第一章　张量场可微性

1.1　张量的范数

1.1.1　赋范线性空间

对于一个**线性空间** $\mathscr{V}$，它总是定义了**线性结构**：

$$\forall\, \boldsymbol{x},\, \boldsymbol{y} \in \mathscr{V} \text{ 和 } \forall\, \alpha,\, \beta \in \mathbb{R}, \quad \alpha\, \boldsymbol{x} + \beta\, \boldsymbol{y} \in \mathscr{V}. \tag{1.1}$$

为了进一步研究的需要，我们还要引入**范数**的概念．所谓“范数”，就是对线性空间中任意元素大小的一种刻画．举个我们熟悉的例子，m 维 Euclid 空间 $\mathbb{R}^m$ 中某个向量的范数，就定义为该向量在 Descartes 坐标下各分量的平方和的平方根．

一般而言，线性空间 $\mathscr{V}$ 中的范数 $\|\cdot\|_{\mathscr{V}}$ 是从 $\mathscr{V}$ 到 $\mathbb{R}$ 的一个映照，并且需要满足以下三个条件：

1. **非负性**

$$\forall\, \boldsymbol{x} \in \mathscr{V}, \quad \|\boldsymbol{x}\|_{\mathscr{V}} \geqslant 0 \tag{1.2}$$

以及**非退化性**

$$\forall\, \boldsymbol{x} \in \mathscr{V}, \quad \|\boldsymbol{x}\|_{\mathscr{V}} = 0 \iff \boldsymbol{x} = \mathbf{0} \in \mathscr{V}, \tag{1.3}$$

这里的 $\mathbf{0}$ 是线性空间 $\mathscr{V}$ 中的**零元素**，它是唯一存在的．

2. 零元是唯一的，线性空间中的元素 $\boldsymbol{x}$ 与从 $\mathbf{0}$ 指向它的向量一一对应．因此，线性空间中的元素也常被称为“向量”．

考虑线性空间中的数乘运算．从几何上看，$\boldsymbol{x}$ 乘上 λ，就是将 $\boldsymbol{x}$ 沿着原来的指向进行伸缩．显然有

$$\forall\, \boldsymbol{x} \in \mathscr{V} \text{ 和 } \forall\, \lambda \in \mathbb{R}, \quad \|\lambda\, \boldsymbol{x}\|_{\mathscr{V}} = |\lambda| \cdot \|\boldsymbol{x}\|_{\mathscr{V}}, \tag{1.4}$$

这称为**正齐次性**．

想要图吗？

3. 线性空间中的加法满足*平行四边形法则*．直观地看，就有

$$\forall\, \boldsymbol{x},\, \boldsymbol{y} \in \mathscr{V}, \quad \|\boldsymbol{x} + \boldsymbol{y}\|_{\mathscr{V}} \leqslant \|\boldsymbol{x}\|_{\mathscr{V}} + \|\boldsymbol{y}\|_{\mathscr{V}}, \tag{1.5}$$

这称为**三角不等式**．

定义了范数的线性空间称为**赋范线性空间**．

1.1.2 张量范数的定义

考虑 p 阶张量 $\boldsymbol{\Phi} \in \mathscr{T}^p(\mathbb{R}^m)$，它可以用逆变分量或协变分量来表示：

$$\boldsymbol{\Phi} = \begin{cases} \Phi^{i_1\cdots i_p}\, \boldsymbol{g}_{i_1} \otimes \cdots \otimes \boldsymbol{g}_{i_p}, & \text{(1.6-a)} \\ \Phi_{i_1\cdots i_p}\, \boldsymbol{g}^{i_1} \otimes \cdots \otimes \boldsymbol{g}^{i_p}, & \text{(1.6-b)} \end{cases}$$

其中

$$\begin{cases} \Phi^{i_1\cdots i_p} = \boldsymbol{\Phi}\left(\boldsymbol{g}^{i_1}, \cdots, \boldsymbol{g}^{i_p}\right). & \text{(1.7-a)} \\ \Phi_{i_1\cdots i_p} = \boldsymbol{\Phi}\left(\boldsymbol{g}_{i_1}, \cdots, \boldsymbol{g}_{i_p}\right), & \text{(1.7-b)} \end{cases}$$

张量的**范数**定义为

$$\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)} \triangleq \sqrt{\Phi^{i_1\cdots i_p}\Phi_{i_1\cdots i_p}} \in \mathbb{R}. \tag{1.8}$$

$i_1 \cdots i_p$ 可独立取值，每个又有 m 种取法，所以根号下共有 m^p 项. 注意 $\Phi^{i_1\cdots i_p}$ 与 $\Phi_{i_1\cdots i_p}$ 未必相等，因而根号下的部分未必是平方和，这与 Euclid 空间中向量的模是不同的.

复习一下 **??** 小节，我们可以用另一组（带括号的）基表示张量 $\boldsymbol{\Phi}$：

$$\begin{cases} \Phi^{i_1\cdots i_p} = c^{i_1}_{(\xi_1)} \cdots c^{i_p}_{(\xi_p)} \Phi^{(\xi_1)\cdots(\xi_p)}, & \text{(1.9-a)} \\ \Phi_{i_1\cdots i_p} = c^{(\eta_1)}_{i_1} \cdots c^{(\eta_p)}_{i_p} \Phi_{(\eta_1)\cdots(\eta_p)}, & \text{(1.9-b)} \end{cases}$$

其中的 $c^i_{(\xi)} = \left\langle \boldsymbol{g}_{(\xi)}, \boldsymbol{g}^i \right\rangle_{\mathbb{R}^m}$，$c^{(\eta)}_i = \left\langle \boldsymbol{g}^{(\eta)}, \boldsymbol{g}_i \right\rangle_{\mathbb{R}^m}$，它们满足

$$c^i_{(\xi)} c^{(\eta)}_i = \delta^{(\eta)}_{(\xi)}. \tag{1.10}$$

于是

$$\begin{aligned} & \Phi^{i_1\cdots i_p}\Phi_{i_1\cdots i_p} \\ = & \left(c^{i_1}_{(\xi_1)} \cdots c^{i_p}_{(\xi_p)} \Phi^{(\xi_1)\cdots(\xi_p)}\right)\left(c^{(\eta_1)}_{i_1} \cdots c^{(\eta_p)}_{i_p} \Phi_{(\eta_1)\cdots(\eta_p)}\right) \\ = & \left(c^{i_1}_{(\xi_1)} c^{(\eta_1)}_{i_1}\right) \cdots \left(c^{i_p}_{(\xi_p)} c^{(\eta_p)}_{i_p}\right) \Phi^{(\xi_1)\cdots(\xi_p)}\Phi_{(\eta_1)\cdots(\eta_p)} \\ = & \delta^{(\eta_1)}_{(\xi_1)} \cdots \delta^{(\eta_p)}_{(\xi_p)} \Phi^{(\xi_1)\cdots(\xi_p)}\Phi_{(\eta_1)\cdots(\eta_p)} \\ = & \Phi^{(\xi_1)\cdots(\xi_p)}\Phi_{(\xi_1)\cdots(\xi_p)}. \end{aligned} \tag{1.11}$$

它是 $\boldsymbol{\Phi}$ 在另一组基下的逆变分量与协变分量乘积之和.

以上结果说明，张量的范数不依赖于基的选取，这就好比用不同的秤来称同一个人的体重，都将获得相同的结果. 既然如此，不妨采用单位正交基来表示张量的范数：

$$\begin{aligned} \|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)} & \triangleq \sqrt{\Phi^{i_1\cdots i_p}\Phi_{i_1\cdots i_p}} \\ & = \sqrt{\Phi^{\langle i_1\rangle\cdots\langle i_p\rangle}\Phi_{\langle i_1\rangle\cdots\langle i_p\rangle}} \\ & =: \sqrt{\sum\nolimits_{i_1,\cdots,i_p=1}^{m} \left(\Phi_{\langle i_1,\cdots,i_p\rangle}\right)^2}. \end{aligned} \tag{1.12}$$

这里的 $\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}$ 表示张量 $\boldsymbol{\Phi}$ 在单位正交基下的分量，它的指标不区分上下．

有了这样的表示，很容易就可以验证张量范数符合之前的三个要求．一组数的平方和开根号，必然是非负的．至于非退化性，若范数为零，则所有分量均为零，自然成为零张量；反之，对于零张量，所有分量为零，范数也为零．将 $\boldsymbol{\Phi}$ 乘上 λ，则有

$$\begin{aligned}\|\lambda\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)} &= \sqrt{\sum\nolimits_{i_1,\cdots,i_p=1}^{m}\left(\lambda\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\right)^2} \\ &= \sqrt{\lambda^2\sum\nolimits_{i_1,\cdots,i_p=1}^{m}\left(\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\right)^2} \\ &= |\lambda|\sqrt{\sum\nolimits_{i_1,\cdots,i_p=1}^{m}\left(\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\right)^2} \\ &= |\lambda|\cdot\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)},\end{aligned}\tag{1.13}$$

