MULTI-INDEX AND MULTI-LEVEL MONTE CARLO EVALUATION OF HJM MODELS

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Abstract. Notes of MIMC and MLMC evaluation of HJM models.

1. Intro

We are interested in analysing the Forward Euler Finite Difference evaluation of forward curve models of HJM models. The main approach was represented with a convergence analysis in **?**, whose notation we largely follow.

We are focused on the evolution of the forward curve $f(t,\tau)$ for $t \in [0, t_{max}]$, $\tau \in [0, \tau_{max}]$ with $\tau_{max} > t_{max}$. In order to define a mesh for the numerical solution of the problem, let us define $\bar{\ell} \in \mathbb{N}^3$.

Let our quantity of interest be defined as the expectation of a functional $\mathcal{F}$ of the forward curve:

$$\begin{aligned}\mathcal{F}(f) =& F\left(\int_0^{t_{max}} f(s,s)\,ds\right) G\left(\int_{\tau_a}^{\tau_{max}} \Psi\left(f\left(t_{max},\tau\right)\right) d\tau\right) \\ &+ \int_0^{t_{max}} F\left(\int_0^s f\left(s',s'\right) ds'\right) U\left(f\left(s,s\right)\right) ds.\end{aligned}$$

Let us first discretise the time interval as

$$\begin{aligned}\Delta t_{\bar{\ell}} &= 2^{-\ell_1} t_{max}, \\ t_{\bar{\ell},n} &= n \Delta t_{\bar{\ell}},\ n \in [0,1,2,\cdots,2^{\ell_1}].\end{aligned}$$

Since there is an additional requirement that t_{max} be in the mesh of maturities, one point in the mesh in $\tau \in [0, \tau_{max}]$. We let ℓ_1 and ℓ_2 define the number of discretisation points to the right and to the left of the fixed point, respectively.

$$\begin{aligned}\Delta\tau_{\bar{\ell},1} &= 2^{-\ell_3} t_{max}, \\ \Delta\tau_{\bar{\ell},2} &= 2^{-\ell_2}\left(\tau_{max} - t_{max}\right), \\ \tau_{\bar{\ell},n} &= n\Delta\tau t_{\bar{\ell},1},\ n \in [0,1,2,\cdots,2^{\ell_3}] \\ \tau_{\bar{\ell},n} &= n\Delta\tau t_{\bar{\ell},1},\ n \in [2^{\ell_3}, 2^{\ell_3}+1, \cdots, 2^{\ell_3}+2^{\ell 2}].\end{aligned}$$

Using this notation, we have the following rates of convergence, following the Multi-Index notation:

$$\bar{\beta} = [2,2,2]^T \tag{1}$$

$$\bar{\gamma} = [1,1,1]^T. \tag{2}$$

Date: April 25, 2016.

Choosing the Simpson method for integration, we have the order of quadrature $p_Q = 3$, which brings the quadrature error to

$$\mathrm{E}\left(\left|\mathcal{G}\left(\bar{\bar{g}}\right) - \mathcal{G}\left(g\right)\right|^{2\kappa}\right) \leq C\left(\Delta\tau_{\bar{\ell}}\right)^{2\kappa p_Q} = \tilde{C}2^{-6\kappa\min(\ell_3,\ell_2)}. \tag{3}$$

On the other hand, the error of the simpson method is the sum of errors for $\tau < t_{max}$ and $\tau \leq t_{max}$. The error for these two domains are $\mathcal{O}\left(2^{-2\ell_2}\right)$ and $\mathcal{O}\left(2^{-2\ell_3}\right)$, respectively. Combining together, we get the weak and the strong rate for the Simpson quadrature:

$$\mathrm{E}\left(\left|\mathcal{G}\left(\bar{\bar{g}}\right) - \mathcal{G}\left(g\right)\right|^{2}\right) \leq \tilde{C}_Q 2^{-6\min(\ell_3,\ell_2)} \tag{4}$$

$$\mathrm{E}\left(\left|\mathcal{G}\left(\bar{\bar{g}}\right) - \mathcal{G}\left(g\right)\right|\right) \leq C_Q 2^{-2\min(\ell_2,\ell_3)}. \tag{5}$$

$$\mathrm{E}\left(\left|\mathcal{G}\left(\bar{\bar{g}}\right) - \mathcal{G}\left(g\right)\right|^{2\kappa}\right) \tag{6}$$

using Theorem 3.3 of **?**.

On the other hand, theorem 3.2 and equation (3.54) states that

$$\mathrm{E}\left(\left|\mathcal{G}\left(g\right) - \mathcal{G}\left(\bar{\bar{g}}\right)\right|^{2\kappa}\right)^{\frac{1}{2\kappa}} \leq C_5^{CV}\left(\left(\Delta t\right)^{\frac{1}{2}} + \Delta\tau\right). \tag{7}$$

The weak rate for the discretisation error can be bounded by eq. (4.6):

$$\mathrm{E}\left(\left|\mathcal{G}\left(g\right) - \mathcal{G}\left(\bar{\bar{g}}\right)\right|\right) = \mathcal{O}\left(\left(\Delta t\right)^2 + \left(\Delta\tau\right)^2\right) = \mathcal{O}\left(2^{-2\ell_1} + 2^{-2\min(\ell_2,\ell_3)}\right) \tag{8}$$

Due to the fact that the discretisation errors relating to τ depend on the minimum of ℓ_2 and ℓ_3, it seems natural to assume $\ell_3 = \ell_2$. This gives us a two-dimensional Multi-Index Monte Carlo formulation with the following effective parameters

$$\bar{\beta} = [2, 2]^T, \tag{9}$$

$$\bar{\gamma} = [1, 1]^T, \tag{10}$$

$$\bar{s} = [1, 2], \tag{11}$$

$$\bar{w} = [1, 2]. \tag{12}$$

On the other hand, one may set $\ell_1 = \ell_2 = \ell_3 = \ell$ and arrive at a one-dimensional MLMC formulation with a weak rate of 2, strong rate of 1, and computational cost per realisation of $W_\ell = 2^{2\ell}$. In the sequel, we will present the results of these two variations for comparison.

