42114: Integer Programming

Solutions

Week 5

Updated: September 29, 2022

Week 5: Branch and Bound I

Wolsey Exercise 7.1

(i) Since the problem is a minimization problem, an upper bound will be a feasible solution and hence any upper bound on any node can be used. The tightest (lowest) upper bound is then:

$$UB = \min\{31, 32\} = 31$$

from node 7.

A lower bound will only be the lower bound for the sub problem of the node. Therefore we propagate the worst (highest) lower bounds up through the tree.
Node 8 is infeasible and can thus be pruned. The lower bound of node 4 is then 31 since node 7 is the only child. The lower bound of node 1 is then 28 since this is the worst of the lower bounds of its children (node 3: 28 and node 4: 31). The lower bound of node 2 is 27 since this is the worst lower bound of its children (node 5: 27 and node 6: 35). Finally the lower bound of node 0 and thus of the optimal value z is 27 since this is the worst lower bound of its children (node 1: 28 and node 2: 27).
So the tightest bounds we can give is $(LB, UB) = (27, 31)$.

(ii) Node 8 can be pruned by infeasibility since it is infeasible.
Node 7 can be pruned by optimality since its bounds are equal: $LB^7 = UB^7 = 31$.
Node 6 can be pruned by bound since the lower bound of node 6 is worse than the current upper bound: $35 = LB^6 \geq UB = 31$.

Nodes 3 and 5 must be explored further since they are not pruned.

Wolsey Exercise 7.2

We draw the constraints graphically in Figure 1 and mark the area of feasible solutions.

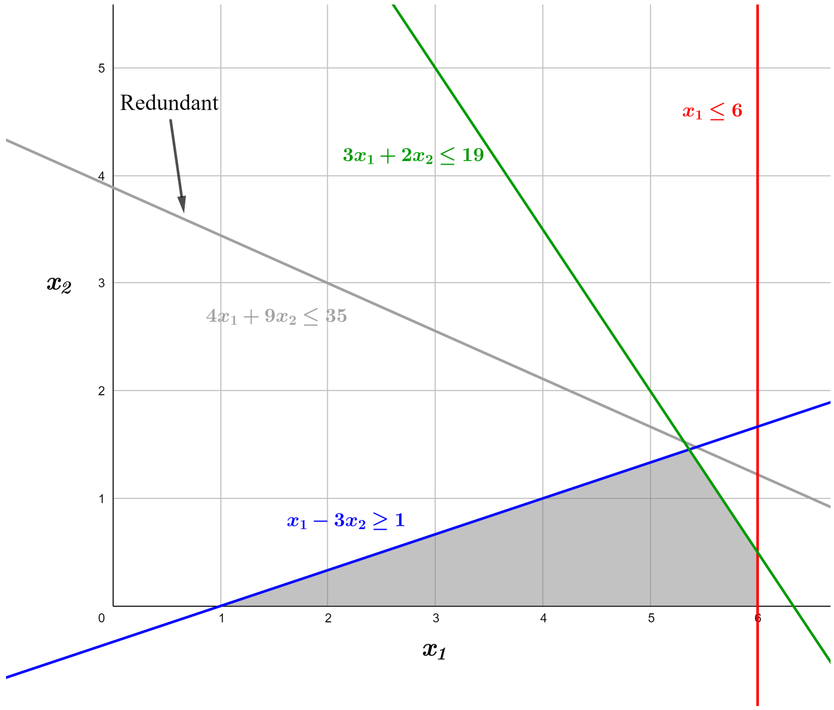

Figure 1: All constraints plotted. The grayed out area is the feasible solution space.

Looking at Figure 1 it seems that the first constraint in the model is redundant since constraints 3 and 4 ($x_1 - 3x_2 \geq 1$ and $3x_1 + 2x_2 \leq 19$) cut off any points that would violate constraint 1. We can therefore ignore the first constraint.

