MULTI-INDEX AND MULTI-LEVEL MONTE CARLO EVALUATION OF HJM MODELS

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Abstract. Notes of MIMC and MLMC evaluation of HJM models.

1. HJM model in Fourier space

Let us focus on the following HJM type SDE:

$$df(t,\tau) = \alpha(t,\tau)\,dt + \beta(t,\tau)\,d\overline{W}(t) \tag{1}$$

$$f(0,\tau) = f_0(\tau), \tag{2}$$

with the diffusion term given as the infinite-dimensional extension of eq. (1.1) in Björk et al. (2013):

$$\beta(t,\tau)\,d\overline{W}(t) = \sum_{k=0}^{\infty} c_n \left(\sin(\nu_k \tau) + \cos(\nu_k \tau)\right) \tag{3}$$

with W_n independent Brownian motions and

$$\nu_k \equiv \frac{k\pi}{L}$$

with $L >> 1$. This gives the following Fourier decomposition for the covariance of increments:

$$\mathrm{E}\left(\beta(t,\tau_1)\,d\overline{W}(t)\,\beta(t,\tau_2)\,d\overline{W}(t)\right) \tag{4}$$

$$= \sum_{k=0}^{\infty} c_k^2 \left(\cos\left(\nu_k(\tau_1 - \tau_2)\right)\right) \tag{5}$$

To keep the exponential HJM model risk neutral, we fix the drift of the equation as

$$\begin{aligned}\alpha(t,\tau) =& c_0^2(\tau - t)\\ &+ \sum_{k=1}^{\infty} \frac{c_k^2}{k} \sin(\nu_k \tau)\left(\cos(\nu_q t) - \sin(\nu_k t)\right)\\ &+ \sum_{k=1}^{\infty} \frac{c_k^2}{k} \cos(\nu_k \tau)\left(\cos(\nu_k t) - \sin(\nu_k t)\right)\\ &+ \sum_{n=1}^{\infty} \frac{c_n^2}{n} \cos(2\nu_k \tau).\end{aligned}$$

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The solution to the SDE can be written as

$$f(t,\tau) - f_0(\tau) = \tilde{f}(t,\tau) + \sum_{n=1}^{N} b_n(t)\cos\left(\frac{n\pi\tau}{L}\right) + a_n(t)\sin\left(\frac{n\pi\tau}{L}\right),$$

with

$$\tilde{f}(t,\tau) = c_0^2 t\left(\tau - \frac{t}{2}\right), \tag{6}$$

$$a_k(t) = b_k(t) = \frac{c_k^2}{\nu_k^2}\left(\cos(\nu_k t) + \sin(\nu_k t) - 1\right) - \frac{\mathbf{1}_{\frac{k}{2}\in\mathbb{Z}_+} c_j^2 t}{\nu_k} + c_k W_k(t). \tag{7}$$

The solution above lends itself to approximate solutions to the quantity of interest. First, the discount factor can be approximated as:

$$\begin{aligned}
&\int_0^T f(s,s)\,ds \\
\approx&\Delta t\sum_{n=1}^{N_t}\left(f_0(t_n) + \tilde{f}(t_n,t_n) + \sum_{k=1}^{N_f} b_k(t_n)\cos(\nu_k t_n) + a_k(t_n)\sin(\nu_k t_n)\right) \\
=&F_{N_t,N_f},
\end{aligned}$$

with $t_n = \frac{nT}{N_t}$. This approximation gives the following approximation error in N_f.

The natural next step is, of coures, estimating the error in approximating the integral by F_{N_t,N_f}. Firstly, there is the case of the frequency cutoff:

$$\begin{aligned}
&\mathrm{E}\left(\left(\int_0^T f(s,s)\,ds - \lim_{N_t\to\infty} F_{N_t,N_f}\right)^2\right) \\
=&\mathrm{E}\left(\int_0^T \sum_{k=N_f+1}^{\infty}\left(b_k(t_n)\cos(\nu_k t_n) + a_k(t_n)\sin(\nu_k t_n)\right)ds\right) \\
\approx&\mathrm{E}\left(\left(\int_0^T \sum_{k=N_f+1}^{\infty} c_k W_k(t)\left(\sin(\nu_k t) + \cos(\nu_k t)\right)\right)^2\right) \\
=&\mathrm{E}\left(\left(\sum_{k=N_f+1}^{\infty}\int_0^T c_k W_k(t)\left(\sin(\nu_k t) + \cos(\nu_k t)\right)ds\right)^2\right) \\
=&\mathcal{O}\left(\sum_{k=N_f+1}^{\infty}\left(\frac{c_k}{k}\right)^2\right).
\end{aligned}$$

The bound for the weak error can be obtained using Jensen's inequality. Setting as a test case the temporal correlation structure as $\rho(\tau_1,\tau_2) \propto \exp(-\kappa|\tau_1 - \tau_2|)$ it follows that $c_k^2 = \frac{\kappa}{L}\frac{(-1)^k(1-\exp(-|\kappa L|))}{\kappa^2+(\nu_k)^2}$. Using this as an example, we plot the empirical strong and weak convergence rates in figure 1.

