第一章 张量场可微性

1.1 张量的范数

1.1.1 赋范线性空间

对于一个**线性空间** $\mathscr{V}$，它总是定义了**线性结构**：

$$\forall\, x, y \in \mathscr{V} \,\text{和}\, \forall\, \alpha, \beta \in \mathbb{R}, \quad \alpha\, x + \beta\, y \in \mathscr{V}. \tag{1.1}$$

为了进一步研究的需要，我们还要引入**范数**的概念. 所谓"范数"，就是对线性空间中任意元素大小的一种刻画. 举个我们熟悉的例子，m 维 Euclid 空间 $\mathbb{R}^m$ 中某个向量的范数，就定义为该向量在 Descartes 坐标下各分量的平方和的平方根.

一般而言，线性空间 $\mathscr{V}$ 中的范数 $\|\cdot\|_{\mathscr{V}}$ 是从 $\mathscr{V}$ 到 $\mathbb{R}$ 的一个映照，并且需要满足以下三个条件：

1. 非负性

$$\forall\, x \in \mathscr{V}, \quad \|x\|_{\mathscr{V}} \geqslant 0 \tag{1.2}$$

以及**非退化性**

$$\forall\, x \in \mathscr{V}, \quad \|x\|_{\mathscr{V}} = 0 \iff x = 0 \in \mathscr{V}, \tag{1.3}$$

这里的 **0** 是线性空间 $\mathscr{V}$ 中的**零元素**，它是唯一存在的.

2. 零元是唯一的，线性空间中的元素 x 与从 **0** 指向它的向量一一对应. 因此，线性空间中的元素也常被称为"向量".

考虑线性空间中的数乘运算. 从几何上看，x 乘上 λ，就是将 x 沿着原来的指向进行伸缩. 显然有

$$\forall\, x \in \mathscr{V} \,\text{和}\, \forall\, \lambda \in \mathbb{R}, \quad \|\lambda\, x\|_{\mathscr{V}} = |\lambda| \cdot \|x\|_{\mathscr{V}}, \tag{1.4}$$

这称为**正齐次性**.

想要图吗?

3. 线性空间中的加法满足平行四边形法则. 直观地看，就有

$$\forall\, x, y \in \mathscr{V}, \quad \|x + y\|_{\mathscr{V}} \leqslant \|x\|_{\mathscr{V}} + \|y\|_{\mathscr{V}}, \tag{1.5}$$

这称为**三角不等式**.

定义了范数的线性空间称为**赋范线性空间**.

1.1.2 张量范数的定义

考虑 p 阶张量 $\boldsymbol{\Phi} \in \mathscr{T}^p(\mathbb{R}^m)$，它可以用逆变分量或协变分量来表示：

$$
\boldsymbol{\Phi} = \begin{cases} \boldsymbol{\Phi}^{i_1 \cdots i_p} \, \boldsymbol{g}_{i_1} \otimes \cdots \otimes \boldsymbol{g}_{i_p}, & \text{(1.6-a)} \\[2mm] \boldsymbol{\Phi}_{i_1 \cdots i_p} \, \boldsymbol{g}^{i_1} \otimes \cdots \otimes \boldsymbol{g}^{i_p}, & \text{(1.6-b)} \end{cases}
$$

其中

$$
\begin{cases} \boldsymbol{\Phi}^{i_1 \cdots i_p} = \boldsymbol{\Phi}\left(\boldsymbol{g}^{i_1}, \cdots, \boldsymbol{g}^{i_p}\right). & \text{(1.7-a)} \\[2mm] \boldsymbol{\Phi}_{i_1 \cdots i_p} = \boldsymbol{\Phi}\left(\boldsymbol{g}_{i_1}, \cdots, \boldsymbol{g}_{i_p}\right), & \text{(1.7-b)} \end{cases}
$$

张量的**范数**定义为

$$
\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)} \triangleq \sqrt{\boldsymbol{\Phi}^{i_1 \cdots i_p} \boldsymbol{\Phi}_{i_1 \cdots i_p}} \in \mathbb{R}. \tag{1.8}
$$

$i_1 \cdots i_p$ 可独立取值，每个又有 m 种取法，所以根号下共有 m^p 项. 注意 $\boldsymbol{\Phi}^{i_1 \cdots i_p}$ 与 $\boldsymbol{\Phi}_{i_1 \cdots i_p}$ 未必相等，因而根号下的部分未必是平方和，这与 Euclid 空间中向量的模是不同的.

复习一下 **??** 小节，我们可以用另一组（带括号的）基表示张量 $\boldsymbol{\Phi}$：

$$
\begin{cases} \boldsymbol{\Phi}^{i_1 \cdots i_p} = c^{i_1}_{(\xi_1)} \cdots c^{i_p}_{(\xi_p)} \boldsymbol{\Phi}^{(\xi_1) \cdots (\xi_p)}, & \text{(1.9-a)} \\[2mm] \boldsymbol{\Phi}_{i_1 \cdots i_p} = c^{(\eta_1)}_{i_1} \cdots c^{(\eta_p)}_{i_p} \boldsymbol{\Phi}_{(\eta_1) \cdots (\eta_p)}, & \text{(1.9-b)} \end{cases}
$$

其中的 $c^i_{(\xi)} = \left\langle \boldsymbol{g}_{(\xi)}, \boldsymbol{g}^i \right\rangle_{\mathbb{R}^m}$，$c^{(\eta)}_i = \left\langle \boldsymbol{g}^{(\eta)}, \boldsymbol{g}_i \right\rangle_{\mathbb{R}^m}$，它们满足

$$
c^i_{(\xi)} c^{(\eta)}_i = \delta^{(\eta)}_{(\xi)}. \tag{1.10}
$$

于是

$$
\begin{aligned}
& \boldsymbol{\Phi}^{i_1 \cdots i_p} \boldsymbol{\Phi}_{i_1 \cdots i_p} \\
&= \left(c^{i_1}_{(\xi_1)} \cdots c^{i_p}_{(\xi_p)} \boldsymbol{\Phi}^{(\xi_1) \cdots (\xi_p)} \right) \left(c^{(\eta_1)}_{i_1} \cdots c^{(\eta_p)}_{i_p} \boldsymbol{\Phi}_{(\eta_1) \cdots (\eta_p)} \right) \\
&= \left(c^{i_1}_{(\xi_1)} c^{(\eta_1)}_{i_1} \right) \cdots \left(c^{i_p}_{(\xi_p)} c^{(\eta_p)}_{i_p} \right) \boldsymbol{\Phi}^{(\xi_1) \cdots (\xi_p)} \boldsymbol{\Phi}_{(\eta_1) \cdots (\eta_p)} \\
&= \delta^{(\eta_1)}_{(\xi_1)} \cdots \delta^{(\eta_p)}_{(\xi_p)} \boldsymbol{\Phi}^{(\xi_1) \cdots (\xi_p)} \boldsymbol{\Phi}_{(\eta_1) \cdots (\eta_p)} \\
&= \boldsymbol{\Phi}^{(\xi_1) \cdots (\xi_p)} \boldsymbol{\Phi}_{(\xi_1) \cdots (\xi_p)}.
\end{aligned} \tag{1.11}
$$

它是 $\boldsymbol{\Phi}$ 在另一组基下的逆变分量与协变分量乘积之和.

以上结果说明，张量的范数不依赖于基的选取，这就好比用不同的秤来称同一个人的体重，都将获得相同的结果. 既然如此，不妨采用单位正交基来表示张量的范数：

$$
\begin{aligned}
\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)} &\triangleq \sqrt{\boldsymbol{\Phi}^{i_1 \cdots i_p} \boldsymbol{\Phi}_{i_1 \cdots i_p}} \\
&= \sqrt{\boldsymbol{\Phi}^{\langle i_1 \rangle \cdots \langle i_p \rangle} \boldsymbol{\Phi}_{\langle i_1 \rangle \cdots \langle i_p \rangle}} \\
&=: \sqrt{\sum_{i_1, \cdots, i_p = 1}^m \left(\boldsymbol{\Phi}_{\langle i_1, \cdots, i_p \rangle} \right)^2}.
\end{aligned} \tag{1.12}
$$

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这里的 $\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}$ 表示张量 $\boldsymbol{\Phi}$ 在单位正交基下的分量，它的指标不区分上下.

有了这样的表示，很容易就可以验证张量范数符合之前的三个要求. 一组数的平方和开根号，必然是非负的. 至于非退化性，若范数为零，则所有分量均为零，自然成为零张量；反之，对于零张量，所有分量为零，范数也为零. 将 $\boldsymbol{\Phi}$ 乘上 λ，则有

$$
\begin{aligned}
\|\lambda\,\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)} &= \sqrt{\sum\nolimits_{i_1,\cdots,i_p=1}^{m}\left(\lambda\,\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\right)^2} \\
&= \sqrt{\lambda^2\sum\nolimits_{i_1,\cdots,i_p=1}^{m}\left(\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\right)^2} \\
&= |\lambda|\sqrt{\sum\nolimits_{i_1,\cdots,i_p=1}^{m}\left(\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\right)^2} \\
&= |\lambda|\cdot\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)},
\end{aligned}
\tag{1.13}
$$

于是正齐次性也得以验证. 最后，利用 Cauchy–Schwarz 不等式，可有

$$
\begin{aligned}
&\|\boldsymbol{\Phi}+\boldsymbol{\Psi}\|_{\mathscr{T}^p(\mathbb{R}^m)}^2 \\
&= \sum\left(\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}+\boldsymbol{\Psi}_{\langle i_1,\cdots,i_p\rangle}\right)^2 \\
&= \sum\left[\left(\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\right)^2+2\,\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\boldsymbol{\Psi}_{\langle i_1,\cdots,i_p\rangle}+\left(\boldsymbol{\Psi}_{\langle i_1,\cdots,i_p\rangle}\right)^2\right] \\
&= \sum\left(\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\right)^2+2\sum\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\boldsymbol{\Psi}_{\langle i_1,\cdots,i_p\rangle}+\sum\left(\boldsymbol{\Psi}_{\langle i_1,\cdots,i_p\rangle}\right)^2 \\
&\leqslant \|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)}^2+2\sqrt{\sum\left(\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\right)^2}\sqrt{\sum\left(\boldsymbol{\Psi}_{\langle i_1,\cdots,i_p\rangle}\right)^2}+\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)}^2 \\
&= \|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)}^2+2\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)}\cdot\|\boldsymbol{\Psi}\|_{\mathscr{T}^p(\mathbb{R}^m)}+\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)}^2 \\
&= \left(\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)}+\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)}\right)^2.
\end{aligned}
\tag{1.14}
$$

两边开方，即为三角不等式.

由此，我们就完整地给出了张量大小的刻画手段. 可以看出，它实际上就是 Euclid 空间中向量模的直接推广.

