Mathland: Constructionist Mathematical Learning in the Real World Using Immersive Mixed Reality

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Abstract. Mathematical experiences are intrinsic to our everyday lives, yet mathematics education is mostly confined to textbooks. Seymour Papert used the term 'Mathland' to propose a world where one would learn mathematics as naturally as one learns French while growing up in France. We built a Mixed Reality application that augments the physical world with interactive mathematical concepts to enable constructionist mathematical learning in the real world. Using Mathland, people can collaboratively explore, experience and experiment with mathematical phenomena in playful, applied and exploratory ways. We implemented Mathland using the Microsoft Hololens and two custom controllers to afford complete immersion through tangible interactions, embodiment and situated learning. Our preliminary study with 30 participants shows that a considerable percentage of participants found Mathland to not only be engaging (83%), but also efficacious in the areas of collaborative learning (92.8%), problem solving (96.6%) and mathematics education (90%).

Keywords: Virtual/augmented reality · Education/learning
Play · Tangible · Wearable computers · Embodied interaction
Situated learning

1 Introduction

Mathematical anxiety is a common problem, but it is not one's actual but perceived ability that hinders one from excelling in mathematics [1,2]. Mathematical anxiety leads to high dropout rates in science, technology, engineering and mathematics (STEM) fields especially for women, as women are more likely than men to drop out of STEM after a math class [3,4]. Mathematical anxiety can be attributed to the way mathematics is taught using abstract symbols that have no inherent meaning for students, in punitive ways such that math becomes a punishment, and as a set of rules to be memorized rather than understood [5–7]. We seek to replace the abstract symbols in mathematics with mathematical applications, the punitive teaching methods with playful experiences, and the rote learning of concepts with exploratory learning.

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Educational environments have been gradually changing in the past few years. In particular, there has been a slow shift from education to learning; learning places more emphasis on the learner as compared to education, which is more teacher-centered. Learning methodologies like active learning, experiential learning, and discovery-based learning are related to the constructivist learning theory [8], which states that learners construct knowledge by making observations. Project-based learning was proposed by Seymour Papert as central in his constructionist learning theory [9], which extends constructivist learning to suggest that learners construct their knowledge while working on meaningful projects. Constructionist learning allows for playful, exploratory, and application-based learning, and has been considered promising for mathematical learning [10,11].

Our goal for Mathland is to create an immersive and constructionist world for learners to learn mathematics through explorations in the real world. Mixed Reality (MR) allows us to augment the physical world with virtual objects and annotations so that learners can explore and experiment with mathematics in the real world. Thus, Mathland serves as a mathematical lens through which the users can explore the hidden mathematical realities of our world, and also a mathematical playground where users can experiment with mathematical phenomena in applied, playful and exploratory ways.

2 Background and Related Work

In late 1960s, Seymour Papert developed the Logo programming language [12], which allowed people to learn coding and use geometry while creating drawings. Users could give movement-related commands to a robotic or digital "turtle". Computer simulations have been popular in education, especially science education as many science concepts are difficult for students to understand because of the lack of real-life visuals [13]. Simulations can explain scientific phenomena in-context and without any physical limitations so that learners can visualize abstract scientific concepts [14]. Bret Victor's Kill Math [15] uses simulations and visualizations to teach math concepts in non-abstract ways. Some experiments show that students who learn math using games outperform those who do not [16]. Armadillo Run [17] is a 2D game that uses engineering concepts in game play. Virtual Reality (VR) has also been explored for immersive science education [18], e.g Project Science Space [19] creates science worlds in VR and Fantastic Contraption [20] is a VR game that uses physics/engineering concepts.

Mathland integrates the constructionist, simulation-based and playful learning approaches, but instead of computer screens or virtual reality worlds, Mathland allows for mathematical learning in the real world through virtual annotations and simulations. There are some Augmented Reality (AR) applications, like Construct3D [21]. But instead of connecting the content to user's real physical environment and creating an immersive experience, these applications simply overlay content on top of the physical world. Also, most MR educational applications only allow learners to visualize educational/textbook content in the real world, instead of facilitating learning through new kinds of playful and constructionist activities in the real world.

