

# MULTI-INDEX AND MULTI-LEVEL MONTE CARLO EVALUATION OF HJM MODELS

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ABSTRACT. Notes of MIMC and MLMC evaluation of HJM models.

## 1. HJM MODEL IN FOURIER SPACE

Let us focus on the following HJM type SDE:

$$(1) \quad df(t, \tau) = \alpha(t, \tau) dt + \beta(t, \tau) d\bar{W}(t)$$

$$(2) \quad f(0, \tau) = f_0(\tau),$$

with the diffusion term given as the infinite-dimensional extension of eq. (1.1) in Björk et al. (2013):

$$(3) \quad \beta(t, \tau) d\bar{W}(t) = \sum_{k=0}^{\infty} c_k (\sin(\nu_k \tau) + \cos(\nu_k \tau))$$

with  $W_n$  independent Brownian motions and

$$\nu_k \equiv \frac{k\pi}{L}$$

with  $L \gg 1$ . This gives the following Fourier decomposition for the covariance of increments:

$$(4) \quad \mathbb{E}(\beta(t, \tau_1) d\bar{W}(t) \beta(t, \tau_2) d\bar{W}(t))$$

$$(5) \quad = \sum_{k=0}^{\infty} c_k^2 (\cos(\nu_k (\tau_1 - \tau_2)))$$

To keep the exponential HJM model risk neutral, we fix the drift of the equation as

$$\begin{aligned} \alpha(t, \tau) = & c_0^2 (\tau - t) \\ & + \sum_{k=1}^{\infty} \frac{c_k^2}{k} \sin(\nu_k \tau) (\cos(\nu_k t) - \sin(\nu_k t)) \\ & + \sum_{k=1}^{\infty} \frac{c_k^2}{k} \cos(\nu_k \tau) (\cos(\nu_k t) - \sin(\nu_k t)) \\ & + \sum_{n=1}^{\infty} \frac{c_n^2}{n} \cos(2\nu_n \tau). \end{aligned}$$

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The solution to the SDE can be written as

$$f(t, \tau) - f_0(\tau) = \tilde{f}(t, \tau) + \sum_{n=1}^N b_n(t) \cos\left(\frac{n\pi\tau}{L}\right) + a_n(t) \sin\left(\frac{n\pi\tau}{L}\right),$$

with

$$(6) \quad \tilde{f}(t, \tau) = c_0^2 t \left( \tau - \frac{t}{2} \right),$$

$$(7) \quad a_k(t) = b_k(t) = \frac{c_k^2}{\nu_k^2} (\cos(\nu_k t) + \sin(\nu_k t) - 1) - \frac{\mathbf{1}_{\frac{k}{2} \in \mathbb{Z}_+} c_j^2 t}{\nu_k} + c_k W_k(t).$$

The solution above lends itself to approximate solutions to the quantity of interest. First, the discount factor can be approximated as:

$$\begin{aligned} & \int_0^T f(s, s) ds \\ & \approx \Delta t \sum_{n=1}^{N_t} \left( f_0(t_n) + \tilde{f}(t_n, t_n) + \sum_{k=1}^{N_f} b_k(t_n) \cos(\nu_k t_n) + a_k(t_n) \sin(\nu_k t_n) \right) \\ & = F_{N_t, N_f}, \end{aligned}$$

with  $t_n = \frac{nT}{N_t}$ . This approximation gives the following approximation error in  $N_f$ .

The natural next step is, of couses, estimating the error in approximating the integral by  $F_{N_t, N_f}$ . Firstly, there is the case of the frequency cutoff:

$$\begin{aligned} & \mathbb{E} \left( \left( \int_0^T f(s, s) ds - \lim_{N_t \rightarrow \infty} F_{N_t, N_f} \right)^2 \right) \\ & = \mathbb{E} \left( \int_0^T \sum_{k=N_f+1}^{\infty} (b_k(t_n) \cos(\nu_k t_n) + a_k(t_n) \sin(\nu_k t_n)) ds \right)^2 \\ & \approx \mathbb{E} \left( \left( \int_0^T \sum_{k=N_f+1}^{\infty} c_k W_k(t) (\sin(\nu_k t) + \cos(\nu_k t)) dt \right)^2 \right) \\ & = \mathbb{E} \left( \left( \sum_{k=N_f+1}^{\infty} \int_0^T c_k W_k(t) (\sin(\nu_k t) + \cos(\nu_k t)) dt \right)^2 \right) \\ & = \mathcal{O} \left( \sum_{k=N_f+1}^{\infty} \left( \frac{c_k}{k} \right)^2 \right). \end{aligned}$$

The bound for the weak error can be obtained using Jensen's inequality. Setting as a test case the temporal correlation structure as  $\rho(\tau_1, \tau_2) \propto \exp(-\kappa |\tau_1 - \tau_2|)$  it follows that  $c_k^2 = \frac{\kappa}{L} \frac{(-1)^k (1 - \exp(-|\kappa L|))}{\kappa^2 + (\nu_k)^2}$ . Using this as an example, we plot the empirical strong and weak convergence rates in figure 1.

