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of the Lognormal distribution

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Abstract

Integral transforms of the lognormal distribution are of great importance in statistics and probability, yet closed-form expressions do not exist. A wide variety of methods have been employed to provide approximations, both analytical and numerical. In this paper, we analyze a closed-form approximation $\tilde{\mathcal{L}}(\theta)$ of the Laplace transform $\mathcal{L}(\theta)$ which is obtained via a modified version of Laplace's method. This approximation, given in terms of the Lambert $\mathcal{W}(\cdot)$ function, is tractable enough for applications. We prove that $\tilde{\mathcal{L}}(\theta)$ is asymptotically equivalent to $\mathcal{L}(\theta)$ as $\theta \rightarrow \infty$. We apply this result to construct a reliable Monte Carlo estimator of $\mathcal{L}(\theta)$ and prove it to be logarithmically efficient in the rare event sense as $\theta \rightarrow \infty$.

Keywords: Lognormal distribution, Laplace transform, Characteristic function, Moment generating function, Laplace's method, Saddlepoint method, Lambert W function, rare event simulation, Monte Carlo method, Efficiency.

MSC: 60E05, 60E10, 90-04

1 Introduction

The lognormal distribution is of major importance in probability and statistics as it arises naturally in a wide variety of applications. For instance, the central limit theorem implies that the limit distribution of a product of random variables often can be well approximated by the lognormal distribution. Hence it is not surprising that the lognormal distribution is frequently employed in disciplines such as engineering, economics, insurance or finance, and it often appears in modeling across the sciences including chemistry, physics, biology, physiology, ecology, environmental sciences and geology; even social sciences and linguistics, see [1–5].

The lognormal distribution F_{μ, σ^2} is defined as the distribution of the exponential of a $N(\mu, \sigma^2)$ random variable and has density

$$f(x) = \frac{1}{x\sqrt{2\pi\sigma}} \exp\left\{-\frac{(\log x - \mu)^2}{2\sigma^2}\right\}, \quad x \in \mathbb{R}^+.$$

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However, closed-form expressions for transforms of the lognormal distribution do not exist and reliable numerical approximations are scarce. In particular, an inspection of the defining integrals

$$\begin{aligned}\mathcal{L}(\theta) &= \int_0^\infty e^{-\theta x} dF_{\mu, \sigma^2}(x) = \int_0^\infty \frac{1}{x\sqrt{2\pi}\sigma} \exp\left\{-\theta x - \frac{(\log x - \mu)^2}{2\sigma^2}\right\} dx, \\ &= \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\theta e^t - \frac{(t - \mu)^2}{2\sigma^2}\right\} dt\end{aligned}\quad (1.1)$$

for the Laplace transform $\mathcal{L}(\theta)$ reveals that there is very little hope in finding a closed form expression for transforms of the lognormal distribution (note that the Laplace transform and the characteristic function φ are connected by $\varphi(\omega) = \mathcal{L}(-i\omega)$, $\mathcal{L}(\theta) = \varphi(i\theta)$). Moreover, the integral defining $\mathcal{L}(\theta)$ diverges when $\Re(\theta) < 0$ and in consequence, the function $\mathcal{L}(\theta)$ is not defined in the left half of the complex plane and fails to be analytic on the imaginary axis (thus, the moment generating function is not defined). Nevertheless, in the absence of a closed-form expression it is desirable to have sharp approximations for the transforms of the lognormal distributions as this paves the road for obtaining the distribution of a sum of i.i.d. lognormal random variables via transform inversion.

In this paper, we further analyze the closed form approximation of the Laplace transform of the Lognormal distribution which we reported in [6] and was obtained via a modified version of Laplace's method. Such expression will be denoted $\tilde{\mathcal{L}}(\theta)$ and it is given by

$$\tilde{\mathcal{L}}(\theta) = \frac{\exp\left\{-\frac{\mathcal{W}^2(\theta e^\mu \sigma^2) + 2\mathcal{W}(\theta e^\mu \sigma^2)}{2\sigma^2}\right\}}{\sqrt{1 + \mathcal{W}(\theta e^\mu \sigma^2)}}, \quad \theta \in \mathbb{R}^+. \quad (1.2)$$

In the expression above, $\mathcal{W}(\cdot)$ is the *Lambert W function* which is defined as the solution of the equation $\mathcal{W}(x)e^{\mathcal{W}(x)} = x$; this function has been widely studied in the last 20 years mainly due to the advent of fast computational methods, cf. [7]. Roughly speaking, the standard Laplace's method [8] states that an integral of the general form

$$\int_a^b e^{-\theta h(t)} g(t) dt = \frac{\sqrt{2\pi} e^{-\theta h(\rho)}}{\sqrt{\theta h''(\rho)}} g(\rho) (1 + O(\theta^{-1})), \quad \theta \rightarrow \infty, \quad (1.3)$$

where $g(t)$ and $h(t)$ are functions such that $g(t)$ is "well behaved" and $h(t)$ has a unique global minimum at $\rho \in (a, b)$. Therefore, the leading term of the expression on the right hand side of (1.3) can be used as an asymptotic approximation of the integral on the left hand side. Surprisingly, this approximation can be very accurate for very general functions $g(t)$ and $h(t)$, not only in asymptotic regions but also for relatively small values of θ . In our case, we employ a variant of Laplace's method by first noting the following

Lemma 1.1. *Let $\theta > 0$. Then the exponent $h_\theta(t) := -\theta e^t - (t - \mu)^2/2\sigma^2$ in the integrand in (1.1) has a unique maximum at $\rho_\theta = -\mathcal{W}(\theta e^\mu \sigma^2) + \mu$.*

We then proceed in similar way as in the standard Laplace's method with $g(t) \equiv 1$ and $h(t) = -h_\theta(t)/\theta$ dependent on θ . However, we also pay a price for such modification: the expression derived is no longer of an asymptotic order $1 + O(\theta^{-1})$ as given in formula (1.3). In the following section we will prove that

$$\mathcal{L}(\theta) = \tilde{\mathcal{L}}(\theta)(1 + O(\log^{-1}(\theta))), \quad \theta \rightarrow \infty,$$

so the modification increases the order of the asymptotic series. Yet, the suggested approximation remains asymptotically equivalent to the true Laplace transform as the value of the argument goes to infinity. Moreover, based on these results we will obtain a probabilistic representation of the error term and construct a Monte Carlo estimator for the Laplace transform. We remark that one must be careful in the implementation of Monte Carlo estimators as naive simulation can lead to unreliable approximations. We show that our proposal corresponds to an importance sampling estimator which delivers reliable estimates for any value of the argument by proving its asymptotic efficiency in a rare-event sense to be defined. We further provide numerical comparisons of our proposal against other methods available in the literature.

We note that the problem of approximating the transforms of a lognormal is of high complexity and it has been long standing. Therefore, a significant number of methods have been developed to approximate both the Laplace and the characteristic functions of the lognormal distribution. We give a more complete literature survey at the end in Section 5 and mention here the work which is most relevant for the present paper. Barakat [9] introduced the term $i\omega(-t-1)$ in the exponent of the integrand defining the characteristic function and by taking a series representation of $e^{i\omega(e^t-t-1)}$, he obtains a representation in terms of Hermite polynomials after integrating term by term. Holgate [10] employed the classical *saddle point method* [11], which consists in applying Cauchy's theorem to deform the path of integration in such a way that it traverses through a saddlepoint of the integrand in the steepest descent direction. In the lognormal case, the saddlepoint is given as the solution $t = \rho(\omega)$ of the equation $te^{-t} = i\sigma^2\omega$, so the saddlepoint approximation of the characteristic function is of the form

$$\varphi(\omega) \sim (1 - \rho(\omega))^{-1/2} \exp \left\{ (\rho(\omega)^2 - 2\rho(\omega))/2\sigma^2 \right\}. \quad (1.4)$$

Gubner [12] employed (as many others) numerical integration techniques and was the first in proposing alternative path contours to reduce the oscillatory behavior of the integrand. This approach was further extended in Tellambura and Senaratne [13] where they proposed specific contours passing through the saddlepoint at a steepest descent rate; this election has the effect that oscillations are removed in a neighborhood around the saddlepoint. In addition, they also addressed the heavy-tailed nature of the lognormal density by proposing a transformation which delivers an integrand with lighter tails.

In this paper we followed a path somewhat different from Holgate to approximate the Laplace transform of a Lognormal distribution (1.1) by using a variant of the *Laplace's method* (related to the saddlepoint method). To the best of our knowledge, the resulting closed form approximation (1.2) derived from this methodology was first reported in [6].

The paper is organized as follows: In Section 2 we compute the approximation of the Laplace transform of the lognormal distribution and analyze its asymptotic properties; in addition, we extend this result to the complex plane via the saddlepoint method and establish the relationships with the results of Holgate. In section 3 we construct an importance sampling estimator for approximating the Laplace transform and prove its efficiency properties; we discuss the disadvantages of using naïve Monte Carlo for estimating this Laplace transform. We verify the sharpness of our approximations and present some numerical comparisons in the analysis presented in Section 4. A discussion and concluding remarks are in Section 5.

2 Approximating the lognormal Laplace transform

We start by proving Lemma 1.1.

