Relaxation and bounds

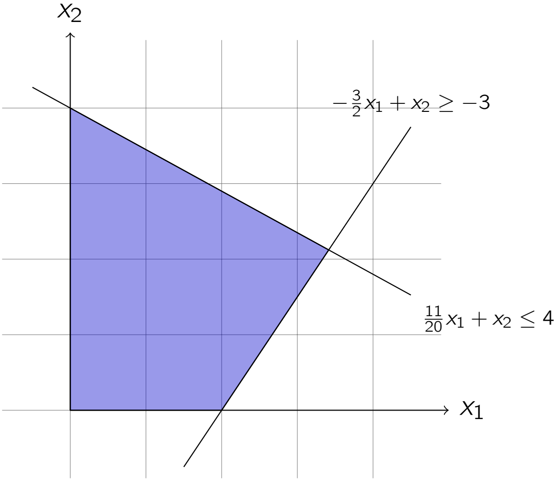
Jesper Larsen[1]

[1]Department of Management Engineering
Technical University of Denmark

Todays lecture:

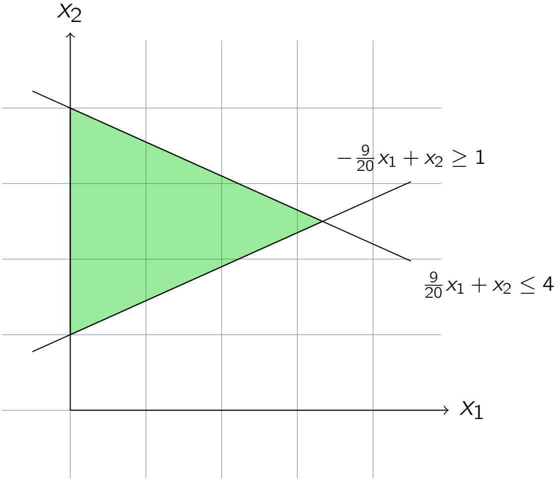
- Introduction to the concept of bounds in the solution process.
- Introduction of a relaxation.
- Examples of relaxation.
- Introduction to duality as an alternative to relaxation.
- Introduction to greedy heuristic to generate bounds.

Using the formulation – feasible but not optimal



$$
\begin{aligned}
\max \quad & 0.6x_1 + 1.01x_2 \\
\text{s.t} \quad & \frac{11}{20}x_1 + x_2 \leq 4 \\
& -\frac{3}{2}x_1 + x_2 \geq -3 \\
& x_1, x_2 \geq 0 \\
& x_1, x_2 \text{ integer}
\end{aligned}
$$

Using the formulation – not even feasible



$$\max\ 0.55x_1 + 1.01x_2$$
$$\text{s.t}\quad \frac{9}{20}x_1 + x_2 \leq 4$$
$$-\frac{9}{20}x_1 + x_2 \geq 1$$
$$x_1, x_2 \geq 0$$
$$x_1, x_2 \text{ integer}$$

Optimality and Relaxation

Basic solution approach to any IP or COP:

$$z = \max\{c(x) : x \in X \subseteq Z^n\}$$

- Find lower bound (LB) $\underline{z}$ s.t. $\underline{z} \leq z$
- Find upper bound (UB) $\bar{z}$ s.t. $\bar{z} \geq z$

Now clearly $\bar{z} \geq \underline{z}$. Furthermore, if we have $\bar{z} = \underline{z} = z$ we are done.

Upper and lower bounds

A general approach will usually be:

$$\underline{z}_1 < \underline{z}_2 < \underline{z}_3 < \ldots \leq z \leq \ldots \bar{z}_3 < \bar{z}_2 < \bar{z}_1$$

If $\bar{z}_t - \underline{z}_s < \epsilon$ then we may stop — if $\bar{z}_t - \underline{z}_s = 0$ we have found an optimum.

Bounds

How do we actually find (upper and lower) bounds?

- Primal bounds: (lower bound for a max problem). Every feasible solution $x \in X$ is a lower bound.
- Dual bounds: (upper bound for a max problem). Most important approach is by relaxation, that is, replace the original problem by a simpler optimization problem whose value is at least as large as z.

