : *, B : *) \parallel \vdash \text{swap} : (p : \text{Product}(A, B)) \rightarrow \text{Product}(B, A)
 \end{array}$$

We show (a) in the following proof. (b) works analog.

$$\begin{array}{c}
 (A : *, B : *) \parallel (x : A) \vdash \text{Snd} : (x : A) \rightarrow \text{Product}(A, B) \quad \frac{(x : A)(x) \rightsquigarrow A}{(x : A) \vdash x : A} \\
 \hline
 (A : *, B : *) \parallel (x : A) \vdash \text{Snd} @ x : \text{Product}(A, B)
 \end{array}$$

For brevity we omitted the beta equality premises and the checking for of the parameters. The beta equality premises wouldn't be interesting because they all already syntactically identical.

The Binary Coproduct corresponds to the Either type in Haskell. It is defined as an inductive type. It is either **A** or **B**. We have one constructor **Left** for **A** and one constructor **Right** for **B**.

```

979 data Coproduct(A,B) : Set where
980     Left : A → Coproduct
981     Right : B → Coproduct

```

6.4. Sigma and Pi Type

The sigma type is a dependent pair of two types. The second type can depend on the value of the first type. It corresponds to exists in logic. We define it as an inductive type and call the constructor **Exists**.

```
data Sigma(A : Set, B : (x : A) → Set) : Set where
  Exists : (x:A) → B x → Sigma
```

The pi type is a generalization of the function type to dependent types. The type of the codomain or result of a function can depend on the value. We define it as a coinductive type. To destruct a function we just apply it to a value. So the destructor is **Apply**.

```
codata Pi(A : Set, B : (x : A) → Set) : Set where
  Apply : (x : A) → Pi x → B
```

To construct a function we use corecursion on **Unit**. The identity function is defined like this

```
id(A : Set) = corec Unit to Pi(A, (v:A).A) where
  { Apply v p = v } @ ◇
```

Evaluation on one goes as follows:

```
apply = Apply(Nat, (v : Nat).Nat)
one = S @ (Z @ )
apply @ id(Nat) @ one
= apply @ one @ ((corec Unit to Pi(Nat, (x:Nat).Nat) where
  Apply v p = v ) @ ◇)
> Nat  $\left( \underbrace{\text{corec Unit to Pi where } \{ \text{Apply } v \_ = v \} @ x}_t \right) [v/x] [one, \diamond]$ 
= (rec Nat to Nat where
  Zero x = Zero @ (Unit(t, x))
  Succ x = Suc @ (Y(t, x)) @ x [v/x] [one, ◇]
= (rec Nat to Nat where
  Zero x = Zero @ (Unit(t))
  Succ x = Suc @ x @ x [v/x] [one, ]
= (rec Nat to Nat where
  Zero x = Zero @ (Unit())
  Succ x = Suc @ x @ x [v/x] [one, ◇]
= (rec Nat to Nat where
  Zero x = Zero @ x
  Succ x = Suc @ x @ x [v/x] [one, ◇]
= (rec Nat to Nat where
  Zero x = Zero @ x
  Succ x = Suc @ x @ v [one, ◇]
= (rec Nat to Nat where
  Zero x = Zero @ x
  Succ x = Suc @ x @ one
= one
```

1024 **6.5. Vectors and Streams**

1025 Vectors are a standard example for dependent types. They are like lists, except their
 1026 type depends on their length. For example a vector `[1;2]` has type `Vector<Nat> 2`,
 1027 because its length is 2. It has 2 constructors `Nil` and `Cons` like lists. `Nil` gives back
 1028 the empty vector. Because the length of the empty vector is zero its return type is
 1029 `Vector 0`. The second constructor `Cons` takes a natural number `k`, a value of type `A`
 1030 and a vector of length `k`, a `Vector k`. It returns a new vector. Its head is the first
 1031 argument and its tail the second. So the results length is one more then the second
 1032 argument . Therefore it is `Vector (Suc k)`. In [BG16] the head and tail are encoded
 1033 in a pair.

```
1034 data Vector<A : Set> : (n:Nat) → Set where
1035   Nil : Unit → Vector zero
1036   Cons : (k:Nat, v:A) → Vector @ k → Vector (Suc @ k)
1037   nil<A : Set> = Nil<A : Set> @ ∅
```