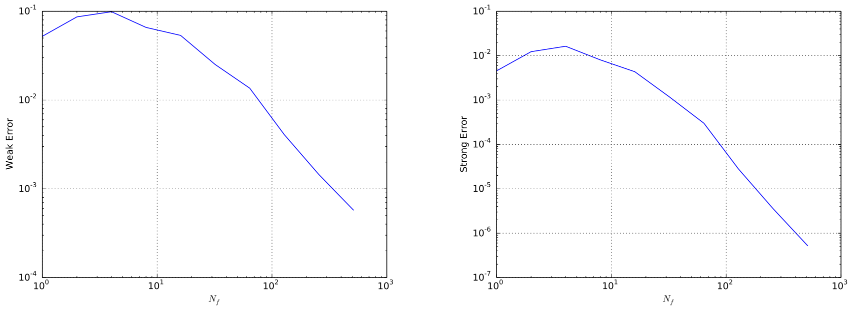

FIGURE 1. Weak (left) and strong (right) error $\left|\left|\int_0^T f(s,s)\,ds - \lim_{N_t\to\infty} F_{N_t,N_f}\right|\right|$. $N_t = 2^9+1$.

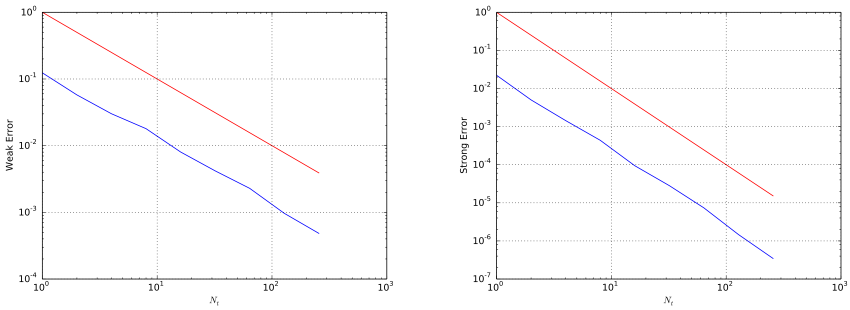

FIGURE 2. Weak (left) and strong (right) error for the temporal discretisation $\left|\left|F_{2N_t-1,N_f} - F_{N_t,N_f}\right|\right|$ along with N_t^{-1} and N_t^{-2} reference lines in red. $N_f = 2^4$.

As for the time discretisation error, we are concerned with a rectangle quadrature error, giving us weak and strong error rate of 1 and 2 respectively. Empirical test of these rates is presented in figure 4.

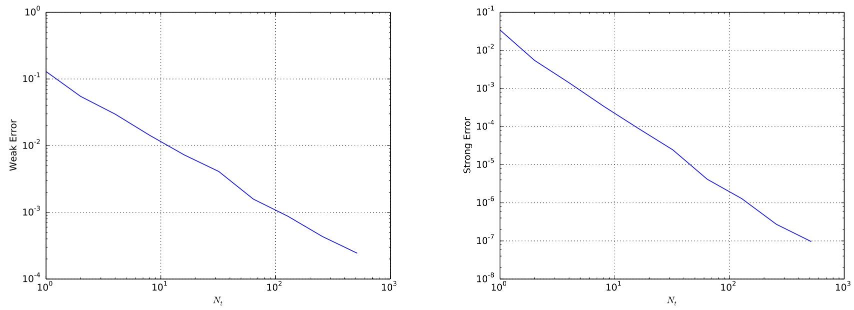

FIGURE 3. Weak (left) and strong (right) error for frequency cutoff $\left|\left|\Psi_{2N_f} - \Psi_{N_f}\right|\right|$.