1.1.3 简单张量的范数

根据 **??** 小节中的定义，简单张量是形如 $\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta}$ 的张量，其中的 $\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta}\in\mathbb{R}^m$，它是三个向量的张量积. 简单张量的范数为

$$
\|\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta}\|_{\mathscr{T}^3(\mathbb{R}^m)}=\|\boldsymbol{\xi}\|_{\mathbb{R}^m}\cdot\|\boldsymbol{\eta}\|_{\mathbb{R}^m}\cdot\|\boldsymbol{\zeta}\|_{\mathbb{R}^m}.
\tag{1.15}
$$

证明： $\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta}$ 的逆变分量为

$$
(\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta})^{ijk}\triangleq\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta}\left(\boldsymbol{g}^i,\boldsymbol{g}^j,\boldsymbol{g}^k\right)=\xi^i\eta^j\zeta^k.
\tag{1.16}
$$

同理，它的协变分量为

$$
(\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta})_{ijk}\triangleq\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta}\left(\boldsymbol{g}_i,\boldsymbol{g}_j,\boldsymbol{g}_k\right)=\xi_i\eta_j\zeta_k.
\tag{1.17}
$$

二者相乘，有

$$(\boldsymbol{\xi} \otimes \boldsymbol{\eta} \otimes \boldsymbol{\zeta})^{ijk} \cdot (\boldsymbol{\xi} \otimes \boldsymbol{\eta} \otimes \boldsymbol{\zeta})_{ijk}$$

$$= \left(\xi^i \eta^j \zeta^k \right) \cdot \left(\xi_i \eta_j \zeta_k \right)$$

$$= \left(\xi^i \xi_i \right) \cdot \left(\eta^j \eta_j \right) \cdot \left(\zeta^k \zeta_k \right). \tag{1.18}$$

注意到

$$\|\xi\|_{\mathbb{R}^m}^2 = \langle \xi, \xi \rangle_{\mathbb{R}^m}$$

分别把二者用协变和逆变分量表示：

$$= \left\langle \xi^i \boldsymbol{g}_i, \xi_j \boldsymbol{g}^j \right\rangle_{\mathbb{R}^m}$$

$$= \xi^i \xi_j \left\langle \boldsymbol{g}_i, \boldsymbol{g}^j \right\rangle_{\mathbb{R}^m}$$

$$= \xi^i \xi_j \delta_i^j = \xi^i \xi_i, \tag{1.19}$$

于是

$$(\boldsymbol{\xi} \otimes \boldsymbol{\eta} \otimes \boldsymbol{\zeta})^{ijk} \cdot (\boldsymbol{\xi} \otimes \boldsymbol{\eta} \otimes \boldsymbol{\zeta})_{ijk} = \|\xi\|_{\mathbb{R}^m}^2 \cdot \|\eta\|_{\mathbb{R}^m}^2 \cdot \|\zeta\|_{\mathbb{R}^m}^2. \tag{1.20}$$

两边开方，即得 (1.15) 式. □

1.2 张量场的偏导数；协变导数

在区域 $\mathfrak{D}_{\boldsymbol{x}} \subset \mathbb{R}^m$ 上，若存在一个自变量用位置刻画的映照

$$\boldsymbol{\Phi} : \mathfrak{D}_{\boldsymbol{x}} \ni \boldsymbol{x} \mapsto \boldsymbol{\Phi}(\boldsymbol{x}) \in \mathscr{T}^r(\mathbb{R}^m), \tag{1.21}$$

则称张量 $\boldsymbol{\Phi}(\boldsymbol{x})$ [1] 是定义在 $\mathfrak{D}_{\boldsymbol{x}}$ 上的一个**张量场**.

下面我们以三阶张量为例. 设在物理域 $\mathfrak{D}_{\boldsymbol{X}} \subset \mathbb{R}^m$ 和参数域 $\mathfrak{D}_{\boldsymbol{x}} \subset \mathbb{R}^m$ 之间已经建立了微分同胚 $\boldsymbol{X}(\boldsymbol{x}) \in \mathscr{C}^p(\mathfrak{D}_{\boldsymbol{x}}; \mathfrak{D}_{\boldsymbol{X}})$. 在 $\boldsymbol{X}(\boldsymbol{x})$ 处，张量场 $\boldsymbol{\Phi}(\boldsymbol{x})$ 可以用分量形式表示为

$$\boldsymbol{\Phi}(\boldsymbol{x}) = \Phi^i{}_j{}^k(\boldsymbol{x}) \, \boldsymbol{g}_i(\boldsymbol{x}) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x}) \in \mathscr{T}^3(\mathbb{R}^m), \tag{1.22}$$

其中的 $\boldsymbol{g}_i(\boldsymbol{x}), \boldsymbol{g}^j(\boldsymbol{x}), \boldsymbol{g}_k(\boldsymbol{x})$ 都是局部基，而张量分量则定义为 [2]

$$\Phi^i{}_j{}^k(\boldsymbol{x}) \triangleq \boldsymbol{\Phi}(\boldsymbol{x}) \big[\boldsymbol{g}_i(\boldsymbol{x}), \boldsymbol{g}^j(\boldsymbol{x}), \boldsymbol{g}_k(\boldsymbol{x}) \big] \in \mathbb{R}. \tag{1.23}$$

类似地，当点沿着 x^μ-线运动到 $\boldsymbol{X}(\boldsymbol{x} + \lambda \, \boldsymbol{e}_\mu)$ 处时，有

$$\boldsymbol{\Phi}(\boldsymbol{x} + \lambda \, \boldsymbol{e}_\mu) = \Phi^i{}_j{}^k(\boldsymbol{x} + \lambda \, \boldsymbol{e}_\mu) \, \boldsymbol{g}_i(\boldsymbol{x} + \lambda \, \boldsymbol{e}_\mu) \otimes \boldsymbol{g}^j(\boldsymbol{x} + \lambda \, \boldsymbol{e}_\mu) \otimes \boldsymbol{g}_k(\boldsymbol{x} + \lambda \, \boldsymbol{e}_\mu). \tag{1.24}$$

现在研究 $\lambda \to 0 \in \mathbb{R}$ 时的极限

$$\lim_{\lambda \to 0} \frac{\boldsymbol{\Phi}(\boldsymbol{x} + \lambda \, \boldsymbol{e}_\mu) - \boldsymbol{\Phi}(\boldsymbol{x})}{\lambda} =: \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}(\boldsymbol{x}) \in \mathscr{T}^3(\mathbb{R}^m). \tag{1.25}$$

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[1] 类似 "$\boldsymbol{\Phi}(\boldsymbol{x})$" 的记号在前文也表示张量 $\boldsymbol{\Phi}$ 作用在向量 $\boldsymbol{x}$ 上（"吃掉"了 $\boldsymbol{x}$），此时有 $\boldsymbol{\Phi}(\boldsymbol{x}) \in \mathbb{R}$，注意不要混淆. 符号有限，难免如此，还望诸位体谅.

[2] 请注意，下式 $\boldsymbol{\Phi}$ 之后的第一个圆括号表示位于 $\boldsymbol{x}$ 处；而后面的方括号则表示作用在这几个向量上.

与之前的向量值映照类似，该极限表示张量场 $\boldsymbol{\Phi}(\boldsymbol{x})$ 作为一个整体，相对于自变量第 μ 个分量的变化率，即 $\boldsymbol{\Phi}$ 关于 x^μ（在 $\boldsymbol{x}$ 处）的**偏导数**. 式中，$\boldsymbol{\Phi}(\boldsymbol{x} + \lambda\,\boldsymbol{e}_\mu)$ 已由 (1.24) 式给出. 注意到张量分量实际上就是一个多元函数，于是

$$\boldsymbol{\Phi}^i{}_j{}^k\bigl(\boldsymbol{x} + \lambda\,\boldsymbol{e}_\mu\bigr) = \boldsymbol{\Phi}^i{}_j{}^k(\boldsymbol{x}) + \frac{\partial\boldsymbol{\Phi}^i{}_j{}^k}{\partial x^\mu}(\boldsymbol{x})\cdot\lambda + o^i{}_j{}^k(\lambda) \in \mathbb{R}. \tag{1.26}$$

另外，三个基向量作为向量值映照，同样可以展开：

$$\begin{cases} \boldsymbol{g}_i\bigl(\boldsymbol{x} + \lambda\,\boldsymbol{e}_\mu\bigr) = \boldsymbol{g}_i(\boldsymbol{x}) + \dfrac{\partial\boldsymbol{g}_i}{\partial x^\mu}(\boldsymbol{x})\cdot\lambda + \boldsymbol{o}_i(\lambda) \in \mathbb{R}^m, & \text{(1.27-a)} \\[3mm] \boldsymbol{g}^j\bigl(\boldsymbol{x} + \lambda\,\boldsymbol{e}_\mu\bigr) = \boldsymbol{g}^j(\boldsymbol{x}) + \dfrac{\partial\boldsymbol{g}^j}{\partial x^\mu}(\boldsymbol{x})\cdot\lambda + \boldsymbol{o}^j(\lambda) \in \mathbb{R}^m, & \text{(1.27-b)} \\[3mm] \boldsymbol{g}_k\bigl(\boldsymbol{x} + \lambda\,\boldsymbol{e}_\mu\bigr) = \boldsymbol{g}_k(\boldsymbol{x}) + \dfrac{\partial\boldsymbol{g}_k}{\partial x^\mu}(\boldsymbol{x})\cdot\lambda + \boldsymbol{o}_k(\lambda) \in \mathbb{R}^m. & \text{(1.27-c)} \end{cases}$$

如果直接展开，一共有 81 项，显然过于繁杂，不便操作. 我们将按 λ 的次数逐次展开. 首先看 λ 的零次项：

$$\boldsymbol{\Phi}^i{}_j{}^k(\boldsymbol{x})\,\boldsymbol{g}_i(\boldsymbol{x}) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x}), \tag{1.28}$$

这就是 $\boldsymbol{\Phi}(\boldsymbol{x})$. 然后是 λ 的一次项：

$$\lambda\cdot\left[\frac{\partial\boldsymbol{\Phi}^i{}_j{}^k}{\partial x^\mu}(\boldsymbol{x})\,\boldsymbol{g}_i(\boldsymbol{x}) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x}) + \boldsymbol{\Phi}^i{}_j{}^k(\boldsymbol{x})\frac{\partial\boldsymbol{g}_i}{\partial x^\mu}(\boldsymbol{x}) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x}) \right.$$
$$\left. + \boldsymbol{\Phi}^i{}_j{}^k(\boldsymbol{x})\,\boldsymbol{g}_i(\boldsymbol{x}) \otimes \frac{\partial\boldsymbol{g}^j}{\partial x^\mu}(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x}) + \boldsymbol{\Phi}^i{}_j{}^k(\boldsymbol{x})\,\boldsymbol{g}_i(\boldsymbol{x}) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \frac{\partial\boldsymbol{g}_k}{\partial x^\mu}(\boldsymbol{x})\right]. \tag{1.29}$$

剩下的至少是 λ 的二次项，我们将其统一写作 "res."（余项）.