1038 The function `extend` takes a value `x` and extends it to a vector.

```
1039 extend<A : Set> =
1040   rec Vec<A> to ((x).Vec< A> @ (Suc x) where
1041     Nil u = Cons<A> @ x @ nil<A>
1042     Cons k v = Cons<A> @ (Suc @ k) @ v
```

1043 The type checking of this function goes as follows:

$$\begin{array}{c}
 (A : \text{Set}) \Vdash (x).(\text{Vec}\langle A \rangle @ (\text{Suc } @ x)) : (k : \text{Nat}) \rightarrow * \\
 (A : \text{Set}) \Vdash (u : A) \vdash \text{Cons}\langle A \rangle @ 0 @ (\text{Nil}\langle A \rangle @ @) : (x).(\text{Vec}\langle A \rangle @ (\text{Suc } @ x)) @ 0 \\
 \hline
 (k : \text{Nat}, v : (x).(\text{Vec } @ (\text{Suc } @ x)) @ k) \vdash \text{Cons}\langle A \rangle @ @ (\text{Suc } @ k) @ v : (x).(\text{Vec } @ (\text{Suc } @ x)) @ (\text{Suc } @ k) \\
 \hline
 \vdash \text{extend}\langle A \rangle : (k : \text{Nat}, y : \text{Vec}\langle A \rangle @ k) \rightarrow (x).(\text{Vec}\langle A \rangle @ (\text{Suc } x)) @ k
 \end{array}$$

As an example we evaluate a vector of length 1 with this function. We choose length one to see all `rec` cases.

```

extend<Nat> @ 1 @ (Cons<Nat> @ 0 @ 0 @ nil<Nat>)
= extend<Nat> @ (Suc @ k • 0) @ (Cons<Nat> @ 0 @ 0 @ nil<Nat>)
> Cons<Nat> @ (Suc @ k) @ v [X̂ @ k (extend<Nat> @ n @ x) / v] [0, nil<Nat>]
= Cons<Nat> @ (Suc @ k) @ v [X̂ (extend<Nat> @ n @ x) [k/n] / v] [0, nil<Nat>]
= Cons<Nat> @ (Suc @ k) @ v [extend @ n @ x [k/n] / v] [0, nil<Nat>]
= Cons<Nat> @ (Suc @ k) @ v [extend @ k @ x / v] [0, nil<Nat>]
= Cons<Nat> @ (Suc @ k) @ (extend @ k @ x) [0, nil<Nat>]
= Cons<Nat> @ (Suc @ 0) @ (extend @ 0 @ (nil<Nat>))
= Cons<Nat> @ 1 @ (extend @ 0 @ (Nil<Nat> @))
> Cons<Nat> @ 1 @ (Cons<Nat> @ 0 @ (Nil<Nat> @)) [Unit(extend @ k @ x) / u] [∅]
= Cons<Nat> @ 1 @ (Cons<Nat> @ 0 @ (Nil<Nat> @ x)) [∅]
= Cons<Nat> @ 1 @ (Cons<Nat> @ 0 @ (Nil<Nat> @))
```

1045 Here we write 1 for `Suc @ (Zero @)` and 0 for `Zero @ ∅`.

1046 With the help of extended naturals, we can define partial streams. These are streams
 1047 which depend on there definition depth. Like non-dependent streams they are coin-
 1048 ductive and have 2 destructors for head and tail.

```
1049 codata PStr(A : Set): (n: ExNat) → Set where
1050   hd : (k : ExNat) → PStr(A) (succE k) → A
1051   tl : (k : ExNat) → PStr(A) (succE k) → PStr(A) @ k
```

1052 These streams are like vectors except they also can be infinite long. This is in contrary
 1053 to non dependent streams. A non dependent stream could not be of length zero.
 1054 Because then a call of **hd** and **tl** on it wouldn't be defined. In the dependent case
 1055 the type checker wouldn't allow such a call because **hd** and **tl** expect streams which
 1056 are at least of length one. We can then define **repeat**.

```
1057 repeat(A : Set)(x : A, n : Conat) =
1058   corec (n : Conat).Unit to PStr(A) where
1059     { Hd k s = x
1060       ; Tl k s = () } @ n @ ()
```

1061 This function gets a value and an extended natural number. It generate an constant
 1062 partial stream of that value with the number as its length.

