MULTI-INDEX AND MULTI-LEVEL MONTE CARLO EVALUATION OF HJM MODELS

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Abstract. Notes of MIMC and MLMC evaluation of HJM models.

1. HJM model in Fourier space

Let us focus on the following HJM type SDE:

$$df(t,\tau) = \alpha(t,\tau)\,dt + \beta(t,\tau)\,d\overline{W}(t) \tag{1}$$

$$f(0,\tau) = f_0(\tau), \tag{2}$$

with the diffusion term given as the infinite-dimensional expansion of eq. (1.1) in Björk et al. (2013):

$$\beta(t,\tau)\,d\overline{W}(t) = \sum_{n=0}^{\infty} c_n \left(\sin\left(\frac{n\pi\tau}{L}\right) + \cos\left(\frac{n\pi\tau}{L}\right)\right) dW_n(t), \tag{3}$$

with W_n independent Brownian motions and $L >> 1$. This gives the following Fourier decomposition for the covariance of interests:

$$\mathrm{E}\left(\beta(t,\tau_1)\,d\overline{W}(t)\,\beta(t,\tau_2)\,d\overline{W}(t)\right) \tag{4}$$

$$= \sum_{n=0}^{\infty} c_n^2 \left(\cos\left(\frac{n\pi(\tau_1-\tau_2)}{L}\right)\right) \tag{5}$$

To keep the exponential HJM model risk neutral, we fix the drift of the equation as

$$\begin{aligned}\alpha(t,\tau) =& c_0^2(\tau-t) \\ &+ \sum_{n=1}^{\infty} \frac{c_n^2}{n} \sin\left(\frac{n\pi\tau}{L}\right)\left(\cos\left(\frac{n\pi t}{L}\right) - \sin\left(\frac{n\pi t}{L}\right)\right) \\ &+ \sum_{n=1}^{\infty} \frac{c_n^2}{n} \cos\left(\frac{n\pi\tau}{L}\right)\left(\cos\left(\frac{n\pi t}{L}\right) - \sin\left(\frac{n\pi t}{L}\right)\right) \\ &+ \sum_{n=1}^{\infty} \frac{c_n^2}{n} \cos\left(\frac{2n\pi\tau}{L}\right).\end{aligned}$$

The solution to the SDE can be written as

$$f(t,\tau) - f_0(\tau) = \tilde{f}(t,\tau) + \sum_{n=1}^{N} b_n(t)\cos\left(\frac{n\pi\tau}{L}\right) + a_n(t)\sin\left(\frac{n\pi\tau}{L}\right),$$

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with

$$\tilde{f}(t,\tau) = c_0^2 t\left(\tau - \frac{t}{2}\right), \tag{6}$$

$$a_n(t) = \frac{c_n^2 L^2}{\pi^2 n^2}\left(\cos\left(\frac{n\pi t}{L}\right) + \sin\left(\frac{n\pi t}{L}\right) - 1\right) + c_n W_n(t) \tag{7}$$

$$b_n(t) = a_n(t) - \frac{\mathbf{1}_{\frac{n}{2}\in\mathbb{Z}_+} c_n^2 L t}{\pi n}. \tag{8}$$

The solution above lends itself to approximate solutions to the quantity of interest. First, the discount factor can be approximated as:

$$\begin{aligned}
&\int_0^{t_{max}} f(s,s)\,ds \\
\approx &\Delta t \sum_{n=1}^{N_t}\left(f_0(t_n) + \tilde{f}(t_n,t_n) + \sum_{k=1}^{N_f} b_k(t_n)\cos\left(\frac{k\pi t_n}{L}\right) + a_k(t_n)\sin\left(\frac{k\pi t_n}{L}\right)\right) \\
= &F_{N_t,N_f},
\end{aligned}$$

with $t_n = \frac{nT}{N_t}$. This approximation gives the follwoing approximation error in N_f:

$$\begin{aligned}
&\int_0^T f(s,s)\,ds - \lim_{N_t\to\infty} F_{N_t,N_f} \\
= &\int_0^T \sum_{k=1}^{N_f} \frac{c_k^2 L^2}{k^2\pi^2}\left(1 + \sin\left(\frac{2k\pi s}{L}\right)\right) ds \\
&+ \int_0^T \sum_{k=1}^{N_f} c_k W_k(s)\left(\sin\left(\frac{k\pi s}{L}\right) + \cos\left(\frac{k\pi s}{L}\right)\right) ds \\
&- \int_0^T \sum_{k=1}^{N_f} \frac{c_k^2 L}{\pi k}\cos\left(\frac{k\pi s}{L}\right), ds,
\end{aligned}$$

Similarly, one may the underlying part of the payoff functional

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2} f(T,\tau)\, d\tau \\
\approx &\int_{\tau_1}^{\tau_2} f_0(\tau) + \tilde{f}(t_n,t_n) + \sum_{k=1}^{N_f} b_k(T)\cos\left(\frac{k\pi\tau}{L}\right) + a_k(T)\sin\left(\frac{k\pi\tau}{L}\right) d\tau. \\
= &+\int_{\tau_1}^{\tau_2} f_0(\tau)\, d\tau + \frac{c_0 T}{2}\left(\tau_2^2 - \tau_2 + \tau_1 - \tau_1^2\right) \\
&+ \sum_{k=1}^{N_f} \frac{b_k(T)\, L}{k\pi}\left(\sin\left(\frac{k\pi\tau_2}{L}\right) - \sin\left(\frac{k\pi\tau_1}{L}\right)\right) \\
&- \sum_{k=1}^{N_f} \frac{a_k(T)\, L}{k\pi}\left(\cos\left(\frac{k\pi\tau_2}{L}\right) - \cos\left(\frac{k\pi\tau_1}{L}\right)\right) \\
= &\Psi_{N_f}.
\end{aligned}$$

