第一章　非完整基理论

1.1　完整基与非完整基的概念

在 **??** 节中，我们利用曲线坐标系 $\boldsymbol{X}(\boldsymbol{x})$ 构造了 $\mathbb{R}^m$ 上的一组（局部协变）基

$$\left\{\boldsymbol{g}_i(\boldsymbol{x}) = \frac{\partial \boldsymbol{X}}{\partial x^i}(\boldsymbol{x})\right\}_{i=1}^{m} \subset \mathbb{R}^m, \tag{1.1}$$

它们称为**完整基**．与之对应，不是由曲线坐标系诱导的基，称为**非完整基**．

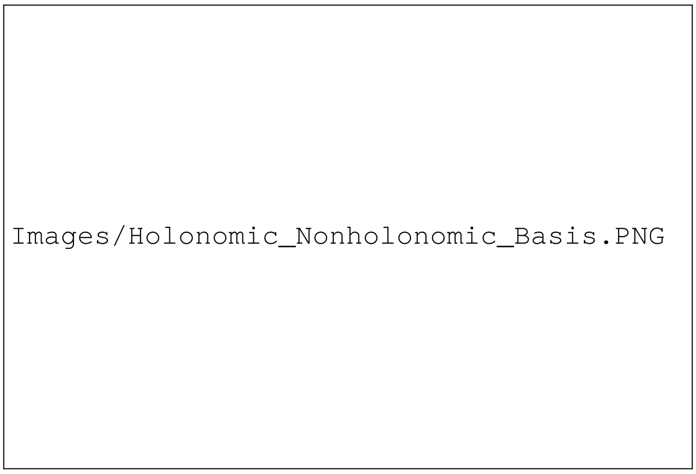

图 1.1: 完整基与非完整基

如图 1.1，x^i-线的切向量构成一组局部协变基 $\left\{\boldsymbol{g}_i(\boldsymbol{x})\right\}_{i=1}^{m}$，它和它的对偶 $\left\{\boldsymbol{g}^i(\boldsymbol{x})\right\}_{i=1}^{m}$ 都是完整基．除此以外，我们当然可以选取另外的基 $\left\{\boldsymbol{g}_{(i)}(\boldsymbol{x})\right\}_{i=1}^{m}$ 和 $\left\{\boldsymbol{g}^{(i)}(\boldsymbol{x})\right\}_{i=1}^{m}$，它们不是由曲线坐标系诱导，因而是非完整基．

1.2　非完整基下的张量梯度

下面我们来考察张量梯度在非完整基下的表达形式．在 **??** 节中，我们已经推导出了张量场的（右）梯度：

$$\left(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\right)(\boldsymbol{x}) \triangleq \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}(\boldsymbol{x}) \otimes \boldsymbol{g}^\mu(\boldsymbol{x}) = \nabla_\mu \Phi^{i\ \,k}_{\ j}(\boldsymbol{x})\, \boldsymbol{g}_i(\boldsymbol{x}) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x}) \otimes \boldsymbol{g}^\mu(\boldsymbol{x}). \tag{1.2}$$

这是一个四阶张量，对应的张量分量可记作

$$\left(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\right)^{i\ \,k}_{\ j\ \,\mu}(\boldsymbol{x}) \coloneqq \nabla_\mu \Phi^{i\ \,k}_{\ j}(\boldsymbol{x}). \tag{1.3}$$

除此以外，其他的基当然也可以用来表示该张量，比如前文提到过的 $\left\{\boldsymbol{g}_{(i)}(\boldsymbol{x})\right\}_{i=1}^{m}$ 和 $\left\{\boldsymbol{g}^{(i)}(\boldsymbol{x})\right\}_{i=1}^{m}$，它们都是非完整基.

非完整基与完整基之间的关系，可以利用 **??** 小节中引入的坐标转换关系来获得：

$$
\begin{cases}
\boldsymbol{g}_{(i)}(\boldsymbol{x}) = c_{(i)}^{k}(\boldsymbol{x})\,\boldsymbol{g}_{k}(\boldsymbol{x}), & (1.4\text{-a})\\
\boldsymbol{g}^{(i)}(\boldsymbol{x}) = c_{k}^{(i)}(\boldsymbol{x})\,\boldsymbol{g}^{k}(\boldsymbol{x}); & (1.4\text{-b})\\
\boldsymbol{g}_{i}(\boldsymbol{x}) = c_{i}^{(k)}(\boldsymbol{x})\,\boldsymbol{g}_{(k)}(\boldsymbol{x}), & (1.4\text{-c})\\
\boldsymbol{g}^{i}(\boldsymbol{x}) = c_{(k)}^{i}(\boldsymbol{x})\,\boldsymbol{g}^{(k)}(\boldsymbol{x}). & (1.4\text{-d})
\end{cases}
$$