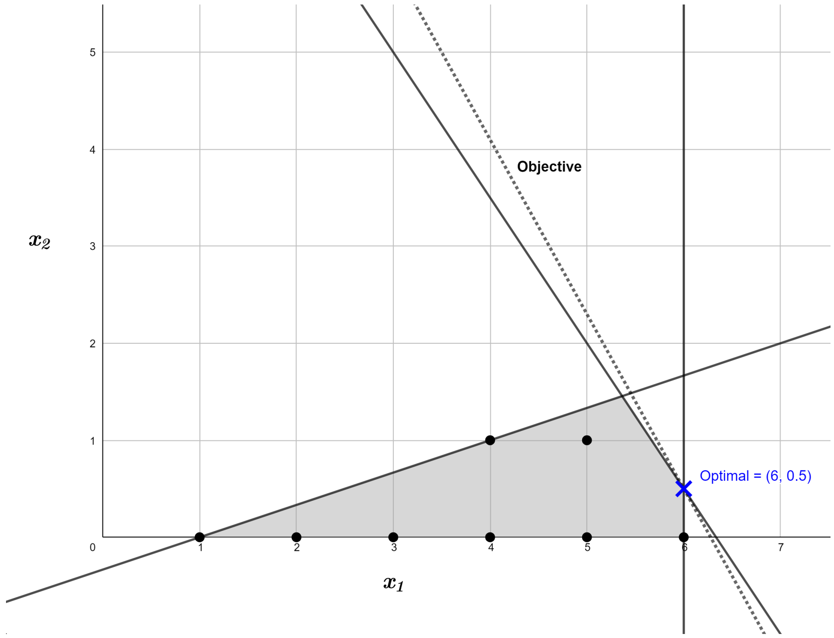

Figure 2: All (non-redundant) constraints plotted. The grayed out area is the feasible solution space. The dashed line is the objective function for the LP optimum. All feasible integer solutions are marked with dots.

Looking at Figure 2 and considering the slope of the objective function, we can quite easily see that $(x_1, x_2) = (6, 0)$ is the optimal integer solution with value 54.
However, to find the optimal integer solution by branch and bound, we first find the LP optimum. In Figure 2 we can see that the LP optimum is in the intersection between the 2nd and 4th constraints: $(x_1, x_2) = (6, \frac{1}{2})$ with a value of 56.5.

Since x_2 is fractional we create two branches for $x_2 \leq 0$ and $x_2 \geq 1$. First we consider the branch with $x_2 \leq 0$ and add this new constraint to the plot as seen in Figure 3.

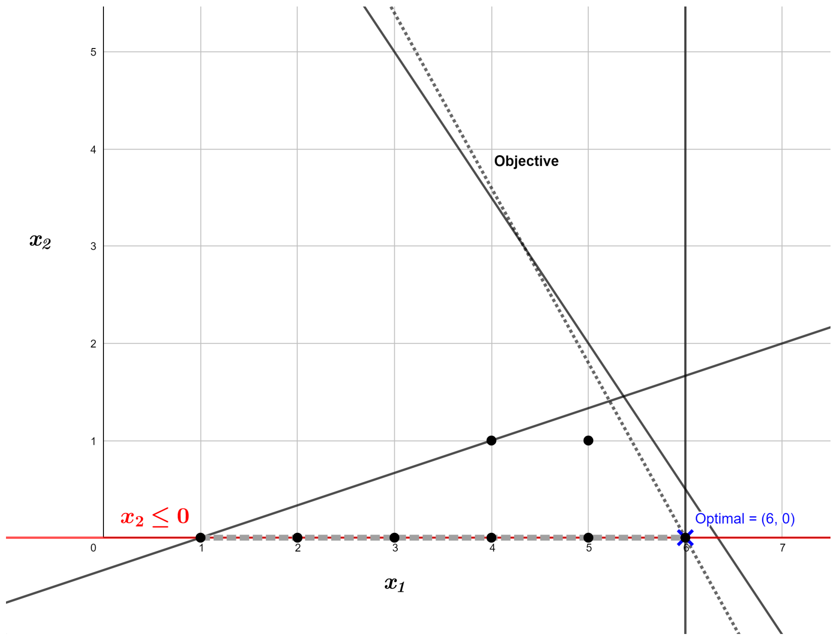

Figure 3: The branch constraint $x_2 \leq 0$ is added to the original non-redundant constraints.

We see in Figure 3 that the feasible solution space now only contains the points on the x_1-axis between 1 and 6 as marked by the gray line. The LP optimum is found to be 6 for the solution $(x_1, x_2) = (6, 0)$ with a value of 54. We see that the solution is integer and can therefore prune the branch by optimality. We also note that we now have a lower bound of 54.