The corresponding empirical rates for the approximations of the quantity of interest

$$\mathcal{G}(f) = \mathrm{E}\left(\exp\left(-\int_0^{t_T} f(s,s)\, ds\right)\exp\left(-\int_{\tau_1}^{\tau_2} f(T,\tau)\, d\tau\right)\right) \approx \mathrm{E}\left(F_{N_t,N_f}\Psi_{N_f}\right)$$

are presented in fig. 1

Setting $\ell = (\ell_1, \ell_2)$, we may define a Monte Carlo estimator using m independent realisations of Ψ_{N_f} and F_{N_t,N_f}:

$$\mathcal{A}_{\ell_1,\ell_2} \equiv \sum_{m=1}^{M} \frac{F_{C_1 2^{\ell_1}+1, C_2 2^{\ell_2}}\Psi_{C_2 2^{\ell_2}}(m)}{M}. \tag{9}$$

Similarly, we may extend the above to a MLMC estimator as

$$\begin{aligned}
\mathcal{A}_{ML} = &\sum_{m=0}^{M_0} \frac{F_{2C_1,C_2}\Psi_{C_2}(m)}{M_0} \\
&+ \sum_{\ell_1=1}^{L}\sum_{m=0}^{M_{\ell_1}} \frac{\left(F_{C_1 2^{\ell_1}+1, C_2 2^{\ell_1}}\Psi_{C_2 2^{\ell_1}} - F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_1-1}}\Psi_{C_2 2^{\ell_1}}\right)(m)}{M_{\ell_1}}, \\
\equiv &\sum_{\ell_1=0}^{L}\sum_{m=0}^{M_{\ell_1}} \frac{\Delta_{\ell_1}(m)}{M_{\ell_1}}
\end{aligned}$$

and, into a MIMC estimator through defining the appropriate two-dimensional difference operators Δ_{ℓ_1,ℓ_2} and a downward-closed index-set $L_K = \left\{(\ell_1,\ell_2) \in \mathbb{Z}_+^2 : I(\ell_1,\ell_2) < K\right\}$:

$$\begin{aligned}
\mathcal{A}_{MI} = &\sum_{\ell \in L_K}\sum_{m=1}^{M_\ell} \frac{\Delta_{\ell_1,\ell_2}(m)}{M_\ell} \\
\Delta_{\ell_1,\ell_2} \equiv &F_{C_1 2^{\ell_1}+1, C_2 2^{\ell_2}}\Psi_{C_2 2^{\ell_2}} - \mathbf{1}_{\ell_1>1} F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2-1}}\Psi_{C_2 2^{\ell_2-1}} \\
&- \mathbf{1}_{\ell_2>1} F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2}}\Psi_{C_2 2^{\ell_2}} + \mathbf{1}_{\ell_1,\ell_2>1} F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2-1}}\Psi_{C_2 2^{\ell_2-1}}.
\end{aligned}$$

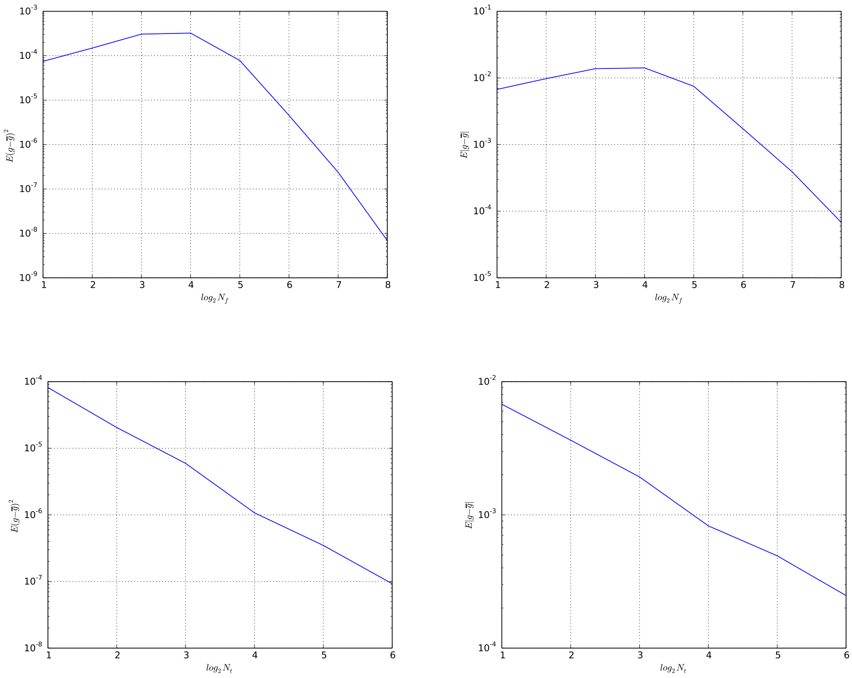

FIGURE 1. Empirical strong and weak convergence rate for the N_t and N_f.

REFERENCES

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