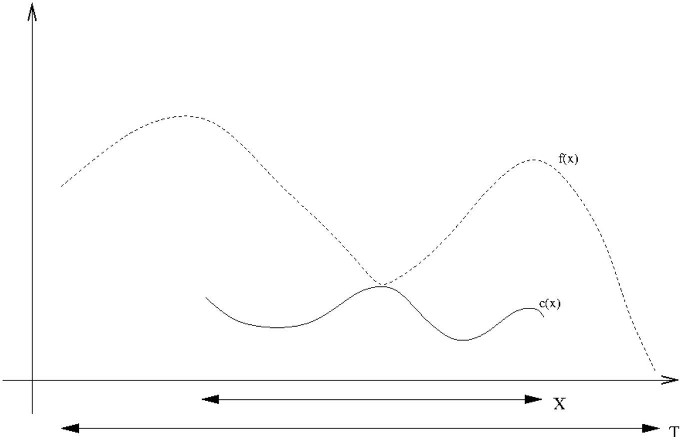
Bounds vs maximization and minimization

	Lower bound	Upper bound
max	Primal	Dual
min	Dual	Primal

Relaxation

- A problem (RP) $z^R = \max\{f(x) : x \in T \subseteq R^n\}$ is a **relaxation** of (IP) $z = \max\{c(x) : x \in X \subseteq R^n\}$ if:
  - *(i)* $X \subseteq T$
  - *(ii)* for all $x \in X$: $c(x) \leq f(x)$

Relaxation



Consequence of relaxation

Let x^* be an optimal solution in IP, and x^R an optimal solution in RP.

$$z = c(x^*) \leq f(x^*) \leq f(x^R) = z^R$$

So in conclusion, we have $z \leq z^R$ (for a maximization problem)

Approaches to relaxation

- Wolsey presents 3 approaches to relaxation:
  1. Linear Programming relaxation
  2. Combinatorial relaxation
  3. Lagrangian relaxation

- **Proposition 2.3**: Let $f(x)$ be the objective function of the relaxed problem, and let $c(x)$ be the objective function of the original Integer Program. Then,
  - *(i)* If a relaxation RP is infeasible, the original problem is infeasible.
  - *(ii)* Let x^* be an optimal solution to RP. If $x^* \in X$ and $f(x^*) = c(x^*)$ then x^* is an optimal solution to IP.

Proof of Prop. 2.3 (ii)

- Start with $f(x^*) = c(x^*)$. Since x^* is an optimal solution of to RP we get:
- $c(x^*) = f(x^*) = z^R$
- As x^* is a feasible solution to IP we have $z \geq c(x^*)$
- Putting the two statements above together we get: $z \geq z^R$
- Since RP is a relaxation of IP we have from the definition of relaxation that $z \leq z^R$
- And combining the last two statements we get $z = z^R$

Linear Programming relaxation

- For the integer program $\max\{cx : x \in P \cap Z^n\}$ with the formulation $P = \{x \in R^n_+ : Ax \leq b\}$ the linear programming relaxation is $\max\{cx : x \in P\}$.
- Recall proposition 2.3: "Let x^* be an optimal solution to RP. If $x^* \in X$ and $f(x^*) = c(x^*)$ then x^* is an optimal solution to IP."

LP relaxation example

Consider the following integer program:

$$\begin{aligned} z = \quad \max \quad & 4x_1 - x_2 \\ \text{s.t} \quad & 7x_1 - 2x_2 && \leq 14 \\ & x_2 && \leq 3 \\ & 2x_1 - 2x_2 && \leq 3 \\ & x \in Z_+^2 \end{aligned}$$

- Lower bound: $(2, 1)$ is a feasible solution.
- Upper bound: LP relaxation with LP optimum being $x^* = (\frac{20}{7}, 3)$

LP relaxation and formulations

The following proposition shows an interesting relationship between the Linear Programming relaxation and the formulation for an integer programming problem.

Proposition 2.2: P_1 and P_2 are two formulations for the same integer programming problem. Let P_1 be a better formulation than P_2. Let $z_i^{\mathrm{LP}} = \max\{cx : x \in P_i\}$. Then $z_1^{\mathrm{LP}} \leq z_2^{\mathrm{LP}}$.

Revisit the TSP formulations

- It can be shown that the cut set formulation (C) is a better formulation than the sequence variable formulation (S).
- For a small constructed 10-city TSP problem the optimal solution is 881.