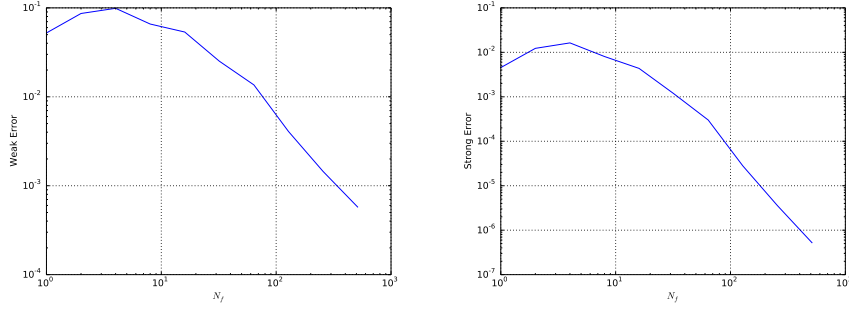


FIGURE 1. Weak (left) and strong (right) error  $\left\| \int_0^T f(s, s) ds - \lim_{N_t \rightarrow \infty} F_{N_t, N_f} \right\|$ .  $N_t = 2^9 + 1$ .

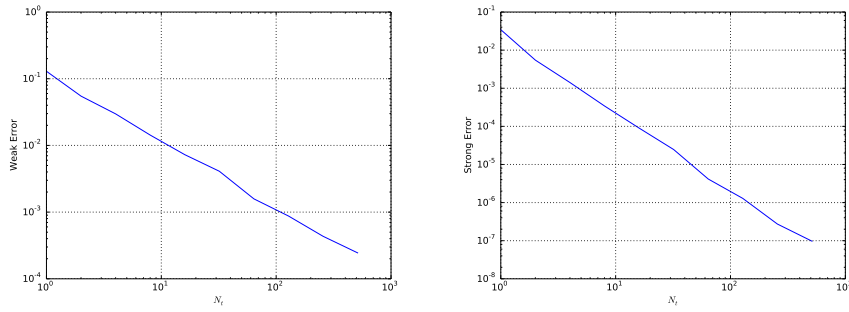


FIGURE 2. Weak (left) and strong (right) error for the temporal discretisation  $\left\| F_{2N_t-1, N_f} - F_{N_t, N_f} \right\|$  along with  $N_t^{-1}$  and  $N_t^{-2}$  reference lines in red.  $N_f = 2^4$ .

As for the time discretisation error, we are concerned with a rectangle quadrature error, giving us weak and strong error rate of 1 and 2 respectively. Empirical test of these rates is presented in figure 2.

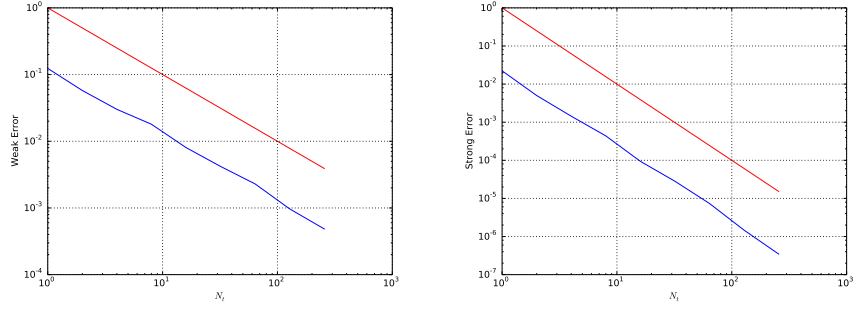


FIGURE 3. Weak (left) and strong (right) error for frequency cutoff  $\|\Psi_{2N_f} - \Psi_{N_f}\|$ .

Similarly, one may estimate the underlying part of the payoff functional

$$\begin{aligned}
 & \int_{\tau_1}^{\tau_2} f(T, \tau) d\tau \\
 & \approx \int_{\tau_1}^{\tau_2} f_0(\tau) + \tilde{f}(t_n, t_n) + \sum_{k=1}^{N_f} b_k(T) \cos(\nu_k \tau) + a_k(T) \sin(\nu_k \tau) d\tau. \\
 & = \int_{\tau_1}^{\tau_2} f_0(\tau) + \frac{c_0 T}{2} (\tau_2^2 - \tau_2 + \tau_1 - \tau_1^2) d\tau \\
 & \quad + \sum_{k=1}^{N_f} \frac{b_k(T)}{\nu_k} (\sin(\nu_k \tau_2) - \sin(\nu_k \tau_1)) \\
 & \quad - \sum_{k=1}^{N_f} \frac{a_k(T)}{\nu_k} (\cos(\nu_k \tau_2) - \cos(\nu_k \tau_1)) \\
 & = \Psi_{N_f}.
 \end{aligned}$$