2017-01-20　坐标转换关系

其中的基转换系数都是已知量，它们的定义如下：[①]

$$
\begin{cases}
c_{(i)}^{j}(\boldsymbol{x}) \coloneqq \left\langle \boldsymbol{g}_{(i)}(\boldsymbol{x}), \boldsymbol{g}^{j}(\boldsymbol{x}) \right\rangle_{\mathbb{R}^m}, & (1.5\text{-a})\\
c_{j}^{(i)}(\boldsymbol{x}) \coloneqq \left\langle \boldsymbol{g}^{(i)}(\boldsymbol{x}), \boldsymbol{g}_{j}(\boldsymbol{x}) \right\rangle_{\mathbb{R}^m}. & (1.5\text{-b})
\end{cases}
$$

代入 (1.2) 式，可有[②]

$$
\begin{aligned}
\boldsymbol{\Phi} \otimes \boldsymbol{\nabla} &= \nabla_{\mu} \Phi^{i}{}_{j}{}^{k} \left(\boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \otimes \boldsymbol{g}^{\mu}\right)\\
&= \nabla_{\mu} \Phi^{i}{}_{j}{}^{k}(\boldsymbol{x}) \left[\left(c_{i}^{(p)} \boldsymbol{g}_{(p)}\right) \otimes \left(c_{(q)}^{j} \boldsymbol{g}^{(q)}\right) \otimes \left(c_{k}^{(r)} \boldsymbol{g}_{(r)}\right) \otimes \left(c_{(\alpha)}^{\mu} \boldsymbol{g}^{(\alpha)}\right)\right]
\end{aligned}
$$

根据线性性，提出系数：

$$
= \left(c_{i}^{(p)} c_{(q)}^{j} c_{k}^{(r)} c_{(\alpha)}^{\mu} \nabla_{\mu} \Phi^{i}{}_{j}{}^{k}\right)\left(\boldsymbol{g}_{(p)} \otimes \boldsymbol{g}^{(q)} \otimes \boldsymbol{g}_{(r)} \otimes \boldsymbol{g}^{(\alpha)}\right)
$$

写成张量分量与简单张量“乘积”的形式，即为

$$
\eqqcolon (\boldsymbol{\Phi} \otimes \boldsymbol{\nabla})^{(p)}{}_{(q)}{}^{(r)}{}_{(\alpha)} \left(\boldsymbol{g}_{(p)} \otimes \boldsymbol{g}^{(q)} \otimes \boldsymbol{g}_{(r)} \otimes \boldsymbol{g}^{(\alpha)}\right). \tag{1.6}
$$

这样，我们就获得了非完整基下张量梯度的表示．再利用式 (1.3)，可知

$$
(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla})^{(p)}{}_{(q)}{}^{(r)}{}_{(\alpha)} = c_{i}^{(p)} c_{(q)}^{j} c_{k}^{(r)} c_{(\alpha)}^{\mu} (\boldsymbol{\Phi} \otimes \boldsymbol{\nabla})^{i}{}_{j}{}^{k}{}_{\mu}. \tag{1.7}
$$

以上结果与 **??** 小节中的推导是完全一致的.

1.3　非完整基的形式运算

在 1.2 节中，我们利用坐标转换关系获得了张量梯度在非完整基下的表示．而在本节，我们将通过定义，建立所谓“形式理论”，获得一套更统一、更连贯的表述.

2017-01-31　统一、连贯？

首先需要给出一些定义.

---

① 只有两个基转换系数的原因是内积具有交换律.

② 这里我们省略了“$(\boldsymbol{x})$”.

1. **形式偏导数**：

$$\frac{\partial}{\partial x^{(\mu)}} \triangleq c^m_{(\mu)} \frac{\partial}{\partial x^m}. \tag{1.8}$$

注意 $\partial/\partial x^{(\mu)}$ 本身是不能用极限形式来定义的，因为曲线坐标系中并不存在有 $x^{(\mu)}$ 坐标线.