于是正齐次性也得以验证．最后，利用 Cauchy–Schwarz 不等式，可有

$$\begin{aligned}&\|\boldsymbol{\Phi}+\boldsymbol{\Psi}\|^2_{\mathscr{T}^p(\mathbb{R}^m)} \\ =&\sum\left(\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}+\boldsymbol{\Psi}_{\langle i_1,\cdots,i_p\rangle}\right)^2 \\ =&\sum\left[\left(\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\right)^2+2\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\boldsymbol{\Psi}_{\langle i_1,\cdots,i_p\rangle}+\left(\boldsymbol{\Psi}_{\langle i_1,\cdots,i_p\rangle}\right)^2\right] \\ =&\sum\left(\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\right)^2+2\sum\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\boldsymbol{\Psi}_{\langle i_1,\cdots,i_p\rangle}+\sum\left(\boldsymbol{\Psi}_{\langle i_1,\cdots,i_p\rangle}\right)^2 \\ \leqslant&\|\boldsymbol{\Phi}\|^2_{\mathscr{T}^p(\mathbb{R}^m)}+2\sqrt{\sum\left(\boldsymbol{\Phi}_{\langle i_1,\cdots,i_p\rangle}\right)^2}\sqrt{\sum\left(\boldsymbol{\Psi}_{\langle i_1,\cdots,i_p\rangle}\right)^2}+\|\boldsymbol{\Phi}\|^2_{\mathscr{T}^p(\mathbb{R}^m)} \\ =&\|\boldsymbol{\Phi}\|^2_{\mathscr{T}^p(\mathbb{R}^m)}+2\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)}\cdot\|\boldsymbol{\Psi}\|_{\mathscr{T}^p(\mathbb{R}^m)}+\|\boldsymbol{\Phi}\|^2_{\mathscr{T}^p(\mathbb{R}^m)} \\ =&\left(\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)}+\|\boldsymbol{\Phi}\|_{\mathscr{T}^p(\mathbb{R}^m)}\right)^2.\end{aligned}\tag{1.14}$$

两边开方，即为三角不等式．

由此，我们就完整地给出了张量大小的刻画手段．可以看出，它实际上就是 Euclid 空间中向量模的直接推广．

1.1.3 简单张量的范数

根据 **??** 小节中的定义，简单张量是形如 $\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta}$ 的张量，其中的 $\boldsymbol{\xi},\boldsymbol{\eta},\boldsymbol{\zeta}\in\mathbb{R}^m$，它是三个向量的张量积．简单张量的范数为

$$\|\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta}\|_{\mathscr{T}^3(\mathbb{R}^m)}=\|\boldsymbol{\xi}\|_{\mathbb{R}^m}\cdot\|\boldsymbol{\eta}\|_{\mathbb{R}^m}\cdot\|\boldsymbol{\zeta}\|_{\mathbb{R}^m}.\tag{1.15}$$

证明： $\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta}$ 的逆变分量为

$$(\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta})^{ijk}\triangleq\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta}\left(\boldsymbol{g}^i,\boldsymbol{g}^j,\boldsymbol{g}^k\right)=\xi^i\eta^j\zeta^k.\tag{1.16}$$

同理，它的协变分量为

$$(\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta})_{ijk}\triangleq\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta}\left(\boldsymbol{g}_i,\boldsymbol{g}_j,\boldsymbol{g}_k\right)=\xi_i\eta_j\zeta_k.\tag{1.17}$$

二者相乘，有

$$
\begin{aligned}
&(\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta})^{ijk}\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta})_{ijk}\\
&=\left(\xi^i\eta^j\zeta^k\right)\cdot\left(\xi_i\eta_j\zeta_k\right)\\
&=\left(\xi^i\xi_i\right)\cdot\left(\eta^j\eta_j\right)\cdot\left(\zeta^k\zeta_k\right).
\end{aligned}
\tag{1.18}
$$

注意到

$$
\|\xi\|^2_{\mathbb{R}^m}=\langle\xi,\xi\rangle_{\mathbb{R}^m}
$$

分别把二者用协变和逆变分量表示：

$$
\begin{aligned}
&=\left\langle\xi^i\boldsymbol{g}_i,\xi_j\boldsymbol{g}^j\right\rangle_{\mathbb{R}^m}\\
&=\xi^i\xi_j\left\langle\boldsymbol{g}_i,\boldsymbol{g}^j\right\rangle_{\mathbb{R}^m}\\
&=\xi^i\xi_j\delta_i^j=\xi^i\xi_i,
\end{aligned}
\tag{1.19}
$$

于是

$$
(\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta})^{ijk}\cdot(\boldsymbol{\xi}\otimes\boldsymbol{\eta}\otimes\boldsymbol{\zeta})_{ijk}=\|\xi\|^2_{\mathbb{R}^m}\cdot\|\boldsymbol{\eta}\|^2_{\mathbb{R}^m}\cdot\|\boldsymbol{\zeta}\|^2_{\mathbb{R}^m}.
\tag{1.20}
$$

两边开方，即得 (1.15) 式. □

1.2 张量场的偏导数；协变导数

在区域 $\mathfrak{D}_x\subset\mathbb{R}^m$ 上，若存在一个自变量用位置刻画的映照

$$
\boldsymbol{\Phi}:\ \mathfrak{D}_x\ni\boldsymbol{x}\mapsto\boldsymbol{\Phi}(\boldsymbol{x})\in\mathscr{T}^r(\mathbb{R}^m),
\tag{1.21}
$$

则称张量 $\boldsymbol{\Phi}(\boldsymbol{x})$ [①] 是定义在 $\mathfrak{D}_x$ 上的一个**张量场**.

下面我们以三阶张量为例. 设在物理域 $\mathfrak{D}_X\subset\mathbb{R}^m$ 和参数域 $\mathfrak{D}_x\subset\mathbb{R}^m$ 之间已经建立了微分同胚 $\boldsymbol{X}(\boldsymbol{x})\in\mathscr{C}^p\left(\mathfrak{D}_x;\mathfrak{D}_X\right)$. 在 $\boldsymbol{X}(\boldsymbol{x})$ 处，张量场 $\boldsymbol{\Phi}(\boldsymbol{x})$ 可以用分量形式表示为

$$
\boldsymbol{\Phi}(\boldsymbol{x})=\Phi^i{}_j{}^k(\boldsymbol{x})\,\boldsymbol{g}_i(\boldsymbol{x})\otimes\boldsymbol{g}^j(\boldsymbol{x})\otimes\boldsymbol{g}_k(\boldsymbol{x})\in\mathscr{T}^3(\mathbb{R}^m),
\tag{1.22}
$$

其中的 $\boldsymbol{g}_i(\boldsymbol{x}),\boldsymbol{g}^j(\boldsymbol{x}),\boldsymbol{g}_k(\boldsymbol{x})$ 都是局部基，而张量分量则定义为[②]

$$
\Phi^i{}_j{}^k(\boldsymbol{x})\triangleq\boldsymbol{\Phi}(\boldsymbol{x})\left[\boldsymbol{g}_i(\boldsymbol{x}),\boldsymbol{g}^j(\boldsymbol{x}),\boldsymbol{g}_k(\boldsymbol{x})\right]\in\mathbb{R}.
\tag{1.23}
$$

类似地，当点沿着 x^μ-线运动到 $\boldsymbol{X}\left(\boldsymbol{x}+\lambda\,\boldsymbol{e}_\mu\right)$ 处时，有

$$
\boldsymbol{\Phi}\left(\boldsymbol{x}+\lambda\,\boldsymbol{e}_\mu\right)=\Phi^i{}_j{}^k\left(\boldsymbol{x}+\lambda\,\boldsymbol{e}_\mu\right)\boldsymbol{g}_i\left(\boldsymbol{x}+\lambda\,\boldsymbol{e}_\mu\right)\otimes\boldsymbol{g}^j\left(\boldsymbol{x}+\lambda\,\boldsymbol{e}_\mu\right)\otimes\boldsymbol{g}_k\left(\boldsymbol{x}+\lambda\,\boldsymbol{e}_\mu\right).
\tag{1.24}
$$

现在研究 $\lambda\to0\in\mathbb{R}$ 时的极限

$$
\lim_{\lambda\to0}\frac{\boldsymbol{\Phi}\left(\boldsymbol{x}+\lambda\,\boldsymbol{e}_\mu\right)-\boldsymbol{\Phi}(\boldsymbol{x})}{\lambda}=:\frac{\partial\boldsymbol{\Phi}}{\partial x^\mu}(\boldsymbol{x})\in\mathscr{T}^3(\mathbb{R}^m).
\tag{1.25}
$$

① 类似“$\boldsymbol{\Phi}(\boldsymbol{x})$”的记号在前文也表示张量 $\boldsymbol{\Phi}$ 作用在向量 $\boldsymbol{x}$ 上（“吃掉”了 $\boldsymbol{x}$），此时有 $\boldsymbol{\Phi}(\boldsymbol{x})\in\mathbb{R}$，注意不要混淆. 符号有限，难免如此，还望诸位体谅.

