MULTI-INDEX AND MULTI-LEVEL MONTE CARLO EVALUATION OF HJM MODELS

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ABSTRACT. Notes of MIMC and MLMC evaluation of HJM models.

1. HJM MODEL IN FOURIER SPACE

Let us focus on the following HJM type SDE:

$$df(t, \tau) = \alpha(t, \tau) \, dt + \beta(t, \tau) \, d\overline{W}(t) \tag{1}$$

$$f(0, \tau) = f_0(\tau), \tag{2}$$

with the diffusion term given as the infinite-dimensional extension of eq. (1.1) in Björk et al. (2013):

$$\beta(t, \tau) \, d\overline{W}(t) = \sum_{k=0}^{\infty} c_n \left(\sin(\nu_k \tau) + \cos(\nu_k \tau) \right) \tag{3}$$

with W_n independent Brownian motions and

$$\nu_k \equiv \frac{k\pi}{L}$$

with $L \gg 1$. This gives the following Fourier decomposition for the covariance of increments:

$$\mathrm{E}\left(\beta(t, \tau_1) \, d\overline{W}(t) \, \beta(t, \tau_2) \, d\overline{W}(t) \right) \tag{4}$$

$$= \sum_{k=0}^{\infty} c_k^2 \left(\cos(\nu_k(\tau_1 - \tau_2)) \right) \tag{5}$$

To keep the exponential HJM model risk neutral, we fix the drift of the equation as

$$
\begin{aligned}
\alpha(t, \tau) = & c_0^2 (\tau - t) \\
& + \sum_{k=1}^{\infty} \frac{c_k^2}{k} \sin(\nu_k \tau) \left(\cos(\nu_q t) - \sin(\nu_k t) \right) \\
& + \sum_{k=1}^{\infty} \frac{c_k^2}{k} \cos(\nu_k \tau) \left(\cos(\nu_k t) - \sin(\nu_k t) \right) \\
& + \sum_{n=1}^{\infty} \frac{c_n^2}{n} \cos(2\nu_k \tau).
\end{aligned}
$$

The solution to the SDE can be written as

$$f(t, \tau) - f_0(\tau) = \tilde{f}(t, \tau) + \sum_{n=1}^{N} b_n(t) \cos\left(\frac{n\pi\tau}{L}\right) + a_n(t) \sin\left(\frac{n\pi\tau}{L}\right),$$

with

$$(6) \qquad \tilde{f}(t, \tau) = c_0^2 t \left(\tau - \frac{t}{2}\right),$$

$$(7) \qquad a_k(t) = b_k(t) = \frac{c_k^2}{\nu_k^2} \left(\cos(\nu_k t) + \sin(\nu_k t) - 1\right) - \frac{\mathbf{1}_{\frac{k}{2} \in \mathbb{Z}_+} c_j^2 t}{\nu_k} + c_k W_k(t).$$

The solution above lends itself to approximate solutions to the quantity of interest. First, the discount factor can be approximated as:

$$\int_0^T f(s, s)\, ds$$

$$\approx \Delta t \sum_{n=1}^{N_t} \left(f_0(t_n) + \tilde{f}(t_n, t_n) + \sum_{k=1}^{N_f} b_k(t_n) \cos(\nu_k t_n) + a_k(t_n) \sin(\nu_k t_n) \right)$$

$$= F_{N_t, N_f},$$

with $t_n = \frac{nT}{N_t}$. This approximation gives the following approximation error in N_f.

The natural next step is, of coures, estimating the error in approximating the integral by F_{N_t, N_f}. Firstly, there is the case of the frequency cutoff:

$$\mathrm{E}\left(\left(\int_0^T f(s, s)\, ds - \lim_{N_t \to \infty} F_{N_t, N_f}\right)^2\right)$$

$$= \mathrm{E}\left(\int_0^T \sum_{k=N_f+1}^{\infty} (b_k(t_n) \cos(\nu_k t_n) + a_k(t_n) \sin(\nu_k t_n))\, ds\right)$$

$$\approx \mathrm{E}\left(\left(\int_0^T \sum_{k=N_f+1}^{\infty} c_k W_k(t) (\sin(\nu_k t) + \cos(\nu_k t))\right)^2\right)$$

$$= \mathrm{E}\left(\left(\sum_{k=N_f+1}^{\infty} \int_0^T c_k W_k(t) (\sin(\nu_k t) + \cos(\nu_k t))\, ds\right)^2\right)$$

$$= \mathcal{O}\left(\sum_{k=N_f+1}^{\infty} \left(\frac{c_k}{k}\right)^2\right).$$