2. **形式 Christoffel 符号**：

$$\Gamma^{(\gamma)}{}_{(\alpha)(\beta)} \triangleq c^i_{(\alpha)} c^j_{(\beta)} c^{(\gamma)}_k \Gamma^k{}_{ij} - c^i_{(\alpha)} c^j_{(\beta)} \frac{\partial c^{(\gamma)}_j}{\partial x^i} = c^i_{(\alpha)} c^j_{(\beta)} \left(c^{(\gamma)}_k \Gamma^k{}_{ij} - \frac{\partial c^{(\gamma)}_j}{\partial x^i} \right). \tag{1.9}$$

2017-01-31　第一类形式 Christoffel 符号

3. **形式协变导数**．我们以三阶张量 $\boldsymbol{\Phi}$ 为例给出定义．$\boldsymbol{\Phi}$ 在非完整基下可以用混合分量表示如下：

$$\Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)} := \boldsymbol{\Phi}\left(\boldsymbol{g}^{(\alpha)}, \boldsymbol{g}_{(\beta)}, \boldsymbol{g}^{(\gamma)}\right). \tag{1.10}$$

它相对 $x^{(\mu)}$ 分量的形式协变导数为

$$\nabla_{(\mu)} \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)} \triangleq \frac{\partial \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)}}{\partial x^{(\mu)}} + \Gamma^{(\alpha)}{}_{(\mu)(\sigma)} \Phi^{(\sigma)}{}_{(\beta)}{}^{(\gamma)} - \Gamma^{(\sigma)}{}_{(\mu)(\beta)} \Phi^{(\alpha)}{}_{(\sigma)}{}^{(\gamma)} + \Gamma^{(\gamma)}{}_{(\mu)(\sigma)} \Phi^{(\alpha)}{}_{(\beta)}{}^{(\sigma)}. \tag{1.11}$$

回顾 **??** 节，(**??**) 式给出了完整基下协变导数的定义：

$$\nabla_\mu \Phi^i{}_j{}^k \triangleq \frac{\partial \Phi^i{}_j{}^k}{\partial x^\mu} + \Gamma^i{}_{\mu s} \Phi^s{}_j{}^k - \Gamma^s{}_{\mu j} \Phi^i{}_s{}^k + \Gamma^k{}_{\mu s} \Phi^i{}_j{}^s. \tag{1.12}$$

可以看出形式协变导数的定义与它几乎一模一样.

接下来我们要证明

$$\nabla_{(\mu)} \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)} = c^m_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \nabla_m \Phi^i{}_j{}^k. \tag{1.13}$$

代入式 (1.3) 和 (1.7)，即得

$$\nabla_{(\mu)} \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)} = \left(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\right)^{(p)}{}_{(q)}{}^{(r)}{}_{(\alpha)}. \tag{1.14}$$

换句话说，此处我们正是要验证这种“形式理论”与 1.2 节中坐标转换关系的一致性.

证明： 左边按照 (1.11) 式展开，第一项为

$$\frac{\partial \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)}}{\partial x^{(\mu)}} = c^m_{(\mu)} \frac{\partial \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)}}{\partial x^m}$$

这里用到了形式偏导数的定义 (1.8) 式．然后利用坐标转换关系展开张量分量：

$$= c^m_{(\mu)} \frac{\partial}{\partial x^m} \left(c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \Phi^i{}_j{}^k \right)$$

按照通常的偏导数法则直接打开：

$$= c^m_{(\mu)} c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial c^{(\alpha)}_i}{\partial x^m} \Phi^i{}_j{}^k + c^m_{(\mu)} c^{(\alpha)}_i c^{(\gamma)}_k \frac{\partial c^j_{(\beta)}}{\partial x^m} \Phi^i{}_j{}^k + c^m_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} \frac{\partial c^{(\gamma)}_k}{\partial x^m} \Phi^i{}_j{}^k$$
$$+ c^m_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial \Phi^i{}_j{}^k}{\partial x^m}$$

$$= c^m_{(\mu)} \boldsymbol{\Phi}^i{}_j{}^k \left(c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial c^{(\alpha)}_i}{\partial x^m} + c^{(\alpha)}_i c^{(\gamma)}_k \frac{\partial c^j_{(\beta)}}{\partial x^m} + c^{(\alpha)}_i c^j_{(\beta)} \frac{\partial c^{(\gamma)}_k}{\partial x^m} \right)$$

$$+ c^m_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial \boldsymbol{\Phi}^i{}_j{}^k}{\partial x^m}. \tag{1.15}$$