Proof of Lemma 1.1. Recall that for $z \in (-e^{-1}, \infty)$ the Lambert W function $w = \mathcal{W}(z)$ is defined as the unique solution $w \in (-1, \infty)$ of the equation $we^w = z$. With the change of variables $t = \mu - y$ we have $h_\theta(y) = -\theta e^\mu e^{-y} - y^2/2\sigma^2$,

$$h'_\theta(y) = \theta e^\mu e^{-y} - \frac{y}{\sigma^2} = \frac{e^{-y}}{\sigma^2} (\theta e^\mu \sigma^2 - ye^y).$$

Here $\theta e^\mu \sigma^2 - ye^y$ is equal to 0 for $y = \mathcal{W}(\theta e^\mu \sigma^2)$, positive for $0 \leq y < \mathcal{W}(\theta e^\mu \sigma^2)$, and negative for $\mathcal{W}(\theta e^\mu \sigma^2) < y < \infty$ since ye^y is strictly increasing on $[0, \infty)$. Also $\theta e^\mu \sigma^2 - ye^y > 0$ for $y < 0$ since then $ye^y > 0$. This shows that there is a unique maximum at $y = \mathcal{W}(\theta e^\mu \sigma^2)$, i.e. at $t = \mu - \mathcal{W}(\theta e^\mu \sigma^2)$. \square

In the rest of this section we will make the approximation (1.2) more precise. In Proposition 2.1, we show that $\mathcal{L}(\theta)/\tilde{\mathcal{L}}(\theta)$ is given in terms of a certain expected value, which in Proposition 2.2 we show goes to 1 as $\theta \rightarrow \infty$. Finally, we note that the approximation derived from Proposition 2.1 can be embedded in a more general result which is valid in the right half of the complex plane including the imaginary axis (Remark 2.7).

We assume that $\mu = 0$. Notice that such assumption is made without loss of generality as $\mathcal{L}_{\mu, \sigma^2}(\theta) = \mathcal{L}_{0, \sigma^2}(e^{-\mu}\theta)$ where $\mathcal{L}_{\mu, \sigma^2}$ stands for the Laplace transform of a lognormal random variable with parameters μ and σ^2 (we also drop the subindexes μ and σ^2 used in the notation above and adopt the notation $\mathcal{L}(\theta)$ for the rest of the paper).

The approximation $\tilde{\mathcal{L}}(\theta)$ given in (1.2) is derived from the following proposition:

Proposition 2.1. *The Laplace transform of F_{0, σ^2} can be written as*

$$\mathcal{L}(\theta) = \frac{\exp\left\{-\frac{\mathcal{W}^2(\theta\sigma^2) + 2\mathcal{W}(\theta\sigma^2)}{2\sigma^2}\right\}}{\sqrt{1 + \mathcal{W}(\theta\sigma^2)}} \mathbb{E}[g(\hat{\sigma}(\theta)Z; \theta)], \quad \theta > 0. \quad (2.1)$$

where Z is an independent normal standard random variable, $\hat{\sigma}^2(\theta) = \sigma^2/(1 + \mathcal{W}(\theta\sigma^2))$ and

$$g(t; \theta) = \exp\left\{-\frac{\mathcal{W}(\theta\sigma^2)}{\sigma^2} \left(e^t - 1 - t - \frac{t^2}{2}\right)\right\}, \quad t \in \mathbb{R}.$$

Observe that equality (2.1) in the previous proposition expresses the Laplace transform of a lognormal distribution as the product of two terms: 1) a closed-form expression in terms of the function Lambert $\mathcal{W}(\cdot)$ and 2) an expected value of a transformed standard normal random variable Z .

The first corresponds to the closed form approximation $\tilde{\mathcal{L}}(\theta)$ given in (1.2) and obtained by the modified version of the Laplace's method. The heuristic arguments used for its construction are illustrated next. Consider the representation

$$\mathcal{L}(\theta) = \mathbb{E}[e^{-\theta X}] = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\theta e^t - \frac{t^2}{2\sigma^2}\right\} dt, \quad \theta > 0, \quad (2.2)$$

and the infinite series representation of $h_\theta(t) = -\theta e^t - t^2/2\sigma^2$ around its maximum $\rho_\theta = -\mathcal{W}(\theta\sigma^2)$ (recall Lemma 1.1). By truncating the series up to terms of second order we can obtain a good approximation of the integral because the exponential transformation makes the contribution from regions away from ρ_θ small. Therefore, most of the total value of the integral will come from a region around the mode point ρ_θ where the two functions have similar values. This truncated integral can be solved explicitly and its value corresponds to the first term in (2.1).

The correction term derived from the use of the Laplace's method comes in the form of an expected value so it provides a probabilistic representation of the error associated to such approximation. A crucial advantage of this representation is that this approximation can be sharpened via a careful implementation of the Monte Carlo method (section 3). Observe that the function $g(\cdot; \theta)$ roughly equals 1 in a neighborhood of 0; in consequence, the value $\mathbb{E}[g(\hat{\sigma}(\theta)Z; \theta)]$ is relatively close to 1. In fact, we will prove that the modified Laplace's method delivers an approximation with an error which is asymptotically negligible. Moreover, it turns out that this approximation is sharp all over the domain of convergence of θ as we will empirically corroborate in the numerical examples in section 4.

The proof of Proposition 2.1 provides transparency to the heuristic arguments of the modified version of the Laplace's method discussed above.

Proof of Proposition 2.1. Consider the expression (2.2) for the Laplace transform. Lemma 1.1 implies that the maximum of the integrand occurs at $\rho_\theta = -\mathcal{W}(\theta\sigma^2)$. To implement the Laplace's method we consider a second order approximation of the exponent around the mode point that is introduced in the exponent of the integrand; then using the identity $-\theta e^{-\mathcal{W}(\theta\sigma^2)} = -\mathcal{W}(\theta\sigma^2)/\sigma^2$ we rewrite the r.h.s. of (2.2) as

$$\frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} \exp\left\{-\frac{\mathcal{W}(\theta\sigma^2)}{\sigma^2} \left[1 + (t + \mathcal{W}(\theta\sigma^2)) + \frac{(t + \mathcal{W}(\theta\sigma^2))^2}{2}\right] - \frac{t^2}{2\sigma^2}\right\} g(t + \mathcal{W}(\theta\sigma^2); \theta) dt \quad (2.3)$$

where the function g collects all the remaining terms, i.e.

$$g(t; \theta) = \exp\left\{-\frac{\mathcal{W}(\theta\sigma^2)}{\sigma^2} \left(e^t - 1 - t - \frac{t^2}{2}\right)\right\}.$$

After some extensive, but otherwise standard algebraic manipulations involving the function $\mathcal{W}(\cdot)$ and its properties, expression (2.3) becomes

$$\frac{\exp\left\{-\frac{\mathcal{W}(\theta\sigma^2)^2 + 2\mathcal{W}(\theta\sigma^2)}{2\sigma^2}\right\}}{\sqrt{1 + \mathcal{W}(\theta\sigma^2)}} \cdot \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\widehat{\sigma}(\theta)} \exp\left\{-\frac{(t + \mathcal{W}(\theta\sigma^2))^2}{2\widehat{\sigma}^2(\theta)}\right\} g(t + \mathcal{W}(\theta\sigma^2); \theta) dt,$$

where $\widehat{\sigma}(\theta)^2 = \sigma^2/(1 + \mathcal{W}(\theta\sigma^2))$. The result of the proposition follows with a change of variable $y = (t + \mathcal{W}(\theta\sigma^2))/\widehat{\sigma}(\theta)$. \square

We refer to $\widetilde{\mathcal{L}}(\theta)$ in (1.2) as the *LM-approximation* of the Laplace transform of the lognormal distribution (this name will be frequently used in our numerical investigations in Section 4). In the following proposition we make use of the asymptotic representation of the standard Laplace's method to prove that the LM-approximation is asymptotically equivalent to the true Laplace transform and demonstrate that the relative error of the approximation of $\widetilde{\mathcal{L}}(\theta)$ vanishes at a rate of convergence of order $\log(\theta)^{-1}$.

Proposition 2.2. $\mathbb{E}[g(\widehat{\sigma}(\theta)Z; \theta)] = 1 + O(\log(\theta)^{-1})$ as $\theta \rightarrow \infty$.

Corollary 2.3. $\mathcal{L}(\theta) = \widetilde{\mathcal{L}}(\theta)(1 + O(\log(\theta)^{-1}))$,

$$\lim_{\theta \rightarrow \infty} \frac{\mathcal{L}(\theta)}{\widetilde{\mathcal{L}}(\theta)} = 1, \quad \lim_{\theta \rightarrow \infty} \mathcal{L}(\theta) = \lim_{\theta \rightarrow \infty} \widetilde{\mathcal{L}}(\theta) = 0.$$

[For the last assertion, just note the obvious fact that the $\mathcal{L}(\theta) \rightarrow 0$ as $\theta \rightarrow \infty$]. It is also appealing to summarize the asymptotic behavior of $\mathcal{L}(\theta)$ in the rough form of logarithmic asymptotics familiar from large deviations theory:

Lemma 2.4.

$$\lim_{\theta \rightarrow \infty} \frac{\mathcal{W}(\theta)}{\log(\theta)} = 1, \quad \lim_{\theta \rightarrow \infty} \frac{\log \mathcal{L}(\theta)}{\log^2 \theta} = -\frac{1}{2\sigma^2}.$$

[For the first assertion, just use L'Hopital and $\mathcal{W}'(\theta) = \mathcal{W}(\theta)/(\theta\mathcal{W}(\theta) + \theta)$].

The proof of Proposition 2.2 is given in two parts. First we prove that the integral over an open interval containing the mode point is asymptotically equal to 1. This is accomplished by employing a standard asymptotic result which also provides information about the asymptotic order of convergence. The second part of the proof consists in proving that the contribution to the integral coming from the tails is negligible as the value of the argument $\theta \rightarrow \infty$. Consequently, we conclude that the expected value goes to 1 as the value of the argument goes to infinity. In addition, we will require both the asymptotic result in the first part of Lemma 2.4 and Lemma 2.5 below, which provides an alternative representation of the expected value in (2.1) in a form that will be useful to construct upper bounds satisfying the required asymptotic conditions. Moreover, Lemma 2.5 will be used later for the construction of efficient Monte Carlo estimators of the Laplace transform.