This approximation has its own approximation error with respect to  $N_f$ :

$$\begin{aligned}
 & \mathbb{E} \left( \left( \int_{\tau_1}^{\tau_2} f(T, \tau) d\tau - \Psi_{N_f} \right)^2 \right) \\
 & = \mathbb{E} \left( \left( \sum_{k=N_f+1}^{\infty} \frac{b_k(T)}{\nu_k} (\sin(\nu_k \tau_2) - \sin(\nu_k \tau_1)) - \frac{a_k(T)}{\nu_k} (\cos(\nu_k \tau_2) - \cos(\nu_k \tau_1)) \right)^2 \right) \\
 & = \mathcal{O} \left( \sum_{k=N_f+1}^{\infty} \frac{c_k^2}{\nu_k^2} \right).
 \end{aligned}$$

Using the exponentially decaying covariance function, we expect order 3 strong convergence and, using Jensen's inequality, we may bound the weak error to order  $\frac{3}{2}$ . Accompanying numerical rates are presented in figure 4.

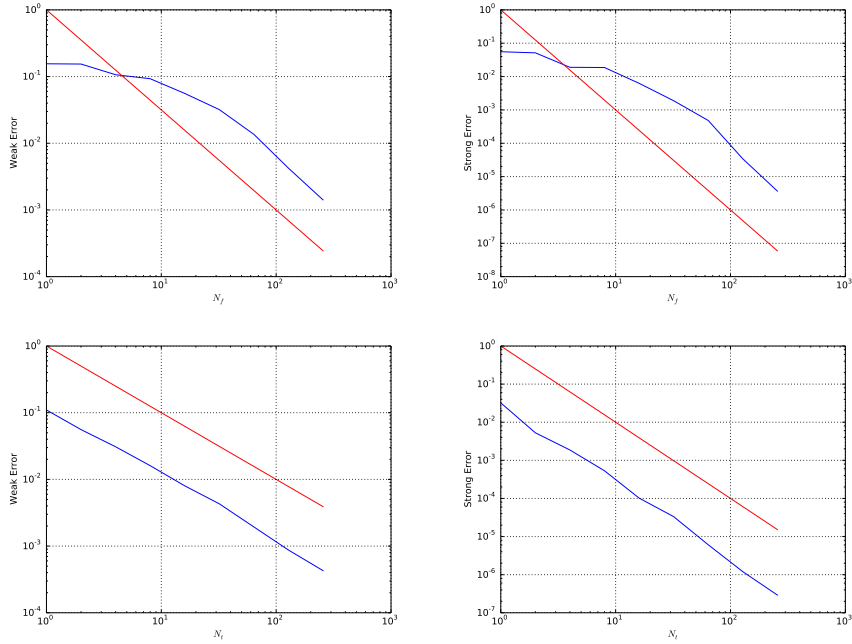


FIGURE 4. Above: Weak (left) and strong (right) error for the cutoff  $||\overline{G}_{65,2N_f} - \overline{G}_{65,N_f}||$ , along with the  $N_f^{-\frac{3}{2}}$  and  $N_f^3$  reference lines (red). Below: Weak (left) and strong (right) error for the temporal discretisation  $||\overline{G}_{2N_t-1,16} - \overline{G}_{N_t,16}||$  with the  $N_t^{-1}$  and  $N_t^{-2}$  reference lines for the weak and strong error, respectively (red).

Overall, we may approximate a future price of a zero-coupon bond as

$$\mathcal{G}(f) = \mathbb{E} \left( \exp \left( - \int_0^{t_T} f(s, s) ds \right) \exp \left( - \int_{\tau_1}^{\tau_2} f(T, \tau) d\tau \right) \right) \approx \underbrace{\mathbb{E} (F_{N_t, N_f} \Psi_{N_f})}_{\overline{G}_{N_t, N_f}}.$$