1063 7. Conclusion

1064 We have implemented a depend type theory with inductive and coinductive types. In
1065 this theory, contrary to coq and agda, coinductive types can also depend on values.
1066 Contrary to the theory of the paper we can define schemata like `Maybe⟨A : Set⟩`
1067 where `A` can be an arbitrary type of kind `Set`.

1068 One downside is that we don't have universes. This prevents type polymorphism.
1069 Further work needs to be done to solve this. Another problem is, that each con-
1070 structor or destructor has at least one argument. The argument with the recursive
1071 occurrence. For example we have to apply an unit to the constructors of a boolean
1072 type. We could allow recursive occurrences in the contexts of the constructors and
1073 destructors. This makes it possible to remove the argument with the recursive oc-
1074 currence. We then have to change the evaluation rules.

1075 Our system allowed us to define the (depended) function type. Therefor we don't
1076 have it as primitive expression. We are hopeful, that in the future we get an more
1077 mainstream language, like Coq or Agda, where the depended function is definable.
1078 As already mentioned in the introduction this would lead to a symmetrical language.

1079 A. Type action proof

1080 **Theorem 1.** $(\Gamma).A@id_\Gamma \leftrightarrow_T A$

1081 *Proof.* We show this by induction on the length of Γ

- $\Gamma = \epsilon$:

$$A \longleftrightarrow_T A$$

- $\Gamma = x : B, \Gamma'$:

$$(x : B, \Gamma').A@x@id_{\Gamma'} \longrightarrow_p (\Gamma').A@id_{\Gamma'}[x/x] = (\Gamma').A@id_{\Gamma'} \xleftrightarrow{IdH}_T A$$

1082

□

1083 **Theorem 2.** *The following rule holds*

$$\frac{x : A \vdash t : B \quad A \longleftrightarrow_T A'}{x : A' \vdash t : B}$$

1085 *Proof.* We show this by induction on t

□

1086 **Theorem 3.** *The typing rule (5) in the paper holds*

$$\frac{X : \Gamma_1 \rightarrow * \mid \Gamma' \vdash C : \Gamma \rightarrow * \quad \Gamma_1, x : A \vdash t : B}{\Gamma', \Gamma, x : \widehat{C}(\vec{A}) \vdash \widehat{C}(\vec{t}) : \widehat{C}(\vec{B})}$$

1088 *Proof.* First we will generalize the rule to

$$\frac{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \Gamma' \vdash C : \Gamma \rightarrow * \quad \Gamma_i, x : A_i \vdash t_i : B_i}{\Gamma', \Gamma, x : \widehat{C}(\vec{A}) \vdash \widehat{C}(\vec{t}) : \widehat{C}(\vec{B})}$$

1090 Then we gonna show it by Induction on the derivation \mathcal{D} of C

- $\mathcal{D} = \frac{}{\vdash \top : *} \text{ (}\top\text{-I)}$

Then the type actions got calculated as follows

$$\widehat{\top}(\vec{A}) = \widehat{\top}() = \top$$

$$\widehat{\top}(\vec{t}) = \widehat{\top}() = x$$

$$\widehat{\top}(\vec{B}) = \widehat{\top}() = \top$$

1092 We than got the following prooftree

$$1093 \quad \frac{\vdash \top : *}{x : \top \vdash x : \top} \text{ (Proj)}$$

$$1094 \quad \bullet \mathcal{D} = \frac{\frac{\mathcal{D}_1}{X_1 : \Gamma_1 \rightarrow *, \dots, X_{n-1} : \Gamma_{n-1}} \text{ TyCtx} \quad \frac{\mathcal{D}_2}{\Gamma_n \text{ Ctx}} \text{ TyVar-I}}{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \emptyset \vdash X_n : \Gamma_n \rightarrow *}$$