2. Numerical tests

For the simplest possible example, let our model be Gaussian with the following exponential covariance structure

$$\mathrm{E}\left(df\left(t,\tau\right)df\left(t,\tau'\right)\right) = \sigma^2\exp\left(-\kappa\left|\tau - \tau'\right|\right)dt. \tag{13}$$

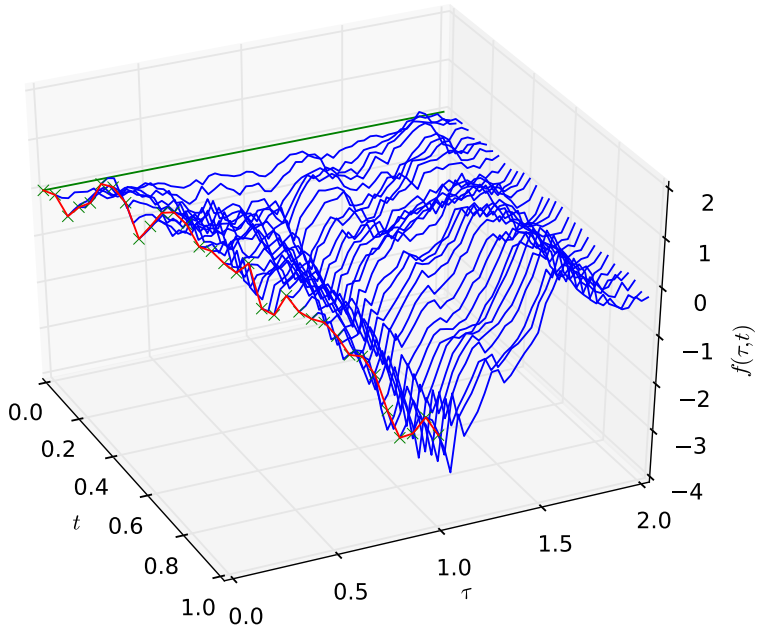

FIGURE 1. Sample realisation of a forward curve with the discretisation level $\overline{\ell} = [5, 5, 5]^T$.

Let our first test case be arguably simplest infinite dimensional one possible,

$$\begin{aligned} F(x) &= 1 - x, \\ U(x) &= 0, \\ G(x) &= x, \\ \Psi(x) &= x, \\ f(0, \tau) &= 0, \\ \kappa &= 2. \end{aligned}$$

In 2 we get see that we have rate one for both the weak and the strong rate along both dimensions for this text example.

It remains to deduce the cost of simulating the cost of a realisation. In order to simulate Gaussian random variables one needs the Cholesky decomposition of the covariance matrix. The cost to generate this $N_\tau \times N_\tau$-matrix is N_τ^3, with $N_\tau = 2^{\ell_3} 2^{\ell_2}$. Using the Cholesky decomposition, the cost to take a a time step becomes N_τ^2. Putting all together, the cost to generate M Monte Carlo realisations with N_t time steps and N_τ

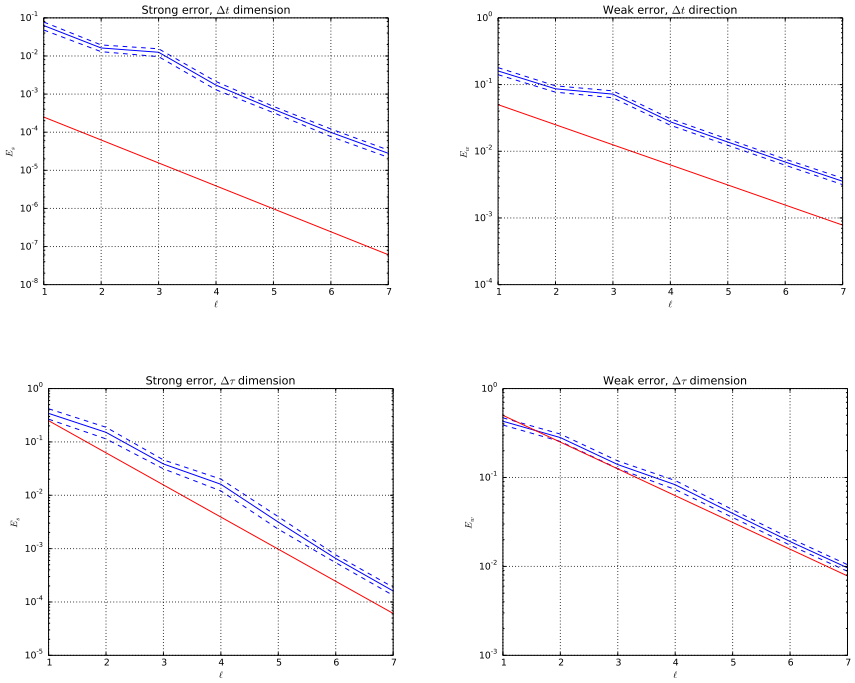

FIGURE 2. Empirical strong and weak convergence rate for the Δt and $\Delta\tau$ discretisation errors.

discretisation steps along the maturity direction becomes

$$C\left(\Delta t, \Delta\tau, M\right) = N_\tau^3 + MN_\tau^2 N_t$$

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