接下来处理含有形式 Christoffel 符号的三项，分别是

$$\Gamma^{(\alpha)}{}_{(\mu)(\sigma)} \boldsymbol{\Phi}^{(\sigma)}{}_{(\beta)}{}^{(\gamma)} = c^p_{(\mu)} c^q_{(\sigma)} \left(c^{(\alpha)}_s \Gamma^s{}_{pq} - \frac{\partial c^{(\alpha)}_q}{\partial x^p} \right) \cdot \boldsymbol{\Phi}^{(\sigma)}{}_{(\beta)}{}^{(\gamma)}$$

$$= c^p_{(\mu)} c^q_{(\sigma)} \left(c^{(\alpha)}_s \Gamma^s{}_{pq} - \frac{\partial c^{(\alpha)}_q}{\partial x^p} \right) \cdot c^{(\sigma)}_i c^j_{(\beta)} c^{(\gamma)}_k \boldsymbol{\Phi}^i{}_j{}^k$$

根据式 (??)，我们有 $c^q_{(\sigma)} c^{(\sigma)}_i = \delta^q_i$，于是

$$= c^p_{(\mu)} \boldsymbol{\Phi}^i{}_j{}^k \left(c^{(\alpha)}_s c^j_{(\beta)} c^{(\gamma)}_k \Gamma^s{}_{pi} - c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial c^{(\alpha)}_i}{\partial x^p} \right); \tag{1.16-a}$$

$$-\Gamma^{(\sigma)}{}_{(\mu)(\beta)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\sigma)}{}^{(\gamma)} = -c^p_{(\mu)} c^q_{(\beta)} \left(c^{(\sigma)}_s \Gamma^s{}_{pq} - \frac{\partial c^{(\sigma)}_q}{\partial x^p} \right) \cdot \boldsymbol{\Phi}^{(\alpha)}{}_{(\sigma)}{}^{(\gamma)}$$

$$= -c^p_{(\mu)} c^q_{(\beta)} \left(c^{(\sigma)}_s \Gamma^s{}_{pq} - \frac{\partial c^{(\sigma)}_q}{\partial x^p} \right) \cdot c^{(\alpha)}_i c^j_{(\sigma)} c^{(\gamma)}_k \boldsymbol{\Phi}^i{}_j{}^k$$

$$= c^p_{(\mu)} \boldsymbol{\Phi}^i{}_j{}^k \left(-c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k \Gamma^j{}_{pq} + c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k c^j_{(\sigma)} \frac{\partial c^{(\sigma)}_q}{\partial x^p} \right); \tag{1.16-b}$$

$$\Gamma^{(\gamma)}{}_{(\mu)(\sigma)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\sigma)} = c^p_{(\mu)} c^q_{(\sigma)} \left(c^{(\gamma)}_s \Gamma^s{}_{pq} - \frac{\partial c^{(\gamma)}_q}{\partial x^p} \right) \cdot \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\sigma)}$$

$$= c^p_{(\mu)} c^q_{(\sigma)} \left(c^{(\gamma)}_s \Gamma^s{}_{pq} - \frac{\partial c^{(\gamma)}_q}{\partial x^p} \right) \cdot c^{(\alpha)}_i c^j_{(\beta)} c^{(\sigma)}_k \boldsymbol{\Phi}^i{}_j{}^k$$

$$= c^p_{(\mu)} \boldsymbol{\Phi}^i{}_j{}^k \left(c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_s \Gamma^s{}_{pk} - c^{(\alpha)}_i c^j_{(\beta)} \frac{\partial c^{(\gamma)}_k}{\partial x^p} \right). \tag{1.16-c}$$

以上三式都有公因子 $c^p_{(\mu)} \boldsymbol{\Phi}^i{}_j{}^k$．为了进一步化简，不妨将哑标 p 换为 m．这样可有

$$\Gamma^{(\alpha)}{}_{(\mu)(\sigma)} \boldsymbol{\Phi}^{(\sigma)}{}_{(\beta)}{}^{(\gamma)} - \Gamma^{(\sigma)}{}_{(\mu)(\beta)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\sigma)}{}^{(\gamma)} + \Gamma^{(\gamma)}{}_{(\mu)(\sigma)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\sigma)}$$

$$= c^m_{(\mu)} \boldsymbol{\Phi}^i{}_j{}^k \left[\left(c^{(\alpha)}_s c^j_{(\beta)} c^{(\gamma)}_k \Gamma^s{}_{mi} - c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k \Gamma^j{}_{mq} + c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_s \Gamma^s{}_{mk} \right) \right.$$

$$\left. - c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial c^{(\alpha)}_i}{\partial x^m} + c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k c^j_{(\sigma)} \frac{\partial c^{(\sigma)}_q}{\partial x^m} - c^{(\alpha)}_i c^j_{(\beta)} \frac{\partial c^{(\gamma)}_k}{\partial x^m} \right]. \tag{1.17}$$

