MULTI-INDEX AND MULTI-LEVEL MONTE CARLO EVALUATION OF HJM MODELS

JUHO HÄPPÖLÄ

Abstract. Notes of MIMC and MLMC evaluation of HJM models.

1. HJM model in Fourier space

Let us focus on the following HJM type SDE:

$$df(t,\tau) = \alpha(t,\tau)\,dt + \beta(t,\tau)\,d\overline{W}(t) \tag{1}$$

$$f(0,\tau) = f_0(\tau), \tag{2}$$

with the diffusion term given as the infinite-dimensional extension of eq. (1.1) in Björk et al. (2013):

$$\beta(t,\tau)\,d\overline{W}(t) = \sum_{k=0}^{\infty} c_n\left(\sin(\nu_k\tau) + \cos(\nu_k\tau)\right) \tag{3}$$

with W_n independent Brownian motions and

$$\nu_k \equiv \frac{k\pi}{L}$$

with $L >> 1$. This gives the following Fourier decomposition for the covariance of increments:

$$\mathrm{E}\left(\beta(t,\tau_1)\,d\overline{W}(t)\,\beta(t,\tau_2)\,d\overline{W}(t)\right) \tag{4}$$

$$= \sum_{k=0}^{\infty} c_k^2\left(\cos\left(\nu_k(\tau_1-\tau_2)\right)\right) \tag{5}$$

To keep the exponential HJM model risk neutral, we fix the drift of the equation as

$$\begin{aligned}\alpha(t,\tau) =& c_0^2(\tau-t)\\ &+\sum_{k=1}^{\infty}\frac{c_k^2}{k}\sin(\nu_k\tau)\left(\cos(\nu_q t)-\sin(\nu_k t)\right)\\ &+\sum_{k=1}^{\infty}\frac{c_k^2}{k}\cos(\nu_k\tau)\left(\cos(\nu_k t)-\sin(\nu_k t)\right)\\ &+\sum_{n=1}^{\infty}\frac{c_n^2}{n}\cos(2\nu_k\tau).\end{aligned}$$

Date: May 7, 2016.

The solution to the SDE can be written as

$$f(t,\tau) - f_0(\tau) = \tilde{f}(t,\tau) + \sum_{n=1}^{N} b_n(t)\cos\left(\frac{n\pi\tau}{L}\right) + a_n(t)\sin\left(\frac{n\pi\tau}{L}\right),$$

with

$$\tilde{f}(t,\tau) = c_0^2 t\left(\tau - \frac{t}{2}\right), \tag{6}$$

$$a_k(t) = b_k(t) = \frac{c_k^2}{\nu_k^2}\left(\cos(\nu_k t) + \sin(\nu_k t) - 1\right) - \frac{\mathbf{1}_{\frac{k}{2}\in\mathbb{Z}_+} c_j^2 t}{\nu_k} + c_k W_k(t). \tag{7}$$

The solution above lends itself to approximate solutions to the quantity of interest. First, the discount factor can be approximated as:

$$\begin{aligned}
&\int_0^T f(s,s)\,ds \\
\approx&\Delta t \sum_{n=1}^{N_t}\left(f_0(t_n) + \tilde{f}(t_n,t_n) + \sum_{k=1}^{N_f} b_k(t_n)\cos(\nu_k t_n) + a_k(t_n)\sin(\nu_k t_n)\right) \\
=&F_{N_t,N_f},
\end{aligned}$$

with $t_n = \frac{nT}{N_t}$. This approximation gives the following approximation error in N_f.

The natural next step is, of coures, estimating the error in approximating the integral by F_{N_t,N_f}. Firstly, there is the case of the frequency cutoff:

$$\begin{aligned}
&\mathrm{E}\left(\left(\int_0^T f(s,s)\,ds - \lim_{N_t\to\infty} F_{N_t,N_f}\right)^2\right) \\
=&\mathrm{E}\left(\int_0^T \sum_{k=N_f+1}^{\infty}\left(b_k(t_n)\cos(\nu_k t_n) + a_k(t_n)\sin(\nu_k t_n)\right)ds\right) \\
\approx&\mathrm{E}\left(\left(\int_0^T \sum_{k=N_f+1}^{\infty} c_k W_k(t)\left(\sin(\nu_k t) + \cos(\nu_k t)\right)\right)^2\right) \\
=&\mathrm{E}\left(\left(\sum_{k=N_f+1}^{\infty}\int_0^T c_k W_k(t)\left(\sin(\nu_k t) + \cos(\nu_k t)\right)ds\right)^2\right) \\
=&\mathcal{O}\left(\sum_{k=N_f+1}^{\infty}\left(\frac{c_k}{k}\right)^2\right).
\end{aligned}$$