Next we consider the branch with $x_2 \geq 1$ and add this constraint to the plot instead, as seen in Figure 4.

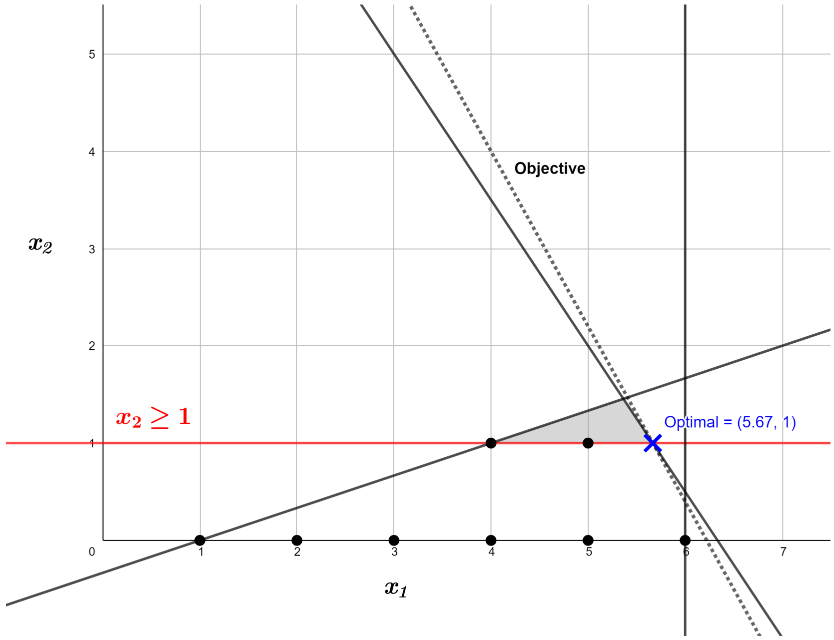

Figure 4: The branch constraint $x_2 \geq 1$ is added to the original non-redundant constraints.

In Figure 4 we find the optimal LP solution to be the intersection between the branch constraint and constraint 4 ($x_2 \geq 1$ and $3x_1 + 2x_2 \leq 19$), that is $(x_1, x_2) = (\frac{17}{3}, 1)$ with value 56.
Since x_1 is fractional we continue branching for $x_1 \leq 5$ and $x_1 \geq 6$. We consider the branch with $x_1 \leq 5$ first and add this constraint as seen in Figure 5.

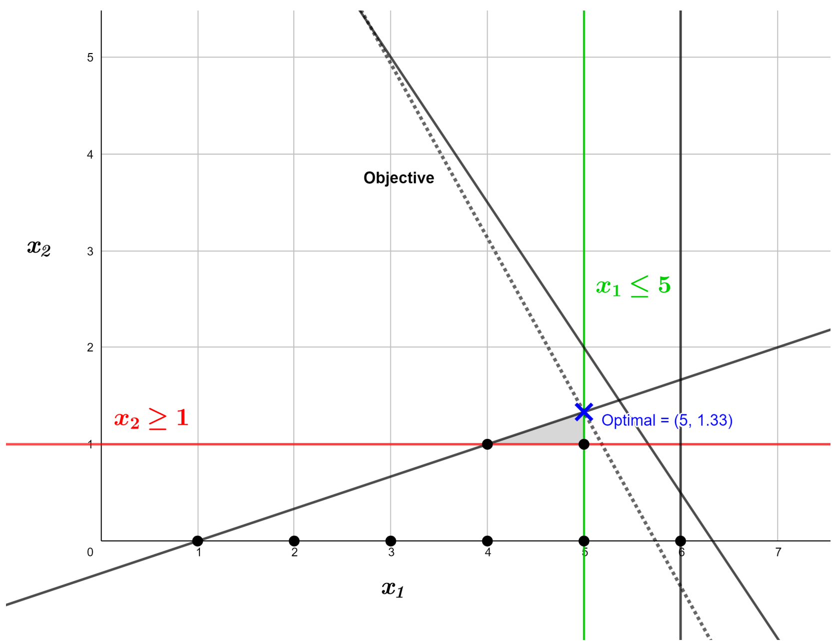

Figure 5: The branch constraints $x_2 \geq 1$ and $x_1 \leq 5$ are added to the original non-redundant constraints.