Mathland creates an immersive Mixed Reality experience through 'physical perception' of virtual objects and connects virtual objects to the user's physical environment, physical body and physical objects to allow for situated learning, embodied cognition [22] and tangible interactions respectively. Situated learning has been shown to be helpful in mathematical learning [23]. Embodied cognition is important in mathematics as mathematical ideas are connected to bodily experiences [24] and students show improved math performance using embodiment [25,26]. Tangible interactions also improve understanding of abstract concepts [27], and physical manipulatives are considered to have cognitive and contextual advantages for mathematical learning in MR [28].

3 Mathland

We experience mathematical phenomena in our everyday lives. With Mathland, we wanted to create a world where learners could visualize and play with the mathematics of their everyday objects e.g. visualize the speed and trajectory of a ball as they play catch with their friends. MR allows us to overlay the physical world with virtual annotations so that the user can learn in the real world with their peers using natural social interactions for collaborative learning [29]. We used the Microsoft Hololens for our MR experience as the Hololens offers 3D visualizations on an untethered, wearable form factor with user interactions like gaze input, voice commands, and hand gestures. The users not only visualize mathematical phenomena, but also use it in constructionist ways. Also, instead of simply overlaying virtual content on top of the real world, we tied the virtual objects to user's physical world to create an immersive MR learning experience.

3.1 Design and Functionality

Designing an Immersive MR Experience
We wanted users to experience mathematics in their real world by seamlessly merging their real and virtual world. One way of merging the physical and virtual is to integrate components of the user's physical environment in Mixed Reality so that real world can be augmented virtually in MR. This not only requires recognition of the user's static environment, but also tracking and recognition of dynamically moving physical objects, e.g. a ball moving through the air, so that we can track the ball and virtually augment it with its speed and trajectory. Real-time object detection and 3D tracking is possible through either external tracking systems, which limit mobility and scalability, or through computer vision, which is computationally expensive and slow, especially with 360° view.

Future advances in computer vision, 360° cameras and computational capabilities will make 3D tracking and object recognition feasible on wearable devices, but until then, instead of tracking and annotating physical objects, we annotate virtual objects and give them physical perception so they 'feel' real. Immersion is defined as giving the "illusion of reality to the senses of the human participant" [30], and thus, giving physical perception to virtual objects allows us to create immersive MR experiences. As a result, we can focus on the design of immersive MR experiences, rather than the technological difficulties.

We give physical perception to virtual objects, where ‘physical perception’ is defined in terms of the following three characteristics: i. Interaction with the physical world, e.g. virtual objects can bounce off a physical wall; ii. Interaction with people’s physical bodies, e.g. people can throw a virtual ball using natural arm gestures, without the need for external controllers; iii. Physical manipulability, e.g. virtual objects can be felt, moved and rotated like physical objects.

Physical perception for virtual objects facilitates better mathematical learning through situated learning using an understanding of the physical world, embodied cognition by interacting with physical bodies, and tangible interactions using physical manipulability. Physical perception for virtual objects not only makes the virtual objects act ‘like’ physical objects, but they also have several advantages for learning mathematics: a. virtual objects are not bound by real world laws, e.g. unlike the physical objects on Earth, they do not have to conform to the laws of gravity on Earth. b. unlike physical objects, virtual objects can be zoomed in or out, resized, moved back in time and then replayed, etc. c. there are no limitations on the type and quantity of objects that can be created, e.g. elephants, black holes, 4D cubes, etc.