现在回头看之前的极限 (1.25) 式. λ 的零次项与 $\boldsymbol{\Phi}(\boldsymbol{x})$ 相互抵消，而一次项就只剩下了系数部分. 至于余项 res.，则要证明它趋于零. 以

$$\boldsymbol{\Phi}^i{}_j{}^k(\boldsymbol{x})\,\boldsymbol{o}_i(\lambda) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x}) \tag{1.30}$$

为例，我们需要证明它等于 $\boldsymbol{o}(\lambda) \in \mathscr{T}^3(\mathbb{R}^m)$，即

$$\lim_{\lambda\to 0}\frac{\left\|\boldsymbol{\Phi}^i{}_j{}^k(\boldsymbol{x})\,\boldsymbol{o}_i(\lambda) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x})\right\|_{\mathscr{T}^3(\mathbb{R}^m)}}{\lambda} = 0 \in \mathbb{R}. \tag{1.31}$$

证明： 这里为了叙述方便，我们将暂时不使用 Einstein 求和约定，而是把求和号显式地写出来. 于是分子部分可以写成

$$\left\|\sum_{i,\,j,\,k=1}^{m}\boldsymbol{\Phi}^i{}_j{}^k(\boldsymbol{x})\,\boldsymbol{o}_i(\lambda) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x})\right\|_{\mathscr{T}^3(\mathbb{R}^m)}$$

根据范数的三角不等式，有

$$\leqslant \sum_{i,\,j,\,k=1}^{m}\left\|\boldsymbol{\Phi}^i{}_j{}^k(\boldsymbol{x})\,\boldsymbol{o}_i(\lambda) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x})\right\|_{\mathscr{T}^3(\mathbb{R}^m)}$$

再利用正齐次性，可得

$$= \sum_{i,j,k=1}^{m} \left| \Phi^{i}_{j}{}^{k}(\boldsymbol{x}) \right| \cdot \left\| \boldsymbol{\sigma}_{i}(\boldsymbol{x}) \otimes \boldsymbol{g}^{j}(\boldsymbol{x}) \otimes \boldsymbol{g}_{k}(\boldsymbol{x}) \right\|_{\mathcal{T}^{3}(\mathbb{R}^{m})}$$

代入简单张量的范数，便有

$$= \sum_{i,j,k=1}^{m} \left| \Phi^{i}_{j}{}^{k}(\boldsymbol{x}) \right| \cdot \left\| \boldsymbol{\sigma}_{i}(\lambda) \right\|_{\mathbb{R}^{m}} \cdot \left\| \boldsymbol{g}^{j}(\boldsymbol{x}) \right\|_{\mathbb{R}^{m}} \cdot \left\| \boldsymbol{g}_{k}(\boldsymbol{x}) \right\|_{\mathbb{R}^{m}}. \tag{1.32}$$

这几项中只有 $\left\| \boldsymbol{\sigma}_{i}(\lambda) \right\|_{\mathcal{T}^{3}(\mathbb{R}^{m})}$ 与 λ 有关. 于是

$$\lim_{\lambda \to 0} \frac{\left\| \Phi^{i}_{j}{}^{k}(\boldsymbol{x}) \, \boldsymbol{\sigma}_{i}(\lambda) \otimes \boldsymbol{g}^{j}(\boldsymbol{x}) \otimes \boldsymbol{g}_{k}(\boldsymbol{x}) \right\|_{\mathcal{T}^{3}(\mathbb{R}^{m})}}{\lambda}$$

$$= \sum_{i,j,k=1}^{m} \left| \Phi^{i}_{j}{}^{k}(\boldsymbol{x}) \right| \cdot \left\| \boldsymbol{g}^{j}(\boldsymbol{x}) \right\|_{\mathbb{R}^{m}} \cdot \left\| \boldsymbol{g}_{k}(\boldsymbol{x}) \right\|_{\mathbb{R}^{m}} \cdot \lim_{\lambda \to 0} \frac{\left\| \boldsymbol{\sigma}_{i}(\lambda) \right\|_{\mathbb{R}^{m}}}{\lambda}$$

根据定义，最后的极限为零，因此

$$= 0. \tag{1.33}$$

$\square$

类似地，其他七十多项也都是 λ 的一阶无穷小量. 而有限个无穷小量之和仍为无穷小量，于是 res. $\to 0$.

综上，我们有

$$\frac{\partial \boldsymbol{\Phi}}{\partial x^{\mu}}(\boldsymbol{x}) := \lim_{\lambda \to 0} \frac{\boldsymbol{\Phi}(\boldsymbol{x} + \lambda \, \boldsymbol{e}_{\mu}) - \boldsymbol{\Phi}(\boldsymbol{x})}{\lambda}$$

$$= \left(\frac{\partial \Phi^{i}_{j}{}^{k}}{\partial x^{\mu}} \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} + \Phi^{i}_{j}{}^{k} \frac{\partial \boldsymbol{g}_{i}}{\partial x^{\mu}} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} + \Phi^{i}_{j}{}^{k} \boldsymbol{g}_{i} \otimes \frac{\partial \boldsymbol{g}^{j}}{\partial x^{\mu}} \otimes \boldsymbol{g}_{k} + \Phi^{i}_{j}{}^{k} \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \frac{\partial \boldsymbol{g}_{k}}{\partial x^{\mu}} \right)(\boldsymbol{x}). \tag{1.34}$$

式中，$\partial \boldsymbol{g}_{i} / \partial x^{\mu}(\boldsymbol{x})$ 可以用 Christoffel 符号表示：

$$\frac{\partial \boldsymbol{g}_{i}}{\partial x^{\mu}}(\boldsymbol{x}) = \Gamma^{s}_{\mu i} \, \boldsymbol{g}_{s}(\boldsymbol{x}). \tag{1.35}$$

因此 (1.34) 式的第二项

$$\Phi^{i}_{j}{}^{k}(\boldsymbol{x}) \frac{\partial \boldsymbol{g}_{i}}{\partial x^{\mu}}(\boldsymbol{x}) \otimes \boldsymbol{g}^{j}(\boldsymbol{x}) \otimes \boldsymbol{g}_{k}(\boldsymbol{x})$$

$$= \Gamma^{s}_{\mu i} \, \Phi^{i}_{j}{}^{k}(\boldsymbol{x}) \, \boldsymbol{g}_{s}(\boldsymbol{x}) \otimes \boldsymbol{g}^{j}(\boldsymbol{x}) \otimes \boldsymbol{g}_{k}(\boldsymbol{x})$$

i 和 s 都是哑标，不妨进行一下交换：

$$= \Gamma^{i}_{\mu s} \, \Phi^{s}_{j}{}^{k}(\boldsymbol{x}) \, \boldsymbol{g}_{i}(\boldsymbol{x}) \otimes \boldsymbol{g}^{j}(\boldsymbol{x}) \otimes \boldsymbol{g}_{k}(\boldsymbol{x}). \tag{1.36}$$

同样，后面的两项也可进行类似的处理. 这样便有

$$\frac{\partial \boldsymbol{\Phi}}{\partial x^{\mu}}(\boldsymbol{x}) := \lim_{\lambda \to 0} \frac{\boldsymbol{\Phi}(\boldsymbol{x} + \lambda \, \boldsymbol{e}_{\mu}) - \boldsymbol{\Phi}(\boldsymbol{x})}{\lambda}$$

$$= \left[\left(\frac{\partial \Phi^i_{\ j}^{\ k}}{\partial x^\mu} + \Gamma^i_{\mu s} \Phi^s_{\ j}^{\ k} - \Gamma^s_{\mu j} \Phi^i_{\ s}^{\ k} + \Gamma^k_{\mu s} \Phi^i_{\ j}^{\ s} \right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right](\boldsymbol{x})$$

$$=: \nabla_\mu \Phi^i_{\ j}^{\ k}(\boldsymbol{x}) \, \boldsymbol{g}_i(\boldsymbol{x}) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x}), \tag{1.37}$$

式中，我们称 $\nabla_\mu \Phi^i_{\ j}^{\ k}(\boldsymbol{x}) \in \mathbb{R}$ 为张量分量 $\Phi^i_{\ j}^{\ k}(\boldsymbol{x})$ 相对于 x^μ 的**协变导数**，其定义为：

$$\nabla_\mu \Phi^i_{\ j}^{\ k}(\boldsymbol{x}) \triangleq \frac{\partial \Phi^i_{\ j}^{\ k}}{\partial x^\mu}(\boldsymbol{x}) + \Gamma^i_{\mu s} \Phi^s_{\ j}^{\ k}(\boldsymbol{x}) - \Gamma^s_{\mu j} \Phi^i_{\ s}^{\ k}(\boldsymbol{x}) + \Gamma^k_{\mu s} \Phi^i_{\ j}^{\ s}(\boldsymbol{x}). \tag{1.38}$$

1.3 张量场的梯度

1.3.1 梯度；可微性

我们在参数域中的内点 $\boldsymbol{x}_0$ 处取一个半径为 δ 的邻域 $\mathfrak{B}_\delta(\boldsymbol{x}_0)$. 若使自变量变化到 $\boldsymbol{x}_0 + \boldsymbol{h}$，则在物理域中，对应的点就将从 $\boldsymbol{X}(\boldsymbol{x}_0)$ 变化到 $\boldsymbol{X}(\boldsymbol{x}_0 + \boldsymbol{h})$. 考察定义在参数域 $\mathfrak{D}_x$ 上的张量场 $\boldsymbol{\Phi}(\boldsymbol{x})$，它的变化为

$$\boldsymbol{\Phi}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{\Phi}(\boldsymbol{x}_0)$$
$$= \Phi^i_{\ j}^{\ k}(\boldsymbol{x}_0 + \boldsymbol{h}) \, \boldsymbol{g}_i(\boldsymbol{x}_0 + \boldsymbol{h}) \otimes \boldsymbol{g}^j(\boldsymbol{x}_0 + \boldsymbol{h}) \otimes \boldsymbol{g}_k(\boldsymbol{x}_0 + \boldsymbol{h})$$
$$- \Phi^i_{\ j}^{\ k}(\boldsymbol{x}_0) \, \boldsymbol{g}_i(\boldsymbol{x}_0) \otimes \boldsymbol{g}^j(\boldsymbol{x}_0) \otimes \boldsymbol{g}_k(\boldsymbol{x}_0). \tag{1.39}$$

与之前一样将第一部分逐项展开，有

$$\begin{cases} \Phi^i_{\ j}^{\ k}(\boldsymbol{x}_0 + \boldsymbol{h}) = \Phi^i_{\ j}^{\ k}(\boldsymbol{x}_0) + \dfrac{\partial \Phi^i_{\ j}^{\ k}}{\partial x^\mu}(\boldsymbol{x}_0) \cdot h^\mu + o^i_{\ j}^{\ k}(\|\boldsymbol{h}\|_{\mathbb{R}^m}) \in \mathbb{R}, & \text{(1.40-a)} \\[2mm] \boldsymbol{g}_i(\boldsymbol{x}_0 + \boldsymbol{h}) = \boldsymbol{g}_i(\boldsymbol{x}_0) + \dfrac{\partial \boldsymbol{g}_i}{\partial x^\mu}(\boldsymbol{x}_0) \cdot h^\mu + \boldsymbol{o}_i(\|\boldsymbol{h}\|_{\mathbb{R}^m}) \in \mathbb{R}^m, & \text{(1.40-b)} \\[2mm] \boldsymbol{g}^j(\boldsymbol{x}_0 + \boldsymbol{h}) = \boldsymbol{g}^j(\boldsymbol{x}_0) + \dfrac{\partial \boldsymbol{g}^j}{\partial x^\mu}(\boldsymbol{x}_0) \cdot h^\mu + \boldsymbol{o}^j(\|\boldsymbol{h}\|_{\mathbb{R}^m}) \in \mathbb{R}^m, & \text{(1.40-c)} \\[2mm] \boldsymbol{g}_k(\boldsymbol{x}_0 + \boldsymbol{h}) = \boldsymbol{g}_k(\boldsymbol{x}_0) + \dfrac{\partial \boldsymbol{g}_k}{\partial x^\mu}(\boldsymbol{x}_0) \cdot h^\mu + \boldsymbol{o}_k(\|\boldsymbol{h}\|_{\mathbb{R}^m}) \in \mathbb{R}^m. & \text{(1.40-d)} \end{cases}$$

不要忘记对哑标 μ 进行求和.