The optimal LP solution is found at the intersection between the new branch constraint and constraint 3 ($x_1 \leq 5$ and $x_1 - 3x_2 \geq 1$), that is $(x_1, x_2) = (5, \frac{4}{3})$ with value $51\frac{2}{3}$. We notice that the value of the solution is worse than our current lower bound of 54. Hence we can prune the branch by bound.

We then consider the other branch with $x_1 \geq 6$ and add this constraint as seen in Figure 6.

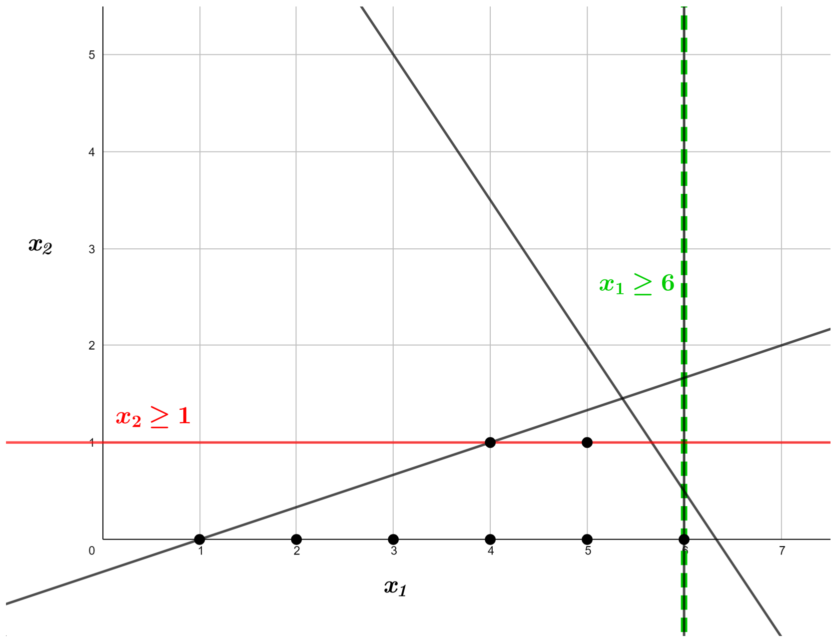

Figure 6: The branch constraints $x_2 \geq 1$ and $x_1 \geq 6$ are added to the original non-redundant constraints.

With the new branch constraint added, there are no feasible solutions left. A solution must have $x_1 = 6$ (since both $x_1 \leq 6$ and $x_1 \geq 6$) and $x_2 \geq 1$. But then constraint 4 in the original problem is violated since it must be at least $3 \cdot 6 + 2 \cdot 1 = 20 \nleq 19$. Hence we can prune the branch by infeasibility.

Then we are done and $(6, 0)$ must be the optimal IP solution with a value of 54.

The branch and bound tree is shown in Figure 7.

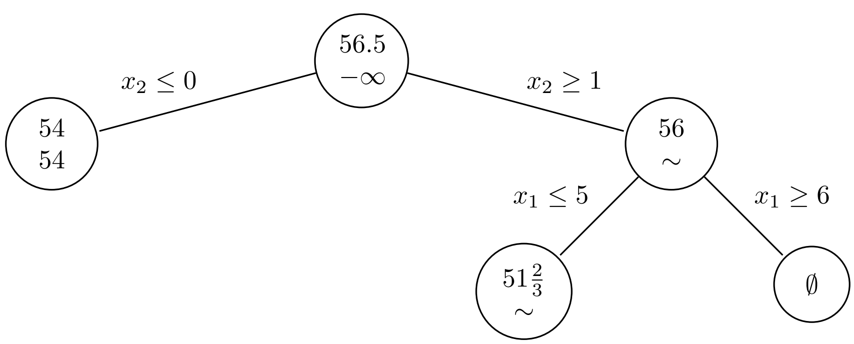

Figure 7: Branch & Bound tree for the problem. Each node contains the upper and lower bound of the sub problem. $\emptyset$ indicates infeasibility.

