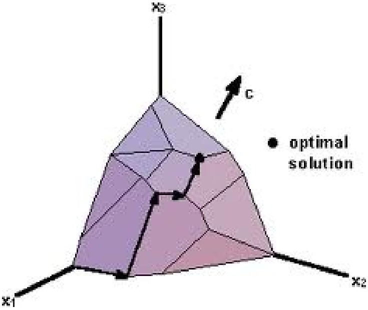
Linear Programming

Jesper Larsen[1]

[1]Department of Management Engineering
Technical University of Denmark

Simplex Method

- Convert $\leq$ inequalities by adding slack variables
- Put data into simplex tableau
- Perform simplex iterations by pivoting
- Entering Variable (*pivot column*)
  - Most negative coefficient in top row
- Leaving Variable (*pivot row*)
  - Minimum ratio: right hand sides and positive pivot column entries
- We disregard complications here
  - Phase 1, no feasible solution, unbounded solutions



First and Final Tableau

$$\begin{aligned} \max \quad & 3x_1 + 4x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 4 \\ & 2x_1 + x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{aligned}$$

Z	x_1	x_2	s_1	s_2	
1	-3	-4			0
	1	1	1		4
	2	1		1	5

Z	x_2	s_2	x_1	s_1	
1	0	0	1	4	16
	1		1	1	4
		1	1	-1	1

Matrix formulation

- General LP

$$\begin{array}{lrl} \text{maximize} & \mathbf{c}^T\mathbf{x} & \\ \text{subject to:} & A\mathbf{x} & \leq \mathbf{b} \\ & \mathbf{x} & \geq 0 \end{array}$$

- becomes ..

$$\begin{array}{lrl} \text{maximize} & \mathbf{c}^T\mathbf{x} + 0\mathbf{s} & \\ \text{subject to:} & A\mathbf{x} + I\mathbf{s} & = \mathbf{b} \\ & \mathbf{x} & \geq 0 \\ & \mathbf{s} & \geq 0 \end{array}$$

Matrix formulation
Continued

$$
\begin{array}{lrl}
\text{maximize} & \mathbf{c}_N^T \mathbf{x}_\mathrm{B} + \mathbf{c}_N^T \mathbf{x}_\mathrm{N} & \\
\text{subject to:} & B\mathbf{x}_\mathrm{B} + N\mathbf{x}_\mathrm{N} & = \mathbf{b} \\
 & \mathbf{x}_\mathrm{B} & \geq 0 \\
 & \mathbf{x}_\mathrm{N} & \geq 0
\end{array}
$$

First and Later Tableau

First tableau ...

Z	$\mathbf{x}$	$\mathbf{s}$	
1	$-c$	0	0
	A	I	b

Later tableau ...

Z	$\mathbf{x}$	$\mathbf{s}$	
1	$c_B B^{-1} A - c$	$c_B B^{-1}$	$c_B B^{-1} b$
	$B^{-1} A$	B^{-1}	$B^{-1} b$

- The current solution is $x_B = B^{-1} b,\ x_N = 0,\ Z = c_B B^{-1} b$
- At optimality we have $c_B B^{-1} \geq 0, c_B B^{-1} A \geq c$
- The dual values are $c_B B^{-1}$

Duality

The *primal* problem

$$
\begin{array}{llrrl}
\max & Z_P = & 3x_1 & +4x_2 & \\
\text{s.t.} & & x_1 & +x_2 & \leq 4 \\
& & 2x_1 & +x_2 & \leq 5 \\
& & x_1 & & \geq 0 \\
& & & x_2 & \geq 0
\end{array}
$$

Corresponding *dual* problem

$$
\begin{array}{llrrl}
\min & Z_D = & 4y_1 & +5y_2 & \\
\text{s.t.} & & y_1 & +2y_2 & \geq 3 \\
& & y_1 & +y_2 & \geq 4 \\
& & y_1 & & \geq 0 \\
& & & y_2 & \geq 0
\end{array}
$$

Weak duality theorem

$$
\begin{array}{rrl}
\text{Primal: } \max & \mathbf{c}^T\mathbf{x} & \\
\text{s.t.} & A\mathbf{x} & \leq \mathbf{b} \\
& \mathbf{x} & \geq 0
\end{array}
$$

$$
\begin{array}{rrl}
\text{Dual: } \min & \mathbf{y}^T\mathbf{b} & \\
\text{s.t.} & \mathbf{y}A & \geq \mathbf{c} \\
& \mathbf{y} & \geq 0
\end{array}
$$

Weak Duality Theorem

If $\mathbf{x}$ is primal feasible and $\mathbf{y}$ is dual feasible, then $c^T\mathbf{x} \leq \mathbf{y}A\mathbf{x} \leq \mathbf{y}^T\mathbf{b}$

Proof?

Weak duality theorem
– consequences

The weak duality theorem has three interesting consequences:

- if x and y are feasible solutions of the primal and dual problems and $\mathbf{c}^T\mathbf{x} = \mathbf{b}^T\mathbf{y}$ then x and y must be optimal.
- if optimal cost of primal is ∞ then dual problem is infeasible
- if optimal cost of dual is $-\infty$ then primal problem is infeasible

Strong duality theorem

Strong Duality Theorem

If one of the problems has an optimal solution the other one also has an optimal solution and the optimal objective function values are equal

- The optimal dual solution appears in the optimal primal tableau, under the slack variables (Proof?)
- The two other possibilities are
  - One problem is infeasible, the other is unbounded
  - Both problems are infeasible

Complementary Slackness

$$
\begin{array}{rrl}
\text{Primal: } \max & \mathbf{c}^T\mathbf{x} & \\
\text{s.t.} & A\mathbf{x}+\mathbf{s} & = \mathbf{b} \\
& \mathbf{x} & \geq 0 \\
& \mathbf{s} & \geq 0 \\
\hline
\end{array}
$$

$$
\begin{array}{rrl}
\text{Dual: } \min & \mathbf{y}^T\mathbf{b} & \\
\text{s.t.} & \mathbf{y}A-\mathbf{e} & = \mathbf{c} \\
& \mathbf{y} & \geq 0 \\
& \mathbf{e} & \geq 0 \\
\hline
\end{array}
$$

Definition

A primal solution and a dual solution exhibit complementary slackness if $\mathbf{e}^T\mathbf{x} = 0$ and $\mathbf{y}^T\mathbf{s} = 0$, i.e., corresponding $\mathbf{x}$- and $\mathbf{y}$-values are not both positive

Complementary Slackness

Complementary Slackness Theorem

Theorem: A primal solution and a dual solution are optimal iff they are feasible and complementary (proof?)