Designing a MR World for Constructionist Mathematical Learning Mathematics is often taught of as a mere manipulation of abstract symbols, but mathematics has widespread applications, e.g. calculus is applied in Newtonian mechanics. By tapping into the applications of mathematics, we aim to enable contextual and constructionist learning of mathematics so learners not only ‘know’ mathematical concepts, but also deeply understand them by applying and playing with those concepts in real world situations.

For the first version of Mathland, we considered different mathematical applications, but we finalized on Newtonian mechanics as people experience Newtonian physics in their everyday lives, e.g. when throwing a ball. Newtonian mechanics is often thought of as physics, but it is also applied mathematics because Newtonian physics concepts are rooted in mathematics, e.g. rate of change of displacement and velocity is linked to calculus. Mathland can not only show people the mathematics of their everyday Newtonian physics experiences, but also allow them to modify the physical laws and explore different Newtonian physics possibilities in MR.

For constructionist learning of Newtonian physics, we were inspired by the Rube Goldberg machine [31,32], which demonstrates how relatively simple objects can be used to create relatively sophisticated systems. Tangible Rube Goldberg machines have also been shown to enable constructionist learning of Newtonian Physics [33]. In order to provide creative tools that users can use to build their own creations using applied math/Newtonian physics concepts, we delved into Newtonian physics concepts. We studied the Force Concept Inventory (FCI) [34], which is commonly used to evaluate the understanding of key Newtonian physics concepts, and noticed that introductory Newtonian physics is about three key concepts: i. constant linear force (e.g. gravity); ii. momentary force/collision (e.g. kicking a hockey puck); and iii. force towards a point (e.g. centripetal force in circular motion).

In order to gamify the Mathland experience, we also have a special Puzzle mode in which users use the items in Mathland's menu to create puzzles for others and solve puzzles created by others. Puzzles help boost social and creative learning in Mathland's constructionist world because users can create and solve puzzles with each other. We employ the four key components of creative learning, i.e. Projects, Peers, Passion and Play [35], in Mathland. Mathland provides a collaborative environment for learners to build projects and solve mathematical puzzles. Mathland's design affords low-floor (easy to get started), high-ceiling (opportunities to increase complexity) and wide-walls (the possibility of several possible solutions) [36] so that users can build the simplest to the most complicated worlds and puzzles using simple virtual tools.

3.2 Implementation

Creating Physical Perception for Virtual Objects. Mathland facilitates physical perception in the following three ways: (i) Spatial mapping for interactions with the physical world: We used the spatial mapping capability of the Hololens to scan the 3D physical environment and integrate it in the Mixed Reality world. Integrating a user's physical world into the MR world allows virtual objects to interact with the physical environment like physical objects do, e.g. a virtual ball can bounce off a physical wall.

(ii) Arm Controller for interactions with the human body: We developed a custom wearable Arm Controller to integrate the user's physical arm movements for gestural congruence [37] in MR. It allows the user to use their natural physical arm movements to interact with virtual objects in the same way they interact with physical objects. For example, users can play catch by catching and throwing the virtual ball in the same way they would play catch/throw with a physical ball. We used the Figur8 [38] sensor, which consists of an IMU and a stretch sensor (Fig. 1a) to implement the Arm Controller. We mounted the Figur8 sensor on a wearable sleeve, such that the IMU is slightly above the user's elbow and the stretch sensor is taut over their elbow to detect the bending and movement of the user's arm (Fig. 1a). Using inverse kinematics on data from the Figur8 sensor, we calculate the movement and motion of the user's physical arm with respect to the user's head. Hololens already tracks the user's head position so we can use the relative position and movement of the arm to calculate the position and motion of the arm in global world coordinates. We use this information about the user's physical arm position and movement to create a virtual arm that is identical to the user's physical arms and that interacts with virtual objects to create the illusion that the user is interacting with the virtual objects using their physical arm. Unlike conventional outside-in trackers as well as computer vision based inside-out tracking, the Arm Controller affords full mobility in the world and also, the user's arms are not limited to just the camera view. Our data processing model for the Figur8 sensor is based on a joint-and-hinge model, which is extensible to other joints on user's body.