不显含 $\boldsymbol{h}$ 的只有每行的第一项；它们组合起来，与式 (1.39) 的第二部分相互抵消. 再看 $\boldsymbol{h}$ 的一次项：[①]

$$\left[\frac{\partial \Phi^i_{\ j}^{\ k}}{\partial x^\mu} \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \Phi^i_{\ j}^{\ k} \left(\frac{\partial \boldsymbol{g}_i}{\partial x^\mu} \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \boldsymbol{g}_i \otimes \frac{\partial \boldsymbol{g}^j}{\partial x^\mu} \otimes \boldsymbol{g}_k + \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \frac{\partial \boldsymbol{g}_k}{\partial x^\mu} \right) \right](\boldsymbol{x}_0) \cdot h^\mu. \tag{1.41}$$

利用 Christoffel 符号又可以把它写成

$$\left[\frac{\partial \Phi^i_{\ j}^{\ k}}{\partial x^\mu} \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \Phi^i_{\ j}^{\ k} \left(\Gamma^s_{\mu i} \boldsymbol{g}_s \right) \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right.$$

$$\left. - \Phi^i_{\ j}^{\ k} \boldsymbol{g}_i \otimes \left(\Gamma^j_{\mu s} \boldsymbol{g}^s \right) \otimes \boldsymbol{g}_k + \Phi^i_{\ j}^{\ k} \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \left(\Gamma^s_{\mu k} \boldsymbol{g}_s \right) \right](\boldsymbol{x}_0) \cdot h^\mu$$

---

① 实际是 h^μ 的一次项. 别忘了求和.

$$
= \left(\frac{\partial \Phi^{i}{}_{j}{}^{k}}{\partial x^{\mu}}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \Gamma^{i}_{\mu s}\, \Phi^{s}{}_{j}{}^{k}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right.
$$

$$
\left. - \Gamma^{s}_{\mu j}\, \Phi^{i}{}_{s}{}^{k}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \Gamma^{k}_{\mu s}\, \Phi^{i}{}_{j}{}^{s}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right) \big(\boldsymbol{x}_0 \big) \cdot h^{\mu}
$$

$$
= \left[\left(\frac{\partial \Phi^{i}{}_{j}{}^{k}}{\partial x^{\mu}} + \Gamma^{i}_{\mu s}\, \Phi^{s}{}_{j}{}^{k} - \Gamma^{s}_{\mu j}\, \Phi^{i}{}_{s}{}^{k} + \Gamma^{k}_{\mu s}\, \Phi^{i}{}_{j}{}^{s} \right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right] \big(\boldsymbol{x}_0 \big) \cdot h^{\mu}. \tag{1.42}
$$

至于高阶项，它们都等于 $\mathscr{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m} \big) \in \mathscr{T}^3(\mathbb{R}^m)$，并且满足

$$
\lim_{\boldsymbol{h} \to \boldsymbol{0} \,\in\, \mathbb{R}^m} \frac{\big\| \mathscr{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m} \big) \big\|_{\mathscr{T}^3(\mathbb{R}^m)}}{\|\boldsymbol{h}\|_{\mathbb{R}^m}} = 0 \in \mathbb{R}. \tag{1.43}
$$

证明与 (1.31) 式类似.

整理一下，我们有

$$
\boldsymbol{\Phi}\big(\boldsymbol{x}_0 + \boldsymbol{h} \big) - \boldsymbol{\Phi}\big(\boldsymbol{x}_0 \big)
$$

$$
= \left[\left(\frac{\partial \Phi^{i}{}_{j}{}^{k}}{\partial x^{\mu}} + \Gamma^{i}_{\mu s}\, \Phi^{s}{}_{j}{}^{k} - \Gamma^{s}_{\mu j}\, \Phi^{i}{}_{s}{}^{k} + \Gamma^{k}_{\mu s}\, \Phi^{i}{}_{j}{}^{s} \right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right] \big(\boldsymbol{x}_0 \big) \cdot h^{\mu} + \mathscr{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m} \big)
$$

利用协变导数，有

$$
=: \left[\nabla_{\mu} \Phi^{i}{}_{j}{}^{k} \big(\boldsymbol{x}_0 \big)\, \boldsymbol{g}_i \big(\boldsymbol{x}_0 \big) \otimes \boldsymbol{g}^j \big(\boldsymbol{x}_0 \big) \otimes \boldsymbol{g}_k \big(\boldsymbol{x}_0 \big) \right] h^{\mu} + \mathscr{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m} \big). \tag{1.44}
$$

至此，从微分学的角度来看，任务已经完成. 但对于张量分析而言，我们还需要再做一点微小的工作. 简单张量部分再并上一个 $\boldsymbol{g}^{\mu}$，从而使张量升一阶；后面则改成 $h^{\nu} \boldsymbol{g}_{\nu}\big(\boldsymbol{x}_0 \big)$，并利用点乘保持总阶数不变：

$$
\left[\nabla_{\mu} \Phi^{i}{}_{j}{}^{k} \big(\boldsymbol{x}_0 \big)\, \boldsymbol{g}_i \big(\boldsymbol{x}_0 \big) \otimes \boldsymbol{g}^j \big(\boldsymbol{x}_0 \big) \otimes \boldsymbol{g}_k \big(\boldsymbol{x}_0 \big) \right] h^{\mu}
$$

$$
= \left[\nabla_{\mu} \Phi^{i}{}_{j}{}^{k} \big(\boldsymbol{x}_0 \big)\, \boldsymbol{g}_i \big(\boldsymbol{x}_0 \big) \otimes \boldsymbol{g}^j \big(\boldsymbol{x}_0 \big) \otimes \boldsymbol{g}_k \big(\boldsymbol{x}_0 \big) \otimes \boldsymbol{g}^{\mu} \big(\boldsymbol{x}_0 \big) \right] \cdot \left[h^{\nu} \boldsymbol{g}_{\nu} \big(\boldsymbol{x}_0 \big) \right]. \tag{1.45}
$$

所谓"点乘"，其实就是 e 点积在 $e = 1$ 时的情况. 实际上，

$$
\boldsymbol{g}^{\mu} \big(\boldsymbol{x}_0 \big) \cdot \left[h^{\nu} \boldsymbol{g}_{\nu} \big(\boldsymbol{x}_0 \big) \right] = h^{\nu} \delta^{\mu}_{\nu} = h^{\mu}. \tag{1.46}
$$

这里用到了局部基的对偶关系 (**??**) 式.

此时，我们获得了一个四阶张量与 $h^{\nu} \boldsymbol{g}_{\nu}\big(\boldsymbol{x}_0 \big)$ 的点积. 接下来讨论该项的意义. 参数域中 $\boldsymbol{x}_0$ 发生 $\boldsymbol{h} = h^i \boldsymbol{e}_i$ 的变化时，根据向量值映照 $\boldsymbol{X}(\boldsymbol{x})$ 的可微性，对应物理域中的变化为

$$
\boldsymbol{X}\big(\boldsymbol{x}_0 + \boldsymbol{h} \big) - \boldsymbol{X}\big(\boldsymbol{x}_0 \big) = \mathsf{D}\boldsymbol{X}\big(\boldsymbol{x}_0 \big)(\boldsymbol{h}) + \mathscr{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m} \big)
$$

$$
=: \frac{\partial \boldsymbol{X}}{\partial x^{\nu}}\big(\boldsymbol{x}_0 \big)\, h^{\nu} + \mathscr{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m} \big)
$$

代入 **??** 小节中局部协变基的定义，可有

$$
=: h^{\nu} \boldsymbol{g}_{\nu}\big(\boldsymbol{x}_0 \big) + \mathscr{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m} \big). \tag{1.47}
$$

代入式 (1.44)，有

$$
\boldsymbol{\Phi}\big(\boldsymbol{x}_0 + \boldsymbol{h}\big) - \boldsymbol{\Phi}\big(\boldsymbol{x}_0\big)
$$

$$
= \left[\nabla_\mu \boldsymbol{\Phi}^{i}_{\ j}{}^{k}\big(\boldsymbol{x}_0\big)\, \boldsymbol{g}_i\big(\boldsymbol{x}_0\big) \otimes \boldsymbol{g}^j\big(\boldsymbol{x}_0\big) \otimes \boldsymbol{g}_k\big(\boldsymbol{x}_0\big) \otimes \boldsymbol{g}^\mu\big(\boldsymbol{x}_0\big) \right]
$$

$$
\cdot \left[\boldsymbol{X}\big(\boldsymbol{x}_0 + \boldsymbol{h}\big) - \boldsymbol{X}\big(\boldsymbol{x}_0\big) + \mathcal{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m}\big) \right] + \mathcal{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m}\big)
$$

合并掉一阶无穷小量[①]，可得

$$
= \left[\nabla_\mu \boldsymbol{\Phi}^{i}_{\ j}{}^{k}\big(\boldsymbol{x}_0\big)\, \boldsymbol{g}_i\big(\boldsymbol{x}_0\big) \otimes \boldsymbol{g}^j\big(\boldsymbol{x}_0\big) \otimes \boldsymbol{g}_k\big(\boldsymbol{x}_0\big) \otimes \boldsymbol{g}^\mu\big(\boldsymbol{x}_0\big) \right]
$$

$$
\cdot \left[\boldsymbol{X}\big(\boldsymbol{x}_0 + \boldsymbol{h}\big) - \boldsymbol{X}\big(\boldsymbol{x}_0\big) \right] + \mathcal{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m}\big)
$$

$$
= \left[\frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}\big(\boldsymbol{x}_0\big) \otimes \boldsymbol{g}^\mu\big(\boldsymbol{x}_0\big) \right] \cdot \left[\boldsymbol{X}\big(\boldsymbol{x}_0 + \boldsymbol{h}\big) - \boldsymbol{X}\big(\boldsymbol{x}_0\big) \right] + \mathcal{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m}\big) \in \mathscr{T}^3(\mathbb{R}^m). \tag{1.48}
$$

引入记号

$$
\boldsymbol{\Phi}\big(\boldsymbol{x}_0\big) \otimes \left[\boldsymbol{g}^\mu \frac{\partial}{\partial x^\mu}\big(\boldsymbol{x}_0\big) \right] := \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}\big(\boldsymbol{x}_0\big) \otimes \boldsymbol{g}^\mu\big(\boldsymbol{x}_0\big), \tag{1.49}
$$

再引入**梯度算子**

$$
\boldsymbol{\nabla} := \boldsymbol{g}^\mu \frac{\partial}{\partial x^\mu}\big(\boldsymbol{x}_0\big), \tag{1.50}
$$

我们得到的结论就可以表述为

$$
\boldsymbol{\Phi}\big(\boldsymbol{x}_0 + \boldsymbol{h}\big) - \boldsymbol{\Phi}\big(\boldsymbol{x}_0\big) = \big(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\big)\big(\boldsymbol{x}_0\big) \cdot \left[\boldsymbol{X}\big(\boldsymbol{x}_0 + \boldsymbol{h}\big) - \boldsymbol{X}\big(\boldsymbol{x}_0\big) \right] + \mathcal{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m}\big) \in \mathscr{T}^3(\mathbb{R}^m), \tag{1.51}
$$

式中的 "·" 表示点乘. 以上结果称之为张量场的**可微性**，它表明，由一点处的位置移动所引起张量场的变化，可以用该点处张量场的**梯度**（即 $\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}$）点乘物理空间中的位置差别来近似，误差为一阶无穷小量. 以上分析基于三阶张量. 但显然，对于 p 阶张量，将会有完全一致的结果.