Similarly, one may estimate the underlying part of the payoff functional

$$
\begin{aligned}
&\int_{\tau_1}^{\tau_2} f(T,\tau)\, d\tau \\
\approx &\int_{\tau_1}^{\tau_2} f_0(\tau) + \tilde{f}(t_n, t_n) + \sum_{k=1}^{N_f} b_k(T)\cos(\nu_k \tau) + a_k(T)\sin(\nu_k\tau)\, d\tau. \\
= &\int_{\tau_1}^{\tau_2} f_0(\tau) + \frac{c_0 T}{2}\left(\tau_2^2 - \tau_2 + \tau_1 - \tau_1^2\right) d\tau \\
&+ \sum_{k=1}^{N_f} \frac{b_k(T)}{\nu_k}\left(\sin(\nu_k\tau_2) - \sin(\nu_k\tau_1)\right) \\
&- \sum_{k=1}^{N_f} \frac{a_k(T)}{\nu_k}\left(\cos(\nu_k\tau_2) - \cos(\nu_k\tau_1)\right) \\
= &\Psi_{N_f}.
\end{aligned}
$$

This approximation has its own approximation error with respect to N_f:

$$
\begin{aligned}
&\mathrm{E}\left(\left(\int_{\tau_1}^{\tau_2} f(T,\tau)\, d\tau - \Psi_{N_f}\right)^2\right) \\
=&\mathrm{E}\left(\left(\sum_{k=N_f+1}^{\infty} \frac{b_k(T)}{\nu_k}\left(\sin(\nu_k\tau_2) - \sin(\nu_k\tau_1)\right) - \frac{a_k(T)}{\nu_k}\left(\cos(\nu_k\tau_2) - \cos(\nu_k\tau_1)\right)\right)^2\right) \\
=&\mathcal{O}\left(\sum_{k=N_f+1}^{\infty} \frac{c_k^2}{\nu_k^2}\right).
\end{aligned}
$$

Using the exponentially decaying covariance function, we expect order 3 strong convergence and, using Jensen's inequality, we may bound the weak error to order $\frac{3}{2}$. Accompanying numerical rates are presetend in figure 4.

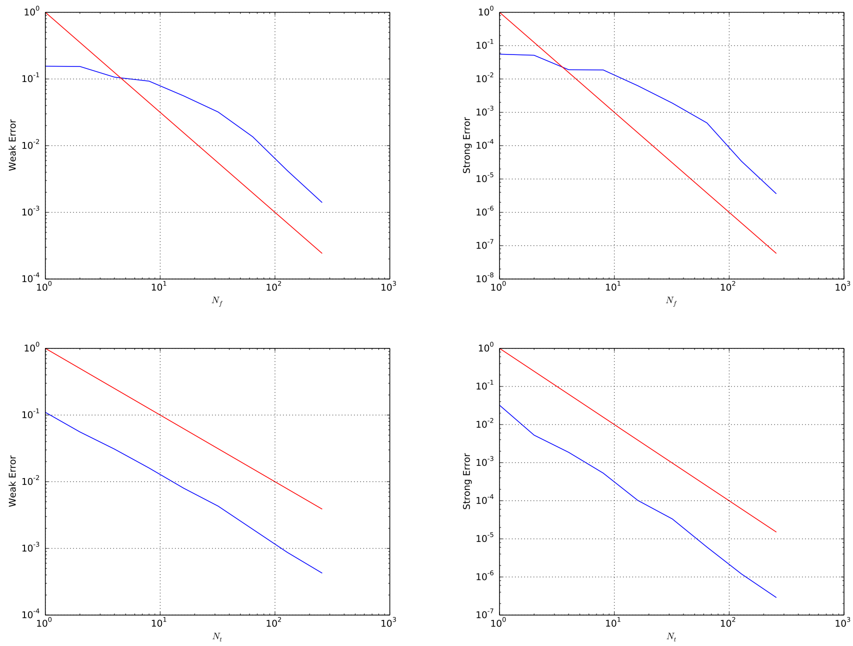

FIGURE 4. Above: Weak (left) and strong (right) error for the cutoff $\left|\left|\overline{G}_{65,2N_f} - \overline{G}_{65,N_f}\right|\right|$, along with the $N_f^{-\frac{3}{2}}$ and N_f^3 reference lines (red). Below: Weak (left) and strong (right) error for the temporal discretisation $\left|\left|\overline{G}_{2N_t-1,16} - \overline{G}_{N_t,16}\right|\right|$ with the N_t^{-1} and N_t^{-2} reference lines for the weak and strong error, respectively (red).

Overall, we may approximate a future price of a zero-coupon bond as

$$\mathcal{G}\left(f\right) = \mathrm{E}\left(\exp\left(-\int_0^{t_T} f\left(s,s\right)ds\right)\exp\left(-\int_{\tau_1}^{\tau_2} f\left(T,\tau\right)d\tau\right)\right) \approx \underbrace{\mathrm{E}\left(F_{N_t,N_f}\Psi_{N_f}\right)}_{\overline{G}_{N_t,N_f}}.$$

Continuing the case of exponential covariance structure where $c_k \sim \mathcal{O}\left(k^{-2}\right)$, we expect, based on the above computations, to observe the following rate

$$\mathrm{E}\left(\left(\mathcal{G} - \overline{G}_{N_t,N_f}\right)^2\right) = \mathcal{O}\left(N_f^{-3}N_t^{-2}\right).$$

To complement the component-wise plots in figures 1, 2 and 4, we plot the overall approximation of $\mathcal{G}$ with $\overline{G}_{N_t,N_f}$ in figure **??**.