(iii) Object controller for physical manipulability: Virtual objects lack the tangible affordances offered by physical objects. On the Hololens, user interactions with virtual objects are limited to air-taps or other hand gestures without tangible feedback. We created a tangible Object Controller (Fig. 1b) for virtual objects in MR so users can interact with virtual objects using tangible interactions, In particular, the Object Controller affords tangible interactions for repositioning, rotating and resizing virtual objects. The Object Controller has three components: (a) Vuforia markers for position tracking; (b) Figur8's [38] IMU for rotation tracking, and its stretch sensor for resizing with haptic feedback; (c) Force Resistive Sensor for touch input and vibration motor for haptic feedback. The Object Controller serves as a tangible placeholder or proxy for virtual objects in MR so users can reposition, rotate and resize virtual objects in the same way as they reposition, rotate and stretch physical objects. The user can gaze and tap on any of the virtual objects to connect the Object Controller to that virtual object. Any subsequent motion, i.e. a change in position and/or rotation, of the Object Controller is reflected in the position and rotation of the virtual object, and there is one-to-one mapping between the controller's and the virtual object's position and rotation. The Object Controller has a stretchable tape that the user can pull to resize the object. The stretchy tape is a stretch sensor that also gives natural haptic feedback to the user as they pull on the tape to resize a virtual object.

3.3 Creating a Game Environment for Constructionist Learning of Newtonian Physics

We have a menu of virtual items in Mathland, and we added three virtual items in Mathland's menu to integrate the aforementioned three key Newtonian physics concepts: (i) 'Force Field' object for creating a constant linear force, (ii) 'Velocity Vector' object for momentary force and (iii) 'Rope' for force towards a point. We also added two more items to the Mathland menu to allow users to create more interesting set ups: (i) 'Ramp; for objects to roll down on and (ii) 'Cube' is an obstacle that objects can bounce off of. Users can create as many instances of the virtual objects as they want, and all the instances of those objects can be independently repositioned, resized and rotated to different physics and physical properties, e.g. resizing the velocity vector changes the magnitude of velocity or rotating the velocity vector and force field changes the direction of velocity and force field respectively.

Mathland's MR starts with a virtual ball that can interact with all other virtual items and the user's physical environment. Users can use the five Newtonian physics menu items to create new worlds. Each of the five virtual items is described in detail below:

(i) Force Field (Fig. 2a): The Force Field is a cube that applies a constant linear force on all the objects within its boundaries, i.e. a local field. On one of its top corners, the Force Field has a white sphere, which can be used to reposition, rotate and resize the Force Field cube. By rotating the Force

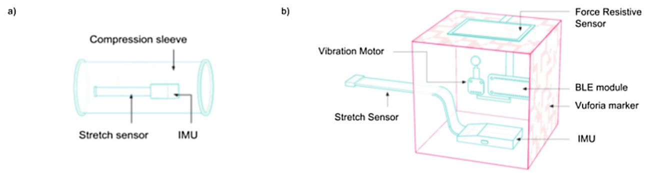

Fig. 1. (a) Arm controller (b) Object controller

Field, users can change the direction of the linear force, and resizing the Force Field changes the volume impacted by the force. The Force Field has directional arrows, which reflect the direction and magnitude of the Force Field. Rotating and resizing the Force Field changes the direction and region of the force field respectively. The users can also change the magnitude of the force using a slider, e.g. adjust the magnitude of a downward facing global Force Field to match the gravity on Earth, Mars, etc.

(ii) Velocity Vector (Fig. 2b): The Velocity Vector has the visual form of a directional arrow. Users can resize and rotate the arrow to change the magnitude and direction of the Velocity Vector respectively. If a user wants to apply an instantaneous force on an object, they can move the Velocity Vector so that it slightly overlaps with that object. The Velocity Vector adds to the instantaneous velocity of every object that comes in contact with it. The user can use the velocity vector to give an initial velocity to objects or change the direction of motion of an object by superimposing one velocity on top of the other.