A bound for the bias error can be obtained using Jensen's inequality. Setting as a test case the temporal correlation structure as $\rho(\tau_1, \tau_2) \propto \exp(-\kappa |\tau_1 - \tau_2|)$ it follows that $c_k^2 = \frac{\kappa}{L} \frac{(-1)^k (1 - \exp(-|\kappa L|))}{\kappa^2 + (\nu_k)^2}$. Using this as an example, we plot the empirical strong and weak convergence rates in figure 1.
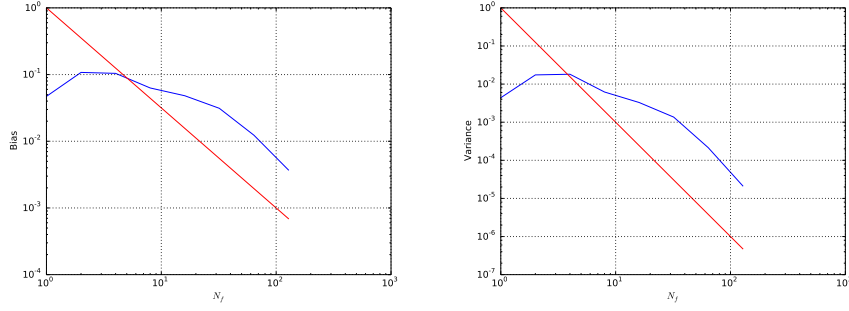
FIGURE 1. Bias (left) and variance (right) error $\left\|F_{513,2N_f} - F_{513,N_f}\right\|$ along with $N_f^{-\frac{3}{2}}$ and N_f^3 reference lines in red.
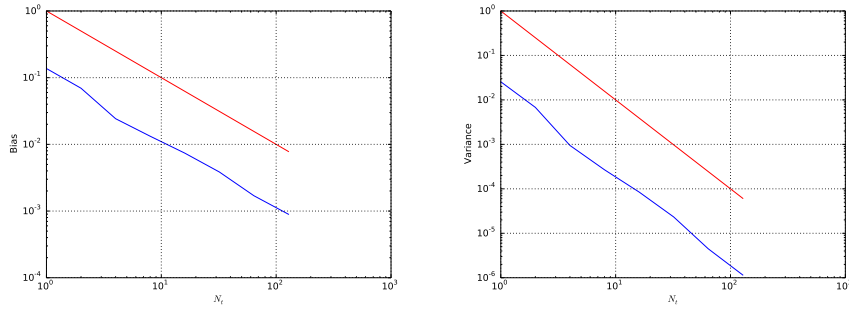

FIGURE 2. Bias (left) and variance (right) error for the temporal discretisation $\|F_{2N_t-1,16} - F_{N_t,16}\|$ along with N_t^{-1} and N_t^{-2} reference lines in red.

As for the time discretisation error, we are concerned with a rectangle quadrature error, giving us weak and strong error rate of 1 and 2 respectively. Empirical test of these rates is presented in figure 2.
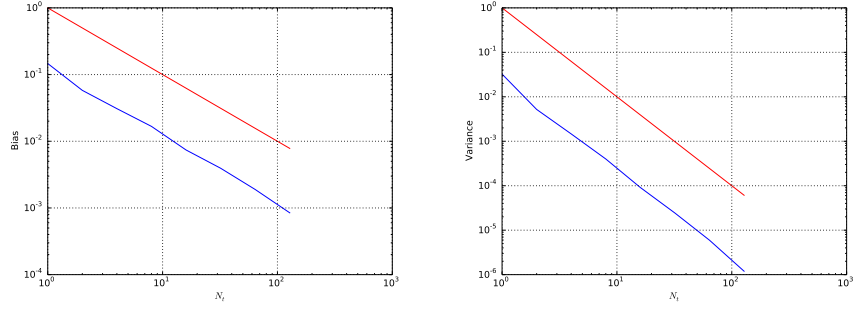
FIGURE 3. Bias (left) and variance (right) error for frequency cutoff $\left\|\Psi_{2N_f} - \Psi_{N_f}\right\|$ together with the $N_f^{-\frac{3}{2}}$ and N_f^{-3} reference lines in red.

