

Basics of Point-Set Topology

Stochastic Batman

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1 Definitions before connectedness and compactness

Topological Space: For a set X , (X, \mathcal{T}) is a *topological space* if \mathcal{T} (topology) is a collection of subsets of X such that:

1. $\emptyset, X \in \mathcal{T}$.
2. $\{U_i\}_{i=1}^n \in \mathcal{T} \implies \bigcap_{i=1}^n U_i \in \mathcal{T}$ (Finite intersection is closed).
3. $\{U_\alpha\}_{\alpha \in A} \subseteq \mathcal{T} \implies \bigcup_{\alpha \in A} U_\alpha \in \mathcal{T}$ (Arbitrary union is closed).

Continuity: Let (X, \mathcal{T}) and (Y, \mathcal{S}) be topological spaces. A function $f : X \rightarrow Y$ is *continuous* if for every open set $V \in \mathcal{S}$ we have $f^{-1}(V) \in \mathcal{T}$.

Homeomorphism: Let (X, \mathcal{T}) and (Y, \mathcal{S}) be topological spaces. A function $f : X \rightarrow Y$ is a homeomorphism if it is bijective; both f and f^{-1} are continuous.

Basis \mathcal{B} of Topological Space (X, \mathcal{T}) : For every open set $U \in \mathcal{T}$, $\exists \mathcal{B}' \subseteq \mathcal{B}$ s.t. $U = \bigcup_{B \in \mathcal{B}'} B$.

Second Countable: A topological space (X, \mathcal{T}) is second countable if it has a countable basis.

Hausdorff Space: (X, \mathcal{T}) is Hausdorff space iff:

$$\forall x, y \in X (x \neq y), \exists U, V \in \mathcal{T} : x \in U \wedge y \in V \wedge U \cap V = \emptyset$$

Locally Euclidean: (X, \mathcal{T}) is locally Euclidean of dimension n if $\forall p \in X$ has a neighborhood U that is homeomorphic to an open subset of \mathbb{R}^n ($\forall p \in X, \exists$ open $U \ni p$ and $\phi : U \rightarrow V$, where $V \subseteq \mathbb{R}^n$ is open).

Convergence of a Sequence: For (X, \mathcal{T}) , $\{x_n\}_{n \in \mathbb{N}}$ of points in X converges to a point $x \in X$ if for every open set $U \in \mathcal{T}$ such that $x \in U, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N, x_n \in U$.

n -dimensional topological manifold: A second countable Hausdorff space (X, \mathcal{T}) that is locally Euclidean of dimension n .

Upper half-space: $\mathbb{R}^n \supseteq \mathbb{H}^n := \{x \in \mathbb{R}^n \mid x_n \geq 0\}$.

n -dimensional manifold with boundary: A second countable Hausdorff space (X, \mathcal{T}) in which every point has a neighborhood homeomorphic to an open subset of \mathbb{R}^n or \mathbb{H}^n .

Let $S \subseteq X$. $\mathcal{T}_S := \{U \subseteq S \mid U = S \cap V \text{ for some } V \in \mathcal{T}\}$ defines **subspace topology** on S . The pair (S, \mathcal{T}_S) is a subspace.

Let $(X_1, \mathcal{T}_1), \dots, (X_n, \mathcal{T}_n)$ be topological spaces. **Product Topology** on $\prod_{i=1}^n X_i$ is generated by the basis $\mathcal{B} = \{\prod_{i=1}^n U_i \mid \forall i, U_i \text{ is open in } \mathcal{T}_i\}$.

Disjoint Union Spaces: Let $\{(X_\alpha, \mathcal{T}_\alpha)\}_{\alpha \in A}$ be an indexed family of topological spaces, and let $\coprod_{\alpha \in A} X_\alpha = \{(x, \alpha) \mid x \in X_\alpha\}$ denote their disjoint union. Subset $U \subseteq \coprod_{\alpha \in A} X_\alpha$ is open in disjoint union topology if $U \cap X_\alpha$ is open in $\mathcal{T}_\alpha, \forall \alpha \in A$.

One might also write a disjoint union of two sets with this symbol: $X \sqcup Y$.

Quotient Topology: Let (X, \mathcal{T}_X) be a topological space and Y a set. Let $q : X \rightarrow Y$ be surjective. Quotient topology on Y via $q : \mathcal{T}_Y = \{U \subseteq Y \mid q^{-1}(U) \in \mathcal{T}_X\}$ (\mathcal{T}_Y is largest topology on Y such that q is continuous).

