**MS-E2122 - Nonlinear optimization** **DEADLINE:**
**Homework 1** **Thursday 27.09.2018 23:55**

Please submit your answers to https://mycourses.aalto.fi/mod/assign/view.php?id=361337 before the given deadline.

## Homework 1.1 Pooling problem

Recall the pooling problem from Exercise 1. In this homework, we consider another pooling problem instance presented in Figure 1 with 16 sources $s \in S$, 2 pools $p \in P$, and 2 target nodes $t \in T$. Let $N = S \cup P \cup T$ and let $A$ denote the arcs between nodes in $N$.

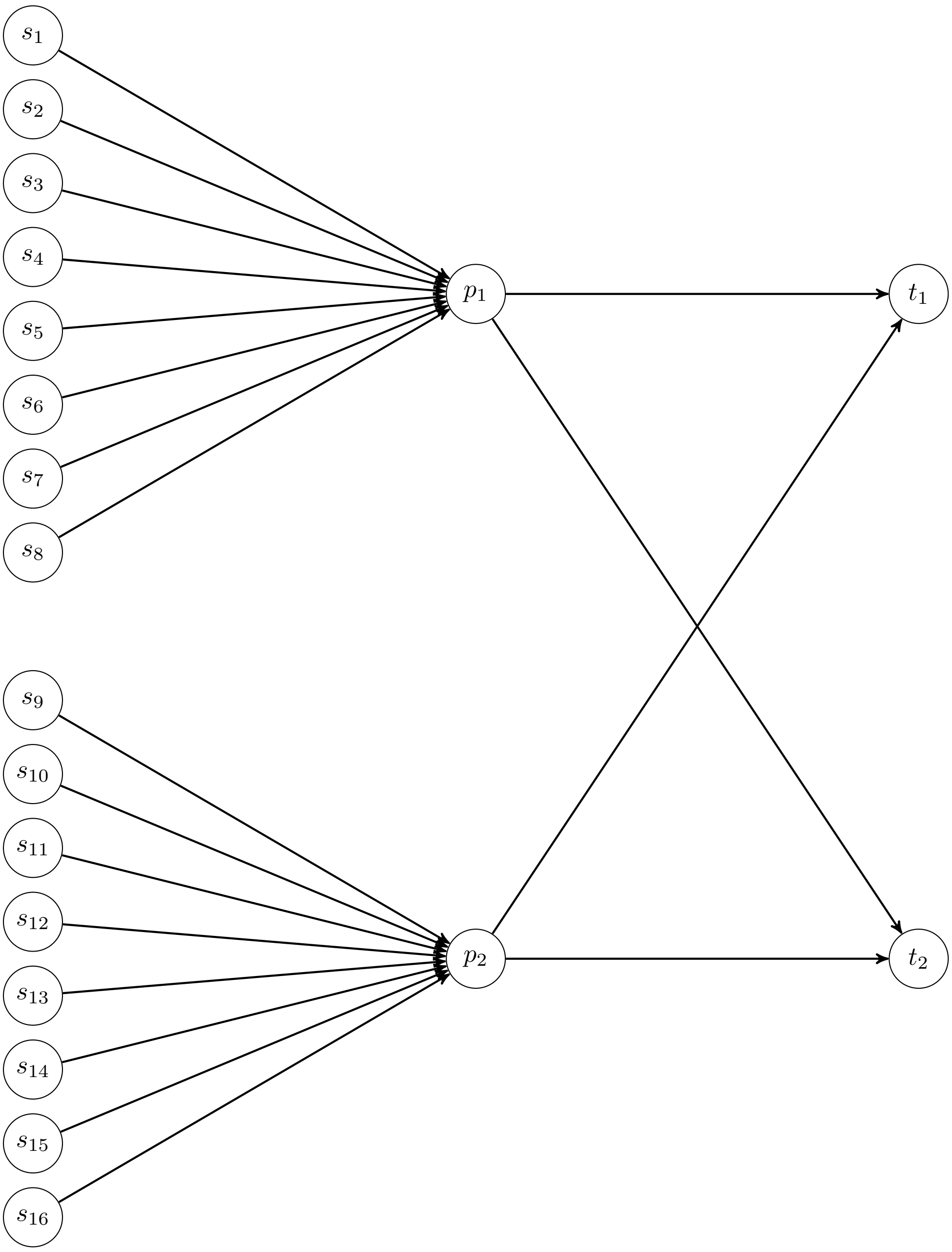


Figure 1: Pooling problem network.