(iii) Rope (Fig. 2c): On one of its ends, the Rope has a white spherical hinge, which can be fixed in the virtual space, while the other end of the Rope is unfixed. The hinge can be used to reposition, rotate and resize the rope in the MR environment, and the other end can be connected to another virtual object. The rope experiences tension (centrifugal force) toward the fixed end (or the hinge.

(iv) Ramp (Fig. 2d): The Ramp can be used by the user to direct the ball along a path.

(v) Cube (Fig. 2e): The Cube can act as a barrier and the ball bounces off of it.

For the Puzzle mode, we added another item, called a Checkpoint, which is a ring shaped object that records if the ball has passed through it. The user can place a series of numbered Checkpoints to create a MR puzzle, and to complete the level, the ball must pass through the checkpoints in the right order. Solving or creating puzzles requires users to think creatively about the Newtonian physics tools and thus encourages constructionist learning in a playful way.

The application has two modes – Launch and Edit – and the user can switch between the two modes using voice commands. The game starts by default in the Edit mode where the user can reposition the ball in the world using the Object Controller. In the Edit mode, the ball is a static object, and does not react to any forces, collisions or velocity vectors. When the user is done editing the ball

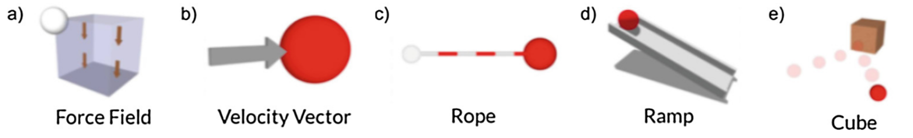

Fig. 2. Five virtual items in Mathland's menu to allow the user to experience different physical forces and interactions.

and objects in the scene, they can say "Launch ball" to enter the Launch mode. Once the user is in the Launch mode, the user cannot edit the ball or any other objects in the scene unless they say "Reset ball" to go back to the Edit mode. In the Launch mode, the ball is no longer a static, but a dynamic object and responds to the force fields, velocity vectors, etc. We provide two visualization for the ball's physics:

(a) Predicted trajectory (Edit mode): In the Edit mode, the ball has a predicted trajectory visualization, which helps the user observe the expected trajectory of the ball if it were launched. The predicted trajectory is based on the MR environment, i.e. the physical surrounding as well as virtual force fields, etc. As the user moves around the virtual ball or other objects in the edit mode, the predicted trajectory of the ball changes so that the user can visualize the effects of different changes on the trajectory of the ball without having to launch it.

(b) Trail and strobe effect (Launch mode): As the ball moves in the Launch mode, it leaves a trajectory (blue line/trail) behind to allow the users to see the path that the ball has traveled through. As the ball moves after it is launched, we create slightly transparent virtual snapshots of the ball at fixed intervals of time, i.e. 'strobe effect'. Using the strobe effect, the users can compare the distance traveled by the ball between fixed time intervals and get an intuition for the velocity, i.e. rate of change of distance/position, and acceleration of the ball. When the ball is traveling fast, the snapshots of the ball are spaced out more, compared to when the ball is traveling slowly. The snapshots produce an effect which is similar to the strobe light pictures commonly used in physics textbooks to help students visualize the motion of an object.