Lemma 2.5. Let $g(\cdot; \theta)$, $\widehat{\sigma}^2(\theta)$ and Z be defined as in Proposition 2.1. Then

$$\mathbb{E}[g(\widehat{\sigma}(\theta) Z; \theta)] = \sqrt{1 + \mathcal{W}(\theta\sigma^2)} \mathbb{E}[\vartheta(\sigma Z; \theta)] \quad (2.4)$$

where

$$\vartheta(t; \theta) = \exp\left\{-\frac{\mathcal{W}(\theta\sigma^2)}{\sigma^2} \left(e^t - 1 - t\right)\right\}, \quad t \in \mathbb{R}.$$

Furthermore, $\mathbb{E}[g(\widehat{\sigma}(\theta) Z; \theta)] \leq \sqrt{1 + \mathcal{W}(\theta\sigma^2)}$ for all $\theta > 0$.

Proof of Lemma 2.5.

$$\begin{aligned} \mathbb{E}[g(\widehat{\sigma}(\theta) Z; \theta)] &= \int_{-\infty}^{\infty} \exp\left\{-\frac{\mathcal{W}(\theta\sigma^2)}{\sigma^2} \left(e^{\widehat{\sigma}(\theta)t} - 1 - \widehat{\sigma}(\theta)t - \frac{\widehat{\sigma}^2(\theta)t^2}{2}\right) - \frac{t^2}{2}\right\} dt \\ &= \int_{-\infty}^{\infty} \vartheta(\widehat{\sigma}(\theta)t; \theta) \exp\left\{-\frac{t^2}{2} \left(-\frac{\mathcal{W}(\theta\sigma^2)\widehat{\sigma}^2(\theta)}{\sigma^2} + 1\right)\right\} dt \\ &= \int_{-\infty}^{\infty} \vartheta(\widehat{\sigma}(\theta)t; \theta) \exp\left\{-\frac{1}{2} \cdot \frac{t^2}{1 + \mathcal{W}(\theta\sigma^2)}\right\} dt. \end{aligned}$$

The change of variable $z = t/\sqrt{1 + \mathcal{W}(\theta\sigma^2)}$ yields $\mathbb{E}[g(\widehat{\sigma}(\theta) Z; \theta)] = \sqrt{1 + \mathcal{W}(\theta\sigma^2)} \cdot \mathbb{E}[\vartheta(\sigma Z; \theta)]$. The upper bound follows from the observation that $\vartheta(t; \theta) \leq 1$ for all $t \in \mathbb{R}$ and $\theta > 0$. This completes the proof of the Lemma. \square

Proof of Proposition 2.2. Consider the representation (2.4) in Lemma 2.5 and rewrite

$$\mathbb{E}[\vartheta(\sigma Z; \theta)] = \int_{-\infty}^a \vartheta(\sigma t; \theta) \phi(t) dt, + \int_a^b \vartheta(\sigma t; \theta) \phi(t) dt, + \int_b^{\infty} \vartheta(\sigma t; \theta) \phi(t) dt, \quad (2.5)$$

where $\phi(t)$ is the density of the standard normal distribution; a, b are selected such that $0 \in (a, b)$. For the second integral we can apply a standard asymptotic result for the standard Laplace's method [cf. p. 44, 14], which says that if a function $h(z)$ has a single minimum (mode point) $\widehat{z} \in (a, b)$ then it holds that

$$\int_a^b e^{-\eta h(z)} \phi(t) dt = \frac{\sqrt{2\pi} e^{-\eta h(\widehat{z})}}{\sqrt{\eta h''(\widehat{z})}} \phi(\widehat{z}) (1 + O(\eta^{-1})), \quad \eta \rightarrow \infty,$$

In our specific case, $\eta = \mathcal{W}(\theta\sigma^2)$, and $\widehat{z} = 0$ is the mode point of the function $h(t) = (e^{\sigma t} - 1 - \sigma t)/\sigma^2$. Since $h(0) = 0$, $h''(0) = 1$ and $\phi(0) = (2\pi)^{-1}$, then it follows that

$$\int_a^b \vartheta(\sigma t; \theta) \phi(t) dt = \frac{1}{\sqrt{\mathcal{W}(\theta\sigma^2)}} (1 + O(\mathcal{W}(\theta\sigma^2)^{-1})), \quad \theta \rightarrow \infty. \quad (2.6)$$

For the first integral we use $\vartheta(\sigma t; \theta) \leq \exp\{-\mathcal{W}(\theta\sigma^2)\sigma^{-2}(-1 - \sigma t)\}$ to obtain an upper bound:

$$\begin{aligned} \int_{-\infty}^a \vartheta(\sigma t; \theta) \phi(z) dt &\leq \int_{-\infty}^a \exp\left\{-\frac{\mathcal{W}(\theta\sigma^2)}{\sigma^2} (-1 - \sigma t)\right\} \phi(t) dt \\ &= \exp\left\{\frac{\mathcal{W}^2(\theta\sigma^2) + 2\mathcal{W}(\theta\sigma^2)}{2\sigma^2}\right\} \Phi\left(\frac{a\sigma - \mathcal{W}(\theta\sigma^2)}{\sigma}\right) \end{aligned}$$

where $\Phi(\cdot)$ corresponds to the normal standard distribution. Using Mill's ratio we find that the previous expression is asymptotically equivalent to

$$\begin{aligned} & \exp\left\{\frac{\mathcal{W}^2(\theta\sigma^2) + 2\mathcal{W}(\theta\sigma^2)}{2\sigma^2}\right\} \frac{\sigma}{\sqrt{2\pi}|a\sigma - \mathcal{W}(\theta\sigma^2)|} \exp\left\{-\frac{(a\sigma - \mathcal{W}(\theta\sigma^2))^2}{2\sigma^2}\right\} \\ & \sim \exp\left\{\frac{(1 + a\sigma)\mathcal{W}(\theta\sigma^2)}{\sigma^2} - \frac{a^2}{2}\right\} \frac{\sigma}{\sqrt{2\pi}\mathcal{W}(\theta\sigma^2)} \end{aligned} \quad (2.7)$$

For the third integral we have that $\vartheta(\sigma t; \theta) < \exp\{-\mathcal{W}(\theta\sigma^2)t^2/2\sigma^2\}$ for $t \geq 0$. Then

$$\begin{aligned} \int_b^\infty \vartheta(\sigma t; \theta) d\Phi(t) & < \int_b^\infty \exp\left\{-\frac{\mathcal{W}(\theta\sigma^2)t^2}{2\sigma^2}\right\} d\Phi(t) \\ & = \int_b^\infty \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{(1 + \mathcal{W}(\theta\sigma^2)\sigma^{-2})t^2}{2}\right\} dt \\ & = 1 - \Phi\left(b\sqrt{1 + \mathcal{W}(\theta\sigma^2)\sigma^{-2}}\right) \end{aligned} \quad (2.8)$$

Using Mill's ratio we obtain the following asymptotically equivalent expression

$$\frac{1}{b\sqrt{2\pi}(1 + \mathcal{W}(\theta\sigma^2)\sigma^{-2})} \exp\left\{-\frac{b^2(1 + \mathcal{W}(\theta\sigma^2)\sigma^{-2})}{2}\right\}.$$

Now, observe that if we select $a < -\sigma^{-1}$, then the asymptotic order of the tail integrals (2.7)-(2.8) is negligible compared to the asymptotic order of the integral in the interval (a, b) given in expression (2.6). Hence, using Lemma 2.5 and the asymptotic equivalence (2.6) we obtain

$$\begin{aligned} \mathbb{E}[g(\hat{\sigma}(\theta) Z; \theta)] & = \sqrt{1 + \mathcal{W}(\theta\sigma^2)} \int_{-\infty}^\infty \vartheta(\sigma t; \theta) d\Phi(t) \\ & = \sqrt{\mathcal{W}(\theta\sigma^2)^{-1} + 1} \cdot (1 + O(\mathcal{W}(\theta\sigma^2)^{-1})). \end{aligned} \quad (2.9)$$

Since $\sqrt{\mathcal{W}(\theta\sigma^2)^{-1} + 1} = 1 + O(\mathcal{W}(\theta\sigma^2)^{-1})$ it follows that $\mathbb{E}[g(\hat{\sigma}(\theta) Z; \theta)] = 1 + O(\mathcal{W}(\theta\sigma^2)^{-1})$. Using the first part of Lemma 2.4 we obtain that $\mathcal{W}(\theta\sigma^2) \sim \log(\theta)$ as $\theta \rightarrow \infty$. This completes the proof. \square

The term $O(\log(\theta)^{-1})$ in Proposition 2.2 arising from the approximation of Laplace's method can be further expanded into an asymptotic series, so we can obtain more information on the asymptotic behavior of the Laplace transform of the Lognormal distribution. For instance, the statement in Remark 2.6 provides a second order term from the asymptotic series. Its proof is similar to that of Proposition 2.2.

Remark 2.6.

$$\mathbb{E}[g(\hat{\sigma}(\theta) Z; \theta)] = 1 + \frac{\sigma^2}{12\log(\theta)} + O(\log(\theta)^{-2}).$$

However, one must be careful when using asymptotic series as approximations of the real Laplace transform because it is common to obtain asymptotic series which are divergent for certain given values. For instance, the second term in the asymptotic series obtained above $\sigma^2/12 \log(\theta) \rightarrow \infty$ as $\theta \rightarrow 0$. One could employ asymptotic techniques to obtain an optimal approximation by truncating the asymptotic series up to a certain term. However, in our numerical results we found that the numerical approximation derived from this strategy delivered poor numerical results even in regions away from $\theta = 0$. An alternative delivering better numerical results is the Monte Carlo method which we explore in the section 3.

