§ 4.

21. (1) $X \sim P(\lambda)$; $Y \sim P(\mu)$; $X+Y \sim P(\lambda+\mu)$

$$P(X=k \mid X+Y=m)$$

$$= \frac{P(X=k,\ X+Y=m)}{P(X+Y=m)}$$

$$= \frac{P(X=k,\ Y=m-k)}{P(X+Y=m)}$$

$$= \frac{\frac{\lambda^k}{k!} \cdot e^{-\lambda} \cdot \frac{\mu^{m-k}}{(m-k)!} e^{-\mu}}{\frac{(\lambda+\mu)^m}{m!} \cdot e^{-(\lambda+\mu)}}$$

$$= \frac{m!}{k!(m-k)!} \cdot \frac{\lambda^k \cdot \mu^{m-k}}{(\lambda+\mu)^m}$$

$$= C_m^k \left(\frac{\lambda}{\lambda+\mu}\right)^k \left(\frac{\mu}{\lambda+\mu}\right)^{m-k}$$

$$X \mid X+Y=m \ \sim b\left(m, \frac{\lambda}{\lambda+\mu}\right)$$

$$E(X \mid X+Y=m) = \frac{m\lambda}{\lambda+\mu}$$

(2) $X \sim b(n,p)$; $Y \sim b(n,p)$; $X+Y \sim b(2n,p)$

$P(X=k \mid X+Y=m)$

$$= \frac{P(X=k,\ X+Y=m)}{P(X+Y=m)}$$

$$= \frac{P(X=k,\ Y=m-k)}{P(X+Y=m)}$$

$$= \frac{C_n^k \cdot p^k (1-p)^{n-k} \cdot C_n^{m-k} \cdot p^{m-k} \cdot (1-p)^{n-m+k}}{C_{2n}^m\ p^m (1-p)^{2n-m}}$$

$$= \frac{C_n^k\ C_n^{m-k}}{C_{2n}^m}$$

$E(X \mid X+Y=m)$

$$= \sum_{k=0}^{m} k \frac{C_n^k \cdot C_n^{m-k}}{C_{2n}^m}$$

$$= \sum_{k=0}^{m} k \cdot \frac{\frac{n!}{k!(n-k)!}\ C_n^{m-k}}{C_{2n}^m}$$

$$= n \cdot \sum_{k=1}^{m} \frac{C_{n-1}^{k-1} \cdot C_n^{m-k}}{C_{2n}^m}$$

$$= n \frac{C_{2n-1}^{m-1}}{C_{2n}^m} = n \frac{\frac{(2n-1)!}{(m-1)!(2n-m)!}}{\frac{(2n)!}{m!(2n-m)!}}$$

$$= \frac{m}{2}$$

注:

$Z = X \mid X+Y=m \sim h(m, 2n, n)$

(超几何分布)

$E(Z) = m \cdot \frac{n}{2n} = \frac{m}{2}$

22.

(1)

$$F_Y(y) = P(Y \le y) = \sum_{k=0}^{2} P(Y \le y \mid X=k) \cdot P(X=k)$$

$$= \frac{1}{3}[P(Y \le y \mid X=0) + P(Y \le y \mid X=1) + P(Y \le y \mid X=2)]$$

$$= \frac{1}{3}[\Phi(y) + \Phi(y-1) + \Phi(y-2)]$$

$$\therefore f_Y(y) = \frac{1}{3}[f(y) + f(y-1) + f(y-2)]$$

$$= \frac{1}{3} \cdot \frac{1}{\sqrt{2\pi}} \left(e^{-\frac{y^2}{2}} + e^{-\frac{(y-1)^2}{2}} + e^{-\frac{(y-2)^2}{2}} \right)$$

注：Φ 为标准正态分布函数

f 为标准正态密度函数

方法一：

$$E(Y) = \frac{1}{3}\int_{-\infty}^{+\infty} y\,(f(y) + f(y-1) + f(y-2))\,dy$$

$$= \frac{1}{3}\left[\int_{-\infty}^{+\infty} y f(y)\,dy + \int_{-\infty}^{+\infty} (t+1) f(t)\,dt + \int_{-\infty}^{+\infty} (u+2) f(u)\,du\right]$$

$$= \frac{1}{3}(0+1+2) = 1$$

方法二：

$$E(Y) = E(E(Y \mid X)) = E(X) = 1$$

(2) 令 $Z = X + Y$

$$P(X+Y \le z)$$

$$= \sum_{k=0}^{2} P(X+Y \le z \mid X=k) \cdot P(X=k)$$

$$= \sum_{k=0}^{2} P(Y \le z-k \mid X=k) \cdot P(X=k)$$

$$= \sum_{k=0}^{2} P(Y-k \le z-2k \mid X=k) \cdot P(X=k)$$

$$= \frac{1}{3}\left[\Phi(z) + \Phi(z-2) + \Phi(z-4)\right]$$

$\therefore F_Z(z) = \frac{1}{3}\left[\Phi(z) + \Phi(z-2) + \Phi(z-4)\right]$

(3) $\mathrm{Cov}(X, Y) = EXY - EX \cdot EY$

$E(XY) = E(E(XY \mid X)) = E(XE(Y \mid X)) = E(X^2) = 5/3$

$\mathrm{Cov}(X, Y) = \frac{5}{3} - 1 = \frac{2}{3}$

26. $X_i \sim U(-1,1) \quad (i=1,2,3)$

$E(X_1+X_2+X_3)=0$

$var(X_1+X_2+X_3)=\sum_{i=1}^{3}var(X_i)+2cov(X_1,X_2)+2cov(X_1,X_3)+2cov(X_2,X_3)$

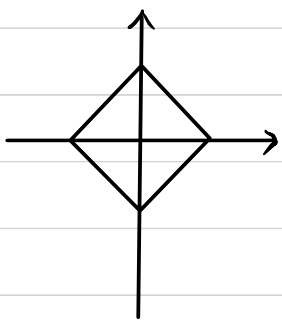
由对称性：$EX_1X_2=0$

$cov(X_1,X_2)=EX_1X_2-EX_1\cdot EX_2=0$

$var(X_i)=\frac{1}{3}$

$\Rightarrow var(X_1+X_2+X_3)=3\times\frac{1}{3}+0+0+0=1$

34.



(1) $f(x,y)=\frac{1}{2},\ (x,y)\in G$ $\qquad EX=\iint_G x f(x,y)d\sigma$ 由对称性 $EXY=EX=EY=0$

$cov(X,Y)=EXY-EXEY=0$

(2) $f_Y(y)=\int_{|y|-1}^{1-|y|}\frac{1}{2}dx=1-|y| \quad (-1<y<1)$

$f_X(x)=\int_{|x|-1}^{1-|x|}\frac{1}{2}dy=1-|x| \quad (-1<x<1)$

$f_X(x)\cdot f_Y(y)\neq f(x,y)$ ∴ X, Y 不相互独立.