No homework this week. Week 1 homework is due no later than **Thursday 27.09.2018 23:55**.

Exercise 2.1 Convexity Properties of Sets

(a) Let $\{S_i\}_{i\in M}$ be a collection of $M = \{1, \dots, m\}$ convex sets in $\mathbf{R}^n$. Show that their intersection $S = \cap_{i\in M} S_i$ is also convex.

(b) Let S_1 and S_2 be closed convex sets in $\mathbf{R}^n$. Show that their Minkowski sum

$$S = S_1 + S_2 = \{x + y : x \in S_1, y \in S_2\}$$

is also convex. Also, show by an example that $S_1 + S_2$ is not necessarily closed.

Exercise 2.2 Weierstrass' Theorem

Consider the following nonlinear optimization problem P:

$$\begin{aligned} (P) \ : \ \underset{x,y}{\text{maximize}} \quad & \frac{1}{x+y} \\ \text{subject to} \quad & xy \geq 1 \\ & x, y \geq 0 \end{aligned}$$

(a) Show that P has a solution by applying Weierstrass' theorem.

(b) Model the problem P with JuMP and try to find the global maximum.

Exercise 2.3 Portfolio Optimization

For this problem, use the data file prices.csv which contains daily prices of $N = \{1, \dots, n\}$ stocks over a time period of $T = \{1, \dots, m\}$ days. Let $x_i \geq 0$ denote the (long) position of stock $i \in N$ in a portfolio throughout the time period. The positions $x = (x_1, \dots, x_n)$ in the portfolio are scaled to represent fractions of the total investment, that is,

$$\sum_{i\in N} x_i = 1$$

Let p_i^t denote the daily price of stock $i \in N$ for all $t \in T$, and let r_i^t be the relative daily return of stock $i \in N$ for all $t \in T \setminus \{m\}$. These are computed as

$$r_i^t = \frac{p_i^{t+1} - p_i^t}{p_i^t}, \quad \forall i \in N, \forall t \in T \setminus \{m\}$$

Let $\mu = (\mu_1, \dots, \mu_n)$ denote the *expected* relative returns of the stocks N, and let $\Sigma \in \mathbf{R}^{n\times n}$ be the corresponding covariance matrix. Thus, the expected average return and variance of a portfolio $x = (x_1, \dots, x_n)$ are $\mu^\top x$ and $x^\top \Sigma x$, respectively. Moreover, let $\sigma \in \mathbf{R}^n$ be the standard deviation vector and $\rho \in \mathbf{R}^{n\times n}$ the correlation matrix of the relative stock returns.

(a) Read the data and plot the price curves of each stock for the whole time period.

(b) Compute the expected average returns μ, the covariance matrix Σ, the correlation matrix ρ, and the standard deviation vector σ using Julia's built-in functions.

(c) Sort the stocks in increasing order with respect to their expected returns. Using this order, plot the standard deviations σ and the expected returns μ of each stock in two different plots but in the same figure by using `subplot`.

(d) Using the same order as in (c), visualize the correlation matrix ρ with `imshow`, and plot the expected returns μ of each stock as a function of their standard deviations σ.

(e) Consider the following portfolio optimization problem

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & x^\top \Sigma x && (1) \\ \text{subject to} \quad & \mu^\top x \geq \mu_{min} && (2) \\ & \sum_{i \in N} x_i = 1 && (3) \\ & x \geq 0 && (4) \end{aligned}$$

where the objective is to minimize the portfolio variance (i.e., risk) $x^\top \Sigma x$ by satisfying a minimum expected return constraint (2). Model the problem (1) – (4) using JuMP and solve the problem with different values of μ_{min}. Use PyPlot's `bar` to plot fractions of capital invested in each stock in the resulting portfolio. You can try values of μ_{min} between

$$0 \leq \mu_{min} \leq 0.000869.$$

(f) Compute the optimal portfolio with 50 different values of μ_{min} between $[0, 0.000869]$ and plot the optimal trade-off curve, i.e., the expected returns or each portfolio as a function of their standard deviations. Plot also the expected values vs standard deviations of each individual stock in the same figure for comparison.