② 请注意，下式 $\boldsymbol{\Phi}$ 之后的第一个圆括号表示位于 $\boldsymbol{x}$ 处；而后面的方括号则表示作用在这几个向量上.

与之前的向量值映照类似，该极限表示张量场 $\boldsymbol{\Phi}(\boldsymbol{x})$ 作为一个整体，相对于自变量第 μ 个分量的变化率，即 $\boldsymbol{\Phi}$ 关于 x^μ（在 $\boldsymbol{x}$ 处）的**偏导数**．式中，$\boldsymbol{\Phi}\left(\boldsymbol{x}+\lambda\,\boldsymbol{e}_\mu\right)$ 已由 (1.24) 式给出．注意到张量分量实际上就是一个多元函数，于是

$$\Phi^{i\ k}_{\ j}\left(\boldsymbol{x}+\lambda\,\boldsymbol{e}_\mu\right)=\Phi^{i\ k}_{\ j}(\boldsymbol{x})+\frac{\partial\Phi^{i\ k}_{\ j}}{\partial x^\mu}(\boldsymbol{x})\cdot\lambda+o^{i\ k}_{\ j}(\lambda)\in\mathbb{R}. \tag{1.26}$$

另外，三个基向量作为向量值映照，同样可以展开：

$$\begin{cases}\boldsymbol{g}_i\left(\boldsymbol{x}+\lambda\,\boldsymbol{e}_\mu\right)=\boldsymbol{g}_i(\boldsymbol{x})+\dfrac{\partial\boldsymbol{g}_i}{\partial x^\mu}(\boldsymbol{x})\cdot\lambda+\boldsymbol{o}_i(\lambda)\in\mathbb{R}^m, & \text{(1.27-a)}\\ \boldsymbol{g}^j\left(\boldsymbol{x}+\lambda\,\boldsymbol{e}_\mu\right)=\boldsymbol{g}^j(\boldsymbol{x})+\dfrac{\partial\boldsymbol{g}^j}{\partial x^\mu}(\boldsymbol{x})\cdot\lambda+\boldsymbol{o}^j(\lambda)\in\mathbb{R}^m, & \text{(1.27-b)}\\ \boldsymbol{g}_k\left(\boldsymbol{x}+\lambda\,\boldsymbol{e}_\mu\right)=\boldsymbol{g}_k(\boldsymbol{x})+\dfrac{\partial\boldsymbol{g}_k}{\partial x^\mu}(\boldsymbol{x})\cdot\lambda+\boldsymbol{o}_k(\lambda)\in\mathbb{R}^m. & \text{(1.27-c)}\end{cases}$$

如果直接展开，一共有 81 项，显然过于繁杂，不便操作．我们将按 λ 的次数逐次展开．首先看 λ 的零次项:

$$\Phi^{i\ k}_{\ j}(\boldsymbol{x})\,\boldsymbol{g}_i(\boldsymbol{x})\otimes\boldsymbol{g}^j(\boldsymbol{x})\otimes\boldsymbol{g}_k(\boldsymbol{x}), \tag{1.28}$$

这就是 $\boldsymbol{\Phi}(\boldsymbol{x})$．然后是 λ 的一次项：

$$\begin{aligned}\lambda\cdot\Bigg[&\frac{\partial\Phi^{i\ k}_{\ j}}{\partial x^\mu}(\boldsymbol{x})\,\boldsymbol{g}_i(\boldsymbol{x})\otimes\boldsymbol{g}^j(\boldsymbol{x})\otimes\boldsymbol{g}_k(\boldsymbol{x})+\Phi^{i\ k}_{\ j}(\boldsymbol{x})\,\frac{\partial\boldsymbol{g}_i}{\partial x^\mu}(\boldsymbol{x})\otimes\boldsymbol{g}^j(\boldsymbol{x})\otimes\boldsymbol{g}_k(\boldsymbol{x})\\&+\Phi^{i\ k}_{\ j}(\boldsymbol{x})\,\boldsymbol{g}_i(\boldsymbol{x})\otimes\frac{\partial\boldsymbol{g}^j}{\partial x^\mu}(\boldsymbol{x})\otimes\boldsymbol{g}_k(\boldsymbol{x})+\Phi^{i\ k}_{\ j}(\boldsymbol{x})\,\boldsymbol{g}_i(\boldsymbol{x})\otimes\boldsymbol{g}^j(\boldsymbol{x})\otimes\frac{\partial\boldsymbol{g}_k}{\partial x^\mu}(\boldsymbol{x})\Bigg].\end{aligned} \tag{1.29}$$

剩下的至少是 λ 的二次项，我们将其统一写作“res.”（余项）.

现在回头看之前的极限 (1.25) 式．λ 的零次项与 $\boldsymbol{\Phi}(\boldsymbol{x})$ 相互抵消，而一次项就只剩下了系数部分．至于余项 res.，则要证明它趋于零．以

$$\Phi^{i\ k}_{\ j}(\boldsymbol{x})\,\boldsymbol{o}_i(\lambda)\otimes\boldsymbol{g}^j(\boldsymbol{x})\otimes\boldsymbol{g}_k(\boldsymbol{x}) \tag{1.30}$$

为例，我们需要证明它等于 $\boldsymbol{o}(\lambda)\in\mathscr{T}^3(\mathbb{R}^m)$，即

$$\lim_{\lambda\to0}\frac{\left\|\Phi^{i\ k}_{\ j}(\boldsymbol{x})\,\boldsymbol{o}_i(\lambda)\otimes\boldsymbol{g}^j(\boldsymbol{x})\otimes\boldsymbol{g}_k(\boldsymbol{x})\right\|_{\mathscr{T}^3(\mathbb{R}^m)}}{\lambda}=0\in\mathbb{R}. \tag{1.31}$$

证明： 这里为了叙述方便，我们将暂时不使用 Einstein 求和约定，而是把求和号显式地写出来．于是分子部分可以写成

$$\left\|\sum_{i,j,k=1}^{m}\Phi^{i\ k}_{\ j}(\boldsymbol{x})\,\boldsymbol{o}_i(\lambda)\otimes\boldsymbol{g}^j(\boldsymbol{x})\otimes\boldsymbol{g}_k(\boldsymbol{x})\right\|_{\mathscr{T}^3(\mathbb{R}^m)}$$

根据范数的三角不等式，有

$$\leqslant\sum_{i,j,k=1}^{m}\left\|\Phi^{i\ k}_{\ j}(\boldsymbol{x})\,\boldsymbol{o}_i(\lambda)\otimes\boldsymbol{g}^j(\boldsymbol{x})\otimes\boldsymbol{g}_k(\boldsymbol{x})\right\|_{\mathscr{T}^3(\mathbb{R}^m)}$$

再利用正齐次性，可得

$$= \sum_{i,j,k=1}^{m} \left|\Phi^{i}{}_{j}{}^{k}(\boldsymbol{x})\right| \cdot \left\|\boldsymbol{o}_{i}(\boldsymbol{x}) \otimes \boldsymbol{g}^{j}(\boldsymbol{x}) \otimes \boldsymbol{g}_{k}(\boldsymbol{x})\right\|_{\mathscr{T}^{3}(\mathbb{R}^{m})}$$

代入简单张量的范数，便有

$$= \sum_{i,j,k=1}^{m} \left|\Phi^{i}{}_{j}{}^{k}(\boldsymbol{x})\right| \cdot \left\|\boldsymbol{o}_{i}(\lambda)\right\|_{\mathbb{R}^{m}} \cdot \left\|\boldsymbol{g}^{j}(\boldsymbol{x})\right\|_{\mathbb{R}^{m}} \cdot \left\|\boldsymbol{g}_{k}(\boldsymbol{x})\right\|_{\mathbb{R}^{m}}. \tag{1.32}$$