Proof of Remark 2.6. Take the function $h(z) = (e^{\sigma z} - 1 - \sigma z)/\sigma^2$, which has a mode point at $z = 0$. Then, the term $O(\mathcal{W}(\theta\sigma^2)^{-1})$ in the decomposition (2.9) has the following expansion [cf. 2.20, 14]

$$\frac{1}{\mathcal{W}(\theta\sigma^2)} \left\{ \frac{5\hat{\kappa}_3^2}{24} - \frac{\hat{\kappa}_4}{8} + \frac{\phi''(0)}{2\phi(0)h''(0)} - \frac{\hat{\kappa}_3\phi'(0)}{2\phi(0)\sqrt{h''(0)}} \right\} + O(\mathcal{W}(\theta\sigma^2)^{-2}).$$

where

$$\hat{\kappa}_3 := \frac{h^{(3)}(0)}{(h''(0))^{3/2}} = \sigma, \quad \hat{\kappa}_4 := \frac{h^{(4)}(0)}{(h''(0))^2} = \sigma^2.$$

Here $\phi'(0) = 0$, $\phi(0) = -\phi''(0) = (2\pi)^{-1/2}$, $h''(0) = 1$, $h^{(3)}(0) = \sigma$ and $h^{(4)}(0) = \sigma^2$. Hence

$$\frac{1}{\mathcal{W}(\theta\sigma^2)} \left\{ \frac{5\hat{\kappa}_3^2}{24} - \frac{\hat{\kappa}_4}{8} + \frac{\phi''(0)}{2\phi(0)h''(0)} - \frac{\hat{\kappa}_3\phi'(0)}{2\phi(0)\sqrt{h''(0)}} \right\} = \frac{1}{\mathcal{W}(\theta\sigma^2)} \left(\frac{\sigma^2}{12} - \frac{1}{2} \right) \quad (2.10)$$

Now observe that $\sqrt{\mathcal{W}(\theta\sigma^2)^{-1} + 1} = 1 + \mathcal{W}(\theta\sigma^2)^{-1}/2 + O(\mathcal{W}(\theta\sigma^2)^{-2})$. We insert these terms into the expression (2.9) and since integrals (2.7)–(2.8) are of negligible order we obtain

$$\mathbb{E}[g(\hat{\sigma}(\theta) Z; \theta)] = 1 + \frac{1}{12\mathcal{W}(\theta\sigma^2)} + O(\mathcal{W}(\theta\sigma^2)^{-2})$$

Apply the first part of Lemma 2.4. This completes the proof. \square

Finally, we show the approximations obtained by using the so called asymptotic saddlepoint methodology [8, 15] and discuss some of its standard theory. For that purpose, we will consider the complex function

$$\mathcal{L}(z) = \int_0^\infty e^{-zx} dF(x) = \int_{-\infty}^\infty \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-ze^t - \frac{t^2}{2\sigma^2}\right\} dt, \quad \Re(z) \geq 0.$$

The saddlepoint method makes use of the Cauchy-Goursat theorem to deform the contour of integration so the new contour traverses the saddlepoint ρ_z of the function

$$h_z(t) := -ze^t - t^2/2\sigma^2, \quad t \in \mathbb{C}.$$

This is possible because the function $h_z(t)$ is entire and there is a unique solution $t = \rho_z$ for the equation $h'_z(t) = 0$ which is also a saddlepoint of the functions

defining the real and imaginary parts of h_z . Under such assumptions, Perron's saddlepoint method indicates that we can select a new contour for which the maximum of $\Re(h_z(t))$ over the contour is reached at the saddlepoint ρ_z and $\Im(h_z(t))$ is approximately constant over the contour in a neighborhood of the saddlepoint. In consequence, the selected contour is such that the maximum of $|e^{h_z(t)}|$ is reached at the saddlepoint; in consequence most of the total value of the integral comes from the section of the contour in the neighborhood of the saddlepoint. Thus, the Laplace's method can be adapted to provide an approximation of this contour integral. The resulting approximation is the complex analogue of (1.2).

Remark 2.7. The approximation of the function $\mathcal{L}(z)$ obtained by applying the saddlepoint methodology is

$$\mathcal{L}(z) \approx \frac{\exp\left\{-\frac{\mathcal{W}^2(z\sigma^2) + 2\mathcal{W}(z\sigma^2)}{2\sigma^2}\right\}}{\sqrt{1 + \mathcal{W}(z\sigma^2)}}, \quad \Re(z) > 0.$$

[cf. 8, p. 84]. This approximation is relevant for the whole domain of convergence of $\mathcal{L}(z)$ in the complex plane. In particular, when restricted to the imaginary axis it coincides with the approximation of the characteristic function given by Holgate [10]. Similarly, when evaluated in the positive reals it coincides with the approximation (1.2) studied in this paper.

3 Efficient Monte Carlo

The approximation of the Laplace transform of the lognormal distribution suggested in the previous section turns out to be reasonable sharp for all positive values of the argument θ when the value of the parameter σ is small; however, the quality of the approximation deteriorates as the value of σ increases (see the numerical results in section 4). When computing transforms it is crucial to count with approximations which remain sharp for all values of σ and all over the domain of the transform; in particular in the tail regions. Hence it is desirable to be able to achieve errors within certain preselected margins. Resorting to numerical integration methods is a natural choice so various proposals employing this approach have emerged across the years [12, 13, 16]. However, the difficulty of approximating the defining integral is such that most of the methods proposed are very complicated and deliver unreliable results as corroborated in our numerical examples.

An alternative is the Monte Carlo method. Such approach has two notable advantages: (1) the approximations can be sharpened at the cost of computational effort; (2) the precision of the estimates can be assessed with accuracy. The basic version, known as *Crude Monte Carlo*, consists in simulating a sequence X_1, \dots, X_R of i.i.d. random variables with common distribution $\text{LN}(0, \sigma^2)$, then applying the transformation $x \mapsto e^{-\theta x e^{\mu}}$ to each random variable and finally returning the arithmetic average of the transformed sequence as an estimator of the Lognormal Laplace transform $\mathcal{L}(\theta)$. The Law of Large Numbers ensures unbiasedness of this estimator while the Central Limit Theorem implies that the *margin of error* can be used as a

measure of performance of the estimator. In our case such quantity is defined as

$$\text{Margin of error} = z_\alpha \sqrt{\frac{\text{Var}(e^{-\theta X})}{R}}, \quad X \sim \text{LN}(\mu, \sigma^2), \quad (3.1)$$

where z_α is the α -quantile associated to the normal distribution and R is the number of replications. Ideally, one would like to obtain a margin of error which is relatively small when compared to the value of the Laplace transform. In principle, one could hope to attain a preselected size of the margin of error by simply selecting a big enough number of replications, in fact, an inspection to the equation (3.1) reveals that this number should be proportional to $\text{Var}(e^{-\theta X})/\mathcal{L}^2(\theta)$. However, this strategy can clearly become unfeasible in the tail regions because the number of replications required to estimate the Laplace transform using Crude Monte Carlo tends to infinity as $\theta \rightarrow \infty$. This is a consequence of the following Proposition.

Proposition 3.1. *Let $X \sim \text{LN}(\mu, \sigma^2)$. Then*

$$\lim_{\theta \rightarrow \infty} \frac{\text{Var}(e^{-\theta X})}{\mathcal{L}^2(\theta)} = \infty.$$

Proof. Without loss of generality we assume $\mu = 0$. We start by observing that $\mathcal{W}(\theta\sigma^2) \rightarrow \infty$, so we can apply the first part of Lemma 2.4 to obtain that

$$\lim_{\theta \rightarrow \infty} -\mathcal{W}(2\theta\sigma^2) + 2\mathcal{W}(\theta\sigma^2) = \infty.$$

By virtue of Proposition 2.2 and the previous limit it follows that

$$\lim_{\theta \rightarrow \infty} \frac{\mathcal{L}(2\theta)}{\mathcal{L}^2(\theta)} = \lim_{\theta \rightarrow \infty} \frac{\tilde{\mathcal{L}}(2\theta)}{\tilde{\mathcal{L}}^2(\theta)} = \lim_{\theta \rightarrow \infty} \frac{\exp\left\{-\frac{\mathcal{W}^2(2\theta\sigma^2) + 2\mathcal{W}(2\theta\sigma^2)}{2\sigma^2}\right\}}{\exp\left\{-\frac{2\mathcal{W}^2(\theta\sigma^2) + 4\mathcal{W}(\theta\sigma^2)}{2\sigma^2}\right\}} \frac{1 + \mathcal{W}(\theta\sigma^2)}{\sqrt{1 + \mathcal{W}(2\theta\sigma^2)}} = \infty.$$

Hence we conclude

$$\lim_{\theta \rightarrow \infty} \frac{\text{Var}(e^{-\theta X})}{\mathcal{L}^2(\theta)} = \lim_{\theta \rightarrow \infty} \frac{\mathbb{E}(e^{-2\theta X})}{\mathcal{L}^2(\theta)} - 1 = \lim_{\theta \rightarrow \infty} \frac{\mathcal{L}(2\theta)}{\mathcal{L}^2(\theta)} - 1 = \infty.$$

□

Since the number of replications needed by the Crude Monte Carlo for achieving a preselected margin of error is unbounded and grows to infinity as $\theta \rightarrow \infty$, it is then clear that we need to construct new estimators without the pitfalls of Crude Monte Carlo, as well as to consider an appropriate set of tools to assess the *efficiency* of these new estimators. The set of advanced techniques used to produce improved Monte Carlo estimators goes under the name of *variance reduction methods* and their performance is conveniently analyzed under the *rare-event framework*. The key ideas are discussed next.