1.3.2 方向导数

现在来研究张量场沿 $\boldsymbol{e}$ 方向的变化率（设 $\|\boldsymbol{e}\|_{\mathbb{R}^m} = 1$）. 取一个与 $\boldsymbol{e}$ 平行的向量 $\lambda \boldsymbol{e}$. 注意到 $\lambda \boldsymbol{e}$ 其实就是物理空间中的位置变化，于是根据张量场的可微性，我们有

$$
\boldsymbol{\Phi}\big(\boldsymbol{x}_0 + \lambda \boldsymbol{e}\big) - \boldsymbol{\Phi}\big(\boldsymbol{x}_0\big) = \big(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\big)\big(\boldsymbol{x}_0\big) \cdot \big(\lambda \boldsymbol{e}\big) + \mathcal{o}(\lambda), \tag{1.52}
$$

该式等价于

$$
\lim_{\lambda \to 0} \frac{\boldsymbol{\Phi}\big(\boldsymbol{x}_0 + \lambda \boldsymbol{e}\big) - \boldsymbol{\Phi}\big(\boldsymbol{x}_0\big)}{\lambda} = \big(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\big)\big(\boldsymbol{x}_0\big) \cdot \boldsymbol{e}. \tag{1.53}
$$

我们把它定义为张量场 $\boldsymbol{\Phi}(\boldsymbol{x})$ 沿 $\boldsymbol{e}$ 方向的**方向导数**：

$$
\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{e}}(\boldsymbol{x}_0) \triangleq \big(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\big)\big(\boldsymbol{x}_0\big) \cdot \boldsymbol{e}. \tag{1.54}
$$

---

① 式中的两个 $\mathcal{o}\big(\|\boldsymbol{h}\|_{\mathbb{R}^m}\big)$ 是不同的，前者属于 $\mathbb{R}^m$，后者属于 $\mathscr{T}^3(\mathbb{R}^m)$.

1.3.3 左梯度与右梯度

我们已经知道，利用梯度算子

$$\nabla := g^{\mu} \frac{\partial}{\partial x^{\mu}}(x_0),\tag{1.55}$$

可以把张量场的梯度表示为

$$(\boldsymbol{\Phi} \otimes \nabla)(x_0) = \boldsymbol{\Phi}(x_0) \otimes \left[g^{\mu} \frac{\partial}{\partial x^{\mu}}(x_0) \right] := \frac{\partial \boldsymbol{\Phi}}{\partial x^{\mu}}(x_0) \otimes g^{\mu}(x_0).\tag{1.56}$$

"∇"在右边，故称之为**右梯度**（简称**梯度**）. 相应地，自然会有**左梯度**：

$$(\nabla \otimes \boldsymbol{\Phi})(x_0) = \left[g^{\mu} \frac{\partial}{\partial x^{\mu}}(x_0) \right] \otimes \boldsymbol{\Phi}(x_0) := g^{\mu}(x_0) \otimes \frac{\partial \boldsymbol{\Phi}}{\partial x^{\mu}}(x_0).\tag{1.57}$$

张量积不存在交换律，因而这两者是不同的. 注意，梯度运算将使张量的阶数增加一阶.

张量场的可微性可以用左梯度来等价表述：

$$\boldsymbol{\Phi}(x_0 + h) - \boldsymbol{\Phi}(x_0) = \left[X(x_0 + h) - X(x_0) \right] \cdot (\nabla \otimes \boldsymbol{\Phi})(x_0) + \mathcal{o}\left(\|h\|_{\mathbb{R}^m} \right).\tag{1.58}$$

类似地，还有方向导数：

$$\frac{\partial \boldsymbol{\Phi}}{\partial e}(x_0) \triangleq (\boldsymbol{\Phi} \otimes \nabla)(x_0) \cdot e = e \cdot (\nabla \otimes \boldsymbol{\Phi})(x_0).\tag{1.59}$$

1.4 场论恒等式（一）

为了给下一节做好铺垫，本节将证明几个重要引理.

1.4.1 Ricci 引理

首先来证明两个结论：

$$\begin{cases} \dfrac{\partial \boldsymbol{G}}{\partial x^{\mu}}(x) = \mathbf{0} \in \mathcal{T}^2(\mathbb{R}^m), & \text{(1.60-a)} \\[2mm] \dfrac{\partial \boldsymbol{\epsilon}}{\partial x^{\mu}}(x) = \mathbf{0} \in \mathcal{T}^3(\mathbb{R}^3), & \text{(1.60-b)} \end{cases}$$

其中的 $\boldsymbol{G}$ 和 $\boldsymbol{\epsilon}$ 分别是度量张量和 Eddington 张量.

证明： 为方便起见，证明中我们将省去"(x)".

先考察度量张量的偏导数：

$$\begin{aligned} \frac{\partial \boldsymbol{G}}{\partial x^{\mu}} &= \frac{\partial}{\partial x^{\mu}} \left(g_{ij} \, \boldsymbol{g}^i \otimes \boldsymbol{g}^j \right) \\ &= \nabla_{\mu} g_{ij} \, \boldsymbol{g}^i \otimes \boldsymbol{g}^j, \end{aligned}\tag{1.61}$$

式中，协变导数定义为

$$\nabla_{\mu} g_{ij} \triangleq \frac{\partial g_{ij}}{\partial x^{\mu}} - \Gamma^s_{\mu i} \, g_{sj} - \Gamma^s_{\mu j} \, g_{is}.\tag{1.62}$$

以下有两种方法证明 $\nabla_{\mu} g_{ij} = 0$.

方法一利用度量的定义：

$$\frac{\partial g_{ij}}{\partial x^{\mu}} \triangleq \frac{\partial}{\partial x^{\mu}} \left\langle \boldsymbol{g}_i, \boldsymbol{g}_j \right\rangle_{\mathbb{R}^m}$$

$$= \left\langle \frac{\partial \boldsymbol{g}_i}{\partial x^\mu}, \boldsymbol{g}_j \right\rangle_{\mathbb{R}^m} + \left\langle \boldsymbol{g}_i, \frac{\partial \boldsymbol{g}_j}{\partial x^\mu} \right\rangle_{\mathbb{R}^m}$$

根据 Christoffel 符号的定义（见 **??** 小节），有

$$= \Gamma_{\mu i, j} + \Gamma_{\mu j, i}. \tag{1.63}$$

另一方面，回忆 (**??**) 式：

$$\Gamma^k_{ij} = \Gamma_{ij, l} g^{kl}, \tag{1.64}$$

可有

$$\begin{cases} \Gamma^s_{\mu i} g_{sj} = \Gamma_{\mu i, k} g^{sk} g_{sj} = \Gamma_{\mu i, k} \delta^k_j = \Gamma_{\mu i, j}, & \text{(1.65-a)} \\ \Gamma^s_{\mu j} g_{is} = \Gamma_{\mu j, k} g^{sk} g_{is} = \Gamma_{\mu j, k} \delta^k_i = \Gamma_{\mu j, i}. & \text{(1.65-b)} \end{cases}$$

于是

$$\nabla_\mu g_{ij} = \Gamma_{\mu i, j} + \Gamma_{\mu j, i} - \Gamma_{\mu i, j} - \Gamma_{\mu j, i} = 0. \tag{1.66}$$

方法二则利用第一类 Christoffel 符号的性质 (**??**) 式：

$$\Gamma_{ij, k} = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right). \tag{1.67}$$

因而

$$\begin{aligned}
& \Gamma^s_{\mu i} g_{sj} + \Gamma^s_{\mu j} g_{is} \\
={} & \Gamma_{\mu i, j} + \Gamma_{\mu j, i} \\
={} & \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^\mu} + \frac{\partial g_{\mu j}}{\partial x^i} - \frac{\partial g_{\mu i}}{\partial x^j} \right) + \frac{1}{2} \left(\frac{\partial g_{ji}}{\partial x^\mu} + \frac{\partial g_{\mu i}}{\partial x^j} - \frac{\partial g_{\mu j}}{\partial x^i} \right) \\
={} & \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^\mu} + \frac{\partial g_{ji}}{\partial x^\mu} \right) = \frac{\partial g_{ij}}{\partial x^\mu}.
\end{aligned} \tag{1.68}$$

显然，立刻就有

$$\nabla_\mu g_{ij} = \frac{\partial g_{ij}}{\partial x^\mu} - \frac{\partial g_{ij}}{\partial x^\mu} = 0. \tag{1.69}$$

综上，因为 $\nabla_\mu g_{ij} = 0 \in \mathbb{R}$，所以

$$\frac{\partial \boldsymbol{G}}{\partial x^\mu}(\boldsymbol{x}) = \boldsymbol{0} \in \mathscr{T}^2(\mathbb{R}^m). \tag{1.70}$$

如果用其他形式的分量来表述这一结果，我们便有

$$\nabla_\mu g_{ij} = \nabla_\mu g^{ij} = \nabla_\mu \delta^i_j = 0. \tag{1.71}$$

此结论称为 **Ricci 引理**.