Similarly, one may estimate the underlying part of the payoff functional

$$\int_{\tau_1}^{\tau_2} f(T,\tau)\, d\tau$$

$$\approx \int_{\tau_1}^{\tau_2} f_0(\tau) + \tilde{f}(t_n, t_n) + \sum_{k=1}^{N_f} b_k(T) \cos(\nu_k \tau) + a_k(T) \sin(\nu_k \tau)\, d\tau.$$

$$= \int_{\tau_1}^{\tau_2} f_0(\tau) + \frac{c_0 T}{2} \left(\tau_2^2 - \tau_2 + \tau_1 - \tau_1^2\right) d\tau$$

$$+ \sum_{k=1}^{N_f} \frac{b_k(T)}{\nu_k} \left(\sin(\nu_k \tau_2) - \sin(\nu_k \tau_1)\right)$$

$$- \sum_{k=1}^{N_f} \frac{a_k(T)}{\nu_k} \left(\cos(\nu_k \tau_2) - \cos(\nu_k \tau_1)\right)$$

$$= \Psi_{N_f}.$$

This approximation has its own approximation error with respect to N_f:

$$\mathrm{E}\left(\left(\int_{\tau_1}^{\tau_2} f(T,\tau)\, d\tau - \Psi_{N_f}\right)^2\right)$$

$$= \mathrm{E}\left(\left(\sum_{k=N_f+1}^{\infty} \frac{b_k(T)}{\nu_k}\left(\sin(\nu_k \tau_2) - \sin(\nu_k \tau_1)\right) - \frac{a_k(T)}{\nu_k}\left(\cos(\nu_k \tau_2) - \cos(\nu_k \tau_1)\right)\right)^2\right)$$

$$= \mathcal{O}\left(\sum_{k=N_f+1}^{\infty} \frac{c_k^2}{\nu_k^2}\right).$$

Using the exponentially decaying covariance function, we expect order 3 strong convergence and, using Jensen's inequality, we may bound the weak error to order $\frac{3}{2}$. Accompanying numerical rates are presentend in figure 3.
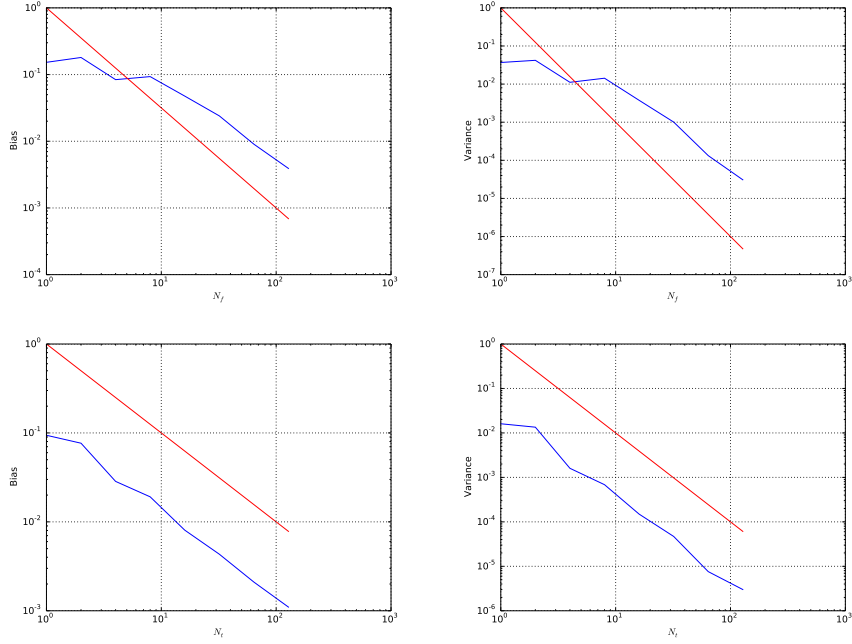
FIGURE 4. Above: Bias (left) and variance (right) error for the cutoff $\left|\left|\overline{G}_{65,2N_f} - \overline{G}_{65,N_f}\right|\right|$, along with the $N_f^{-\frac{3}{2}}$ and N_f^3 reference lines (red). Below: Bias (left) and variance (right) error for the temporal discretisation $\left|\left|\overline{G}_{2N_t-1,16} - \overline{G}_{N_t,16}\right|\right|$ with the N_t^{-1} and N_t^{-2} reference lines for the bias and variance error, respectively (red).