Again we calculate the type actions

$$\begin{aligned} \widehat{X_n}(\vec{A}) &= X_n[(\Gamma_i).\vec{A}/\vec{X}]@id_{\Gamma_n} = X_n[(\Gamma_n).A_n/X_n]@id_{\Gamma_n} = (\Gamma_n).A_n@id_{\Gamma_n} \\ \widehat{X_n}(\vec{t}) &= t_n \\ \widehat{X_n}(\vec{B}) &= X_n[(\Gamma_i).\vec{B}/\vec{X}]@id_{\Gamma_n} = X_n[(\Gamma_n).B_n/X_n]@id_{\Gamma_n} = (\Gamma_n).B_n@id_{\Gamma_n} \end{aligned}$$

1095 We know from the first premise that $\Gamma = \Gamma_n$ and $\Gamma' = \emptyset$

1096 Here we got the prooftree

$$1097 \quad \frac{\frac{\Gamma_n, x : A \vdash t : B}{\Gamma_n, x : (\Gamma_n).A@id_{\Gamma_n} \vdash t : B} \text{Thrm. 1} \quad \frac{A \longleftrightarrow_T (\Gamma_n).A@id_{\Gamma_n}}{\Gamma_n, x : (\Gamma_n).A@id_{\Gamma_n} \vdash t_n : (\Gamma_n).B@id_{\Gamma_n}} \text{Thrm. 2}}{B \longleftrightarrow_T (\Gamma_n).B@id_{\Gamma_n}} \text{Thrm. 1 Conv}$$

$$1098 \quad \bullet \mathcal{D} = \frac{\frac{\mathcal{D}_1}{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \mid \Gamma' \vdash C : \Gamma \rightarrow *} \quad \frac{\mathcal{D}_2}{\Gamma_n \text{ Ctx}} \text{ (TyVar-Weak)}}{X_1 : \Gamma_1 \rightarrow *, \dots, X_{n+1} : \Gamma_{n+1} \rightarrow * \mid \Gamma' \vdash C : \Gamma \rightarrow *}$$

1099 Here we got the prooftree

$$1100 \quad \frac{\frac{X_1 : \Gamma_1 \rightarrow *, \dots, X_{n+1} : \Gamma_{n+1} \rightarrow * \mid \Gamma' \vdash C : \Gamma \rightarrow *}{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \Gamma' \vdash C : \Gamma \rightarrow *} (*) \quad \Gamma_i, x : A_i \vdash t_i : B_i}{\Gamma', \Gamma, x : \underbrace{\widehat{C}(\vec{A})}_{\equiv \widehat{C}(\vec{A}, A_{n+1})} \vdash \underbrace{\widehat{C}(\vec{t})}_{\equiv \widehat{C}(\vec{t}, t_{n+1})} : \underbrace{\widehat{C}(\vec{B})}_{\equiv \widehat{C}(\vec{B}, B_{n+1})}} \text{IdH.}$$

1101 (*) Here we undo **(TyVar-Weak)**

1102 (**) X_{n+1} doesn't occur free in C, otherwise \mathcal{D}_1 wouldn't be possible

1103 (***) Case for **(TyVar-Weak)** of type actions on terms

1104 $\bullet \mathcal{D} =$

$$1105 \quad \frac{\frac{\mathcal{D}_1}{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \mid \Gamma' \vdash C : \Gamma \rightarrow *} \quad \frac{\mathcal{D}_2}{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \mid \Gamma' \vdash D : *}}{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \Gamma', y : D \vdash C : \Gamma \rightarrow *} \text{ (Ty-Weak)}$$

1106 Here we got the prooftree

$$1107 \quad \frac{\frac{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \Gamma', y : D \vdash C : \Gamma \rightarrow *}{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \Gamma' \vdash C : \Gamma \rightarrow *} (*) \quad \Gamma_i, x : A_i \vdash t_i : B_i}{\Gamma', \Gamma, x : \widehat{C}(\vec{A}) \vdash \widehat{C}(\vec{t}) : \widehat{C}(\vec{B})} \text{IdH.} \quad \frac{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \mid \Gamma' \vdash D : *}{\Gamma', \Gamma, x : \widehat{C}(\vec{A})y \vdash \widehat{C}(\vec{t}) : \widehat{C}(\vec{B})} \text{ (Term-Weak)}$$