这几项中只有 $\left\|\boldsymbol{o}_{i}(\lambda)\right\|_{\mathscr{T}^{3}(\mathbb{R}^{m})}$ 与 λ 有关．于是

$$\lim_{\lambda\to 0} \frac{\left\|\Phi^{i}{}_{j}{}^{k}(\boldsymbol{x})\, \boldsymbol{o}_{i}(\lambda) \otimes \boldsymbol{g}^{j}(\boldsymbol{x}) \otimes \boldsymbol{g}_{k}(\boldsymbol{x})\right\|_{\mathscr{T}^{3}(\mathbb{R}^{m})}}{\lambda}$$
$$= \sum_{i,j,k=1}^{m} \left|\Phi^{i}{}_{j}{}^{k}(\boldsymbol{x})\right| \cdot \left\|\boldsymbol{g}^{j}(\boldsymbol{x})\right\|_{\mathbb{R}^{m}} \cdot \left\|\boldsymbol{g}_{k}(\boldsymbol{x})\right\|_{\mathbb{R}^{m}} \cdot \lim_{\lambda\to 0} \frac{\left\|\boldsymbol{o}_{i}(\lambda)\right\|_{\mathbb{R}^{m}}}{\lambda}$$

根据定义，最后的极限为零，因此

$$= 0. \tag{1.33}$$

□

类似地，其他七十多项也都是 λ 的一阶无穷小量．而有限个无穷小量之和仍为无穷小量，于是 res. $\to 0$.

综上，我们有

$$\frac{\partial \boldsymbol{\Phi}}{\partial x^{\mu}}(\boldsymbol{x}) := \lim_{\lambda\to 0} \frac{\boldsymbol{\Phi}\left(\boldsymbol{x} + \lambda\, \boldsymbol{e}_{\mu}\right) - \boldsymbol{\Phi}(\boldsymbol{x})}{\lambda}$$
$$= \left(\frac{\partial \Phi^{i}{}_{j}{}^{k}}{\partial x^{\mu}}\, \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} + \Phi^{i}{}_{j}{}^{k} \frac{\partial \boldsymbol{g}_{i}}{\partial x^{\mu}} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} + \Phi^{i}{}_{j}{}^{k}\, \boldsymbol{g}_{i} \otimes \frac{\partial \boldsymbol{g}^{j}}{\partial x^{\mu}} \otimes \boldsymbol{g}_{k} + \Phi^{i}{}_{j}{}^{k}\, \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \frac{\partial \boldsymbol{g}_{k}}{\partial x^{\mu}} \right)(\boldsymbol{x}). \tag{1.34}$$

式中，$\partial \boldsymbol{g}_{i} / \partial x^{\mu}\ (\boldsymbol{x})$ 可以用 Christoffel 符号表示：

$$\frac{\partial \boldsymbol{g}_{i}}{\partial x^{\mu}}(\boldsymbol{x}) = \Gamma^{s}{}_{\mu i}\, \boldsymbol{g}_{s}(\boldsymbol{x}). \tag{1.35}$$

因此 (1.34) 式的第二项

$$\Phi^{i}{}_{j}{}^{k}(\boldsymbol{x}) \frac{\partial \boldsymbol{g}_{i}}{\partial x^{\mu}}(\boldsymbol{x}) \otimes \boldsymbol{g}^{j}(\boldsymbol{x}) \otimes \boldsymbol{g}_{k}(\boldsymbol{x})$$
$$= \Gamma^{s}{}_{\mu i}\, \Phi^{i}{}_{j}{}^{k}(\boldsymbol{x})\, \boldsymbol{g}_{s}(\boldsymbol{x}) \otimes \boldsymbol{g}^{j}(\boldsymbol{x}) \otimes \boldsymbol{g}_{k}(\boldsymbol{x})$$

i 和 s 都是哑标，不妨进行一下交换：

$$= \Gamma^{i}{}_{\mu s}\, \Phi^{s}{}_{j}{}^{k}(\boldsymbol{x})\, \boldsymbol{g}_{i}(\boldsymbol{x}) \otimes \boldsymbol{g}^{j}(\boldsymbol{x}) \otimes \boldsymbol{g}_{k}(\boldsymbol{x}). \tag{1.36}$$

同样，后面的两项也可进行类似的处理．这样便有

$$\frac{\partial \boldsymbol{\Phi}}{\partial x^{\mu}}(\boldsymbol{x}) := \lim_{\lambda\to 0} \frac{\boldsymbol{\Phi}\left(\boldsymbol{x} + \lambda\, \boldsymbol{e}_{\mu}\right) - \boldsymbol{\Phi}(\boldsymbol{x})}{\lambda}$$

$$= \left[\left(\frac{\partial \Phi^i{}_j{}^k}{\partial x^\mu} + \Gamma^i{}_{\mu s} \Phi^s{}_j{}^k - \Gamma^s{}_{\mu j} \Phi^i{}_s{}^k + \Gamma^k{}_{\mu s} \Phi^i{}_j{}^s \right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right] (\boldsymbol{x})$$

$$=: \nabla_\mu \Phi^i{}_j{}^k(\boldsymbol{x})\, \boldsymbol{g}_i(\boldsymbol{x}) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x}), \tag{1.37}$$

式中，我们称 $\nabla_\mu \Phi^i{}_j{}^k(\boldsymbol{x}) \in \mathbb{R}$ 为张量分量 $\Phi^i{}_j{}^k(\boldsymbol{x})$ 相对于 x^μ 的**协变导数**，其定义为：

$$\nabla_\mu \Phi^i{}_j{}^k(\boldsymbol{x}) \triangleq \frac{\partial \Phi^i{}_j{}^k}{\partial x^\mu}(\boldsymbol{x}) + \Gamma^i{}_{\mu s} \Phi^s{}_j{}^k(\boldsymbol{x}) - \Gamma^s{}_{\mu j} \Phi^i{}_s{}^k(\boldsymbol{x}) + \Gamma^k{}_{\mu s} \Phi^i{}_j{}^s(\boldsymbol{x}). \tag{1.38}$$

1.3 张量场的梯度

1.3.1 梯度；可微性

我们在参数域中的内点 $\boldsymbol{x}_0$ 处取一个半径为 δ 的邻域 $\mathfrak{B}_\delta(\boldsymbol{x}_0)$．若使自变量变化到 $\boldsymbol{x}_0 + \boldsymbol{h}$，则在物理域中，对应的点就将从 $\boldsymbol{X}(\boldsymbol{x}_0)$ 变化到 $\boldsymbol{X}(\boldsymbol{x}_0 + \boldsymbol{h})$．考察定义在参数域 $\mathfrak{D}_x$ 上的张量场 $\boldsymbol{\Phi}(\boldsymbol{x})$，它的变化为

$$\begin{aligned} &\boldsymbol{\Phi}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{\Phi}(\boldsymbol{x}_0) \\ =\ &\Phi^i{}_j{}^k(\boldsymbol{x}_0 + \boldsymbol{h})\, \boldsymbol{g}_i(\boldsymbol{x}_0 + \boldsymbol{h}) \otimes \boldsymbol{g}^j(\boldsymbol{x}_0 + \boldsymbol{h}) \otimes \boldsymbol{g}_k(\boldsymbol{x}_0 + \boldsymbol{h}) \\ &- \Phi^i{}_j{}^k(\boldsymbol{x}_0)\, \boldsymbol{g}_i(\boldsymbol{x}_0) \otimes \boldsymbol{g}^j(\boldsymbol{x}_0) \otimes \boldsymbol{g}_k(\boldsymbol{x}_0). \end{aligned} \tag{1.39}$$

与之前一样将第一部分逐项展开，有

$$\begin{cases} \Phi^i{}_j{}^k(\boldsymbol{x}_0 + \boldsymbol{h}) = \Phi^i{}_j{}^k(\boldsymbol{x}_0) + \dfrac{\partial \Phi^i{}_j{}^k}{\partial x^\mu}(\boldsymbol{x}_0) \cdot h^\mu + o^i{}_j{}^k(\|\boldsymbol{h}\|_{\mathbb{R}^m}) \in \mathbb{R}, & \text{(1.40-a)} \\ \boldsymbol{g}_i(\boldsymbol{x}_0 + \boldsymbol{h}) = \boldsymbol{g}_i(\boldsymbol{x}_0) + \dfrac{\partial \boldsymbol{g}_i}{\partial x^\mu}(\boldsymbol{x}_0) \cdot h^\mu + \boldsymbol{o}_i(\|\boldsymbol{h}\|_{\mathbb{R}^m}) \in \mathbb{R}^m, & \text{(1.40-b)} \\ \boldsymbol{g}^j(\boldsymbol{x}_0 + \boldsymbol{h}) = \boldsymbol{g}^j(\boldsymbol{x}_0) + \dfrac{\partial \boldsymbol{g}^j}{\partial x^\mu}(\boldsymbol{x}_0) \cdot h^\mu + \boldsymbol{o}^j(\|\boldsymbol{h}\|_{\mathbb{R}^m}) \in \mathbb{R}^m, & \text{(1.40-c)} \\ \boldsymbol{g}_k(\boldsymbol{x}_0 + \boldsymbol{h}) = \boldsymbol{g}_k(\boldsymbol{x}_0) + \dfrac{\partial \boldsymbol{g}_k}{\partial x^\mu}(\boldsymbol{x}_0) \cdot h^\mu + \boldsymbol{o}_k(\|\boldsymbol{h}\|_{\mathbb{R}^m}) \in \mathbb{R}^m. & \text{(1.40-d)} \end{cases}$$