Wolsey Exercise 7.3

(i) The items are ordered in non-increasing order of "value per weight" and the items with the highest value per weight are selected first. One element, denoted r, is selected partially, which is allowed since we are considering the LP relaxation.

Let us assume that the solution is not optimal.
Then since it is an LP problem, there must exist a pair of variables, x_i and x_j, such that reducing x_i (removing some of a selected item) and increasing x_j (adding some of the item which would be better to select) would improve the solution. All variables must be non-negative, so the variable to reduce must be one of the selected variables (i.e. x_i, $1 \le i \le r$). Also the variable to increase can only be a variable that is less than 1, (i.e. x_j, $r \le j \le n$). But then $\frac{c_i}{a_i} \ge \frac{c_j}{a_j}$ (from the assumptions in the exercise).
Decreasing x_i by an amount $\delta > 0$ will free up $a_i\delta$ space in the knapsack and thus allow us to increase x_j by at most $\frac{a_i}{a_j}\delta$ using the free space.

This will make a change in the objective value of $-c_i\delta + c_j\frac{a_i}{a_j}\delta = (-c_i + c_j\frac{a_i}{a_j})\delta$. Since $\delta > 0$ we can remove it without changing the sign of the change: $-c_i + c_j\frac{a_i}{a_j}$. From the prior assumptions in the exercise we know that $\frac{c_i}{a_i} \ge \frac{c_j}{a_j} \to c_i \ge \frac{c_j a_i}{a_j}$. But then $-c_i + c_j\frac{a_i}{a_j} \le 0 \to (-c_i + c_j\frac{a_i}{a_j})\delta \le 0$.

So the resulting objective function value is at most as good as the previous one and hence the previous must have been optimal.

(ii) We notice that the x_i's are actually sorted in non-increasing order with respect to the "value per weight":

i	1	2	3	4
c_i/a_i	3.4	3.33	3.125	2.429

This means that we can use (i) for finding the LP solutions.
The branch and bound tree is shown in Figure 8.

The LP optimum in node 0 is:
$x_1 = x_2 = 1$, $x_3 = \dfrac{12-(5+3)}{8} = \frac{1}{2}$, $x_4 = 0$
with value 39.5. Since x_3 is the only non-integer variable we branch on it. This gives nodes 1 and 2 for $x_3 = 0$ and $x_3 = 1$.

Node 1 with $x_3 = 0$ has the solution:
$x_1 = x_2 = 1$, $x_3 = 0$, $x_4 = \dfrac{12-(5+3)}{7} = \frac{4}{7}$
with value $36\frac{4}{7}$. Since x_4 is non-integer we branch on it, creating nodes 3 and 4 for $x_4 = 0$ and $x_4 = 1$.

Node 2 with $x_3 = 1$ has the solution:
$x_1 = \dfrac{12-8}{5} = \frac{4}{5}$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$
with value $38\frac{3}{5}$. Since x_1 is non-integer we branch on it, creating nodes 5 and 6 for $x_1 = 0$ and $x_1 = 1$.

Node 3 with $x_3 = x_4 = 0$ has the solution:
$x_1 = x_2 = 1$, $x_3 = x_4 = 0$

with value 27. Since all variables are integer, this is a feasible solution and thus a globally valid lower bound. Since the node is optimal we can prune it by optimality.

Node 4 with $x_3 = 0$, $x_4 = 1$ has the solution:
$x_1 = 1$, $x_2 = 0$, $x_3 = 0$, $x_4 = 1$
with value 34. Since all variables are integer, this is a feasible solution and thus also a globally valid lower bound. This is a better lower bound than node 3. Since the node is optimal we can prune it by optimality.

Node 5 with $x_1 = 0$, $x_3 = 1$ has the solution:
$x_1 = 0,\ x_2 = x_3 = 1,\ x_4 = \dfrac{12 - (3 + 8)}{7} = \frac{1}{7}$
with value $37\frac{3}{7}$. Since x_4 is non-integer we branch on it, creating nodes 7 and 8 for $x_4 = 0$ and $x_4 = 1$.

Node 6 with $x_1 = 1$, $x_3 = 1$ is infeasible since the knapsack constraint is violated: $5 \cdot x_1 + 8 \cdot x_3 = 13 \nleq 12$. Hence the node is pruned by infeasibility.