Type	# Var	# Constr	LP opt	Gap (%)
C	90	502	878	0.34
S	99	92	773.6	12.2

- This is of course just one example. The only thing we know is that $z_C^{\mathrm{LP}} \geq z_S^{\mathrm{LP}}$ (it is a lower bound for a **min** problem).

Lagrangian Relaxation

Consider an integer programming problem:

$$\begin{array}{lrcl} \max & cx & & \\ \text{s.t.} & Ax & \leq & b \\ & x & \in & X \end{array}$$

Now assume that if we dropped $Ax \leq b$ the problem

$$\begin{array}{lrcl} \max & cx & & \\ \text{s.t.} & x & \in & X \end{array}$$

would be "easy".

Lagrangian relaxation

Now go one step further and add a penalty term (to the objective function) that is "active" when $Ax \leq b$ is violated, that is,

$$\begin{array}{ll} \max & cx + u(b - Ax) \\ \text{s.t.} & x \in X \end{array}$$

$z(u) = \max\{cx + u(b - Ax) : x \in X\}$ is called the Lagrangian relaxation of $z = \max\{cx : Ax \leq b, x \in X\}$.

Knapsack example of Lagrangian Relaxation

Recall our 0-1-Knapsack problem from the first lecture:

	1	2	3	4	5
c_i	5	3	2	7	4
w_i	2	8	4	2	5

and with capacity $b = 10$.

$$\begin{aligned} \max \quad & 5x_1 + 3x_2 + 2x_3 + 7x_4 + 4x_5 \\ \text{s.t.} \quad & 2x_1 + 8x_2 + 4x_3 + 2x_4 + 5x_5 \leq 10 \\ & x_1, x_2, x_3, x_4, x_5 \in \{0, 1\} \end{aligned}$$

LR of 0-1-Knapsack

In the 0-1-Knapsack problem there is only one constraint. We "relax" the constraint and get an Lagrangian Relaxation of the 0-1-Knapsack problem:

$$\begin{array}{ll} \text{max} & 5x_1 + 3x_2 + 2x_3 + 7x_4 + 4x_5 + u(10 - 2x_1 - 8x_2 - 4x_3 - 2x_4 - 5x_5) \\ \text{s.t.} & x_1, x_2, x_3, x_4, x_5 \in \{0, 1\} \end{array}$$

Now we reshuffle the terms in the objective function and get

$$\begin{array}{ll} \text{max} & (5 - 2u)x_1 + (3 - 8u)x_2 + (2 - 4u)x_3 + (7 - 2u)x_4 + (4 - 5u)x_5 + 10u \\ \text{s.t.} & x_1, x_2, x_3, x_4, x_5 \in \{0, 1\} \end{array}$$

LR of 0-1-Knapsack (cont.)

Notice:

- $10u$ is a constant
- For $u \geq 0$ we have a feasible LR to the 0-1-Knapsack problem.
- The only "constraint" left is that our variables are binary.

As an example, let $u = 1$, we then get:

$$\begin{aligned} \max \quad & 3x_1 - 5x_2 - 2x_3 + 5x_4 - 1x_5 + 10 \\ \text{s.t.} \quad & x_1, x_2, x_3, x_4, x_5 \in \{0, 1\} \end{aligned}$$

Optimal solution to this problem is $x_1 = x_4 = 1$ and the rest equal to zero with a value of 18.

Duality

- The two problems $z = \max\{c(x) : x \in X\}$ and $w = \min\{w(u) : u \in U\}$ form a (weak-)dual pair if $c(x) \leq w(u)$ for all $x \in X, u \in U$.
- If $z = w$, that is, there exists $x^* \in X$ and $u^* \in U$ s.t. $c(x^*) = w(u^*)$ they form a (strong-)dual pair.

Linear programming relaxations immediately leads to a weak dual.

- The integer program $z = \max\{c(x) : Ax \leq b, x \in Z^n_+\}$ and the linear program $w^{LP} = \min\{ub : uA \geq c, u \in R^m_+\}$ form a weak dual pair.
- Suppose that IP and D are a weak-dual pair.
  1. If D is unbounded, IP is infeasible.
  2. If $x^* \in X, u^* \in U$ satisfy $c(x^*) = w(u^*)$ then x^* is optimal for IP and u^* is optimal for D.