不要忘记对哑标 μ 进行求和．

不显含 $\boldsymbol{h}$ 的只有每行的第一项；它们组合起来，与式 (1.39) 的第二部分相互抵消．再看 $\boldsymbol{h}$ 的一次项：[①]

$$\left[\frac{\partial \Phi^i{}_j{}^k}{\partial x^\mu} \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \Phi^i{}_j{}^k \left(\frac{\partial \boldsymbol{g}_i}{\partial x^\mu} \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \boldsymbol{g}_i \otimes \frac{\partial \boldsymbol{g}^j}{\partial x^\mu} \otimes \boldsymbol{g}_k + \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \frac{\partial \boldsymbol{g}_k}{\partial x^\mu} \right) \right] (\boldsymbol{x}_0) \cdot h^\mu. \tag{1.41}$$

利用 Christoffel 符号又可以把它写成

$$\begin{aligned} &\left[\frac{\partial \Phi^i{}_j{}^k}{\partial x^\mu} \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \Phi^i{}_j{}^k \left(\Gamma^s{}_{\mu i} \boldsymbol{g}_s \right) \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right. \\ &\left. \quad - \Phi^i{}_j{}^k \boldsymbol{g}_i \otimes \left(\Gamma^j{}_{\mu s} \boldsymbol{g}^s \right) \otimes \boldsymbol{g}_k + \Phi^i{}_j{}^k \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \left(\Gamma^s{}_{\mu k} \boldsymbol{g}_s \right) \right] (\boldsymbol{x}_0) \cdot h^\mu \end{aligned}$$

① 实际是 h^μ 的一次项．别忘了求和．

$$
\begin{aligned}
&= \Bigg(\frac{\partial \Phi^{i\ k}_{\ j}}{\partial x^\mu} \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \Gamma^i_{\mu s} \Phi^{s\ k}_{\ j} \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \\
&\quad - \Gamma^s_{\mu j} \Phi^{i\ k}_{\ s} \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \Gamma^k_{\mu s} \Phi^{i\ s}_{\ j} \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \Bigg) (\boldsymbol{x}_0) \cdot h^\mu \\
&= \left[\left(\frac{\partial \Phi^{i\ k}_{\ j}}{\partial x^\mu} + \Gamma^i_{\mu s} \Phi^{s\ k}_{\ j} - \Gamma^s_{\mu j} \Phi^{i\ k}_{\ s} + \Gamma^k_{\mu s} \Phi^{i\ s}_{\ j} \right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right] (\boldsymbol{x}_0) \cdot h^\mu .
\end{aligned} \tag{1.42}
$$

至于高阶项，它们都等于 $\mathscr{o}\left(\|\boldsymbol{h}\|_{\mathbb{R}^m}\right) \in \mathscr{T}^3(\mathbb{R}^m)$，并且满足

$$
\lim_{\boldsymbol{h} \to \boldsymbol{0} \in \mathbb{R}^m} \frac{\left\| \mathscr{o}\left(\|\boldsymbol{h}\|_{\mathbb{R}^m}\right) \right\|_{\mathscr{T}^3(\mathbb{R}^m)}}{\|\boldsymbol{h}\|_{\mathbb{R}^m}} = 0 \in \mathbb{R}. \tag{1.43}
$$

证明与 (1.31) 式类似.

整理一下，我们有

$$
\begin{aligned}
&\Phi(\boldsymbol{x}_0 + \boldsymbol{h}) - \Phi(\boldsymbol{x}_0) \\
&= \left[\left(\frac{\partial \Phi^{i\ k}_{\ j}}{\partial x^\mu} + \Gamma^i_{\mu s} \Phi^{s\ k}_{\ j} - \Gamma^s_{\mu j} \Phi^{i\ k}_{\ s} + \Gamma^k_{\mu s} \Phi^{i\ s}_{\ j} \right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right] (\boldsymbol{x}_0) \cdot h^\mu + \mathscr{o}\left(\|\boldsymbol{h}\|_{\mathbb{R}^m}\right)
\end{aligned}
$$

利用协变导数，有

$$
=: \left[\nabla_\mu \Phi^{i\ k}_{\ j}(\boldsymbol{x}_0)\, \boldsymbol{g}_i(\boldsymbol{x}_0) \otimes \boldsymbol{g}^j(\boldsymbol{x}_0) \otimes \boldsymbol{g}_k(\boldsymbol{x}_0) \right] h^\mu + \mathscr{o}\left(\|\boldsymbol{h}\|_{\mathbb{R}^m}\right). \tag{1.44}
$$

至此，从微分学的角度来看，任务已经完成．但对于张量分析而言，我们还需要再做一点微小的工作．简单张量部分再并上一个 $\boldsymbol{g}^\mu$，从而使张量升一阶；后面则改成 $h^\nu \boldsymbol{g}_\nu(\boldsymbol{x}_0)$，并利用点乘保持总阶数不变：

$$
\begin{aligned}
&\left[\nabla_\mu \Phi^{i\ k}_{\ j}(\boldsymbol{x}_0)\, \boldsymbol{g}_i(\boldsymbol{x}_0) \otimes \boldsymbol{g}^j(\boldsymbol{x}_0) \otimes \boldsymbol{g}_k(\boldsymbol{x}_0) \right] h^\mu \\
&= \left[\nabla_\mu \Phi^{i\ k}_{\ j}(\boldsymbol{x}_0)\, \boldsymbol{g}_i(\boldsymbol{x}_0) \otimes \boldsymbol{g}^j(\boldsymbol{x}_0) \otimes \boldsymbol{g}_k(\boldsymbol{x}_0) \otimes \boldsymbol{g}^\mu(\boldsymbol{x}_0) \right] \cdot \left[h^\nu \boldsymbol{g}_\nu(\boldsymbol{x}_0) \right].
\end{aligned} \tag{1.45}
$$

所谓“点乘”，其实就是 e 点积在 $e = 1$ 时的情况．实际上，

$$
\boldsymbol{g}^\mu(\boldsymbol{x}_0) \cdot \left[h^\nu \boldsymbol{g}_\nu(\boldsymbol{x}_0) \right] = h^\nu \delta^\mu_\nu = h^\mu. \tag{1.46}
$$

这里用到了局部基的对偶关系 (**??**) 式.

此时，我们获得了一个四阶张量与 $h^\nu \boldsymbol{g}_\nu(\boldsymbol{x}_0)$ 的点积．接下来讨论该项的意义．参数域中 $\boldsymbol{x}_0$ 发生 $\boldsymbol{h} = h^i \boldsymbol{e}_i$ 的变化时，根据向量值映照 $\boldsymbol{X}(\boldsymbol{x})$ 的可微性，对应物理域中的变化为

$$
\begin{aligned}
\boldsymbol{X}(\boldsymbol{x}_0 + \boldsymbol{h}) - \boldsymbol{X}(\boldsymbol{x}_0) &= \mathrm{D}\boldsymbol{X}(\boldsymbol{x}_0)(\boldsymbol{h}) + \mathscr{o}\left(\|\boldsymbol{h}\|_{\mathbb{R}^m}\right) \\
&=: \frac{\partial \boldsymbol{X}}{\partial x^\nu}(\boldsymbol{x}_0)\, h^\nu + \mathscr{o}\left(\|\boldsymbol{h}\|_{\mathbb{R}^m}\right)
\end{aligned}
$$

代入 **??** 小节中局部协变基的定义，可有

$$
=: h^\nu \boldsymbol{g}_\nu(\boldsymbol{x}_0) + \mathscr{o}\left(\|\boldsymbol{h}\|_{\mathbb{R}^m}\right). \tag{1.47}
$$