再来看 Eddington 张量的偏导数：

$$\begin{aligned}
\frac{\partial \boldsymbol{\epsilon}}{\partial x^\mu} &= \frac{\partial}{\partial x^\mu} \left(\epsilon^i{}_j{}^k \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right) \\
&= \nabla_\mu \epsilon^i{}_j{}^k \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k,
\end{aligned} \tag{1.72}$$

式中，

$$\nabla_\mu \epsilon^i_{\ j}^{\ k} \triangleq \frac{\partial \epsilon^i_{\ j}^{\ k}}{\partial x^\mu} + \Gamma^i_{\mu s} \epsilon^s_{\ j}^{\ k} - \Gamma^s_{\mu j} \epsilon^i_{\ s}^{\ k} + \Gamma^k_{\mu s} \epsilon^i_{\ j}^{\ s}. \tag{1.73}$$

根据定义，$\epsilon^i_{\ j}^{\ k} = \det\left[\boldsymbol{g}^i, \boldsymbol{g}_j, \boldsymbol{g}^k\right]$. 因此

$$\begin{aligned}
\frac{\partial \epsilon^i_{\ j}^{\ k}}{\partial x^\mu} &= \frac{\partial}{\partial x^\mu}\left(\det\left[\boldsymbol{g}^i, \boldsymbol{g}_j, \boldsymbol{g}^k\right]\right) \\
&= \det\left[\frac{\partial \boldsymbol{g}^i}{\partial x^\mu}, \boldsymbol{g}_j, \boldsymbol{g}^k\right] + \det\left[\boldsymbol{g}^i, \frac{\partial \boldsymbol{g}_j}{\partial x^\mu}, \boldsymbol{g}^k\right] + \det\left[\boldsymbol{g}^i, \boldsymbol{g}_j, \frac{\partial \boldsymbol{g}^k}{\partial x^\mu}\right]
\end{aligned}$$

利用标架运动方程，有

$$= \det\left[-\Gamma^i_{\mu s}\boldsymbol{g}^s, \boldsymbol{g}_j, \boldsymbol{g}^k\right] + \det\left[\boldsymbol{g}^i, \Gamma^s_{\mu j}\boldsymbol{g}_s, \boldsymbol{g}^k\right] + \det\left[\boldsymbol{g}^i, \boldsymbol{g}_j, -\Gamma^k_{\mu s}\boldsymbol{g}^s\right]$$

再利用行列式的线性性，提出系数：

$$= -\Gamma^i_{\mu s}\det\left[\boldsymbol{g}^s, \boldsymbol{g}_j, \boldsymbol{g}^k\right] + \Gamma^s_{\mu j}\det\left[\boldsymbol{g}^i, \boldsymbol{g}_s, \boldsymbol{g}^k\right] - \Gamma^k_{\mu s}\det\left[\boldsymbol{g}^i, \boldsymbol{g}_j, \boldsymbol{g}^s\right]$$

代回 Eddington 张量的定义，可得

$$= -\Gamma^i_{\mu s}\epsilon^s_{\ j}^{\ k} + \Gamma^s_{\mu j}\epsilon^i_{\ s}^{\ k} - \Gamma^k_{\mu s}\epsilon^i_{\ j}^{\ s}. \tag{1.74}$$

这与式 (1.73) 的后三项恰好抵消. 于是便有 $\nabla_\mu \epsilon^i_{\ j}^{\ k} = 0$. 进而

$$\frac{\partial \boldsymbol{\epsilon}}{\partial x^\mu}(\boldsymbol{x}) = \boldsymbol{0} \in \mathscr{T}^3\left(\mathbb{R}^3\right). \tag{1.75}$$

和度量张量类似，Eddington 张量其他分量的偏导数，如 $\nabla_\mu \epsilon_{ijk}$、$\nabla_\mu \epsilon^{ijk}$ 等，也都等于零. 此结论同样称为 **Ricci 引理**.   □

1.4.2 Leibniz 法则

协变导数满足 **Leibniz 法则**：

$$\nabla_\mu\left(\Phi^i_{\ j}^{\ k}\Psi_p^{\ q}\right) = \left(\nabla_\mu\Phi^i_{\ j}^{\ k}\right)\Psi_p^{\ q} + \Phi^i_{\ j}^{\ k}\left(\nabla_\mu\Psi_p^{\ q}\right). \tag{1.76}$$

式中，张量分量的形式可以是任意的.

证明： 显然，$\Phi^i_{\ j}^{\ k}\Psi_p^{\ q} \in \mathscr{T}^5(\mathbb{R}^m)$. 不妨令

$$\Omega^i_{\ j}^{\ k}_{\ p}^{\ q} = \Phi^i_{\ j}^{\ k}\Psi_p^{\ q}. \tag{1.77}$$

则

$$\begin{aligned}
\nabla_\mu\left(\Phi^i_{\ j}^{\ k}\Psi_p^{\ q}\right) &= \nabla_\mu\Omega^i_{\ j}^{\ k}_{\ p}^{\ q} \\
&\triangleq \frac{\partial \Omega^i_{\ j}^{\ k}_{\ p}^{\ q}}{\partial x^\mu} + \Gamma^i_{\mu s}\Omega^s_{\ j}^{\ k}_{\ p}^{\ q} - \Gamma^s_{\mu j}\Omega^i_{\ s}^{\ k}_{\ p}^{\ q} + \Gamma^k_{\mu s}\Omega^i_{\ j}^{\ s}_{\ p}^{\ q} - \Gamma^s_{\mu p}\Omega^i_{\ j}^{\ k}_{\ s}^{\ q} + \Gamma^q_{\mu s}\Omega^i_{\ j}^{\ k}_{\ p}^{\ s} \\
&= \frac{\partial}{\partial x^\mu}\left(\Phi^i_{\ j}^{\ k}\Psi_p^{\ q}\right) + \left(\Gamma^i_{\mu s}\Phi^s_{\ j}^{\ k} - \Gamma^s_{\mu j}\Phi^i_{\ s}^{\ k} + \Gamma^k_{\mu s}\Phi^i_{\ j}^{\ s}\right)\Psi_p^{\ q} + \Phi^i_{\ j}^{\ k}\left(-\Gamma^s_{\mu p}\Psi_s^{\ q} + \Gamma^q_{\mu s}\Psi_p^{\ s}\right). \tag{1.78}
\end{aligned}$$

第一项偏导数自然满足乘积法则：

$$\frac{\partial}{\partial x^\mu}\left(\varPhi^i_{\ j}{}^k \varPsi_p{}^q\right) = \frac{\partial \varPhi^i_{\ j}{}^k}{\partial x^\mu}\varPsi_p{}^q + \varPhi^i_{\ j}{}^k\frac{\partial \varPsi_p{}^q}{\partial x^\mu}. \tag{1.79}$$

代回前一式，即有

$$\nabla_\mu\left(\varPhi^i_{\ j}{}^k \varPsi_p{}^q\right) = \left(\frac{\partial \varPhi^i_{\ j}{}^k}{\partial x^\mu} + \varGamma^i_{\ \mu s}\varPhi^s_{\ j}{}^k - \varGamma^s_{\ \mu j}\varPhi^i_{\ s}{}^k + \varGamma^k_{\ \mu s}\varPhi^i_{\ j}{}^s\right)\varPsi_p{}^q + \varPhi^i_{\ j}{}^k\left(\frac{\partial \varPsi_p{}^q}{\partial x^\mu} - \varGamma^s_{\ \mu p}\varPsi_s{}^q + \varGamma^q_{\ \mu s}\varPsi_p{}^s\right)$$

$$\triangleq \left(\nabla_\mu\varPhi^i_{\ j}{}^k\right)\varPsi_p{}^q + \varPhi^i_{\ j}{}^k\left(\nabla_\mu\varPsi_p{}^q\right). \tag{1.80}$$

$\square$

现在来考虑 $\nabla_\mu\left(\varPhi^i_{\ j}{}^k \varPsi_k{}^q\right)$，注意其中的 k 是哑标．若按照 Leibniz 法则，似乎有

$$\nabla_\mu\left(\varPhi^i_{\ j}{}^k \varPsi_k{}^q\right) = \left(\nabla_\mu\varPhi^i_{\ j}{}^k\right)\varPsi_k{}^q + \varPhi^i_{\ j}{}^k\left(\nabla_\mu\varPsi_k{}^q\right)$$

$$= \left(\frac{\partial \varPhi^i_{\ j}{}^k}{\partial x^\mu} + \varGamma^i_{\ \mu s}\varPhi^s_{\ j}{}^k - \varGamma^s_{\ \mu j}\varPhi^i_{\ s}{}^k + \varGamma^k_{\ \mu s}\varPhi^i_{\ j}{}^s\right)\varPsi_k{}^q + \varPhi^i_{\ j}{}^k\left(\frac{\partial \varPsi_k{}^q}{\partial x^\mu} - \varGamma^s_{\ \mu k}\varPsi_s{}^q + \varGamma^q_{\ \mu s}\varPsi_k{}^s\right)$$

$$= \frac{\partial}{\partial x^\mu}\left(\varPhi^i_{\ j}{}^k \varPsi_k{}^q\right) + \left(\varGamma^i_{\ \mu s}\varPhi^s_{\ j}{}^k - \varGamma^s_{\ \mu j}\varPhi^i_{\ s}{}^k\right)\varPsi_k{}^q + \varPhi^i_{\ j}{}^k\left(\varGamma^q_{\ \mu s}\varPsi_k{}^s\right)$$

$$+ \varGamma^k_{\ \mu s}\varPhi^i_{\ j}{}^s\varPsi_k{}^q - \varGamma^s_{\ \mu k}\varPhi^i_{\ j}{}^k\varPsi_s{}^q. \tag{1.81}$$

而根据定义，则

$$\nabla_\mu\left(\varPhi^i_{\ j}{}^k \varPsi_k{}^q\right) \triangleq \frac{\partial}{\partial x^\mu}\left(\varPhi^i_{\ j}{}^k \varPsi_k{}^q\right) + \left(\varGamma^i_{\ \mu s}\varPhi^s_{\ j}{}^k - \varGamma^s_{\ \mu j}\varPhi^i_{\ s}{}^k\right)\varPsi_k{}^q + \varPhi^i_{\ j}{}^k\left(\varGamma^q_{\ \mu s}\varPsi_k{}^s\right). \tag{1.82}$$

很明显，式 (1.81) 中多了两项．不过稍作计算，就可知道

$$\varGamma^k_{\ \mu s}\varPhi^i_{\ j}{}^s\varPsi_k{}^q - \varGamma^s_{\ \mu k}\varPhi^i_{\ j}{}^k\varPsi_s{}^q$$

k 和 s 都是哑标，不妨在第二项中将二者交换：

$$= \varGamma^k_{\ \mu s}\varPhi^i_{\ j}{}^s\varPsi_k{}^q - \varGamma^k_{\ \mu s}\varPhi^i_{\ j}{}^s\varPsi_k{}^q = 0. \tag{1.83}$$

可见，Leibniz 法则经受住了考验．

把 Ricci 引理和 Leibniz 法则联合起来，便有

$$\begin{cases} \nabla_\mu\left(g_{ij}\varPsi^p{}_q\right) = \left(\nabla_\mu g_{ij}\right)\varPsi^p{}_q + g_{ij}\left(\nabla_\mu\varPsi^p{}_q\right) = g_{ij}\,\nabla_\mu\varPsi^p{}_q, & \text{(1.84-a)} \\[2mm] \nabla_\mu\left(\epsilon^i_{\ j}{}^k\varPsi^p{}_q\right) = \left(\nabla_\mu\epsilon^i_{\ j}{}^k\right)\varPsi^p{}_q + \epsilon^i_{\ j}{}^k\left(\nabla_\mu\varPsi^p{}_q\right) = \epsilon^i_{\ j}{}^k\,\nabla_\mu\varPsi^p{}_q. & \text{(1.84-b)} \end{cases}$$

这说明度量张量和 Eddington 张量类似常数，可以提到协变导数的外面．

1.4.3 混合协变导数

与混合偏导数定理类似，协变导数满足

$$\nabla_\nu\nabla_\mu\varPhi^i_{\ j}{}^k = \nabla_\mu\nabla_\nu\varPhi^i_{\ j}{}^k. \tag{1.85}$$