The bound for the weak error can be obtained using Jensen's inequality. Setting as a test case the temporal correlation structure as $\rho(\tau_1,\tau_2) \propto \exp(-\kappa|\tau_1-\tau_2|)$ it follows that $c_k^2 = \frac{\kappa}{L}\frac{(-1)^k(1-\exp(-|\kappa L|))}{\kappa^2+(\nu_k)^2}$. Using this as an example, we plot the empirical strong and weak convergence rates in figure 1.

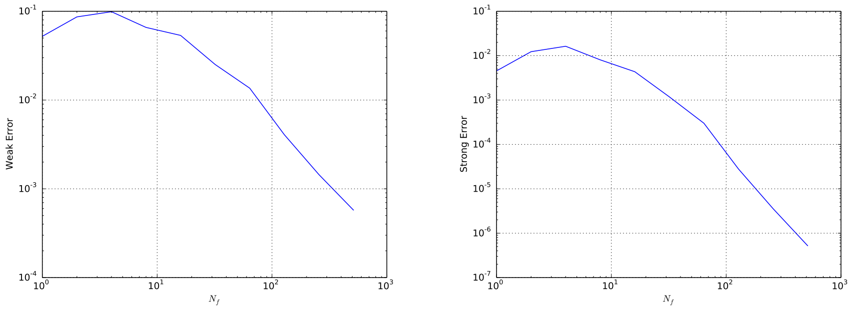

FIGURE 1. Weak (left) and strong (right) error $\left|\left|\int_0^T f\left(s,s\right)ds - \lim_{N_t\to\infty} F_{N_t,N_f}\right|\right|$. $N_t = 2^9+1$.

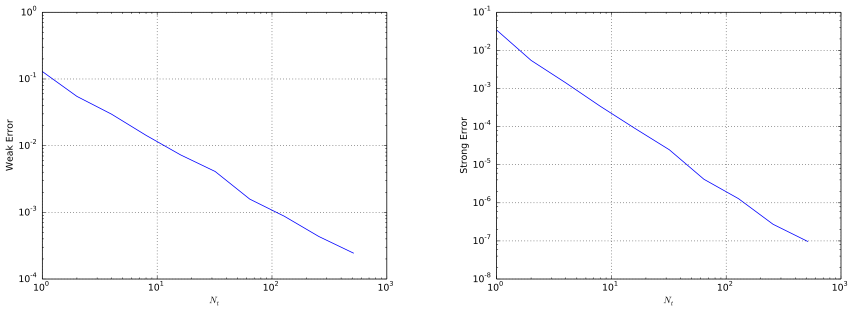

FIGURE 2. Weak (left) and strong (right) error for the temporal discretisation $\left|\left|F_{2N_t-1,N_f} - F_{N_t,N_f}\right|\right|$. $N_f = 2^4$.

As for the time discretisation error, we are concerned with a rectangle quadrature error, giving us weak and strong error rate of 1 and 2 respectively. Empirical test of these rates is presented in figure 2.

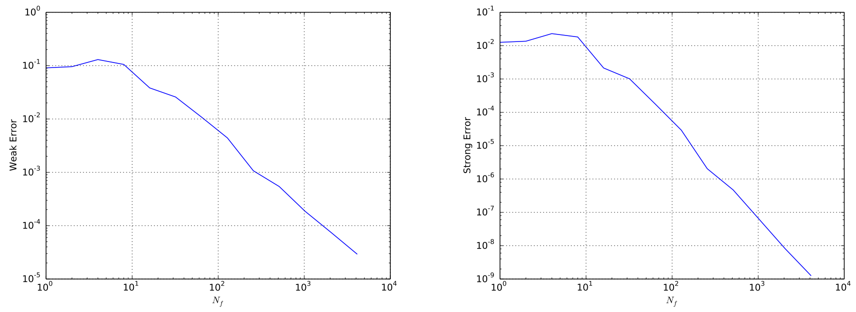

FIGURE 3. Weak (left) and strong (right) error for the temporal discretisation $\left|\left|\Psi_{2N_f} - \Psi_{N_f}\right|\right|$.