代入式 (1.44)，有

$$
\begin{aligned}
&\boldsymbol{\Phi}(\boldsymbol{x}_0+\boldsymbol{h})-\boldsymbol{\Phi}(\boldsymbol{x}_0)\\
=&\left[\nabla_\mu \Phi^{i\ \ k}_{\ j}(\boldsymbol{x}_0)\,\boldsymbol{g}_i(\boldsymbol{x}_0)\otimes\boldsymbol{g}^j(\boldsymbol{x}_0)\otimes\boldsymbol{g}_k(\boldsymbol{x}_0)\otimes\boldsymbol{g}^\mu(\boldsymbol{x}_0)\right]\\
&\cdot\left[\boldsymbol{X}(\boldsymbol{x}_0+\boldsymbol{h})-\boldsymbol{X}(\boldsymbol{x}_0)+\boldsymbol{o}(\|\boldsymbol{h}\|_{\mathbb{R}^m})\right]+\boldsymbol{o}(\|\boldsymbol{h}\|_{\mathbb{R}^m})
\end{aligned}
$$

合并掉一阶无穷小量[①]，可得

$$
\begin{aligned}
=&\left[\nabla_\mu \Phi^{i\ \ k}_{\ j}(\boldsymbol{x}_0)\,\boldsymbol{g}_i(\boldsymbol{x}_0)\otimes\boldsymbol{g}^j(\boldsymbol{x}_0)\otimes\boldsymbol{g}_k(\boldsymbol{x}_0)\otimes\boldsymbol{g}^\mu(\boldsymbol{x}_0)\right]\\
&\cdot\left[\boldsymbol{X}(\boldsymbol{x}_0+\boldsymbol{h})-\boldsymbol{X}(\boldsymbol{x}_0)\right]+\boldsymbol{o}(\|\boldsymbol{h}\|_{\mathbb{R}^m})\\
=&\left[\frac{\partial\boldsymbol{\Phi}}{\partial x^\mu}(\boldsymbol{x}_0)\otimes\boldsymbol{g}^\mu(\boldsymbol{x}_0)\right]\cdot\left[\boldsymbol{X}(\boldsymbol{x}_0+\boldsymbol{h})-\boldsymbol{X}(\boldsymbol{x}_0)\right]+\boldsymbol{o}(\|\boldsymbol{h}\|_{\mathbb{R}^m})\in\mathscr{T}^3(\mathbb{R}^m).
\end{aligned}
\tag{1.48}
$$

引入记号

$$
\boldsymbol{\Phi}(\boldsymbol{x}_0)\otimes\left[\boldsymbol{g}^\mu\frac{\partial}{\partial x^\mu}(\boldsymbol{x}_0)\right]\coloneqq\frac{\partial\boldsymbol{\Phi}}{\partial x^\mu}(\boldsymbol{x}_0)\otimes\boldsymbol{g}^\mu(\boldsymbol{x}_0),\tag{1.49}
$$

再引入**梯度算子**

$$
\boldsymbol{\nabla}\coloneqq\boldsymbol{g}^\mu\frac{\partial}{\partial x^\mu}(\boldsymbol{x}_0),\tag{1.50}
$$

我们得到的结论就可以表述为

$$
\boldsymbol{\Phi}(\boldsymbol{x}_0+\boldsymbol{h})-\boldsymbol{\Phi}(\boldsymbol{x}_0)=(\boldsymbol{\Phi}\otimes\boldsymbol{\nabla})(\boldsymbol{x}_0)\cdot\left[\boldsymbol{X}(\boldsymbol{x}_0+\boldsymbol{h})-\boldsymbol{X}(\boldsymbol{x}_0)\right]+\boldsymbol{o}(\|\boldsymbol{h}\|_{\mathbb{R}^m})\in\mathscr{T}^3(\mathbb{R}^m),\tag{1.51}
$$

式中的“$\cdot$”表示点乘．以上结果称之为张量场的**可微性**，它表明，由一点处的位置移动所引起张量场的变化，可以用该点处张量场的**梯度**（即 $\boldsymbol{\Phi}\otimes\boldsymbol{\nabla}$）点乘物理空间中的位置差别来近似，误差为一阶无穷小量．以上分析基于三阶张量．但显然，对于 p 阶张量，将会有完全一致的结果．

1.3.2 方向导数

现在来研究张量场沿 $\boldsymbol{e}$ 方向的变化率（设 $\|\boldsymbol{e}\|_{\mathbb{R}^m}=1$）．取一个与 $\boldsymbol{e}$ 平行的向量 $\lambda\boldsymbol{e}$．注意到 $\lambda\boldsymbol{e}$ 其实就是物理空间中的位置变化，于是根据张量场的可微性，我们有

$$
\boldsymbol{\Phi}(\boldsymbol{x}_0+\lambda\boldsymbol{e})-\boldsymbol{\Phi}(\boldsymbol{x}_0)=(\boldsymbol{\Phi}\otimes\boldsymbol{\nabla})(\boldsymbol{x}_0)\cdot(\lambda\boldsymbol{e})+\boldsymbol{o}(\lambda),\tag{1.52}
$$

该式等价于

$$
\lim_{\lambda\to0}\frac{\boldsymbol{\Phi}(\boldsymbol{x}_0+\lambda\boldsymbol{e})-\boldsymbol{\Phi}(\boldsymbol{x}_0)}{\lambda}=(\boldsymbol{\Phi}\otimes\boldsymbol{\nabla})(\boldsymbol{x}_0)\cdot\boldsymbol{e}.\tag{1.53}
$$

我们把它定义为张量场 $\boldsymbol{\Phi}(\boldsymbol{x})$ 沿 $\boldsymbol{e}$ 方向的**方向导数**：

$$
\frac{\partial\boldsymbol{\Phi}}{\partial\boldsymbol{e}}(\boldsymbol{x}_0)\triangleq(\boldsymbol{\Phi}\otimes\boldsymbol{\nabla})(\boldsymbol{x}_0)\cdot\boldsymbol{e}.\tag{1.54}
$$

① 式中的两个 $\boldsymbol{o}(\|\boldsymbol{h}\|_{\mathbb{R}^m})$ 是不同的，前者属于 $\mathbb{R}^m$，后者属于 $\mathscr{T}^3(\mathbb{R}^m)$.

1.3.3 左梯度与右梯度

我们已经知道，利用梯度算子

$$\boldsymbol{\nabla} \coloneqq \boldsymbol{g}^{\mu} \frac{\partial}{\partial x^{\mu}}\left(\boldsymbol{x}_0\right), \tag{1.55}$$

可以把张量场的梯度表示为

$$\left(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\right)\left(\boldsymbol{x}_0\right) = \boldsymbol{\Phi}\left(\boldsymbol{x}_0\right) \otimes \left[\boldsymbol{g}^{\mu} \frac{\partial}{\partial x^{\mu}}\left(\boldsymbol{x}_0\right)\right] \coloneqq \frac{\partial \boldsymbol{\Phi}}{\partial x^{\mu}}\left(\boldsymbol{x}_0\right) \otimes \boldsymbol{g}^{\mu}\left(\boldsymbol{x}_0\right). \tag{1.56}$$

“$\boldsymbol{\nabla}$”在右边，故称之为**右梯度**（简称**梯度**）. 相应地，自然会有**左梯度**：

$$\left(\boldsymbol{\nabla} \otimes \boldsymbol{\Phi}\right)\left(\boldsymbol{x}_0\right) = \left[\boldsymbol{g}^{\mu} \frac{\partial}{\partial x^{\mu}}\left(\boldsymbol{x}_0\right)\right] \otimes \boldsymbol{\Phi}\left(\boldsymbol{x}_0\right) \coloneqq \boldsymbol{g}^{\mu}\left(\boldsymbol{x}_0\right) \otimes \frac{\partial \boldsymbol{\Phi}}{\partial x^{\mu}}\left(\boldsymbol{x}_0\right). \tag{1.57}$$

张量积不存在交换律，因而这两者是不同的. 注意，梯度运算将使张量的阶数增加一阶.