该式与 (1.15) 式相加，得

$$\nabla_{(\mu)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\gamma)} \triangleq \frac{\partial \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\gamma)}}{\partial x^{(\mu)}} + \Gamma^{(\alpha)}{}_{(\mu)(\sigma)} \boldsymbol{\Phi}^{(\sigma)}{}_{(\beta)}{}^{(\gamma)} - \Gamma^{(\sigma)}{}_{(\mu)(\beta)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\sigma)}{}^{(\gamma)} + \Gamma^{(\gamma)}{}_{(\mu)(\sigma)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\sigma)}$$

$$
\begin{aligned}
&= c^m_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial \boldsymbol{\Phi}^i{}_j{}^k}{\partial x^m} + c^m_{(\mu)} \boldsymbol{\Phi}^i{}_j{}^k \Bigg[c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial c^{(\alpha)}_i}{\partial x^m} + c^{(\alpha)}_i c^{(\gamma)}_k \frac{\partial c^j_{(\beta)}}{\partial x^m} + c^{(\alpha)}_i c^j_{(\beta)} \frac{\partial c^{(\gamma)}_k}{\partial x^m} \\
&\qquad + \left(c^{(\alpha)}_s c^j_{(\beta)} c^{(\gamma)}_k \Gamma^s{}_{mi} - c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k \Gamma^j{}_{mq} + c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_s \Gamma^s{}_{mk} \right) \\
&\qquad - c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial c^{(\alpha)}_i}{\partial x^m} + c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k c^j_{(\sigma)} \frac{\partial c^{(\sigma)}_q}{\partial x^m} - c^{(\alpha)}_i c^j_{(\beta)} \frac{\partial c^{(\gamma)}_k}{\partial x^m} \Bigg]
\end{aligned}
$$

高亮部分相互抵消：

$$
\begin{aligned}
&= c^m_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial \boldsymbol{\Phi}^i{}_j{}^k}{\partial x^m} + c^m_{(\mu)} \boldsymbol{\Phi}^i{}_j{}^k \Bigg[\left(c^{(\alpha)}_s c^j_{(\beta)} c^{(\gamma)}_k \Gamma^s{}_{mi} - c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k \Gamma^j{}_{mq} + c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_s \Gamma^s{}_{mk} \right) \\
&\qquad + c^{(\alpha)}_i c^{(\gamma)}_k \frac{\partial c^j_{(\beta)}}{\partial x^m} + c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k c^j_{(\sigma)} \frac{\partial c^{(\sigma)}_q}{\partial x^m} \Bigg]
\end{aligned} \tag{1.18}
$$

注意到 $c^j_{(\beta)} = c^q_{(\beta)} \delta^j_q = c^q_{(\beta)} c^j_{(\sigma)} c^{(\sigma)}_q$，因此

$$
\frac{\partial c^j_{(\beta)}}{\partial x^m} = \frac{\partial}{\partial x^m} \left(c^q_{(\beta)} c^j_{(\sigma)} c^{(\sigma)}_q \right) = c^j_{(\sigma)} c^{(\sigma)}_q \frac{\partial c^q_{(\beta)}}{\partial x^m} + c^q_{(\beta)} c^{(\sigma)}_q \frac{\partial c^j_{(\sigma)}}{\partial x^m} + c^q_{(\beta)} c^j_{(\sigma)} \frac{\partial c^{(\sigma)}_q}{\partial x^m}. \tag{1.19}
$$