Similarly, one may estimate the underlying part of the payoff functional

$$
\begin{aligned}
&\int_{\tau_1}^{\tau_2} f\left(T,\tau\right) d\tau \\
\approx &\int_{\tau_1}^{\tau_2} f_0\left(\tau\right) + \tilde{f}\left(t_n,t_n\right) + \sum_{k=1}^{N_f} b_k\left(T\right)\cos\left(\nu_k\tau\right) + a_k\left(T\right)\sin\left(\nu_k\tau\right) d\tau. \\
= &\int_{\tau_1}^{\tau_2} f_0\left(\tau\right) + \frac{c_0 T}{2}\left(\tau_2^2 - \tau_2 + \tau_1 - \tau_1^2\right) d\tau \\
&+ \sum_{k=1}^{N_f} \frac{b_k\left(T\right)}{\nu_k}\left(\sin\left(\nu_k\tau_2\right) - \sin\left(\nu_k\tau_1\right)\right) \\
&- \sum_{k=1}^{N_f} \frac{a_k\left(T\right)}{\nu_k}\left(\cos\left(\nu_k\tau_2\right) - \cos\left(\nu_k\tau_1\right)\right) \\
= &\Psi_{N_f}.
\end{aligned}
$$

This approximation has its own approximation error with respect to N_f:

$$
\begin{aligned}
&\mathrm{E}\left(\left(\int_{\tau_1}^{\tau_2} f\left(T,\tau\right) d\tau - \Psi_{N_f}\right)^2\right) \\
=&\mathrm{E}\left(\left(\sum_{k=N_f+1}^{\infty} \frac{b_k\left(T\right)}{\nu_k}\left(\sin\left(\nu_k\tau_2\right) - \sin\left(\nu_k\tau_1\right)\right) - \frac{a_k\left(T\right)}{\nu_k}\left(\cos\left(\nu_k\tau_2\right) - \cos\left(\nu_k\tau_1\right)\right)\right)^2\right) \\
=&\mathcal{O}\left(\sum_{k=N_f+1}^{\infty} \frac{c_k^2}{\nu_k^2}\right).
\end{aligned}
$$

Using the exponentially decaying covariance function, we expect order 3 strong convergence and, using Jensen's inequality, we may bound the weak error to order $\frac{3}{2}$. Accompanying numerical rates are presetend in figure 3.

Overall, we may approximate a future price of a zero-coupon bond as

$$\mathcal{G}(f) = \mathrm{E}\left(\exp\left(-\int_0^{t_T} f(s,s)\,ds\right)\exp\left(-\int_{\tau_1}^{\tau_2} f(T,\tau)\,d\tau\right)\right) \approx \underbrace{\mathrm{E}\left(F_{N_t,N_f}\Psi_{N_f}\right)}_{\overline{G}_{N_t,N_f}}.$$

Continuing the case of exponential covariance structure where $c_k \sim \mathcal{O}\left(k^{-2}\right)$, we expect, based on the above computations, to observe the following rate

$$\mathrm{E}\left(\left(\mathcal{G} - \overline{G}_{N_t,N_f}\right)^2\right) = \mathcal{O}\left(N_f^{-3}N_t^{-2}\right).$$

To complement the component-wise plots in figures 1, 2 and 3, we plot the overall approximation of $\mathcal{G}$ with $\overline{G}$ in figure **??**.

Setting $\ell = (\ell_1, \ell_2)$, we may define a Monte Carlo estimator using m independent realisations of Ψ_{N_f} and F_{N_t,N_f}:

$$\mathcal{A}_{\ell_1,\ell_2} \equiv \sum_{m=1}^{M} \frac{F_{C_1 2^{\ell_1}+1, C_2 2^{\ell_2}}\Psi_{C_2 2^{\ell_2}}(m)}{M}. \tag{8}$$