1109 (*) Here we undo **(Ty-Weak)**

$$1110 \bullet \mathcal{D} = \frac{X_1 : \Gamma_1, \dots, X_n : \Gamma_n \mid \Gamma' \vdash C' : (y : D, \Gamma) \rightarrow * \quad \Gamma' \vdash s : D}{X_1 : \Gamma_1, \dots, X_n : \Gamma_n \mid \Gamma' \vdash C' @s : \Gamma \rightarrow *} \text{ (Ty-Inst)}$$

1111 Then we got the following induction hypothesis

$$1112 \frac{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \Gamma' \vdash C' : (y : D, \Gamma) \rightarrow * \quad \Gamma_i, x : A_i \vdash t_i : B_i}{\Gamma', y : D, \Gamma, x : \widehat{C'}(\vec{A}) \vdash \widehat{C'}(\vec{t}) : \widehat{C'}(\vec{B})}$$

Calculated type actions:

$$\widehat{C' @s}(\vec{A}) = C' @s[(\Gamma_i). \vec{A} / \vec{X}] @id_\Gamma = C'[(\Gamma_i). \vec{A} / \vec{X}] @s @id_\Gamma = \widehat{C'}(\vec{A})[s/y]$$

$$\widehat{C' @s}(\vec{t}) = \widehat{C'}(\vec{t})[s/y]$$

$$\widehat{C' @s}(\vec{B}) = C' @s[(\Gamma_i). \vec{B} / \vec{X}] @id_\Gamma = C'[(\Gamma_i). \vec{B} / \vec{X}] @s @id_\Gamma = \widehat{C'}(\vec{B})[s/y]$$

1113 We then got the following proof tree

$$1114 \frac{\frac{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \Gamma'_2 \vdash C' @s : \Gamma_2[s/y] \rightarrow *}{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \Gamma'_2 \vdash C' : (y : D, \Gamma_2) \rightarrow *} (*) \quad \Gamma_i, x : A_i \vdash t_i : B_i}{\frac{\Gamma'_2, y : D, \Gamma_2, x : \widehat{C'}(\vec{A}) \vdash \widehat{C'}(\vec{t}) : \widehat{C'}(\vec{B})}{\Gamma'_2, \Gamma_2[s/y], x : \widehat{C'}(\vec{A})[s/y] \vdash \widehat{C'}(\vec{t})[s/y] : \widehat{C'}(\vec{B})[s/y]}} \text{ IdH.}$$

1115 (*) This is the reverse of **(Ty-Inst)**.

$$1116 \bullet \mathcal{D} = \frac{X_1 : \Gamma_1, \dots, X_n : \Gamma_n \mid \Gamma', y : D \vdash C' : \Gamma \rightarrow *}{X_1 : \Gamma_1, \dots, X_n : \Gamma_n \mid \Gamma' \vdash (y). C' : (y : D, \Gamma) \rightarrow *} \text{ (Param-Abstr)}$$

Calculated type actions:

$$\begin{aligned} (\widehat{y}). \widehat{C'}(\vec{A}) &= (y). C'[(\Gamma_i). \vec{A} / \vec{X}] @id_\Gamma \\ &= (y). (C'[(\Gamma_i). \vec{A} / \vec{X}]) @y @id_\Gamma \\ &\longleftrightarrow_T (C'[(\Gamma_i). \vec{A} / \vec{X}]) @id_\Gamma \\ &= \widehat{C'}(\vec{A}) \\ (\widehat{y}). \widehat{C'}(\vec{t}) &= \widehat{C'}(\vec{t}) \\ (\widehat{y}). \widehat{C'}(\vec{B}) &= (y). C'[(\Gamma_i). \vec{B} / \vec{X}] @id_\Gamma \\ &= (y). (C'[(\Gamma_i). \vec{B} / \vec{X}]) @y @id_\Gamma \\ &\longleftrightarrow_T (C'[(\Gamma_i). \vec{B} / \vec{X}]) @id_\Gamma \\ &= \widehat{C'}(\vec{B}) \end{aligned}$$

1117 The proof tree then becomes the following

Appendix A. Type action proof

$$\begin{array}{c}
1118 \quad \frac{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \Gamma' \vdash (y).C' : (y : D, \Gamma) \rightarrow *}{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid y : D, \Gamma' \vdash C' : \Gamma \rightarrow *} (*) \\
1119 \quad \frac{\Gamma_i, x : A_i \vdash t_i : B_i}{y : D, \Gamma', \Gamma, x : \widehat{C'}(\vec{A}) \vdash \widehat{C'}(\vec{t}) : \widehat{C'}(\vec{B})} \text{IdH.}
\end{array}$$

1120 (*) This is the reverse of **(Param-Abstr)**.