The definition of quotient topology might resemble the definition of continuity, but they differ: continuity is defined between two topological spaces and is a condition that a map must satisfy, while the quotient topology is defined from a topological space to a set - we construct a topology on that set so that the given surjection becomes continuous.

Equivalence relation. An equivalence relation \sim on a set X partitions X into equivalence classes $[x] = \{y \in X : y \sim x\}$.

Quotient space (set). Given \sim on X , define the quotient set

$$X/\sim = \{[x] \mid x \in X\}$$

Wedge sum. For a family of topological spaces $\{(X_i, \mathcal{T}_i)\}_{i \in I}$ with chosen basepoints $x_i \in X_i$,

$$\bigvee_{i \in I} X_i = (\coprod_{i \in I} X_i) / \sim,$$

where all basepoints x_i are identified (i.e., $x_i \sim x_j$ for all $i, j \in I$ and no other identifications).

Let $q : (X, \mathcal{T}) \rightarrow Y$ be any map. $q^{-1}(y)$ for $y \in Y$ is called a **fiber** of q . A subset $U \subseteq X$ is called **saturated** w.r.t. q , if $U = q^{-1}(V)$ for some $V \subseteq Y$.

Let $f : (X, \mathcal{T}) \rightarrow (Y, \mathcal{S})$ be a continuous map that is either open or closed.

- (a) If f is injective, it is a topological embedding.
- (b) If f is surjective, it is a quotient map.
- (c) If f is bijective, it is a homeomorphism.

Adjunction Spaces: Let (X, \mathcal{T}_X) and (Y, \mathcal{T}_Y) be topological spaces, $A \subseteq X$ closed, $f : A \rightarrow Y$ a continuous function. Define adjunction space (where $a \sim f(a), \forall a \in A$):

$$X \cup_f Y = X \sqcup Y / \sim$$

2 Connectedness

Disconnected: A topological space (X, \mathcal{T}) is disconnected iff it can be expressed as the disjoint union of two non-empty open subset $U \sqcup V = X$.

Connected: A topological space (X, \mathcal{T}) that is NOT disconnected is said to be connected.

(X, \mathcal{T}) is connected $\iff X, \emptyset$ are the only subsets of X that are simultaneously open and closed (clopen for short).

Proof:

(\implies) (X, \mathcal{T}) connected. Suppose $U \subseteq X$ is clopen. U open and $U^c = X \setminus U$ open. Then $U \cup U^c = X \implies U = \emptyset$ or $U^c = \emptyset \implies U = X$.

(\impliedby) X, \emptyset are the only clopen sets of X . If $X = U \cup V$, U, V open $\implies U^c = X \setminus U = (U \cup V) \setminus U = V$ (so open) $\implies U$ clopen $\implies U = X$ or $U = \emptyset$ ($\implies V = \emptyset$). \square

Path: Let (X, \mathcal{T}) be a topological space and $p, q \in X$. A *path* in X from p to q is a continuous map $f : [0, 1] \rightarrow X$ with $f(0) = p$ and $f(1) = q$.

Path-connectedness: A space (X, \mathcal{T}) is called *path-connected* iff for all $p, q \in X$, there is a path in X from p to q .

Component: Let (X, \mathcal{T}) be a topological space. A *component* of (X, \mathcal{T}) is a maximal (to make sure subsets of a connected set do not count as separate components) nonempty connected subset of X .

Path Component: A path component of (X, \mathcal{T}) is a maximal nonempty path-connected subset.

Locally (Path) connected: A topological space (X, \mathcal{T}) is locally (path) connected if it admits a basis of (path) connected open subsets.

This means any open neighborhood of $p \in X$ contains a (path) connected open set containing p .

3 Compactness

Open Cover: An *open cover* of a space X is a collection \mathcal{U} of open subsets of X whose union is X .

Subcover: A *subcover* of \mathcal{U} is a subcollection of elements of \mathcal{U} that still covers X .

Compact: A space X is *compact* if every open cover of X has a finite subcover.

A subset $A \subseteq X$ is compact if it is compact as a subspace.

Compact Subspace Lemma: A subset $A \subseteq X$ is compact iff every collection $\{U_\alpha\}$ of open subsets of X with $\bigcup_\alpha U_\alpha \supseteq A$ has a finite subcollection $\{U_{\alpha_k}\}_{k=1}^n$ satisfying $\bigcup_{k=1}^n U_{\alpha_k} \supseteq A$.