We say that a given estimator $\hat{\mathcal{L}}(\theta)$ of $\mathcal{L}(\theta)$ is *strongly efficient* or has *bounded relative error* if

$$\limsup_{\theta \rightarrow \infty} \frac{\text{Var} \hat{\mathcal{L}}(\theta)}{\mathcal{L}^2(\theta)} < \infty.$$

This efficiency property implies that the number of replications required to estimate $\mathcal{L}(\theta)$ with certain fixed relative precision remains bounded as $\theta \rightarrow \infty$. A weaker criterion is *logarithmic efficiency* defined as

$$\limsup_{\theta \rightarrow \infty} \frac{\text{Var } \widehat{\mathcal{L}}(\theta)}{\mathcal{L}^{2-\epsilon}(\theta)} = 0, \quad \forall \epsilon > 0.$$

This criterion implies that the number of replications needed for achieving certain relative precision grows at most at rate of order $|\log(\mathcal{L}(\theta))|$. While bounded relative error is clearly a stronger form of efficiency, it is widely accepted that for practical purposes (numerical implementations), there is no substantial difference between these two criteria, although it is often more involved to prove bounded relative error than logarithmic efficiency.

Our objective is to construct an efficient estimator of the Lognormal Laplace transform $\mathcal{L}(\theta)$. For that purpose we will construct a new estimator employing the probabilistic representation of $\mathcal{L}(\theta)$ obtained in Proposition 2.1. We will apply a variance reduction technique known as *Importance Sampling* which consists in sampling from an alternative distribution and then applying an appropriate transformation to remove the bias. In general, this method requires a careful analysis in order to be effective as it not always produces a reduction in variance. We proceed to discuss these ideas in detail:

Recall Proposition 2.1 which says that for any $\theta > 0$, it holds that

$$\mathcal{L}_X(\theta) = \widetilde{\mathcal{L}}_X(\theta) \mathbb{E}[g(\widehat{\sigma}(\theta) Z; \theta)]$$

where $\widetilde{\mathcal{L}}_X(\theta)$ is the LM-approximation (1.2) of the Laplace transform, Z is a normal standard random variable and

$$\widehat{\sigma}^2(\theta) = \frac{\sigma^2}{1 + \mathcal{W}(\theta\sigma^2)}, \quad g(w; \theta) = \exp\left\{-\frac{\mathcal{W}(\theta\sigma^2)}{\sigma^2} \left(e^w - (1 + w + w^2/2)\right)\right\}.$$

A naïve approach is to use a Crude Monte Carlo estimator of $\mathbb{E}[g(\widehat{\sigma}(\theta) Z; \theta)]$, i.e. simulate $Z \sim N(0, 1)$ and return $g(\widehat{\sigma}(\theta) Z; \theta)$. We refer to this estimator as *Naïve Monte Carlo* and denote it $\widehat{\mathcal{L}}_N(\theta)$ in order to distinguish from the Crude Monte Carlo estimator discussed previously.

The Naïve Monte Carlo estimator $\widehat{\mathcal{L}}_N(\theta)$ is still highly unreliable (in spite of the apparent sharpness observed in the numerical examples in section 4) as it turns out it has infinite variance when $\theta > e^1\sigma^{-2}$. For proving this consider the second moment

$$\begin{aligned} \mathbb{E}[g^2(\widehat{\sigma}(\theta) Z; \theta)] &= \int_{-\infty}^{\infty} \exp\left\{-\frac{2\mathcal{W}(\theta\sigma^2)}{\sigma^2} \left(e^{\widehat{\sigma}(\theta)t} - 1 - \widehat{\sigma}(\theta)t - \frac{\widehat{\sigma}(\theta)^2 t^2}{2}\right) - \frac{t^2}{2}\right\} dt \\ &= \int_{-\infty}^{\infty} \exp\left\{-\frac{2\mathcal{W}(\theta\sigma^2)}{\sigma^2} \left(e^{\widehat{\sigma}(\theta)t} - 1 - \widehat{\sigma}(\theta)t\right) + \frac{\mathcal{W}(\theta\sigma^2) - 1}{\mathcal{W}(\theta\sigma^2) + 1} \cdot \frac{t^2}{2}\right\} dt \end{aligned} \quad (3.2)$$

If $t \rightarrow -\infty$ the exponential term $e^{\widehat{\sigma}t}$ vanishes and we are left with a second order polynomial with leading coefficient

$$\frac{\mathcal{W}(\theta\sigma^2) - 1}{2\mathcal{W}(\theta\sigma^2) + 1}.$$

The last takes positive values if $\mathcal{W}(\theta\sigma^2) > 1$ which occurs if and only if $\theta > e^1\sigma^{-2}$. In such case, then the integrand goes to infinity as $t \rightarrow -\infty$ (Figure 1).

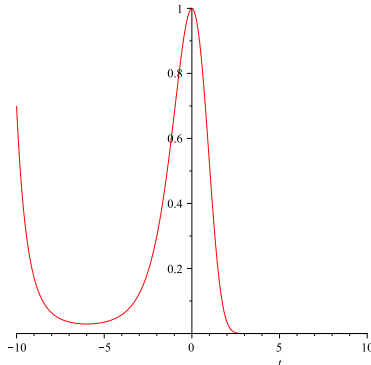


Figure 1: Graph of the integrand in (3.2) with $\sigma = 1$ and $\theta = 30$.

The argument above, shows that even when the random variable $g^2(\widehat{\sigma}(\theta) Z; \theta)$ has a finite expected value which coincides with the value of the error term, the naïve Monte Carlo estimator is fated to deliver unreliable estimates as it will have infinite variance. The result above is not surprising as exponential transformations of light tailed random variables (as the one obtained by applying the function g^2 to a normal random variable) often yields heavy-tailed distributions with infinite moments. Now, to fix this problem we propose a second estimator which is based on a change of measure suggested by Lemma 2.5. Recall that such lemma says that

$$\begin{aligned} \mathcal{L}(\theta) &:= \exp\left\{-\frac{\mathcal{W}^2(\theta\sigma^2) + 2\mathcal{W}(\theta\sigma^2)}{2\sigma^2}\right\} \mathbb{E}[\vartheta(\sigma Z; \theta)], \\ \vartheta(t; \theta) &= \exp\left\{-\frac{\mathcal{W}(\theta\sigma^2)}{\sigma^2}(e^t - 1 - t)\right\}. \end{aligned}$$

where $\sigma Z \sim N(0, \sigma^2)$. Hence, it follows that if $Y \sim N(0, \sigma^2)$ then the following is an unbiased estimator of $\mathcal{L}(\theta)$

$$\widehat{\mathcal{L}}_{IS}(\theta) := \exp\left\{-\frac{\mathcal{W}^2(\theta\sigma^2) + 2\mathcal{W}(\theta\sigma^2)}{2\sigma^2}\right\} \cdot \vartheta(Y, \theta). \quad (3.3)$$

For reasons to be explained next, we refer to this estimator as the *Importance Sampling estimator* of the Lognormal Laplace transform $\mathcal{L}(\theta)$ and denote it with $\widehat{\mathcal{L}}_{IS}(\theta)$. The name of the new estimator is due to the fact that it can be constructed by using *importance sampling*. Such method relies on the existence of a Radon-Nykodym derivative with respect to a probability measure, say \mathbb{Q} . If we are interested in estimating $\mathbb{E}[g(Y)]$ where Y is a random variable, \mathbb{E} is the expectation operator under the measure \mathbb{P} and \mathbb{Q} is an absolutely continuous measure with respect to \mathbb{P} , then it holds that

$$\mathbb{E}[g(Y)] = \mathbb{E}^{\mathbb{Q}}[L \cdot g(Y)],$$

where $\mathbb{E}^{\mathbb{Q}}$ is the expectation operator under the measure \mathbb{Q} and $L := d\mathbb{P}/d\mathbb{Q}$ is the Radon-Nykodym derivative of \mathbb{P} with respect to \mathbb{Q} (the last also goes under the name *likelihood ratio* in the simulation community). Moreover, if the measures \mathbb{P} and

\mathbb{Q} are absolutely continuous, then the Radon-Nykodym derivative/likelihood ratio is simply the ratio of the corresponding density functions. Hence, if Y is simulated according to \mathbb{Q} , then $L \cdot g(Y)$ serves as an unbiased estimator of the quantity $\mathbb{E}[g(Y)]$.

Notice that the importance sampling methodology ensures that the new estimator is unbiased; also in most of the cases it will have a different variance. However, there is no guarantee that the new variance is smaller and that of course depends on an adequate selection of the importance distribution.