1121 • $\mathcal{D} =$

$$\begin{array}{c}
1122 \quad \frac{\mathcal{D}_1 \quad \mathcal{D}_2 \quad \sigma_k : \Delta_k \triangleright \Gamma \quad X_1 : \Gamma_1 \rightarrow *, \dots, X_n \rightarrow *, X : \Gamma \rightarrow * \mid \Delta_k \vdash D_k : *}{X_1 : \Gamma_1 \rightarrow *, \dots, X_n \rightarrow * \mid \emptyset \vdash \mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D}) : \Gamma \rightarrow *} \text{(FP-Ty)}
\end{array}$$

1123 From this we know $\Gamma' = \emptyset$

Calculated type actions:

$$\begin{aligned}
& \mu(Y : \widehat{\Gamma \rightarrow *}; \vec{\sigma}; \vec{D})(\vec{A}) \\
&= \mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\overline{\Gamma_i}).\vec{A} / \vec{X}]@_{\text{id}_\Gamma} \\
&= \mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\overline{\Gamma_i}).\vec{A} / \vec{X}]@_{\text{id}_\Gamma} \\
& \mu(Y : \widehat{\Gamma \rightarrow *}; \vec{\sigma}; \vec{D})(\vec{t}) \\
&= \text{rec}^{\mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\overline{\Gamma_i}).\vec{A} / \vec{X}]}(\Delta_k, x). \alpha_k @_{\text{id}_{\Delta_k}} @_{\widehat{D_k}}(\vec{t}, x) @_{\text{id}_\Gamma} @_x \\
& \mu(Y : \widehat{\Gamma \rightarrow *}; \vec{\sigma}; \vec{D})(\vec{B}) \\
&= \mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\overline{\Gamma_i}).\vec{B} / \vec{X}]@_{\text{id}_\Gamma} \\
&= \mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\overline{\Gamma_i}).\vec{B} / \vec{X}]@_{\text{id}_\Gamma}
\end{aligned}$$

From the assumptions

$$\begin{array}{c}
X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \emptyset \vdash \mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D}) : \Gamma \rightarrow * \\
\Gamma_i, x : A_i \vdash t_i : B_i
\end{array}$$

We have to proof that in **Ctx**

$$\Gamma, x : \mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\overline{\Gamma_i}).\vec{A} / \vec{B}]@_{\text{id}_\Gamma}$$

the expression

$$\text{rec}^{\mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\overline{\Gamma_i}).\vec{A} / \vec{X}]}(\Delta_k, y). \alpha_k @_{\text{id}_{\Delta_k}} @_{\widehat{D_k}}(t, y) @_{\text{id}_\Gamma} @_x$$

has type

$$\mu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\overline{\Gamma_i}).\vec{B} / \vec{X}]@_{\text{id}_\Gamma}$$

1124 We can use the induction hypothesis

$$\frac{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow *, Y : \Gamma_{n+1} \rightarrow * \mid \Delta_k \vdash D_k : * \quad \Gamma_i, x : A_i \vdash t_i : B_i}{\Delta_k, x : \widehat{D}_k(\vec{A}, A_{n+1}) \vdash \widehat{D}_k(\vec{t}, y) : \widehat{D}_k(\vec{B}, B_{n+1})} \quad 1125$$

See A.1 for a proof of it. 1126

• $\mathcal{D} =$ 1127

$$\frac{\begin{array}{c} \mathcal{D}_1 \\ \sigma_k : \Delta_k \triangleright \Gamma \end{array} \quad \begin{array}{c} \mathcal{D}_2 \\ X_1 : \Gamma_1 \rightarrow *, \dots, X_n \rightarrow *, X : \Gamma \rightarrow * \mid \Delta_k \vdash D_k : * \end{array}}{X_1 : \Gamma_1 \rightarrow *, \dots, X_n \rightarrow * \mid \emptyset \vdash \nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D}) : \Gamma \rightarrow *} \text{ (FP-Ty)} \quad 1128$$