Matchings and Coverings

- **Matching:** Given a graph, a matching is a subgraph with the property that no two edges are incident with the same node.
- **Covering:** Given a graph, a cover is a subset of nodes such that at least one endpoint for all edges belongs to the cover.

Greedy heuristic

- A heuristic is a method for finding a feasible but not necessarily optimal solution to a problem.
- A greedy heuristic generates a feasible solution by making a sequence of choices. Each time the choice that brings the "best" immediate reward is taken.

Example 1: The 0-1 Knapsack problem

Consider our instance of the 0-1 Knapsack problem from earlier (remember $b = 10$).

	1	2	3	4	5
c_i	5	3	2	7	4
w_i	2	8	4	2	5

- One greedy heuristic could be to choose the most profitable item for inclusion as long as there is space left in the knapsack.
  - First select item 4 for the knapsack. Capacity left: 8
  - Then select item 1 for the knapsack. Capacity left: 6
  - Then select item 5 for the knapsack. Capacity left: 1
  - There is not room for more items.

The solution is identical to the optimal solution, but this is not guaranteed. And we have no proof!!

Example 1B: The 0-1 Knapsack problem

Consider the following instance of the 0-1 Knapsack problem (with $b = 40$).

item	1	2	3	4	5	6	7
c_i	80	35	45	60	11	15	10
a_i	20	10	15	30	6	10	6

- One greedy heuristic could be to choose the most profitable item for inclusion as long as there is space left in the knapsack.
  - I leave it up to you to verify that the greedy heuristic returns the following solution: $x_1 = x_3 = 1$, all others zero with solution value 125.

Example 1B: The 0-1 Knapsack problem (cont.)

item	1	2	3	4	5	6	7
c_i	80	35	45	60	11	15	10
a_i	20	10	15	30	6	10	6

- An alternative greedy heuristic could be to calculate $\frac{c_i}{a_i}$ (denoted the "ratio") and choose the item with the largest ratio for inclusion as long as there is space left in the knapsack.

item	1	2	3	4	5	6	7
$\frac{c_i}{a_i}$	4	3.5	3	2	1.83	1.5	1.67

Example 1B: The 0-1-Knapsack problem (cont. again)

item	1	2	3	4	5	6	7
$\frac{c_i}{a_i}$	4	3.5	3	2	1.83	1.5	1.67

- Select item 1. Capacity left: 20
- Select item 2. Capacity left: 10
- Cannot select item 3 due to capacity violation.
- Cannot select item 4 due to capacity violation.
- Select item 5. Capacity left: 4

No more items fits in the knapsack and we have found a feasible solution: $x_1 = x_2 = x_5 = 1$ and the rest is zero. Value of solution is 126.

Optimal solution for the problem is 130

Example 1B: Collecting all information

- Optimal solution: 130
- Best upper bound (LP-relaxation): 160
- Best lower bound (greedy heuristic): 126
- Absolute Gap ($UB - LB$): 34
- (Relative) Gap ($\frac{UB-LB}{UB}$): 21.25%

Example 2: The Symmetric TSP

Consider an instance of the symmetric TSP with the following distance matrix:

$$\begin{pmatrix} - & 9 & 2 & 8 & 12 & 11 \\ & - & 7 & 19 & 10 & 32 \\ & & - & 29 & 18 & 6 \\ & & & - & 24 & 3 \\ & & & & - & 19 \\ & & & & & - \end{pmatrix}$$

- A greedy heuristic here could be to always select the edges in nondecreasing length as long as the selected set of edges remains a subset of a feasible tour.

Example 2: The symmmetric TSP

- Start from city 1: $1 \to 3 \to 6 \to 4 \to 2 \to 5 \to 1$ with a total cost of 52
- But if I had started from city 2 I would get a different solution:
  - $2 \to 3 \to 1 \to 4 \to 6 \to 5 \to 2$ with a total cost of 49
  - ... which is better since TSP is a minimization problem.

Topics we have been through:

- Introduction to the concept of bounds in the solution process.
- Introduction of a relaxation.
- Examples of relaxation.
- Introduction to duality as an alternative to relaxation.
- Introduction to greedy heuristic to generate bounds.