不必多说，张量分量依然可以任意选取．只是需要注意，该定理只在 <mark>体积上张量场场论</mark> 成立．

证明： 首先计算张量场整体的一阶偏导数：

$$\frac{\partial \boldsymbol{\Phi}}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\left(\Phi^{i\ k}_{\ j}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right) = \nabla_\mu \Phi^{i\ k}_{\ j}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \in \mathscr{T}^3(\mathbb{R}^m). \tag{1.86}$$

再求一次偏导数，可有

$$\frac{\partial^2 \boldsymbol{\Phi}}{\partial x^\nu \partial x^\mu} := \frac{\partial}{\partial x^\nu}\left(\frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}\right) = \frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^{i\ k}_{\ j}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right). \tag{1.87}$$

请注意，括号里的张量带有一个独立指标 μ. 按照极限分析，有

$$\frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^{i\ k}_{\ j}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right)$$

$$= \frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^{i\ k}_{\ j}\right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \nabla_\mu \Phi^{i\ k}_{\ j}\left(\frac{\partial \boldsymbol{g}_i}{\partial x^\nu} \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \boldsymbol{g}_i \otimes \frac{\partial \boldsymbol{g}^j}{\partial x^\nu} \otimes \boldsymbol{g}_k + \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \frac{\partial \boldsymbol{g}_k}{\partial x^\nu}\right)$$

$$= \frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^{i\ k}_{\ j}\right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \nabla_\mu \Phi^{i\ k}_{\ j}\left(\Gamma^s_{\nu i}\, \boldsymbol{g}_s \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k - \Gamma^j_{\nu s}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^s \otimes \boldsymbol{g}_k + \Gamma^s_{\nu k}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_s\right)$$

$$= \left[\frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^{i\ k}_{\ j}\right) + \Gamma^i_{\nu s} \nabla_\mu \Phi^{s\ k}_{\ j} - \Gamma^s_{\nu j} \nabla_\mu \Phi^{i\ k}_{\ s} + \Gamma^k_{\nu s} \nabla_\mu \Phi^{i\ s}_{\ j}\right] \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k. \tag{1.88}$$

但是 $\nabla_\mu \Phi^{i\ k}_{\ j}$ 本身带有 4 个指标，因而

$$\nabla_\nu\left(\nabla_\mu \Phi^{i\ k}_{\ j}\right) \triangleq \frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^{i\ k}_{\ j}\right) + \Gamma^i_{\nu s} \nabla_\mu \Phi^{s\ k}_{\ j} - \Gamma^s_{\nu j} \nabla_\mu \Phi^{i\ k}_{\ s} + \Gamma^k_{\nu s} \nabla_\mu \Phi^{i\ s}_{\ j} - \Gamma^s_{\nu \mu} \nabla_s \Phi^{i\ k}_{\ j}. \tag{1.89}$$

代入 (1.88) 式，可得

$$\frac{\partial^2 \boldsymbol{\Phi}}{\partial x^\nu \partial x^\mu} = \frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^{i\ k}_{\ j}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right) = \left(\nabla_\nu \nabla_\mu \Phi^{i\ k}_{\ j} + \Gamma^s_{\nu \mu} \nabla_s \Phi^{i\ k}_{\ j}\right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k. \tag{1.90}$$

同理，

$$\frac{\partial^2 \boldsymbol{\Phi}}{\partial x^\mu \partial x^\nu} = \left(\nabla_\mu \nabla_\nu \Phi^{i\ k}_{\ j} + \Gamma^s_{\mu \nu} \nabla_s \Phi^{i\ k}_{\ j}\right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k. \tag{1.91}$$

根据 Christoffel 符号的性质，

$$\Gamma^s_{\nu \mu} = \Gamma^s_{\mu \nu}; \tag{1.92}$$

而按照 一般赋范线性空间上的微分学，当张量场具有足够正则性时，成立

$$\frac{\partial^2 \boldsymbol{\Phi}}{\partial x^\nu \partial x^\mu} = \frac{\partial^2 \boldsymbol{\Phi}}{\partial x^\mu \partial x^\nu}. \tag{1.93}$$

这样就可得到

$$\nabla_\nu \nabla_\mu \Phi^{i\ k}_{\ j} = \nabla_\mu \nabla_\nu \Phi^{i\ k}_{\ j}. \tag{1.94}$$

$\square$

1.5　场论恒等式（二）

本节将给出微分形式张量场场论中的若干恒等式，以及它们的推演过程．

1.5.1 微分算子

在 1.3.3 小节中，我们已经定义了**左梯度**

$$(\boldsymbol{\nabla} \otimes \boldsymbol{\Phi})(\boldsymbol{x}) \triangleq \left[\boldsymbol{g}^\mu \frac{\partial}{\partial x^\mu}(\boldsymbol{x})\right] \otimes \boldsymbol{\Phi}(\boldsymbol{x}) := \boldsymbol{g}^\mu(\boldsymbol{x}) \otimes \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}(\boldsymbol{x}) \tag{1.95-a}$$

和（右）**梯度**

$$(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla})(\boldsymbol{x}) \triangleq \boldsymbol{\Phi}(\boldsymbol{x}) \otimes \left[\boldsymbol{g}^\mu \frac{\partial}{\partial x^\mu}(\boldsymbol{x})\right] := \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}(\boldsymbol{x}) \otimes \boldsymbol{g}^\mu(\boldsymbol{x}). \tag{1.95-b}$$

如果 $\boldsymbol{\Phi}(\boldsymbol{x})$ 是 r 阶张量，则左右梯度都是 $r+1$ 阶张量.

类似地，我们还可以定义**左散度**

$$(\boldsymbol{\nabla} \cdot \boldsymbol{\Phi})(\boldsymbol{x}) \triangleq \left[\boldsymbol{g}^\mu \frac{\partial}{\partial x^\mu}(\boldsymbol{x})\right] \cdot \boldsymbol{\Phi}(\boldsymbol{x}) := \boldsymbol{g}^\mu(\boldsymbol{x}) \cdot \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}(\boldsymbol{x}) \tag{1.96-a}$$

和**右散度**

$$(\boldsymbol{\Phi} \cdot \boldsymbol{\nabla})(\boldsymbol{x}) \triangleq \boldsymbol{\Phi}(\boldsymbol{x}) \cdot \left[\boldsymbol{g}^\mu \frac{\partial}{\partial x^\mu}(\boldsymbol{x})\right] := \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}(\boldsymbol{x}) \cdot \boldsymbol{g}^\mu(\boldsymbol{x}). \tag{1.96-b}$$

如果 $\boldsymbol{\Phi}(\boldsymbol{x})$ 是 r 阶张量，则左右散度都是 $r-1$ 阶张量.

当然，如果底空间是 $\mathbb{R}^3$，即 $\boldsymbol{\Phi}(\boldsymbol{x}) \in \mathscr{T}^r(\mathbb{R}^3)$，还不能忘了定义**左旋度**

$$(\boldsymbol{\nabla} \times \boldsymbol{\Phi})(\boldsymbol{x}) \triangleq \left[\boldsymbol{g}^\mu \frac{\partial}{\partial x^\mu}(\boldsymbol{x})\right] \times \boldsymbol{\Phi}(\boldsymbol{x}) := \boldsymbol{g}^\mu(\boldsymbol{x}) \times \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}(\boldsymbol{x}) \tag{1.97-a}$$

和**右旋度**

$$(\boldsymbol{\Phi} \times \boldsymbol{\nabla})(\boldsymbol{x}) \triangleq \boldsymbol{\Phi}(\boldsymbol{x}) \times \left[\boldsymbol{g}^\mu \frac{\partial}{\partial x^\mu}(\boldsymbol{x})\right] := \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}(\boldsymbol{x}) \times \boldsymbol{g}^\mu(\boldsymbol{x}). \tag{1.97-b}$$

旋度不改变张量的阶数，即左右旋度仍属于 $\mathscr{T}^r(\mathbb{R}^3)$.

左右梯度、散度和梯度都是张量场中常用的微分算子. 向量微积分中的梯度、散度和梯度，其实就是一阶张量的特殊情况.

1.5.2 推演举例

首先是为人熟知的"梯度场无旋，旋度场无源"：

$$\forall \boldsymbol{\Phi} \in \mathscr{T}^r(\mathbb{R}^3), \quad \text{有} \begin{cases} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \otimes \boldsymbol{\Phi}) = \boldsymbol{0} \in \mathscr{T}^{r+1}(\mathbb{R}^3), & \text{(1.98-a)} \\ \boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\Phi}) = \boldsymbol{0} \in \mathscr{T}^{r-1}(\mathbb{R}^3). & \text{(1.98-b)} \end{cases}$$

证明： 不失一般性，我们设 $\boldsymbol{\Phi}$ 是一个三阶张量. 代入上一小节中梯度和旋度的定义，可有

$$\begin{aligned} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \otimes \boldsymbol{\Phi}) &= \left(\boldsymbol{g}^\nu \frac{\partial}{\partial x^\nu}\right) \times \left(\boldsymbol{g}^\mu \otimes \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}\right) \\ &= \left(\boldsymbol{g}^\nu \frac{\partial}{\partial x^\nu}\right) \times \left(\nabla_\mu \Phi^i{}_j{}^k \boldsymbol{g}^\mu \otimes \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right) \\ &= \boldsymbol{g}^\nu \times \frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^i{}_j{}^k \boldsymbol{g}^\mu \otimes \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right) \end{aligned}$$

与 (1.88) 式不同，第二个括号中的 μ、i、j、k 都是哑标，所以偏导数可以直接用协变导数表示，而不会出现多余的 Christoffel 符号：

$$= \boldsymbol{g}^\nu \times \left[\left(\nabla_\nu \nabla_\mu \boldsymbol{\Phi}^{i\,k}_{\ j} \right) \boldsymbol{g}^\mu \otimes \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right]$$

按照叉乘的定义（见 **??** 节），$\boldsymbol{g}^\nu$ 将与构成简单张量的第一个基向量相乘，即

$$= \left(\nabla_\nu \nabla_\mu \boldsymbol{\Phi}^{i\,k}_{\ j} \right) (\boldsymbol{g}^\nu \times \boldsymbol{g}^\mu) \otimes \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k$$

利用 Levi-Civita 记号展开叉乘项，有

$$= \epsilon^{\mu\nu s} \left(\nabla_\nu \nabla_\mu \boldsymbol{\Phi}^{i\,k}_{\ j} \right) \boldsymbol{g}_s \otimes \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k. \tag{1.99}$$

考虑交换哑标 μ、ν，结果必然保持不变. 但是 Eddington 张量关于指标 $\mu\nu$ 反对称[①]，即 $\epsilon^{\mu\nu s} = -\epsilon^{\nu\mu s}$；另一方面，混合协变导数却又满足 $\nabla_\nu \nabla_\mu \boldsymbol{\Phi}^{i\,k}_{\ j} = \nabla_\mu \nabla_\nu \boldsymbol{\Phi}^{i\,k}_{\ j}$，因此总的结果将变为其相反数. 这样，结果只可能为零，即

$$\boldsymbol{\nabla} \times (\boldsymbol{\nabla} \otimes \boldsymbol{\Phi}) = \boldsymbol{0}. \tag{1.100}$$