From this we know $\Gamma' = \emptyset$. 1129

Calculated type actions:

$$\begin{aligned} & \nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})(\vec{A}) \\ &= \nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\Gamma_i). \vec{A} / \vec{X}]@id_\Gamma \\ &= \nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\Gamma_i). \vec{A} / \vec{X}]@id_\Gamma \\ & \nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})(\vec{t}) \\ &= \text{corec}^{\nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\Gamma_i). \vec{B} / \vec{X}]}(\Delta_k, x) \widehat{D}_k(\vec{t}, x)[(\xi_k @ id_{\Delta_k} @ x) / x]@id_\Gamma @ x \\ & \nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})(\vec{B}) \\ &= \nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\Gamma_i). \vec{B} / \vec{X}]@id_\Gamma \\ &= \nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\Gamma_i). \vec{B} / \vec{X}]@id_\Gamma \end{aligned}$$

From the assumptions

$$\begin{array}{l} X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow * \mid \emptyset \vdash \nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D}) : \Gamma \rightarrow * \\ \Gamma_i, x : A_i \vdash t_i : B_i \end{array}$$

We have to proof that in **Ctx**

$$\Gamma, x : \nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\Gamma_1). A / X]@id_\Gamma$$

the expression

$$\text{corec}^{\nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\Gamma_i). \vec{B} / \vec{X}]}(\Delta_k, x) \widehat{D}_k(\vec{t}, x)[(\xi_k @ id_{\Delta_k} @ x) / x]@id_\Gamma @ x$$

has type

$$\nu(Y : \Gamma \rightarrow *; \vec{\sigma}; \vec{D})[(\Gamma_i). \vec{B} / \vec{X}]@id_\Gamma$$

We can use the induction hypothesis 1130

$$\frac{X_1 : \Gamma_1 \rightarrow *, \dots, X_n : \Gamma_n \rightarrow *, Y : \Gamma_{n+1} \rightarrow * \mid \Delta_k \vdash D_k : * \quad \Gamma_i, y_k : A_i \vdash t_i : B_i}{\Delta_k, y_k : \widehat{D}_k(\vec{A}, A_{n+1}) \vdash \widehat{D}_k(\vec{t}, y) : \widehat{D}_k(\vec{B}, B_{n+1})} \quad 1131$$

See A.1 for this proof. 1132

1133

□

Bibliography

- [Abe10] Andreas Abel. Miniagda: Integrating sized and dependent types. *arXiv preprint arXiv:1012.4896*, 2010.
- [agd] Universe levels - agda 2.6.1.1 documentation.
- [APTS13] Andreas Abel, Brigitte Pientka, David Thibodeau, and Anton Setzer. Copatterns: programming infinite structures by observations. *ACM SIGPLAN Notices*, 48(1):27–38, 2013.
- [BG16] Henning Basold and Herman Geuvers. Type theory based on dependent inductive and coinductive types. In *Proceedings of the 31st Annual ACM/IEEE Symposium on Logic in Computer Science*, pages 327–336, 2016.
- [BJSO19] David Binder, Julian Jabs, Ingo Skupin, and Klaus Ostermann. Decomposition diversity with symmetric data and codata. *Proceedings of the ACM on Programming Languages*, 4(POPL):1–28, 2019.
- [Chl13] Adam Chlipala. *Certified programming with dependent types: a pragmatic introduction to the Coq proof assistant*. MIT Press, 2013.
- [DB72] Nicolaas Govert De Bruijn. Lambda calculus notation with nameless dummies, a tool for automatic formula manipulation, with application to the church-rosser theorem. In *Indagationes Mathematicae (Proceedings)*, volume 75, pages 381–392. North-Holland, 1972.
- [Our08] Nicolas Oury, 06 2008. Message on the coq-clup mailing list.
- [ST14] Matthieu Sozeau and Nicolas Tabareau. Universe polymorphism in coq. In *International Conference on Interactive Theorem Proving*, pages 499–514. Springer, 2014.