In our setting, we want to estimate the quantity $\mathbb{E}[g(Y; \theta)]$ where $Y \sim N(0, \hat{\sigma}^2(\theta))$ and g is the function given in Proposition 2.1. Thus, we select \mathbb{Q} in such way that $Y \sim N(0, \sigma^2)$. It turns out that the likelihood ratio is equal to

$$L(y) := \frac{f_{0, \hat{\sigma}^2(\theta)}(y)}{f_{0, \sigma^2}(y)} = \sqrt{1 + \mathcal{W}(\theta\sigma^2)} \exp\left\{-\mathcal{W}(\theta\sigma^2) \frac{y^2}{2\sigma^2}\right\}$$

where $f_{\mu, \sigma^2}(y)$ denotes the density of the normal distribution $N(\mu, \sigma^2)$. Then it follows that

$$\begin{aligned} L \cdot g(y; \theta) &= \sqrt{1 + \mathcal{W}(\theta\sigma^2)} \exp\left\{-\mathcal{W}(\theta\sigma^2) \frac{y^2}{2\sigma^2}\right\} \\ &\quad \cdot \exp\left\{-\frac{\mathcal{W}(\theta\sigma^2)}{\sigma^2} (e^y - (1 + y + y^2/2))\right\} \\ &= \sqrt{1 + \mathcal{W}(\theta\sigma^2)} \exp\left\{-\frac{\mathcal{W}(\theta\sigma^2)}{\sigma^2} (e^y - (1 + y))\right\} \\ &= \sqrt{1 + \mathcal{W}(\theta\sigma^2)} \cdot \vartheta(y; \theta). \end{aligned}$$

This argument confirms that the estimator $\widehat{\mathcal{L}}_{IS}(\theta)$ is an importance sampling estimator with respect to the Naïve Monte Carlo estimator $\widehat{\mathcal{L}}_N(\theta)$ and with $N(0, \sigma^2)$ as importance sampling distribution. Moreover, it turns out that $\widehat{\mathcal{L}}_{IS}(\theta)$ achieves logarithmic efficiency:

Proposition 3.2. *$\widehat{\mathcal{L}}_{IS}(\theta)$ is an unbiased estimator of the Laplace transform of the lognormal distribution $\text{LN}(0, \sigma^2)$. Moreover, it is logarithmic efficient as $\theta \rightarrow \infty$.*

Proof of Proposition 3.2. By construction, $\widehat{\mathcal{L}}_{IS}(\theta)$ is an unbiased estimator of $\mathcal{L}(\theta)$. To prove logarithmic efficiency we use the equivalence in Lemma 2.5 and the asymptotic relation in Corollary 2.3 to verify the following equivalence

$$\begin{aligned} \lim_{\theta \rightarrow \infty} \frac{\text{Var}[\widehat{\mathcal{L}}_2(\theta)]}{\mathcal{L}^{2-\epsilon}(\theta)} &= \lim_{\theta \rightarrow \infty} \frac{\widetilde{\mathcal{L}}^2(\theta)(1 + \mathcal{W}(\theta\sigma^2)) \cdot \text{Var}[\vartheta(Y; \theta)]}{\widetilde{\mathcal{L}}^{2-\epsilon}(\theta)} \\ &= \lim_{\theta \rightarrow \infty} \widetilde{\mathcal{L}}^\epsilon(\theta)(1 + \mathcal{W}(\theta\sigma^2)) \cdot \text{Var}[\vartheta(Y; \theta)], \end{aligned}$$

where $\widetilde{\mathcal{L}}(\theta)$ is the asymptotically equivalent approximation of $\mathcal{L}(\theta)$ given by the Laplace's method. Since $\vartheta^2(y; \theta) \leq 1$ we have the following bounds

$$\lim_{\theta \rightarrow \infty} \widetilde{\mathcal{L}}^\epsilon(\theta)(1 + \mathcal{W}(\theta\sigma^2)) \mathbb{E}[\vartheta^2(\sigma Z; \theta)] \leq \lim_{\theta \rightarrow \infty} \widetilde{\mathcal{L}}^\epsilon(\theta)(1 + \mathcal{W}(\theta\sigma^2)) = 0.$$

The last limit is straightforward to verify by inserting the formula for $\widetilde{\mathcal{L}}(\theta)$ given in (1.2) and checking that the term $1 + \mathcal{W}(\theta\sigma^2)$ is asymptotically bounded by $\widetilde{\mathcal{L}}^\epsilon(\theta)$ for all $\epsilon > 0$. \square

Algorithm. The following generates a single replicate of the IS estimator.

1. Simulate $Y \sim N(0, \sigma^2)$.
2. Compute $\vartheta(Y; \theta) = \exp \left\{ -\frac{\mathcal{W}(\theta\sigma^2)}{\sigma^2} \left(e^Y - 1 - Y \right) \right\}$.
3. Return $\widehat{\mathcal{L}}_{IS}(\theta) := \exp \left\{ -\frac{\mathcal{W}^2(\theta\sigma^2) + 2\mathcal{W}(\theta\sigma^2)}{2\sigma^2} \right\} \vartheta(Y; \theta)$.

4 Numerical examples

In this section we investigate the numerical performance of our proposals and compare them against several approximations available in the literature; we further discuss the quality of the approximates. First we conduct separate investigations on approximations of the Laplace transform and the characteristic function. Secondly, we corroborate empirically the efficiency properties of the Monte Carlo estimators considered in this paper.

4.1 Approximations of the Laplace transform

We first considered approximations of the Laplace transform of the lognormal distributions. For that purpose we examined the proposals of Barakat [9], Gubner [12] and Tellambura/Senaratne [13] (further discussion on these results can be found in the last section of this paper). These proposals were originally designed for approximating the characteristic function; however, one can make the appropriate comparisons by using the relation $\mathcal{L}(i\theta) = \varphi(\theta)$. We compared these approximations against our closed form expression (1.2) which was referred as *LM-approximation*. We decided to use the importance sampling Monte Carlo estimator suggested in section 3 as benchmark of comparison; recall that we employed the name *IS estimator* to refer to it. We have employed 10^8 replications for each estimate. In the examples below, we present tables with the values of both the approximation and the relative errors with respect to the IS estimator. The latter is defined as

$$\text{Relative Error} = \frac{\text{Approximation} - \text{IS estimate}}{\text{IS estimate}}.$$

It is important to remark that for the approximations suggested by Gubner and Tellambura-Senaratne we employed the Matlab codes provided by the authors.

Example 1. For our first example we have used a value of $\sigma = 0.25$ (this value is employed in the numerical results presented in the paper of Barakat [9] and used here for corroboration purposes). The results are presented in Table 1.

For this example, the results of Gubner were excluded since the corresponding algorithm delivered values which were much larger than 1. The method of Tellambura/Senaratne also produced unreliable results: notice for instance that it produces an estimate of $\mathcal{L}(0)$ which has an error of about 5% (recall that $\mathcal{L}(0) = 1$). In contrast, our two proposals delivered very similar results, thus confirming that the Laplace's method can produce indeed very sharp approximations of the Laplace

Table 1: Approximated Values of $\mathcal{L}(\theta)$ with $\sigma = 0.25$.

θ	IS Monte Carlo	LM approximation	Tellambura/Senaratne	Barakat
0.0	1.000000	1.000000	0.945115	1.000000
0.4	0.670006	0.670006	0.633173	0.670006
0.8	0.449189	0.449190	0.424455	0.449189
1.2	0.301335	0.301336	0.284715	0.301335
1.6	0.202274	0.202275	0.191099	0.202274
2.0	0.135862	0.135862	0.128344	0.135862

transform of the lognormal distribution. The results of Barakat coincide with our results, thus reassuring the sharpness of our methods for small values of σ . Table 2 below shows the relative errors of the estimators. It is noted that the method of Tellambura/Senaratne underestimates the real values with errors of about 5.5% not only in a neighborhood of 0 but all across the domain of the transform.

Table 2: Relative Errors of $\mathcal{L}(\theta)$ with $\sigma = 0.25$.

θ	LM approximation	Tellambura/Senaratne	Barakat
0.0	0.000000×10^0	-0.054885	0.000000×10^0
0.4	7.986949×10^{-7}	-0.054974	4.080170×10^{-8}
0.8	2.009580×10^{-6}	-0.055064	5.047547×10^{-7}
1.2	2.381985×10^{-6}	-0.055154	1.410071×10^{-7}
1.6	2.876162×10^{-6}	-0.055245	-9.037186×10^{-8}
2.0	2.906645×10^{-6}	-0.055337	-7.750253×10^{-7}

Example 2. In the second example we consider a value of $\sigma = 1$. The results are presented in Table 3.

Table 3: Approximated Values of $\mathcal{L}(\theta)$ with $\sigma = 1$.

θ	IS Monte Carlo	LM approximation	Tellambura	Barakat	Gubner
0.0	1.000000	1.000000	1.000000	1.000000	1.000000
0.4	0.616293	0.624119	0.616284	0.642997	0.615747
0.8	0.440037	0.445053	0.440053	0.673959	0.434538
1.2	0.335214	0.338399	0.335201	-1.753636	0.319962
1.6	0.265606	0.267730	0.265610	-50.847370	0.231162
2.0	0.216313	0.217758	0.216309	-596.161998	0.279931

One of the most notorious aspects of this case is that the approximation of Barakat deteriorates rapidly. The method of Gubner delivers better results but still with large errors. The algorithm of Tellambura seems to be the one delivering the best results. In the case of our LM approximation, it delivers sensible results which does not seem to deteriorate as the value of θ increases; nevertheless, it seems that this closed form approximation loses accuracy as σ increases. These observations are reaffirmed after examining the relative errors in Table 4.

Table 4: Relative Errors of $\mathcal{L}(\theta)$ with $\sigma = 1$.

θ	LM approximation	Tellambura/Senaratne	Barakat	Gubner
0.0	0.000000×10^0	0.000000×10^0	0.000000×10^0	0.000000×10^0
0.4	1.269892×10^{-2}	-1.396821×10^{-5}	4.333043×10^{-2}	-8.854354×10^{-4}
0.8	1.139909×10^{-2}	3.720053×10^{-5}	5.315953×10^{-1}	-1.249595×10^{-2}
1.2	9.502939×10^{-3}	-3.903671×10^{-5}	-6.231395×10^0	-4.549968×10^{-2}
1.6	7.997354×10^{-3}	1.402916×10^{-5}	-1.924392×10^2	-1.296799×10^{-1}
2.0	6.681295×10^{-3}	-1.925373×10^{-5}	-2.757016×10^3	2.941032×10^{-1}

Example 3. Next we consider a larger value $\sigma = 4$. The results are shown in Table 5. The results of Gubner and Barakat deteriorated rapidly so these were excluded from our analysis.