Overall, we may approximate a future price of a zero-coupon bond as

$$\mathcal{G}\left(f\right) = \mathrm{E}\left(\exp\left(-\int_0^{t_T} f\left(s,s\right)ds\right)\exp\left(-\int_{\tau_1}^{\tau_2} f\left(T,\tau\right)d\tau\right)\right)$$
$$\approx \underbrace{\mathrm{E}\left(\exp\left(-F_{N_t,N_f} - \Psi_{N_f}\right)\right)}_{\overline{G}_{N_t,N_f}}.$$

Continuing the case of exponential covariance structure where $c_k \sim \mathcal{O}\left(k^{-2}\right)$, we expect, based on the above computations, to observe the following rate

$$\mathrm{E}\left(\left(\mathcal{G} - \overline{G}_{N_t,N_f}\right)^2\right) = \mathcal{O}\left(N_f^{-3}N_t^{-2}\right).$$

To complement the component-wise plots in figures 1, 3 and 2, we plot the overall approximation of $\mathcal{G}$ with $\overline{G}_{N_t,N_f}$ in figure 4.

Setting $\ell = (\ell_1, \ell_2)$, we may define a Monte Carlo estimator using m independent realisations of $\overline{G}_{N_t,N_f}$:

$$(8) \qquad \mathcal{A}_{\ell_1,\ell_2} \equiv \sum_{m=1}^{M} \frac{\overline{G}_{2^{\ell_1}+1,2^{\ell_2}}\left(m\right)}{M}.$$

Similarly, we may extend the above to a MLMC estimator as

$$
\begin{aligned}
\mathcal{A}_{ML} = & \sum_{m=0}^{M_0} \frac{\overline{G}_{2C_1,C_2}(m)}{M_0} \\
& + \sum_{\ell_1=1}^{L} \sum_{m=0}^{M_{\ell_1}} \frac{\left(\overline{GF}_{C_1 2^{\ell_1}+1, C_2 2^{\ell_1}} - \overline{G}_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_1-1}} \right)(m)}{M_{\ell_1}}, \\
\equiv & \sum_{\ell_1=0}^{L} \sum_{m=0}^{M_{\ell_1}} \frac{\Delta_{\ell_1}(m)}{M_{\ell_1}}
\end{aligned}
$$

and, into a MIMC estimator through defining the appropriate two-dimensional difference operators Δ_{ℓ_1,ℓ_2} and a downward-closed index-set $L_K = \left\{ (\ell_1, \ell_2) \in \mathbb{Z}_+^2 : I(\ell_1, \ell_2) < K \right\}$:

$$
\begin{aligned}
\mathcal{A}_{MI} = & \sum_{\ell \in L_K} \sum_{m=1}^{M_\ell} \frac{\Delta_{\ell_1,\ell_2}(m)}{M_\ell} \\
\Delta_{\ell_1,\ell_2} \equiv & \overline{G}_{C_1 2^{\ell_1}+1, C_2 2^{\ell_2}} - \mathbf{1}_{\ell_1>1} \overline{G}_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2-1}} \\
& - \mathbf{1}_{\ell_2>1} \overline{G}_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2}} + \mathbf{1}_{\ell_1,\ell_2>1} \overline{G}_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2-1}}.
\end{aligned}
$$

References

Björk, T., Szepessy, A., Tempone, R. & Zouraris, G. E. (2013), 'Monte carlo euler approximations of hjm term structure financial models', *BIT Numerical Mathematics* **53**(2), 341–383.

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