Node 7 with $x_1 = 0$, $x_3 = 1$, $x_4 = 0$ has the solution:
$x_1 = 0$, $x_2 = x_3 = 1$, $x_4 = 0$
with value 35. Since all variables are integer, this is a feasible solution and thus also a globally valid lower bound. This is a better lower bound than node 3 and node 4. Since the node is optimal we can prune it by optimality.

Node 8 with $x_1 = 0$, $x_3 = 1$, $x_4 = 1$ is infeasible since the knapsack constraint is violated: $8 \cdot x_3 + 7 \cdot x_4 = 15 \nleq 12$. Hence the node is pruned by infeasibility.

All nodes have been considered an so, since node 7 has the best integer solution it is optimal.
I.e. the optimal integer solution is $x_2 = x_3 = 1$, $x_1 = x_4 = 0$ with value 35.

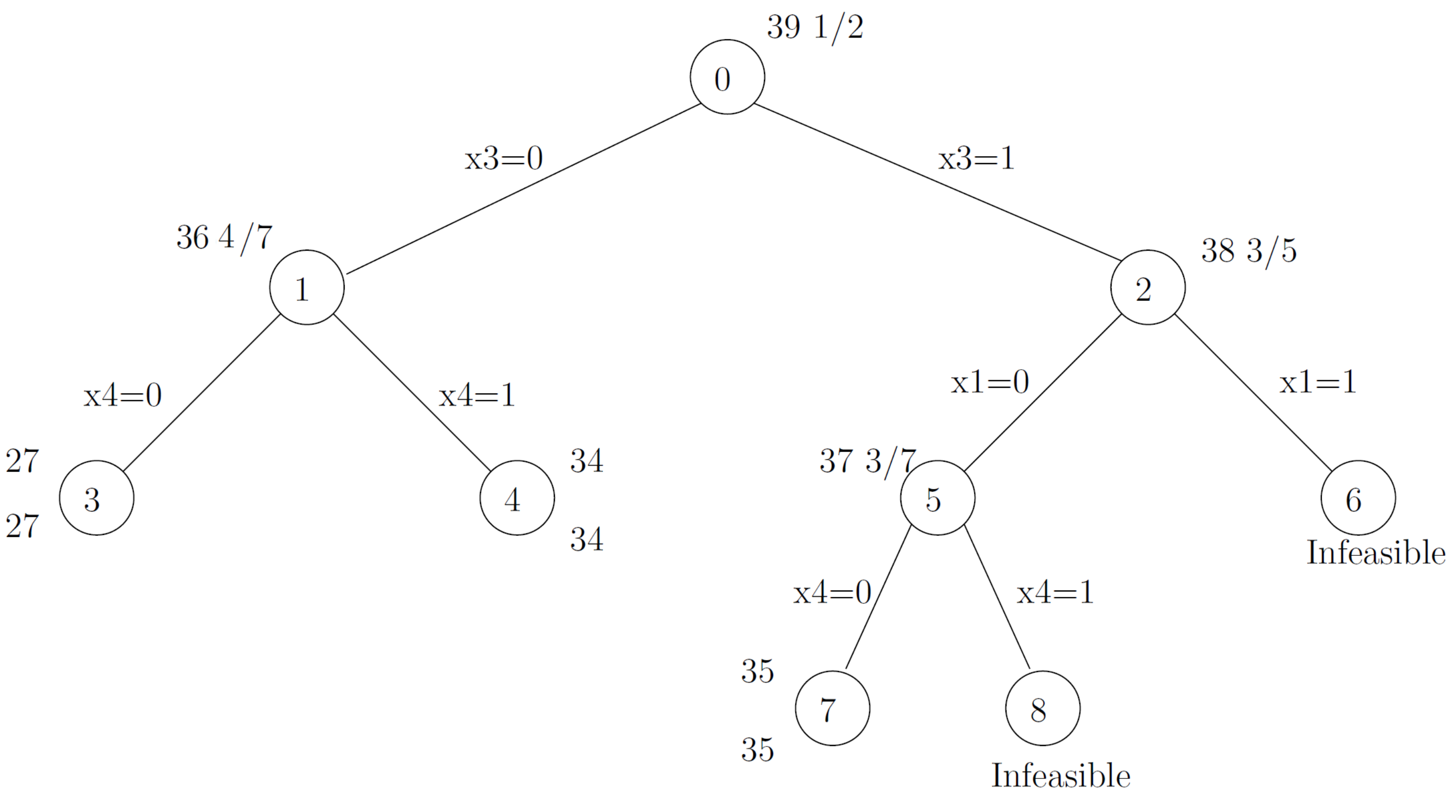

Figure 8: Branch and bound tree for exercise 7.3.