Table 5: Approximated Values of $\mathcal{L}(\theta)$ with $\sigma = 4$.

θ	IS Monte Carlo	LM approximation	Tellambura
2	0.382672	0.371296	0.382685
4	0.321163	0.307613	0.321194
6	0.287195	0.273413	0.287219
8	0.264198	0.250553	0.264181
10	0.246962	0.233637	0.246974

In this case Tellambura/Senaratne provided the best approximations. Our approximation delivered significantly larger errors. These observations are further corroborated after an inspection of the relative errors. The approximation of Tellambura/Senaratne seems to be remain accurate for large values of θ . These results are given in Table 6.

Table 6: Relative Errors of $\mathcal{L}(\theta)$ with $\sigma = 4$.

θ	Our Proposal	Tellambura/Senaratne
2	-0.029728	3.36825×10^{-5}
4	-0.042190	9.65352×10^{-5}
6	-0.047989	8.44206×10^{-5}
8	-0.051645	-6.49857×10^{-5}
10	-0.053957	4.96133×10^{-5}

4.2 Approximations of the Characteristic Function

Next we analyzed the approximations for the Characteristic function of the log-normal distribution. In addition to the results of Barakat, Gubner and Tellambura/Senaratne we inspected the results of Leipnik [17] who proposed a series representations given in terms of Hermite polynomials. However, we could not obtain values which could be considered reliable (other authors have faced the same challenges when trying to implement this algorithm [5]).

Example 4. For our single example we considered a value of $\sigma = 0.5$. The graphs of the real and imaginary part of $\varphi(\omega)$ are shown in Figure 2. Our approximation and the Tellambura/Senaratne are indistinguishable in the graphs as these are very accurate. However, the approximation of Barakat deteriorates in regions away from 0. This phenomena could be explained by the fact that the Barakat approximation is obtained by truncating a converging series. In particular, we observed that the Barakat approximation also deteriorates as the value of σ increases thus requiring an even larger amount of series terms.

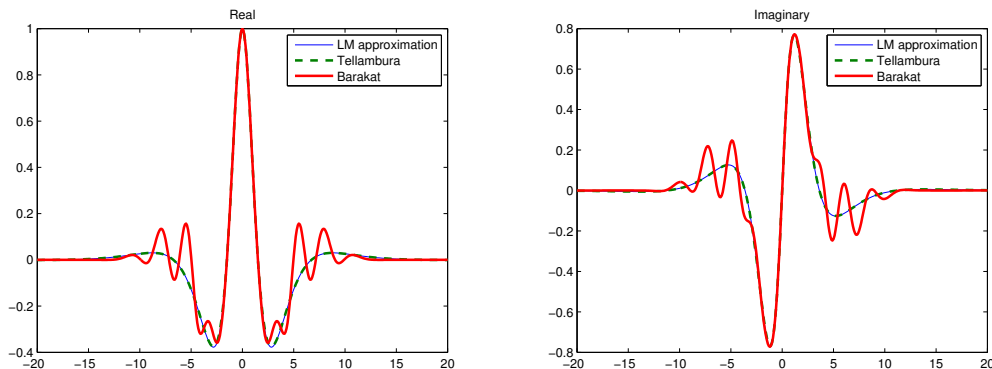


Figure 2: Real and imaginary parts of the Characteristic function of the Lognormal for $\sigma = 0.5$.

We conducted similar experiments for alternative values of σ and we obtained similar conclusions as in the case of the Laplace transform. For very small values of the σ , the approximation of Tellambura/Senaratne delivers unreliable approximations and sometimes it fails to converge. In the same case, the algorithm of Barakat and our proposal deliver very sharp approximates. As the value of σ increases, the proposal of Barakat deteriorates rapidly. Our proposal loses some precision but still delivered sensible approximations. For moderately large values of σ it is the algorithm of Tellambura/Seranarte the one which produces the best results.

4.3 Efficient Monte Carlo

Example 5. In this example we contrast the three estimators discussed in this paper: Crude Monte Carlo, Naïve Monte Carlo and Efficient Monte Carlo. We selected a value of $\sigma = 1$. In the three cases we have employed $R = 10^8$ replications. Figure 3 shows the estimates provided by the three methods in logarithmic scale. It is observed that Crude Monte Carlo provides a reasonable approximation of the true Laplace transform for small values of the argument; however, as the value of θ increases, the quality of the crude estimate deteriorates as the relative error increases (this explains the jiggled nature of the curve). On the other hand, it appears that the two importance sampling methods discussed here provide sharp approximations as their values are very close to each other (the curves are indistinguishable from each other).

However, one of the two importance sampling estimators (Naïve Monte Carlo) has an infinite variance, so it can provide unreliable estimates. This effect is noted

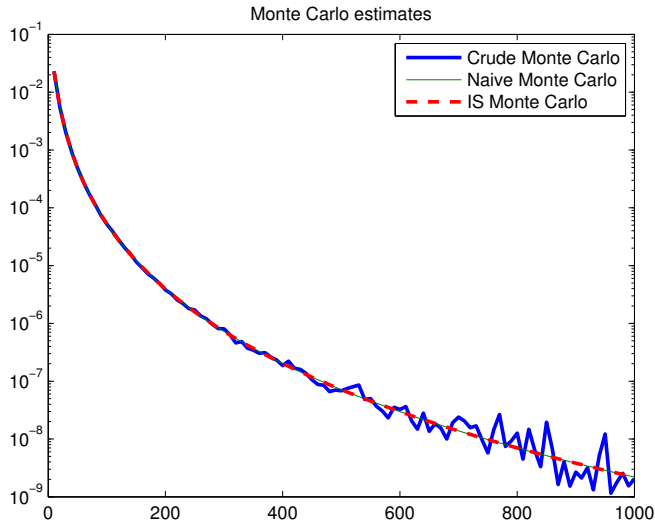


Figure 3: Monte Carlo Estimates for the Laplace transform with $\sigma = 1$.

in the left panel of the Figure 4 where the variances of these two estimators are plotted. It appears that the first estimator has a lower variance but this is only due to the fact that the variance is underestimated (the random “peaks” are a clear symptom of this problem). On the other hand, the efficient algorithm has a sharp estimate of the real variance which is reflected in the smoothness of the curve. The relative errors are plotted in the panel on the right of Figure 4. The relative error of the IS estimator increases at a rate which appears to be at least logarithmic thus corroborating its efficiency. In the case of the Naïve Monte Carlo estimator (infinite variance) the estimators of the relative error are unreliable.

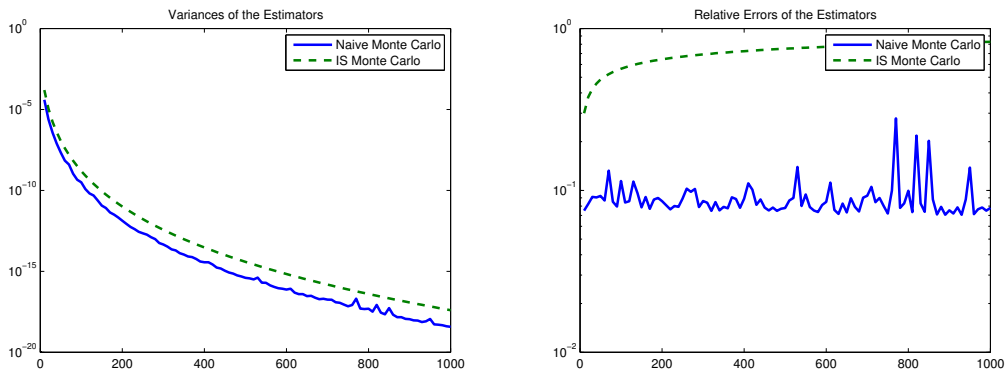


Figure 4: Variance and Relative Errors of the two importance sampling algorithms suggested.