张量场的可微性可以用*左梯度*来等价表述：

$$\boldsymbol{\Phi}\left(\boldsymbol{x}_0 + \boldsymbol{h}\right) - \boldsymbol{\Phi}\left(\boldsymbol{x}_0\right) = \left[\boldsymbol{X}\left(\boldsymbol{x}_0 + \boldsymbol{h}\right) - \boldsymbol{X}\left(\boldsymbol{x}_0\right)\right] \cdot \left(\boldsymbol{\nabla} \otimes \boldsymbol{\Phi}\right)\left(\boldsymbol{x}_0\right) + o\left(\|\boldsymbol{h}\|_{\mathbb{R}^m}\right). \tag{1.58}$$

类似地，还有方向导数：

$$\frac{\partial \boldsymbol{\Phi}}{\partial \boldsymbol{e}}(\boldsymbol{x}_0) \triangleq \left(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\right)\left(\boldsymbol{x}_0\right) \cdot \boldsymbol{e} = \boldsymbol{e} \cdot \left(\boldsymbol{\nabla} \otimes \boldsymbol{\Phi}\right)\left(\boldsymbol{x}_0\right). \tag{1.59}$$

1.4 场论恒等式（一）

为了给下一节做好铺垫，本节将证明几个重要引理.

1.4.1 Ricci 引理

首先来证明两个结论：

$$\begin{cases} \dfrac{\partial \boldsymbol{G}}{\partial x^{\mu}}(\boldsymbol{x}) = \boldsymbol{0} \in \mathscr{T}^2(\mathbb{R}^m), & \text{(1.60-a)} \\ \dfrac{\partial \boldsymbol{\epsilon}}{\partial x^{\mu}}(\boldsymbol{x}) = \boldsymbol{0} \in \mathscr{T}^3\left(\mathbb{R}^3\right), & \text{(1.60-b)} \end{cases}$$

其中的 $\boldsymbol{G}$ 和 $\boldsymbol{\epsilon}$ 分别是度量张量和 Eddington 张量.

证明： 为方便起见，证明中我们将省去“$(\boldsymbol{x})$”.

先考察度量张量的偏导数：

$$\begin{aligned} \frac{\partial \boldsymbol{G}}{\partial x^{\mu}} &= \frac{\partial}{\partial x^{\mu}}\left(g_{ij}\, \boldsymbol{g}^i \otimes \boldsymbol{g}^j\right) \\ &= \nabla_{\mu} g_{ij}\, \boldsymbol{g}^i \otimes \boldsymbol{g}^j, \end{aligned} \tag{1.61}$$

式中，协变导数定义为

$$\nabla_{\mu} g_{ij} \triangleq \frac{\partial g_{ij}}{\partial x^{\mu}} - \Gamma^{s}_{\ \mu i}\, g_{sj} - \Gamma^{s}_{\ \mu j}\, g_{is}. \tag{1.62}$$

以下有两种方法证明 $\nabla_{\mu} g_{ij} = 0$.

方法一利用度量的定义：

$$\frac{\partial g_{ij}}{\partial x^{\mu}} \triangleq \frac{\partial}{\partial x^{\mu}} \left\langle \boldsymbol{g}_i, \boldsymbol{g}_j \right\rangle_{\mathbb{R}^m}$$

$$
= \left\langle \frac{\partial \boldsymbol{g}_i}{\partial x^\mu}, \boldsymbol{g}_j \right\rangle_{\mathbb{R}^m} + \left\langle \boldsymbol{g}_i, \frac{\partial \boldsymbol{g}_j}{\partial x^\mu} \right\rangle_{\mathbb{R}^m}
$$

根据 Christoffel 符号的定义（见 **??** 小节），有

$$
= \Gamma_{\mu i,\, j} + \Gamma_{\mu j,\, i}\,. \tag{1.63}
$$

另一方面，回忆 (**??**) 式：

$$
\Gamma^k{}_{ij} = \Gamma_{ij,\, l}\, g^{kl}\ , \tag{1.64}
$$

可有

$$
\begin{cases}
\Gamma^s{}_{\mu i}\, g_{sj} = \Gamma_{\mu i,\, k}\, g^{sk} g_{sj} = \Gamma_{\mu i,\, k}\, \delta^k_j = \Gamma_{\mu i,\, j}\,, & (1.65\text{-a}) \\
\Gamma^s{}_{\mu j}\, g_{is} = \Gamma_{\mu j,\, k}\, g^{sk} g_{is} = \Gamma_{\mu j,\, k}\, \delta^k_i = \Gamma_{\mu j,\, i}\,. & (1.65\text{-b})
\end{cases}
$$

于是

$$
\nabla_\mu\, g_{ij} = \Gamma_{\mu i,\, j} + \Gamma_{\mu j,\, i} - \Gamma_{\mu i,\, j} - \Gamma_{\mu j,\, i} = 0. \tag{1.66}
$$

方法二则利用第一类 Christoffel 符号的性质 (**??**) 式：

$$
\Gamma_{ij,\, k} = \frac{1}{2} \left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k} \right). \tag{1.67}
$$

因而

$$
\begin{aligned}
&\Gamma^s{}_{\mu i}\, g_{sj} + \Gamma^s{}_{\mu j}\, g_{is} \\
&= \Gamma_{\mu i,\, j} + \Gamma_{\mu j,\, i} \\
&= \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^\mu} + \frac{\partial g_{\mu j}}{\partial x^i} - \frac{\partial g_{\mu i}}{\partial x^j} \right) + \frac{1}{2} \left(\frac{\partial g_{ji}}{\partial x^\mu} + \frac{\partial g_{\mu i}}{\partial x^j} - \frac{\partial g_{\mu j}}{\partial x^i} \right) \\
&= \frac{1}{2} \left(\frac{\partial g_{ij}}{\partial x^\mu} + \frac{\partial g_{ji}}{\partial x^\mu} \right) = \frac{\partial g_{ij}}{\partial x^\mu}.
\end{aligned} \tag{1.68}
$$

显然，立刻就有

$$
\nabla_\mu\, g_{ij} = \frac{\partial g_{ij}}{\partial x^\mu} - \frac{\partial g_{ij}}{\partial x^\mu} = 0. \tag{1.69}
$$

综上，因为 $\nabla_\mu\, g_{ij} = 0 \in \mathbb{R}$，所以

$$
\frac{\partial \boldsymbol{G}}{\partial x^\mu}(\boldsymbol{x}) = \boldsymbol{0} \in \mathscr{T}^2(\mathbb{R}^m). \tag{1.70}
$$

如果用其他形式的分量来表述这一结果，我们便有

$$
\nabla_\mu\, g_{ij} = \nabla_\mu\, g^{ij} = \nabla_\mu\, \delta^i_j = 0. \tag{1.71}
$$

此结论称为 **Ricci 引理**.

再来看 Eddington 张量的偏导数：

$$
\begin{aligned}
\frac{\partial \boldsymbol{\epsilon}}{\partial x^\mu} &= \frac{\partial}{\partial x^\mu} \left(\epsilon^i{}_j{}^k\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \right) \\
&= \nabla_\mu\, \epsilon^i{}_j{}^k\, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\,,
\end{aligned} \tag{1.72}
$$

式中，

$$\nabla_\mu \epsilon^{i\ k}_{\ j} \triangleq \frac{\partial \epsilon^{i\ k}_{\ j}}{\partial x^\mu} + \Gamma^i_{\mu s} \epsilon^{s\ k}_{\ j} - \Gamma^s_{\mu j} \epsilon^{i\ k}_{\ s} + \Gamma^k_{\mu s} \epsilon^{i\ s}_{\ j}. \tag{1.73}$$

根据定义，$\epsilon^{i\ k}_{\ j} = \det\left[\boldsymbol{g}^i, \boldsymbol{g}_j, \boldsymbol{g}^k\right]$．因此

$$\begin{aligned}\frac{\partial \epsilon^{i\ k}_{\ j}}{\partial x^\mu} &= \frac{\partial}{\partial x^\mu}\left(\det\left[\boldsymbol{g}^i, \boldsymbol{g}_j, \boldsymbol{g}^k\right]\right) \\ &= \det\left[\frac{\partial \boldsymbol{g}^i}{\partial x^\mu}, \boldsymbol{g}_j, \boldsymbol{g}^k\right] + \det\left[\boldsymbol{g}^i, \frac{\partial \boldsymbol{g}_j}{\partial x^\mu}, \boldsymbol{g}^k\right] + \det\left[\boldsymbol{g}^i, \boldsymbol{g}_j, \frac{\partial \boldsymbol{g}^k}{\partial x^\mu}\right]\end{aligned}$$

利用标架运动方程，有

$$= \det\left[-\Gamma^i_{\mu s} \boldsymbol{g}^s, \boldsymbol{g}_j, \boldsymbol{g}^k\right] + \det\left[\boldsymbol{g}^i, \Gamma^s_{\mu j} \boldsymbol{g}_s, \boldsymbol{g}^k\right] + \det\left[\boldsymbol{g}^i, \boldsymbol{g}_j, -\Gamma^k_{\mu s} \boldsymbol{g}^s\right]$$