Setting $\ell = (\ell_1, \ell_2)$, we may define a Monte Carlo estimator using m independent realisations of Ψ_{N_f} and F_{N_t,N_f}:

$$\mathcal{A}_{\ell_1,\ell_2} \equiv \sum_{m=1}^{M} \frac{F_{C_1 2^{\ell_1}+1, C_2 2^{\ell_2}}\Psi_{C_2 2^{\ell_2}}\left(m\right)}{M}. \tag{8}$$

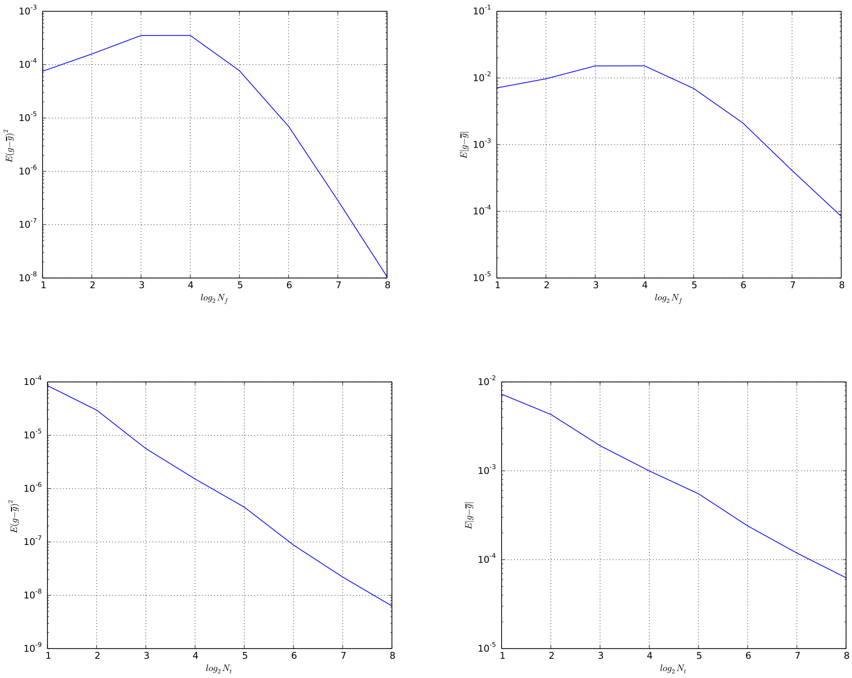

FIGURE 5. Empirical strong and weak convergence rate for the N_t and N_f.

Similarly, we may extend the above to a MLMC estimator as

$$\begin{aligned}
\mathcal{A}_{ML} =& \sum_{m=0}^{M_0} \frac{F_{2C_1,C_2}\Psi_{C_2}(m)}{M_0} \\
&+ \sum_{\ell_1=1}^{L}\sum_{m=0}^{M_{\ell_1}} \frac{\left(F_{C_1 2^{\ell_1}+1, C_2 2^{\ell_1}}\Psi_{C_2 2^{\ell_1}} - F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_1-1}}\Psi_{C_2 2^{\ell_1}}\right)(m)}{M_{\ell_1}}, \\
\equiv& \sum_{\ell_1=0}^{L}\sum_{m=0}^{M_{\ell_1}} \frac{\Delta_{\ell_1}(m)}{M_{\ell_1}}
\end{aligned}$$

and, into a MIMC estimator through defining the appropriate two-dimensional difference operators Δ_{ℓ_1,ℓ_2} and a downward-closed index-set $L_K = \left\{(\ell_1,\ell_2) \in \mathbb{Z}_+^2 : I(\ell_1,\ell_2) < K\right\}$:

$$\begin{aligned}
\mathcal{A}_{MI} =& \sum_{\ell \in L_K}\sum_{m=1}^{M_\ell} \frac{\Delta_{\ell_1,\ell_2}(m)}{M_\ell} \\
\Delta_{\ell_1,\ell_2} \equiv& F_{C_1 2^{\ell_1}+1, C_2 2^{\ell_2}}\Psi_{C_2 2^{\ell_2}} - \mathbf{1}_{\ell_1>1} F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2-1}}\Psi_{C_2 2^{\ell_2-1}} \\
&- \mathbf{1}_{\ell_2>1} F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2}}\Psi_{C_2 2^{\ell_2}} + \mathbf{1}_{\ell_1,\ell_2>1} F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2-1}}\Psi_{C_2 2^{\ell_2-1}}.
\end{aligned}$$

References

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