5 Discussion and Conclusion

More on earlier literature

The use of infinite series representations has been one of the most used approaches to deal with the transforms of the Lognormal distribution. One of the most obvious attempts in the study of $\mathcal{L}(\theta)$ is using formal series representations as follows: consider the following integral expression for the transform of the Lognormal distribution

$$\mathcal{L}(\theta) = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\theta e^t - \frac{(t - \mu)^2}{2\sigma^2}\right\} dt, \quad \Re(\theta) > 0,$$

replace the term $e^{-\theta t}$ with its Taylor series, interchange the integral and sum and perform a term by term integration, thus obtaining a formal series representation with the moments of the lognormal distribution as coefficients. This attempt turns to be invalid as the resulting series diverges. This is not surprising because the procedure described above is equivalent to deriving a Taylor series of the function $\mathcal{L}(\theta)$ around the origin, but as noted before, this function is not analytic in the imaginary axis. This pathology is also related to the well known fact that the lognormal distribution is not uniquely determined by its moment sequence (Heyde, 1963, [18]). Other methods using series representations appear as early as 1976. The manuscript of Barouch and Kaufman [19] provides various approximations in terms of series representations which are valid in specific regions; for instance, a series expansion of the lognormal density is employed to produce a closed-form asymptotic approximation in terms of both the Gamma function and its derivatives. However, none of these expressions can deliver reliable estimates in the whole domain of the characteristic function. The Laplace transform representation proposed by Barakat has the following closed form

$$\mathcal{L}(\theta) = e^{-\theta} e^{\theta^2 \sigma^2 / 2} \sum_{n=0}^{\infty} \frac{(-1)^n \sigma^n}{n!} a_n(\theta) H_n(\sigma \theta), \quad (5.1)$$

where $a_n(\theta)$ is the n -th coefficient in the MacLaurin series representation of $e^{-\theta(e^y - 1 - y)}$ and H_n is the n -th Hermite polynomial (notice that we have employed the *probabilist Hermite* polynomials instead of the *physicist Hermite* polynomials in the definition [cf. 20]). We found that the approximation (5.1) of Barakat [9] is sharp for small values of σ^2 , but rapidly deteriorates for large values of σ^2 in regions away from the origin. A similar expression is obtained by Leipnik [17], but instead he shows that the characteristic function satisfies a functional differential equation of the form $\varphi'(\omega) = i e^{\mu + \sigma^2 / 2} \varphi(e^{\sigma^2} \omega)$ (an observation that appeared previously in [19]). Leipnik employs a method due to de Bruijn to solve this functional differential equation: the solution (given as an integral involving the gamma function) is proven to have an explicit convergent infinite series representation in terms of Hermite polynomials which is of the form

$$\varphi(\theta) = \sqrt{\frac{\pi}{2\sigma^2}} \exp\left\{-\frac{\log^2(\theta + i\pi/2)}{2\sigma^2}\right\} \sum_{n=0}^{\infty} \frac{i^n}{\sigma^n} d_n H_n(\log(\theta + i\pi/2)/\sigma)$$

where d_n are the coefficients in the MacLaurin series representation of the reciprocal of the Gamma function $\Gamma^{-1}(y + 1)$. Leinik includes a recursive formula for calculating the coefficients d_n in terms of the *Euler's constant* and the Riemann Zeta function; this recursion facilitates the calculation of this series representation. However, the solution of the functional differential equation cannot be extended to the whole complex plane, so it appears that this approximation only applies for the characteristic function (in fact, we were not able to obtain a reliable numerical estimate using any of these formulae).

In his study of the characteristic function, cf. [10], Holgate applied the *Lagrange inversion theorem* to the equivalence $te^{-t} = i\sigma^2\omega$ to obtain an asymptotic series representation of the saddle point function $\rho(\omega)$, which inserted into expression (1.4) provides a representation of the function $\varphi(\omega)$ in terms of an asymptotic infinite series. However, the resulting series oscillates wildly and cannot provide a reliable numerical approximations. Finally, another interesting and somewhat different approach which delivers closed-form expressions is given by Rossberg [21], who provides a representation of general Laplace transforms in terms of a 2-fold convolution involving the cdf of the random variable of interest.

Numerical integration methods have also received a good deal of attention and most of these have been developed in parallel with the analytic approximations discussed above. One of the earliest references is [16], where the performance of various standard integration methods is analyzed. It is remarked there that approximating the characteristic function via numerical integration is very challenging due to the oscillatory nature of the term $e^{i\omega t}$ and the heavy-tailed nature of the lognormal density [cf. 22]. This fact has been further discussed in several other papers [9, 12, 16]). An obvious approach to deal with the oscillations is to employ complex analytic techniques: besides the paper of Holgate [10] where the saddlepoint methodology is exploited, it seems that Gubner [12] was the first in proposing alternative path contours to reduce the oscillatory behavior of the integrand, as followed up by Telambura and Senaratne [13] where they proposed specific contours passing through the saddlepoint at a steepest descent rate; this choice has the effect that oscillations are removed in a neighborhood around the saddlepoint. In addition, they also address the heavy-tailed nature of the lognormal density by proposing a transformation which delivers an integrand with lighter tails.

Summary of the paper

A closed form expression of the Laplace transform of the lognormal distribution does not exist. Providing a reliable approximation is a difficult problem since traditional approximation methods fail mainly due to the fact that the Lognormal distribution is heavy-tailed and its transforms are not analytic in the origin. In this paper we proposed a closed form approximation of the Laplace transform which is obtained via a modified version of the classic asymptotic Laplace's method. The main result is a decomposition of the Laplace transform which delivers a closed form approximation of the Laplace transform and an expression of the exact error. The last turns to be useful to prove the asymptotic equivalence of the proposed approximation. Moreover, since the error term is given in a probabilistic representation it turns out to be convenient for analysis.

In addition, we constructed a Monte Carlo estimator of the Laplace transform of the Lognormal distribution. This estimator is based on the probabilistic representation of the error term obtained via the modified version of the Laplace’s method. We prove the efficiency of this estimator. In contrast, we illustrate that the Crude and Naïve Monte Carlo implementations can deliver unreliable estimates for large values of the argument.

Finally, we conducted numerical experiments where we compared our proposals against other approximations available in the literature. We found that most approximations are very sensible for different values of σ . The method of Tellambura is one of the most precise; however, it delivers unreliable results for small values of σ and sometimes it fails to converge. The proposal of Barakat can deliver sharp results for small values of σ but fails for large values of the argument. Our closed-form LM expression delivers approximations which remain precise all over the domain of the transform; in particular, it tends to be more precise for small values of σ . An attractive feature of our proposal is its simple closed-form.

In contrast, we showed that our efficient IS Monte Carlo estimator is the only method which delivered reliable sharp results for any combination of values of the parameters σ and θ . In particular, it remains sharp in asymptotic regions as it is based on an asymptotic method. In addition it has a simple form and it is easy to implement. Moreover, it does not have convergence issues. Overall, this seems to be the best available option to approximate the Laplace transform of the Lognormal distribution.

Work in progress

One obvious way to go is to apply our results to approximate the density and cumulative distribution function of a sum of independent random variables. The obvious idea is via transform inversion, but we could also apply the saddlepoint methodology to obtain approximations in the left tail via Escher and Legendre-Fenchel transforms (note that the left tail is of particular importance in finance because of its interpretation related to, say, portfolio loss). In addition, one could also construct reliable efficient Monte Carlo estimators based on this approximation. Finally, Laplace transforms of correlated lognormals could also be explored.

References

- [1] I. Aitchison and J. A. C. Brown, *The lognormal distribution with special reference to its uses in economics*. Cambridge: Cambridge University Press, 1957.
- [2] E. L. Crow and K. Shimizu, *Lognormal distributions: theory and applications*. New York: Marcel Dekker Inc., 1988.
- [3] N. L. Johnson, S. Kotz, and N. Balakrishnan, *Continuous Univariate Distributions*, vol. 1. New York: Wiley, 2 ed., 1994.
- [4] E. Limpert, W. A. Stahel, and M. Abbt, “Log-normal distributions across the sciences: Keys and clues,” *BioScience*, vol. 51, pp. 341–352, 2001.

- [5] D. Dufresne, “Sums of lognormals,” in *Actuarial Research Conference Proceedings*, 2008.
- [6] L. Rojas-Nandayapa, *Risk probabilities: Asymptotics and Simulation*. PhD thesis, Aarhus University, 2008.
- [7] R. M. Corless, G. H. Gonnet, D. E. G. Hare, D. J. Jeffrey, and D. E. Knuth, “On the Lambert W function,” *Advances in Computational Mathematics*, vol. 5, pp. 329–359, 1996.
- [8] N. G. de Bruijn, *Asymptotic Methods in Analysis*. Courier Dover Publications, 1970.
- [9] R. Barakat, “Sums of independent lognormally distributed random variables,” *Journal of the Optical Society of America*, vol. 66, pp. 211–216, 1976.
- [10] P. Holgate, “The lognormal characteristic function,” *Communications in Statistical-Theory and Methods*, vol. 18, pp. 4539–4548, 1989.
- [11] J. L. Jensen, *Saddlepoint Approximations*. Oxford: Oxford Science Publications, 1994.
- [12] J. A. Gubner, “A new formula for lognormal characteristic functions,” *IEEE Transactions on Vehicular Technology*, vol. 55, no. 5, pp. 1668–1671, 2006.
- [13] C. Tellambura and D. Senaratne, “Accurate computation of the MGF of the lognormal distribution and its application to sum of lognormals,” *IEEE Transactions on Communications*, vol. 58, pp. 1568–1577, 2010.
- [14] R. W. Butler, *Saddlepoint Approximations with Applications*. Cambridge University Press, 2007.
- [15] C. G. Small, *Expansions and Asymptotics for Statistics*. Chapman & Hall/CRC, 2013.
- [16] N. C. Beaulieu and Q. Xie, “An optimal lognormal approximation to lognormal sum distributions,” *IEEE Transactions on Vehicular Technology*, vol. 53, no. 2, pp. 479–489, 2004.
- [17] R. B. Leipnik, “On lognormal random variables: I. the characteristic function,” *Journal of the Australian Mathematical Society Series B*, vol. 32, pp. 327–347, 1991.
- [18] C. Heyde, “On a property of the lognormal distribution,” *J. Roy. Statist. Soc. Ser. B*, vol. 29, pp. 392–393, 1963.
- [19] E. Barouch and G. M. Kaufman, “On sums of lognormal random variables.” Working paper. A. P. Sloan School of Management, MIT., 1976.
- [20] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions: With Formulas, Graphs, and Mathematical Tables*. Applied mathematics series, Dover Publications, 1964.
- [21] A. G. Rossberg, “Laplace transforms or probability distributions and their inversions are easy on logarithmic scales,” *Journal of Applied Probability*, vol. 45, pp. 531–541, 2008.
- [22] S. Foss, D. Korshunov, and S. Zachary, *An introduction to heavy-tailed and subexponential distributions*. Springer, 2011.