4 Experiment Design

We designed a preliminary experiment to evaluate constructionist learning in Mathland's immersive MR world. We particularly focused on evaluating the creative learning and problem-solving experiences in Mathland. We used the Puzzle mode of Mathland to build three puzzles, which targeted different motions that are taught in basic Newtonian physics – Circular Motion (Puzzle 1), Linear Motion (Puzzle 2) and Projectile Motion (Puzzle 3). The participants had to solve the puzzles by applying Newtonian physics concepts using the five virtual items provided in the Mathland's menu. The three puzzles and their respective solutions are shown in Fig. 3. Each puzzle involved the understanding of different Newtonian physics concepts, and could be solved in multiple ways. We also included an optional question at the end of the experiment for our users to design their own Mathland puzzles. The puzzles and one of their possible solutions is as follows: (i) For Puzzle 1 (Circular Motion), the user had to notice that since the checkpoints are arranged in a circular orbit, the ball needs a centripetal force. This centripetal force can be created by attaching the ball to a Rope object, and giving the ball an initial velocity that is perpendicular to the rope so that the virtual ball starts swinging in a circular motion. (ii) To solve Puzzle 2 (Linear Motion), the user could place inclined ramps between the checkpoints and create a downward Force Field to resemble gravity (since there is no gravity by default in the world, and without a downward force, the ball does not move down the ramps). (iii) For Puzzle 3 (Projectile Motion), in addition to the downward gravity-like Force Field, the user needs a Velocity Vector to launch the ball at an angle such as that it follows a parabolic path, i.e. projectile motion.

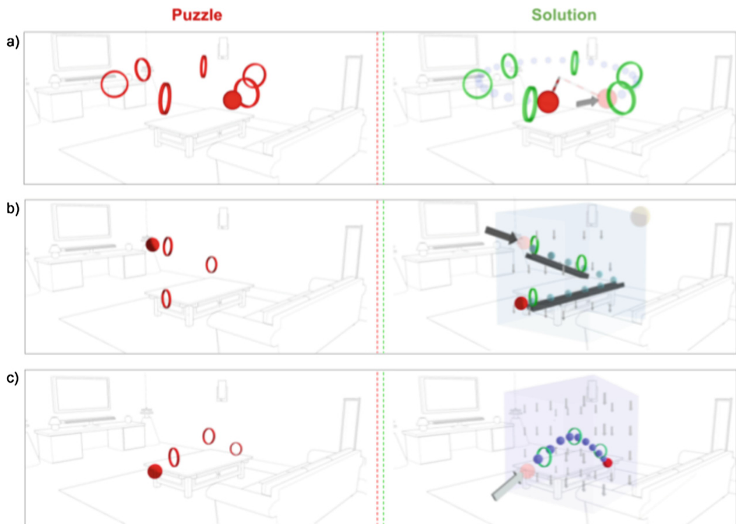

Fig. 3. Puzzles created by placing three virtual checkpoints. The left side images show the puzzle whereas the right side ones show the solution. Each puzzle corresponds to a specific physics concept: (a) Circular Motion (b) Linear Motion (c) Projectile Motion

We conducted the experiment with two participants in one trial to evaluate the collaborative learning experience in Mathland. The two participants shared a common Mixed Reality world, and had a total of 30 min to solve all three of the puzzles. We gave only one Object controller to each participant pair to encourage more collaboration and interaction between participants. We did not include the virtual arm in the experiment as the virtual arm was for scenarios like catching, throwing, pushing the ball, etc, whereas solving the puzzles mostly involved manipulating virtual objects using the Object Controller. The goal of the experiment was not to evaluate a specific type of interaction, but to analyze the overall constructionist learning in MR. Therefore, it was simpler and more efficient for users to solve the puzzles using the just the Object Controller.