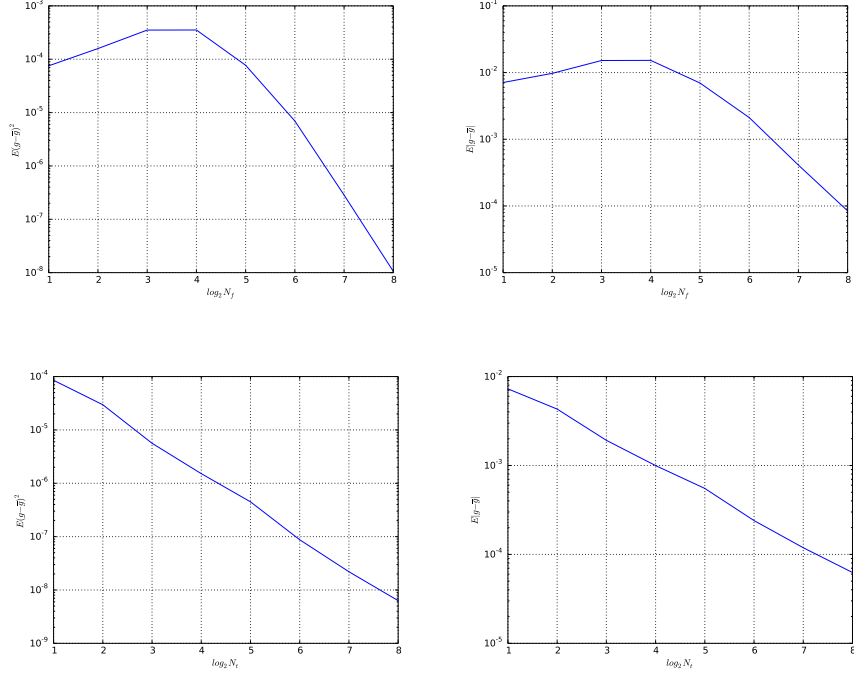
Continuing the case of exponential covariance structure where  $c_k \sim \mathcal{O}(k^{-2})$ , we expect, based on the above computations, to observe the following rate

$$\mathbb{E} \left( (\mathcal{G} - \overline{G}_{N_t, N_f})^2 \right) = \mathcal{O} \left( N_f^{-3} N_t^{-2} \right).$$

To complement the component-wise plots in figures 1, 2 and 4, we plot the overall approximation of  $\mathcal{G}$  with  $\overline{G}_{N_t, N_f}$  in figure ??.

Setting  $\ell = (\ell_1, \ell_2)$ , we may define a Monte Carlo estimator using  $m$  independent realisations of  $\Psi_{N_f}$  and  $F_{N_t, N_f}$ :

$$(8) \quad \mathcal{A}_{\ell_1, \ell_2} \equiv \sum_{m=1}^M \frac{F_{C_1 2^{\ell_1+1}, C_2 2^{\ell_2}} \Psi_{C_2 2^{\ell_2}}(m)}{M}.$$

FIGURE 5. Empirical strong and weak convergence rate for the  $N_t$  and  $N_f$ .

Similarly, we may extend the above to a MLMC estimator as

$$\begin{aligned}
 \mathcal{A}_{ML} &= \sum_{m=0}^{M_0} \frac{F_{2C_1, C_2} \Psi_{C_2}(m)}{M_0} \\
 &\quad + \sum_{\ell_1=1}^L \sum_{m=0}^{M_{\ell_1}} \frac{\left( F_{C_1 2^{\ell_1}+1, C_2 2^{\ell_1}} \Psi_{C_2 2^{\ell_1}} - F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_1-1}} \Psi_{C_2 2^{\ell_1}} \right)(m)}{M_{\ell_1}}, \\
 &\equiv \sum_{\ell_1=0}^L \sum_{m=0}^{M_{\ell_1}} \frac{\Delta_{\ell_1}(m)}{M_{\ell_1}}
 \end{aligned}$$

and, into a MIMC estimator through defining the appropriate two-dimensional difference operators  $\Delta_{\ell_1, \ell_2}$  and a downward-closed index-set  $L_K = \{(\ell_1, \ell_2) \in \mathbb{Z}_+^2 : I(\ell_1, \ell_2) < K\}$ :

$$\begin{aligned}
 \mathcal{A}_{MI} &= \sum_{\ell \in L_K} \sum_{m=1}^{M_\ell} \frac{\Delta_{\ell_1, \ell_2}(m)}{M_\ell} \\
 \Delta_{\ell_1, \ell_2} &\equiv F_{C_1 2^{\ell_1}+1, C_2 2^{\ell_2}} \Psi_{C_2 2^{\ell_2}} - \mathbf{1}_{\ell_1 > 1} F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2-1}} \Psi_{C_2 2^{\ell_2-1}} \\
 &\quad - \mathbf{1}_{\ell_2 > 1} F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2}} \Psi_{C_2 2^{\ell_2}} + \mathbf{1}_{\ell_1, \ell_2 > 1} F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2-1}} \Psi_{C_2 2^{\ell_2-1}}.
 \end{aligned}$$

## REFERENCES

Björk, T., Szepessy, A., Tempone, R. & Zouraris, G. E. (2013), ‘Monte carlo euler approximations of hjm term structure financial models’, *BIT Numerical Mathematics* **53**(2), 341–383.

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