The problem has 2 different properties $q^1$ and $q^2$. Let $K = \{1, 2\}$ be the property index set. Each source node $s \in S$ has flow bounds and fixed property values. Each pool node $p \in P$ has flow bounds and unknown property values. Each target node has flow bounds and values bounds for both properties. The data for this problem is presented in Tables 1 and 2.

Table 1: Flow bounds and property values sources $s \in S$

| | Flow bounds | | Property values | |
|---|---|---|---|---|
| $s \in S$ | $\underline{b}_s$ | $\overline{b}_s$ | $\underline{q}^1_s$ | $\underline{q}^2_s$ |
| $s_1$ | 1 | 3 | 5.84 | 43.7 |
| $s_2$ | 0 | 3 | 5.40 | 36.8 |
| $s_3$ | 0 | 3 | 0.24 | 12.8 |
| $s_4$ | 0 | 3 | 2.01 | 15.4 |
| $s_5$ | 1 | 3 | 5.85 | 47.3 |
| $s_6$ | 0 | 3 | 5.38 | 39.2 |
| $s_7$ | 0 | 3 | 0.26 | 13.1 |
| $s_8$ | 0 | 3 | 2.04 | 15.9 |
| $s_9$ | 1 | 3 | 0.64 | 39.9 |
| $s_{10}$ | 0 | 3 | 0.57 | 38.2 |
| $s_{11}$ | 0 | 3 | 0.02 | 13.5 |
| $s_{12}$ | 0 | 3 | 0.14 | 16.3 |
| $s_{13}$ | 1 | 3 | 0.93 | 38.1 |
| $s_{14}$ | 0 | 3 | 0.85 | 34.1 |
| $s_{15}$ | 0 | 3 | 0.03 | 13.2 |
| $s_{16}$ | 0 | 3 | 0.26 | 15.5 |

Table 2: Flow bounds of pools $p \in P$ and flow and property bounds of targets $t \in T$

| | Flow bounds | |
|---|---|---|
| $p \in P$ | $\underline{b}_p$ | $\overline{b}_p$ |
| $p_1$ | 0 | 15 |
| $p_2$ | 0 | 15 |

| | Flow bounds | | Property bounds | | | |
|---|---|---|---|---|---|---|
| $t \in T$ | $\underline{b}_t$ | $\overline{b}_t$ | $\underline{q}^1_t$ | $\overline{q}^1_t$ | $\underline{q}^2_t$ | $\overline{q}^2_t$ |
| $t_1$ | 10 | 11 | 0 | 1.50 | 30.0 | 34.0 |
| $t_2$ | 11 | 17 | 0 | 3.50 | 32.0 | 40.0 |

The source nodes have no costs in this example, and the costs $c_t$ at target nodes $t \in T$ representing profit are nonlinear and based on the final values of property $q^1$. The costs are defined as follows:

$$c_{t_1} = 100\left(2 - \frac{q^1_{t_1}}{\overline{q}^1_{t_1}}\right) \quad \text{and} \quad c_{t_2} = 150\left(2 - \frac{q^1_{t_2}}{\overline{q}^1_{t_2}}\right) \tag{1}$$

The problem can be formulated as follows.

$$
\begin{aligned}
\underset{x,q}{\text{maximize}} \quad & \sum_{t \in T} c_t \sum_{j \in N_t^-} x_{jt} && && (2)\\
\text{subject to} \quad & \sum_{j \in N_s^+} x_{sj} \geq \underline{b}_s, && \forall s \in S && (3)\\
& \sum_{j \in N_s^+} x_{sj} \leq \overline{b}_s, && \forall s \in S && (4)\\
& \sum_{j \in N_p^-} x_{jp} = \sum_{j \in N_p^+} x_{pj}, && \forall p \in P && (5)\\
& \sum_{j \in N_p^-} x_{jp} \leq \overline{b}_p, && \forall p \in P && (6)\\
& \sum_{j \in N_t^-} x_{jt} \geq \overline{b}_t, && \forall t \in T && (7)\\
& \sum_{j \in N_t^-} x_{jt} \leq \underline{b}_t, && \forall t \in T && (8)\\
& \sum_{j \in N_p^-} q_j^k x_{jp} = q_p^k \sum_{j \in N_p^+} x_{pj}, && \forall p \in P, \forall k \in K && (9)\\
& \sum_{j \in N_t^-} q_j^k x_{jt} = q_t^k \sum_{j \in N_t^-} x_{jt}, && \forall t \in T, \forall k \in K && (10)\\
& q_t^k \geq \underline{q}_t^k, && \forall t \in T, \forall k \in K && (11)\\
& q_t^k \leq \overline{q}_t^k, && \forall t \in T, \forall k \in K && (12)\\
& q_s^k = \underline{q}_s, && \forall s \in S, \forall k \in K && (13)\\
& q_p^k \geq 0, && \forall p \in P, \forall k \in K && (14)\\
& x_{ij} \geq 0, && \forall (i,j) \in A && (15)
\end{aligned}
$$

(a) Model and solve the problem (2) – (15) with Julia using JuMP using the data in Tables 1 and 2. Use the target costs defined in (1) in the objective. Compare with the pooling problem presented Exercise 1 if you get stuck.