同理，也可证明"旋度场无源"：

$$\begin{aligned}
\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\Phi}) &= \left(\boldsymbol{g}^\nu \frac{\partial}{\partial x^\nu} \right) \cdot \left(\boldsymbol{g}^\mu \times \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu} \right) \\
&= \left(\boldsymbol{g}^\nu \frac{\partial}{\partial x^\nu} \right) \cdot \left[\boldsymbol{g}^\mu \times \left(\nabla_\mu \boldsymbol{\Phi}^{i\,k}_{\ j} \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right) \right] \\
&= \boldsymbol{g}^\nu \cdot \frac{\partial}{\partial x^\nu} \left[\boldsymbol{g}^\mu \times \left(\nabla_\mu \boldsymbol{\Phi}^{i\,k}_{\ j} \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right) \right] \\
&= \boldsymbol{g}^\nu \cdot \frac{\partial}{\partial x^\nu} \left[\nabla_\mu \boldsymbol{\Phi}^{i\,k}_{\ j} \left(\boldsymbol{g}^\mu \times \boldsymbol{g}_i \right) \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right] \\
&= \boldsymbol{g}^\nu \cdot \frac{\partial}{\partial x^\nu} \left(\nabla_\mu \boldsymbol{\Phi}^{i\,k}_{\ j} \epsilon^{\mu\ s}_{\ i} \boldsymbol{g}_s \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right)
\end{aligned}$$

μ、s、i、j、k 全部是哑标，因此

$$= \boldsymbol{g}^\nu \cdot \left[\left(\nabla_\nu \nabla_\mu \boldsymbol{\Phi}^{i\,k}_{\ j} \epsilon^{\mu\ s}_{\ i} \right) \boldsymbol{g}_s \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right]$$

点乘之后出来 Kronecker δ：

$$\begin{aligned}
&= \left(\nabla_\nu \nabla_\mu \boldsymbol{\Phi}^{i\,k}_{\ j} \epsilon^{\mu\ s}_{\ i} \right) \delta^\nu_s \boldsymbol{g}^j \otimes \boldsymbol{g}_k \\
&= \epsilon^{\mu\ \nu}_{\ i} \left(\nabla_\nu \nabla_\mu \boldsymbol{\Phi}^{i\,k}_{\ j} \right) \boldsymbol{g}^j \otimes \boldsymbol{g}_k. \tag{1.101}
\end{aligned}$$

交换哑标 μ、ν，仿上，便有

$$\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} \times \boldsymbol{\Phi}) = \boldsymbol{0}. \tag{1.102}$$

$\square$

接下来回忆一下向量场旋度的复合

$$\forall \boldsymbol{A} \in \mathbb{R}^3, \quad \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla} (\boldsymbol{\nabla} \cdot \boldsymbol{A}) - \boldsymbol{\nabla}^2 \boldsymbol{A} \in \mathbb{R}^3, \tag{1.103}$$

[①] 根据式 (**??**)，Levi-Civita 记号由行列式定义，而行列式交换两列将改变符号.

式中，∇^2 称为 **Laplace 算子**，其定义为

$$\nabla^2 \boldsymbol{A} \triangleq \nabla \cdot \left(\nabla \otimes \boldsymbol{A}\right). \tag{1.104}$$

推广到张量场上，我们有如下两式：

$$\forall \boldsymbol{\Phi} \in \mathscr{T}^r\left(\mathbb{R}^3\right), \quad \begin{cases} \nabla \times \left(\nabla \times \boldsymbol{\Phi}\right) = \nabla \otimes \left(\nabla \cdot \boldsymbol{\Phi}\right) - \nabla^2 \boldsymbol{\Phi}, & \text{(1.105-a)} \\[2mm] \left(\boldsymbol{\Phi} \times \nabla\right) \times \nabla = \left(\boldsymbol{\Phi} \cdot \nabla\right) \otimes \nabla - \boldsymbol{\Phi}\nabla^2, & \text{(1.105-b)} \end{cases}$$

其中，Laplace 算子的定义为

$$\begin{cases} \nabla^2 \boldsymbol{\Phi} \triangleq \nabla \cdot \left(\nabla \otimes \boldsymbol{\Phi}\right), & \text{(1.106-a)} \\[2mm] \boldsymbol{\Phi}\nabla^2 \triangleq \left(\boldsymbol{\Phi} \otimes \nabla\right) \cdot \nabla. & \text{(1.106-b)} \end{cases}$$

证明： 同样，我们以三阶张量为例进行计算. 代入旋度的定义，有

$$\begin{aligned}
\nabla \times \left(\nabla \times \boldsymbol{\Phi}\right) &= \left(\boldsymbol{g}^\nu \frac{\partial}{\partial x^\nu}\right) \times \left(\boldsymbol{g}^\mu \times \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}\right) \\
&= \boldsymbol{g}^\nu \times \frac{\partial}{\partial x^\nu}\left[\boldsymbol{g}^\mu \times \left(\nabla_\mu \Phi^{i\ k}_{\ j}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right)\right] \\
&= \boldsymbol{g}^\nu \times \frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^{i\ k}_{\ j}\, \epsilon^{\mu\ s}_{\ i}\, \boldsymbol{g}_s \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right) \\
&= \boldsymbol{g}^\nu \times \left[\left(\nabla_\nu \nabla_\mu \Phi^{i\ k}_{\ j}\, \epsilon^{\mu\ s}_{\ i}\right) \boldsymbol{g}_s \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right] \\
&= \epsilon^{\mu\ s}_{\ i}\, \epsilon^{\nu\ t}_{\ s}\left(\nabla_\nu \nabla_\mu \Phi^{i\ k}_{\ j}\right) \boldsymbol{g}_t \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k
\end{aligned}$$

为了使用"前前后后，里里外外"之法则（见 **??** 小节），需要交换第二个 Eddington 张量的指标 s 挪到最后，不要忘了添上负号：

$$= -\epsilon^{\mu\ s}_{\ i}\, \epsilon^{\nu t}_{\ \ s}\left(\nabla_\nu \nabla_\mu \Phi^{i\ k}_{\ j}\right) \boldsymbol{g}_t \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k$$

这样就可以顺利用上口诀：

$$= -\left(g^{\mu\nu}\delta^t_i - \delta^\nu_i g^{\mu t}\right)\left(\nabla_\nu \nabla_\mu \Phi^{i\ k}_{\ j}\right) \boldsymbol{g}_t \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k. \tag{1.107}$$

接下来，形式上引入**逆变导数**

$$\nabla^\mu := g^{\mu\nu}\nabla_\nu, \tag{1.108}$$

则上式可化为

$$\begin{aligned}
\nabla \times \left(\nabla \times \boldsymbol{\Phi}\right) &= \delta^\nu_i g^{\mu t}\left(\nabla_\nu \nabla_\mu \Phi^{i\ k}_{\ j}\right) \boldsymbol{g}_t \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k - g^{\mu\nu}\delta^t_i\left(\nabla_\nu \nabla_\mu \Phi^{i\ k}_{\ j}\right) \boldsymbol{g}_t \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \\
&= \left(\nabla_i \nabla^t \Phi^{i\ k}_{\ j}\right) \boldsymbol{g}_t \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k - \left(\nabla^\mu \nabla_\mu \Phi^{i\ k}_{\ j}\right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k. \tag{1.109}
\end{aligned}$$

等式右边的第一项为

$$\begin{aligned}
\nabla \otimes \left(\nabla \cdot \boldsymbol{\Phi}\right) &= \left(\boldsymbol{g}^\nu \frac{\partial}{\partial x^\nu}\right) \otimes \left(\boldsymbol{g}^\mu \cdot \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}\right) \\
&= \boldsymbol{g}^\nu \otimes \frac{\partial}{\partial x^\nu}\left[\boldsymbol{g}^\mu \cdot \left(\nabla_\mu \Phi^{i\ k}_{\ j}\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right)\right]
\end{aligned}$$

$$= \boldsymbol{g}^{\nu} \otimes \frac{\partial}{\partial x^{\nu}} \left(\nabla_{\mu} \boldsymbol{\Phi}^{i\ k}_{\ j} \delta^{\mu}_{i} \, \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \right)$$

$$= \boldsymbol{g}^{\nu} \otimes \left[\left(\nabla_{\nu} \nabla_{\mu} \boldsymbol{\Phi}^{i\ k}_{\ j} \delta^{\mu}_{i} \right) \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \right]$$

$$= \delta^{\mu}_{i} g^{\nu t} \left(\nabla_{\nu} \nabla_{\mu} \boldsymbol{\Phi}^{i\ k}_{\ j} \right) \boldsymbol{g}_{t} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k}$$

$$= \left(\nabla^{t} \nabla_{i} \boldsymbol{\Phi}^{i\ k}_{\ j} \right) \boldsymbol{g}_{t} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k}$$

根据式 (1.85)，协变导数可以交换顺序；而逆变导数无非就是利用度量玩了下"指标升降游戏"，理应可以交换：

$$= \left(\nabla_{i} \nabla^{t} \boldsymbol{\Phi}^{i\ k}_{\ j} \right) \boldsymbol{g}_{t} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k}, \tag{1.110}$$

而第二项为

$$\boldsymbol{\nabla}^{2} \boldsymbol{\Phi} \triangleq \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \otimes \boldsymbol{\Phi} \right) = \left(\boldsymbol{g}^{\nu} \frac{\partial}{\partial x^{\nu}} \right) \cdot \left(\boldsymbol{g}^{\mu} \otimes \frac{\partial \boldsymbol{\Phi}}{\partial x^{\mu}} \right)$$

$$= \boldsymbol{g}^{\nu} \cdot \frac{\partial}{\partial x^{\nu}} \left[\boldsymbol{g}^{\mu} \otimes \left(\nabla_{\mu} \boldsymbol{\Phi}^{i\ k}_{\ j} \, \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \right) \right]$$

$$= \boldsymbol{g}^{\nu} \cdot \frac{\partial}{\partial x^{\nu}} \left(\nabla_{\mu} \boldsymbol{\Phi}^{i\ k}_{\ j} \, \boldsymbol{g}^{\mu} \otimes \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \right)$$

$$= \boldsymbol{g}^{\nu} \cdot \left[\left(\nabla_{\nu} \nabla_{\mu} \boldsymbol{\Phi}^{i\ k}_{\ j} \right) \boldsymbol{g}^{\mu} \otimes \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \right]$$

$$= g^{\mu\nu} \left(\nabla_{\nu} \nabla_{\mu} \boldsymbol{\Phi}^{i\ k}_{\ j} \right) \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k}$$

$$= \left(\nabla^{\mu} \nabla_{\mu} \boldsymbol{\Phi}^{i\ k}_{\ j} \right) \boldsymbol{g}_{u} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k}. \tag{1.111}$$

将它们与 (1.109) 式比较，可得

$$\boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \boldsymbol{\Phi} \right) = \boldsymbol{\nabla} \otimes \left(\boldsymbol{\nabla} \cdot \boldsymbol{\Phi} \right) - \boldsymbol{\nabla}^{2} \boldsymbol{\Phi}. \tag{1.112}$$

式 (1.105-b) 可以完全类似地证明，此处不再赘述. $\qquad\square$