Wolsey Exercise 7.9

Looking at the first constraint we notice that only x_3 has a negative coefficient. So if x_1, x_2, x_4 or x_5 is 1 and then x_3 has to be 1 as well for the lhs to be non-positive and satisfy the constraint. Therefore we can make four inequalities:

$$x_1 \leq x_3, \quad x_2 \leq x_3, \quad x_4 \leq x_3, \quad x_5 \leq x_3$$

Furthermore we notice that if both x_2 and x_5 are 1, then even with $x_3 = 1$ the constraint will be violated. Hence at most one of them can be 1:

$$x_2 + x_5 \leq 1$$

Looking at the second constraint we notice that if $x_3 = 1$ then even if $x_1 = x_4 = x_5 = 1$ the lhs would only be -3 and the constraint would still be violated. So if $x_3 = 1$ then we must also have $x_2 = 1$:

$$x_2 \geq x_3$$

Looking at the last constraint, we see that for the lhs to be less than -2, x_3 has to be 1, and x_2 has to be 0. So:

$$x_2 = 0, \quad x_3 = 1$$

But then from the above we have that $\begin{cases} x_2 \leq x_3 \\ x_2 \geq x_3 \end{cases} \rightarrow x_2 = x_3$, which means that $x_2 = x_3 = 1$ but also that $x_3 = x_2 = 0$ which is a contradiction.
Hence the problem is infeasible and has no feasible solutions.

Another way to reach this conclusion is since $x_2 = 0$ from the third constraint, and $x_3 = x_2 = 0$ from the second and first constraints, then from the first constraint we have that $x_1 = x_4 = x_5 = 0$ since they should all be at most x_3. But then the second and third constraints are not satisfied and the problem is again infeasible.

IP Exam 2014: Question 2 (15%)

Subquestion 2.1

For a maximization problem that is branched into $k = 1, ..., K$ leaves, we have that

- $\bar{z} = \max_k \bar{z}^k$ is a global upper bound and hence an upper bound on z.
- $\underline{z} = \max_k \underline{z}^k$ is the best/tightest lower bound and hence the best/tightest lower bound on z.

So we find the tightest upper bound:

$$\bar{z} = \max_{k \in \{P_3, P_4, P_5, P_6\}} \bar{z}^k = \max\{72, -, 69, 70\} = 72$$

where we use the upper bound of P_2 on P_6 as a substitute for the missing bound calculation, and the tightest lower bound:

$$\underline{z} = \max_{k \in \{P_1, \ldots, P_6\}} \underline{z}^k = \max\{56, 56, 58, 60, 0, 61\} = 61$$

Subquestion 2.2

We can prune the tree below node P_6 in three ways:

- Prune by optimality: If $58 \leq \underline{z}^{P_6} = \bar{z}^{P_6} \leq 70$ the solution is an integer solution and is thus optimal for the sub problem. Then we can prune by optimality.
- Prune by bound: If $\bar{z}^{P_6} \leq \underline{z}$, the best solution in the tree below node P_6 will be worse than the best feasible integer solution currently found. Then we can prune by bound.
- Prune by infeasibility: If the branching on P_2 leaves the set of feasible solutions for P_6 empty, i.e. the sub problem P_6 is not feasible, then we can prune by infeasibility.

Subquestion 2.3

Since nodes P_3 and P_5 are still open, to close them the lower bound at node P_6 must be at least $\underline{z}^{P_6} \geq 72$, s.t. we can prune nodes P_3 and P_5 by bound. To also prune P_6, it must be an optimal solution hence $\bar{z}^{P_6} = \underline{z}^{P_6} \geq 72$.
But since P_6 is a branch of P_2 the optimal solution can at most be $\bar{z}^{P_6} = \underline{z}^{P_6} \leq 70$. Hence it is not possible to close all nodes of the tree from examining node P_6.

Julia Code

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