所以 (1.18) 式中最后一步的第二行就能够写成

$$
\begin{aligned}
& c^{(\alpha)}_i c^{(\gamma)}_k \frac{\partial c^j_{(\beta)}}{\partial x^m} + c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k c^j_{(\sigma)} \frac{\partial c^{(\sigma)}_q}{\partial x^m} \\
&= c^{(\alpha)}_i c^{(\gamma)}_k \left(\frac{\partial c^j_{(\beta)}}{\partial x^m} + c^q_{(\beta)} c^j_{(\sigma)} \frac{\partial c^{(\sigma)}_q}{\partial x^m} \right) \\
&= c^{(\alpha)}_i c^{(\gamma)}_k \left(c^j_{(\sigma)} c^{(\sigma)}_q \frac{\partial c^q_{(\beta)}}{\partial x^m} + c^q_{(\beta)} c^{(\sigma)}_q \frac{\partial c^j_{(\sigma)}}{\partial x^m} + c^q_{(\beta)} c^j_{(\sigma)} \frac{\partial c^{(\sigma)}_q}{\partial x^m} + c^q_{(\beta)} c^j_{(\sigma)} \frac{\partial c^{(\sigma)}_q}{\partial x^m} \right)
\end{aligned}
$$

合并同类项：

$$
\begin{aligned}
&= c^{(\alpha)}_i c^{(\gamma)}_k \left[c^j_{(\sigma)} \left(c^{(\sigma)}_q \frac{\partial c^q_{(\beta)}}{\partial x^m} + c^q_{(\beta)} \frac{\partial c^{(\sigma)}_q}{\partial x^m} \right) + c^q_{(\beta)} \left(c^{(\sigma)}_q \frac{\partial c^j_{(\sigma)}}{\partial x^m} + c^j_{(\sigma)} \frac{\partial c^{(\sigma)}_q}{\partial x^m} \right) \right] \\
&= c^{(\alpha)}_i c^{(\gamma)}_k \left[c^j_{(\sigma)} \frac{\partial}{\partial x^m} \left(c^{(\sigma)}_q c^q_{(\beta)} \right) + c^q_{(\beta)} \frac{\partial}{\partial x^m} \left(c^{(\sigma)}_q c^j_{(\sigma)} \right) \right]
\end{aligned}
$$

再次利用式 (**??**)，可得

$$
= c^{(\alpha)}_i c^{(\gamma)}_k \left(c^j_{(\sigma)} \frac{\partial \delta^\sigma_\beta}{\partial x^m} + c^q_{(\beta)} \frac{\partial \delta^j_q}{\partial x^m} \right) = 0. \tag{1.20}
$$

代回式 (1.18)，有

$$
\begin{aligned}
\nabla_{(\mu)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\gamma)} &= c^m_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial \boldsymbol{\Phi}^i{}_j{}^k}{\partial x^m} + c^m_{(\mu)} \boldsymbol{\Phi}^i{}_j{}^k \left(c^{(\alpha)}_s c^j_{(\beta)} c^{(\gamma)}_k \Gamma^s{}_{mi} - c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k \Gamma^j{}_{mq} + c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_s \Gamma^s{}_{mk} \right) \\
&= c^m_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial \boldsymbol{\Phi}^i{}_j{}^k}{\partial x^m} + c^m_{(\mu)} \left(c^{(\alpha)}_s c^j_{(\beta)} c^{(\gamma)}_k \Gamma^s{}_{mi} \boldsymbol{\Phi}^i{}_j{}^k - c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k \Gamma^j{}_{mq} \boldsymbol{\Phi}^i{}_j{}^k + c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_s \Gamma^s{}_{mk} \boldsymbol{\Phi}^i{}_j{}^k \right)
\end{aligned}
$$

下面要对哑标进行重排．括号里的第一项：$s \leftrightarrow i$；第二项：$j \to s, q \to j$；第三项：$s \leftrightarrow k$．于是

$$
\begin{aligned}
&= c^m_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial \boldsymbol{\Phi}^i{}_j{}^k}{\partial x^m} + c^m_{(\mu)} \left(c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \Gamma^i{}_{ms} \boldsymbol{\Phi}^s{}_j{}^k - c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \Gamma^s{}_{mj} \boldsymbol{\Phi}^i{}_s{}^k + c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \Gamma^k{}_{ms} \boldsymbol{\Phi}^i{}_j{}^s \right) \\
&= c^m_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \left(\frac{\partial \boldsymbol{\Phi}^i{}_j{}^k}{\partial x^m} + \Gamma^i{}_{ms} \boldsymbol{\Phi}^s{}_j{}^k - \Gamma^s{}_{mj} \boldsymbol{\Phi}^i{}_s{}^k + \Gamma^k{}_{ms} \boldsymbol{\Phi}^i{}_j{}^s \right) \\
&= c^m_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \nabla_m \boldsymbol{\Phi}^i{}_j{}^k. \qquad (1.21)
\end{aligned}
$$

这就完成了证明． □