Let X and Y be spaces and $f : X \rightarrow Y$ be continuous. If X is compact, then so is $f(X)$.

Some compactness results:

- (a) Closed subsets of compact spaces are compact.
- (b) Compact subsets of Hausdorff spaces are closed.
- (c) Compact subsets of metric spaces are bounded.
- (d) Finite products of compact spaces are compact.
- (e) Quotients of compact spaces are compact.

Closed and bounded intervals in \mathbb{R} are compact.

Heine-Borel Theorem: A subset $S \subseteq \mathbb{R}^n$ is compact iff it is closed and bounded.

Extreme Value Theorem: If X is a compact space and $f : X \rightarrow \mathbb{R}$ continuous, then f is bounded and attains its maximum and minimum values on X .

Closed Map Lemma: Suppose f is continuous map from a compact space to a Hausdorff space. Then:

- (a) f is a closed map.
- (b) f injective $\implies f$ topological embedding.
- (c) f surjective $\implies f$ quotient map.
- (d) f bijective $\implies f$ homeomorphism.

4 Riemannian Geometry

Chart: Let (M, \mathcal{T}) be a topological manifold. A *chart* is a pair (U, ϕ) where $U \in \mathcal{T}$ is an open set and $\phi : U \rightarrow \hat{U} \subseteq \mathbb{R}^n$ is a homeomorphism onto an open subset of \mathbb{R}^n .

Transition Map: Let (U_α, ϕ_α) and (U_β, ϕ_β) be charts with $U_\alpha \cap U_\beta \neq \emptyset$. The map $\tau_{\alpha, \beta} = \phi_\beta \circ \phi_\alpha^{-1} : \phi_\alpha(U_\alpha \cap U_\beta) \rightarrow \phi_\beta(U_\alpha \cap U_\beta)$ is called a *transition map*.

Unlike maps defined on the manifold, transition maps are functions between open subsets of Euclidean space \mathbb{R}^n , so we can use standard calculus to differentiate them.

Atlas: An *atlas* \mathcal{A} on M is a collection of charts $\{(U_\alpha, \phi_\alpha)\}_{\alpha \in A}$ that covers M (i.e., $\bigcup_{\alpha \in A} U_\alpha = M$).

Smooth Structure: An atlas is called *smooth* if all its transition maps are C^∞ (infinitely differentiable). A *smooth structure* on M is a maximal smooth atlas (one that contains every possible chart compatible with it).

Smooth Manifold: A topological manifold (M, \mathcal{T}) is a *smooth manifold* if it is equipped with a smooth structure.

Tangent Space: For a point p in a smooth manifold M , the *tangent space* $T_p M$ is the vector space consisting of all tangent vectors at p . (Intuitively, it is the space of all possible velocity vectors $\gamma'(0)$ of smooth curves $\gamma : (-\epsilon, \epsilon) \rightarrow M$ passing through p at $t = 0$).

Riemannian Metric: Let M be a smooth manifold. A *Riemannian metric* g on M is a field of inner products, assigning to each point $p \in M$ a positive-definite inner product $g_p : T_p M \times T_p M \rightarrow \mathbb{R}$ on the tangent space $T_p M$, varying smoothly with p .

Riemannian Manifold: A pair (M, g) consisting of a smooth manifold M and a Riemannian metric g .

Geodesic: A *geodesic* is a smooth curve $\gamma : I \rightarrow M$ (where $I \subseteq \mathbb{R}$ is an interval) that is "straight" with respect to the metric g . Formally, it satisfies the geodesic equation $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ (zero acceleration).

Maximal Geodesic: A geodesic $\gamma : I \rightarrow M$ is *maximal* if its domain I cannot be extended to any larger interval $J \supsetneq I$ while preserving the geodesic property.

Geodesic Completeness: A Riemannian manifold (M, g) is *geodesically complete* if every maximal geodesic is defined on all of \mathbb{R} (i.e., the domain is $(-\infty, \infty)$).

Complete Riemannian Manifold: A connected Riemannian manifold (M, g) is called *complete* if it is geodesically complete.

By the **Hopf-Rinow Theorem**, for a connected Riemannian manifold, the following are equivalent:

- (a) (M, g) is geodesically complete.
- (b) The metric space (M, d_g) induced by the Riemannian metric is complete (every Cauchy sequence converges).
- (c) The closed and bounded subsets of M are compact (Heine-Borel property).