Similarly, we may extend the above to a MLMC estimator as

$$\begin{aligned}\mathcal{A}_{ML} = &\sum_{m=0}^{M_0} \frac{F_{2C_1,C_2}\Psi_{C_2}(m)}{M_0}\\ &+ \sum_{\ell_1=1}^{L}\sum_{m=0}^{M_{\ell_1}} \frac{\left(F_{C_1 2^{\ell_1}+1, C_2 2^{\ell_1}}\Psi_{C_2 2^{\ell_1}} - F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_1-1}}\Psi_{C_2 2^{\ell_1}}\right)(m)}{M_{\ell_1}},\\ \equiv &\sum_{\ell_1=0}^{L}\sum_{m=0}^{M_{\ell_1}} \frac{\Delta_{\ell_1}(m)}{M_{\ell_1}}\end{aligned}$$

and, into a MIMC estimator through defining the appropriate two-dimensional difference operators Δ_{ℓ_1,ℓ_2} and a downward-closed index-set $L_K = \left\{(\ell_1,\ell_2) \in \mathbb{Z}_+^2 : I(\ell_1,\ell_2) < K\right\}$:

$$\begin{aligned}\mathcal{A}_{MI} = &\sum_{\ell \in L_K}\sum_{m=1}^{M_\ell} \frac{\Delta_{\ell_1,\ell_2}(m)}{M_\ell}\\ \Delta_{\ell_1,\ell_2} \equiv &F_{C_1 2^{\ell_1}+1, C_2 2^{\ell_2}}\Psi_{C_2 2^{\ell_2}} - \mathbf{1}_{\ell_1>1}F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2-1}}\Psi_{C_2 2^{\ell_2-1}}\\ &- \mathbf{1}_{\ell_2>1}F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2}}\Psi_{C_2 2^{\ell_2}} + \mathbf{1}_{\ell_1,\ell_2>1}F_{C_1 2^{\ell_1-1}+1, C_2 2^{\ell_2-1}}\Psi_{C_2 2^{\ell_2-1}}.\end{aligned}$$

References

Björk, T., Szepessy, A., Tempone, R. & Zouraris, G. E. (2013), 'Monte carlo euler approximations of hjm term structure financial models', *BIT Numerical Mathematics* **53**(2), 341–383.

King Abdullah University of Science and Technology, Thuwal, Kingdom of Saudi Arabia
E-mail address: juho.happola@iki.fi

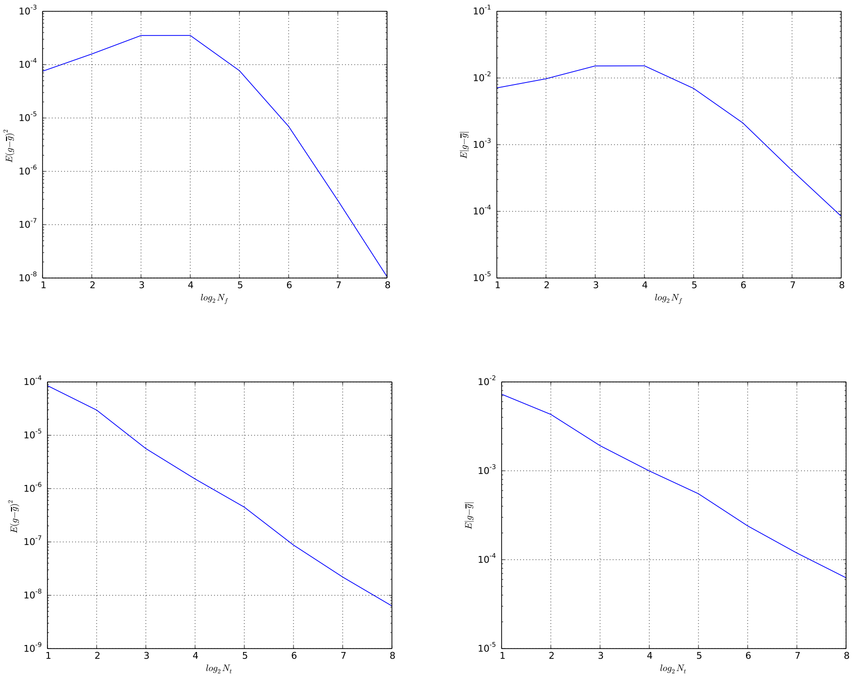

FIGURE 4. Empirical strong and weak convergence rate for the N_t and N_f.