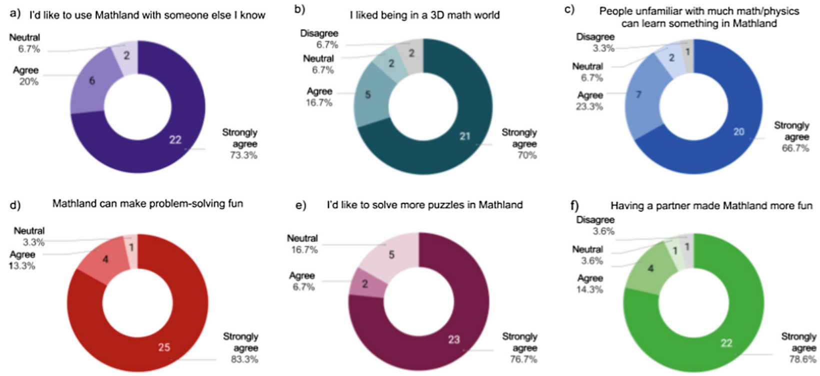

Fig. 4. Our survey responses based on a 5-point Likert scale

5 Results

We had a total of 30 participants (22 female and 8 male; 8 in the age group 16–20, 21 in the age group 20–35, and 1 above 35). 28 participants performed the study in randomly assigned pairs and the remaining 2 performed the study individually because of scheduling conflicts. On the 5-point Likert scale pre-study questions, our participants indicated that 40% of them did not find math/physics fun, 43% did not feel confident about their math/physics skills, and 48.5% did not ace their math/physics tests. Yet, all of our participants were able to solve all the three puzzles, which were designed to allow users to experience and apply Newtonian physics concepts. We presented our participants with a post-study survey, which had six five-point Likert scale type questions about user engagement, collaboration, creative learning, and problem-solving, which are all key to constructionism. Our results (Fig. 4) for each metric are as follows:

(i) Engagement: Our users were thoroughly engaged in Mathland, 28 (93.3%) out of the 30 participants agreed (20%) or strongly agreed (73.3%) that they 'would like to use Mathland with someone else they know' (Fig. 4a). Moreover, 26 (86.7%) of our participants agreed (16.7%) or strongly agreed (70%) that they 'liked being in a 3D Mathland' (Fig. 4b).

(ii) Learning: The participants found the puzzles to have educational value as 27 (90%) of them agreed (23.3%) or strongly agreed (66.7%) that 'people unfamiliar with much mathematics/physics can learn something in Mathland' (Fig. 4c).

iii. Problem-solving: All of our participants not only solved all three puzzles, but also enjoyed solving those puzzles. 29 (96.6%) of them agreed (13.3%) or strongly agreed (83.3%) that ‘Mathland can make problem-solving fun’ (Fig. 4d), and 25 (83.3%) agreed (6.7%) or strongly agreed (76.7%) that they ‘would like to solve more puzzles in Mathland’, (Fig. 4e).

(iii) Collaboration During the study we observed that partners collaborated effectively to problem-solve. They discussed the puzzles and built the solutions collaboratively. Out of the 28 participants who did the study with another partner, 26 (92.8%) agreed (14%) or strongly agreed (78.6%) that ‘having a partner made Mathland fun’ (Fig. 4f).

(iv) Creative Learning In response to our optional question of creating own puzzles, the participants suggested everything from magnetic fields to anti-gravity objects, from pong games to tornado-like spirals and vortices.

We also had two open-ended optional questions for people to mention their most and least favorite parts. For most people, the least favorite part was the narrow field of view of the Hololens. The favorite parts include graphics (‘ball trajectories’, ‘3D holograms’), simulations (‘seeing the ball launch!’, ‘controlling aspects of physics’), MR (‘virtual and real objects’; ‘more colorful and interactive’), problem solving (‘planning the solution’, ‘thinking about different solutions to the same puzzle’, ‘having absolute freedom and being creative’), and collaboration (‘Working in teams’).