(b) Try to find at least 2 different local optima by trying different values for the unknown quality variables $q_p$ of the pool nodes $p \in P$.

(c) Remove all pool nodes $p \in P$ from the model and investigate how it affects the objective function values. Without pool nodes, the different crude oils sent from source nodes $s \in S$ are blended at the target nodes $t \in T$ only.

(d) **Remember to return your Julia code file when submitting your solutions**.

## Homework 1.2 Portfolio optimization

For this problem, use the data file prices.csv which contains daily prices of $N = \{1, \dots, n\}$ assets over a time period of $T = \{1, \dots, m\}$ days. Let $x_i \geq 0$ denote the (long) position of asset $i \in N$ in a portfolio throughout the time period. The positions $x = (x_1, \dots, x_n)$ in the portfolio are scaled to represent fractions of the total investment, that is,

$$\sum_{i \in N} x_i = 1$$

Let $p_i^t$ denote the daily price of asset $i \in N$ for all $t \in T$, and let $r_i^t$ be the relative daily return of asset $i \in N$ for all $t \in T \setminus \{m\}$. These are computed as

$$r_i^t = \frac{p_i^{t+1} - p_i^t}{p_i^t}, \quad \forall i \in N, \forall t \in T \setminus \{m\}$$

Let $\mu = (\mu_1, \dots, \mu_n)$ denote the *expected* relative returns of the assets $N$, and let $\Sigma \in \mathbb{R}^{n \times n}$ be the corresponding covariance matrix. Thus, the expected relative return and variance of the portfolio $x = (x_1, \dots, x_n)$ are $\mu^\top x$ and $x^\top \Sigma x$, respectively. You can compute the initial data using the following Julia code presented in Lecture 1:

```
# Read the daily prices data from a .csv file.
data = readcsv("prices.csv")

# First row has the names of the stocks
stocks = data[1, 1:end-2]

# Last two columns has data and US$ rate
prices = data[2:end, 1:end-2]

# Returns are calculated as (p(t+1) - p(t))/p(t)
returns = diff(prices) ./ prices[1:end-1,:]

# Number of days and stocks in data
T, n = size(returns)

# Calculates expected return and covariance
μ, Σ = mean(returns, 1), cov(returns)
```

Your task is to analyze the performance of the portfolio $x$ by considering different trade-offs between the expected return $\mu^\top x$ and variance $x^\top \Sigma x$. The variance $x^\top \Sigma x$ measures exposition to risk, and ideally we would like to find a portfolio in which the expected return is large but the variance is small. These two conflicting objectives are combined in the following portfolio optimization problem proposed by Markowitz (1952):

$$\begin{aligned} \underset{x}{\text{maximize}} \quad & \lambda(\mu^\top x) - (1-\lambda) x^\top \Sigma x && (16) \\ \text{subject to} \quad & \sum_{i \in N} x_i = 1 && (17) \\ & x \geq 0 && (18) \end{aligned}$$

The parameter $\lambda$ with $0 \leq \lambda \leq 1$ scalarizes the two objectives into a single objective function (16). The value of $\lambda$ depends on the investor's preference: to reduce risk of the investment, the investor would select a smaller value of $\lambda$. Conversely, a risk seeking investor with hopes of higher return would select a larger value of $\lambda$.

(a) Model and solve the problem (16) – (18) with Julia using Jump with different values of $\lambda$. Use the values $\lambda \in (0, 0.1, 0.2, \dots, 1)$. Discuss briefly how the return $\mu^\top x$ and risk $x^\top \Sigma x$ change with different values of $\lambda$.

(b) Plot the values of $\mu^\top x$ and $x^\top \Sigma x$ at the different points of $\lambda$ from part (a). Plot both cases separately.

(c) Plot the values of $\mu^\top x$ at the corresponding points $x^\top \Sigma x$ using the results of part (a) obtained with different values of $\lambda$.

(d) **Remember to return your Julia code file when submitting your solutions**.

## References

Markowitz, H. (1952). Portfolio selection. *The journal of finance*, 7(1), 77–91.