6 Discussion

We observed that the participants were really engaged in the experience and having fun with their partner while solving the puzzles – they were lying down on the floor to view objects from different angles, playing around with elements, etc. It really paid off to give physical perception to virtual objects, instead of digitizing physical objects because physical objects are bounded by physical laws and for most of our users, their favorite part was the limitless nature of our virtual objects, e.g. creating multiple copies of objects, rewinding objects in time, resizing objects, etc. Users were connected with the virtual objects because of their ‘physical perception’ and really enjoyed seeing the trajectory of the ball in 3D, modifying the physics of objects, and experiencing the Newtonian physics phenomena unfold in front of them. Participants particularly liked exploring the physics/mathematical phenomena as they were solving puzzles. This shows us that to constructionist learning as learners created new worlds using Newtonian physics tools to solve puzzles. The MR environment afforded natural social interactions and users loved sharing the puzzle solving experience. We observed that the users did not only share ideas, but also had a playful time with high-fiving each other, laughing with each other, etc. This sort of natural social interaction is unlike VR, where users interact with user avatars, not real users. Moreover, in response to our optional open-ended question, several users enjoyed imagining and creating new physics puzzles and explained the different ways they would solve each puzzle without being prompted to do so. Participants also enjoyed coming up with different solutions for the three Mathland puzzles, even though they were required to produce only one solution. The experiment results show that Mathland's constructionist setup triggered creative thinking and exploration.

7 Future Work

Our current version of Mathland focuses more on constructionist learning using visualizations and spatial understanding rather than learning about mathematical equations. For example, as the user is rotating the virtual Ramp in Mathland, we do not show the slope or equation of the Ramp. Instead, we allow the users to observe the relative positions, angles and sizes of the objects. As our virtual ball moves through space, instead of showing the value of the ball's velocity, we show the direction of the ball's motion using the trail left behind by the ball, and the magnitude of velocity using the distances between the snapshots of the ball taken at equal time intervals. Going forward we aim to add more support for mathematical understanding using interactive graphs and equations. Interactive equations for trajectory, velocity and acceleration are such that changing the equations for each changes the trajectory, velocity and acceleration of the ball respectively, and vice versa. Interactive equations will allow users to understand mathematical equations with respect to their physical representations in the real world. Using interactive graphs, we will demystify the relationship between an object's distance, velocity, and acceleration, which are connected through calculus. Allowing users to visualize these relationships graphically as the area under graphs (integrals) and slope of graphs (differentials) might help users connect the real life behavior of objects to their mathematical representations. We also plan to build assessments for further experimental evaluations of learning in Mathland. Users will be required to answer multiple-choice questions that evaluate their mathematical and conceptual understanding of Newtonian physics. The users will answer the question and then play with the relevant concepts of the questions in Mathland, and then reanswer the question and explain their choice.

8 Conclusion

We live in a mathematical universe, but are unable to see the mathematics underlying our natural world. Mathland aims to unveil the hidden mathematical realities of nature. In Mathland, learners not only visualize mathematics in immersive ways, but also use it to construct new projects, i.e. constructionist learning. Unlike traditional education, we do not confine mathematics to its symbolic representations. Instead, we show the applications of mathematics so learners are motivated to use mathematics to construct new worlds. We use MR to situate learning in the user's real world context, but unlike existing MR applications, which simply overlay virtual content on top of the real world, we created an immersive MR experience where users can explore and play with mathematics using virtual objects, which have 'physical perception', i.e. they are connected to user's physical environment, human body and physical objects and thus, afford situated learning, embodied cognition and tangible interactions.

The goal of Mathland is not necessarily to explicitly test mathematical learning, but to engage learners with mathematical phenomena in constructionist ways so they can constructively and creatively explore the mathematics underlying it. We will continue to delve deeper into Newtonian physics and allow users to develop a deeper understanding of its graphs and equations. We do not confine Mathland to just pure mathematics, but also integrate applied mathematics, so the user can engage with the mathematical language of the universe. We envision a world where people can interactively and seamlessly experience mathematical phenomena in their real world, e.g. hold up a virtual prism and see the sunlight get refracted into a virtual rainbow, and play with it. Therefore, Mathland is both a mathematical lens and a mathematical playground in the real world. It affords immersive and constructionist learning of mathematical phenomena to support applied, playful and exploratory ways of mathematical concepts.

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