再利用行列式的线性性，提出系数：

$$= -\Gamma^i_{\mu s} \det\left[\boldsymbol{g}^s, \boldsymbol{g}_j, \boldsymbol{g}^k\right] + \Gamma^s_{\mu j} \det\left[\boldsymbol{g}^i, \boldsymbol{g}_s, \boldsymbol{g}^k\right] - \Gamma^k_{\mu s} \det\left[\boldsymbol{g}^i, \boldsymbol{g}_j, \boldsymbol{g}^s\right]$$

代回 Eddington 张量的定义，可得

$$= -\Gamma^i_{\mu s} \epsilon^{s\ k}_{\ j} + \Gamma^s_{\mu j} \epsilon^{i\ k}_{\ s} - \Gamma^k_{\mu s} \epsilon^{i\ s}_{\ j}. \tag{1.74}$$

这与式 (1.73) 的后三项恰好抵消．于是便有 $\nabla_\mu \epsilon^{i\ k}_{\ j} = 0$．进而

$$\frac{\partial \boldsymbol{\epsilon}}{\partial x^\mu}(\boldsymbol{x}) = \boldsymbol{0} \in \mathscr{T}^3\left(\mathbb{R}^3\right). \tag{1.75}$$

和度量张量类似，Eddington 张量其他分量的偏导数，如 $\nabla_\mu \epsilon_{ijk}$、$\nabla_\mu \epsilon^{ijk}$ 等，也都等于零．此结论同样称为 **Ricci 引理**． □

1.4.2 Leibniz 法则

协变导数满足 **Leibniz 法则**：

$$\nabla_\mu\left(\Phi^{i\ k}_{\ j} \Psi_p^{\ q}\right) = \left(\nabla_\mu \Phi^{i\ k}_{\ j}\right) \Psi_p^{\ q} + \Phi^{i\ k}_{\ j}\left(\nabla_\mu \Psi_p^{\ q}\right). \tag{1.76}$$

式中，张量分量的形式可以是任意的．

证明： 显然，$\Phi^{i\ k}_{\ j} \Psi_p^{\ q} \in \mathscr{T}^5(\mathbb{R}^m)$．不妨令

$$\Omega^{i\ k\ q}_{\ j\ p} = \Phi^{i\ k}_{\ j} \Psi_p^{\ q}. \tag{1.77}$$

则

$$\begin{aligned}&\nabla_\mu\left(\Phi^{i\ k}_{\ j} \Psi_p^{\ q}\right) = \nabla_\mu \Omega^{i\ k\ q}_{\ j\ p} \\ &\triangleq \frac{\partial \Omega^{i\ k\ q}_{\ j\ p}}{\partial x^\mu} + \Gamma^i_{\mu s} \Omega^{s\ k\ q}_{\ j\ p} - \Gamma^s_{\mu j} \Omega^{i\ k\ q}_{\ s\ p} + \Gamma^k_{\mu s} \Omega^{i\ s\ q}_{\ j\ p} - \Gamma^s_{\mu p} \Omega^{i\ k\ q}_{\ j\ s} + \Gamma^q_{\mu s} \Omega^{i\ k\ s}_{\ j\ p} \\ &= \frac{\partial}{\partial x^\mu}\left(\Phi^{i\ k}_{\ j} \Psi_p^{\ q}\right) + \left(\Gamma^i_{\mu s} \Phi^{s\ k}_{\ j} - \Gamma^s_{\mu j} \Phi^{i\ k}_{\ s} + \Gamma^k_{\mu s} \Phi^{i\ s}_{\ j}\right) \Psi_p^{\ q} + \Phi^{i\ k}_{\ j}\left(-\Gamma^s_{\mu p} \Psi_s^{\ q} + \Gamma^q_{\mu s} \Psi_p^{\ s}\right).\end{aligned} \tag{1.78}$$

第一项偏导数自然满足乘积法则：

$$\frac{\partial}{\partial x^\mu}\left(\boldsymbol{\Phi}^i{}_j{}^k \boldsymbol{\Psi}_p{}^q\right) = \frac{\partial \boldsymbol{\Phi}^i{}_j{}^k}{\partial x^\mu}\boldsymbol{\Psi}_p{}^q + \boldsymbol{\Phi}^i{}_j{}^k \frac{\partial \boldsymbol{\Psi}_p{}^q}{\partial x^\mu}. \tag{1.79}$$

代回前一式，即有

$$\begin{aligned}\nabla_\mu\left(\boldsymbol{\Phi}^i{}_j{}^k \boldsymbol{\Psi}_p{}^q\right) &= \left(\frac{\partial \boldsymbol{\Phi}^i{}_j{}^k}{\partial x^\mu} + \Gamma^i{}_{\mu s}\boldsymbol{\Phi}^s{}_j{}^k - \Gamma^s{}_{\mu j}\boldsymbol{\Phi}^i{}_s{}^k + \Gamma^k{}_{\mu s}\boldsymbol{\Phi}^i{}_j{}^s\right)\boldsymbol{\Psi}_p{}^q + \boldsymbol{\Phi}^i{}_j{}^k\left(\frac{\partial \boldsymbol{\Psi}_p{}^q}{\partial x^\mu} - \Gamma^s{}_{\mu p}\boldsymbol{\Psi}_s{}^q + \Gamma^q{}_{\mu s}\boldsymbol{\Psi}_p{}^s\right)\\ &\triangleq \left(\nabla_\mu \boldsymbol{\Phi}^i{}_j{}^k\right)\boldsymbol{\Psi}_p{}^q + \boldsymbol{\Phi}^i{}_j{}^k\left(\nabla_\mu \boldsymbol{\Psi}_p{}^q\right).\end{aligned} \tag{1.80}$$

□

现在来考虑 $\nabla_\mu\left(\boldsymbol{\Phi}^i{}_j{}^k \boldsymbol{\Psi}_k{}^q\right)$，注意其中的 k 是哑标．若按照 Leibniz 法则，似乎有

$$\begin{aligned}\nabla_\mu\left(\boldsymbol{\Phi}^i{}_j{}^k \boldsymbol{\Psi}_k{}^q\right) &= \left(\nabla_\mu \boldsymbol{\Phi}^i{}_j{}^k\right)\boldsymbol{\Psi}_k{}^q + \boldsymbol{\Phi}^i{}_j{}^k\left(\nabla_\mu \boldsymbol{\Psi}_k{}^q\right)\\ &= \left(\frac{\partial \boldsymbol{\Phi}^i{}_j{}^k}{\partial x^\mu} + \Gamma^i{}_{\mu s}\boldsymbol{\Phi}^s{}_j{}^k - \Gamma^s{}_{\mu j}\boldsymbol{\Phi}^i{}_s{}^k + \Gamma^k{}_{\mu s}\boldsymbol{\Phi}^i{}_j{}^s\right)\boldsymbol{\Psi}_k{}^q + \boldsymbol{\Phi}^i{}_j{}^k\left(\frac{\partial \boldsymbol{\Psi}_k{}^q}{\partial x^\mu} - \Gamma^s{}_{\mu k}\boldsymbol{\Psi}_s{}^q + \Gamma^q{}_{\mu s}\boldsymbol{\Psi}_k{}^s\right)\\ &= \frac{\partial}{\partial x^\mu}\left(\boldsymbol{\Phi}^i{}_j{}^k \boldsymbol{\Psi}_k{}^q\right) + \left(\Gamma^i{}_{\mu s}\boldsymbol{\Phi}^s{}_j{}^k - \Gamma^s{}_{\mu j}\boldsymbol{\Phi}^i{}_s{}^k\right)\boldsymbol{\Psi}_k{}^q + \boldsymbol{\Phi}^i{}_j{}^k\left(\Gamma^q{}_{\mu s}\boldsymbol{\Psi}_k{}^s\right)\\ &\quad + \Gamma^k{}_{\mu s}\boldsymbol{\Phi}^i{}_j{}^s\boldsymbol{\Psi}_k{}^q - \Gamma^s{}_{\mu k}\boldsymbol{\Phi}^i{}_j{}^k\boldsymbol{\Psi}_s{}^q.\end{aligned} \tag{1.81}$$

而根据定义，则

$$\nabla_\mu\left(\boldsymbol{\Phi}^i{}_j{}^k \boldsymbol{\Psi}_k{}^q\right) \triangleq \frac{\partial}{\partial x^\mu}\left(\boldsymbol{\Phi}^i{}_j{}^k \boldsymbol{\Psi}_k{}^q\right) + \left(\Gamma^i{}_{\mu s}\boldsymbol{\Phi}^s{}_j{}^k - \Gamma^s{}_{\mu j}\boldsymbol{\Phi}^i{}_s{}^k\right)\boldsymbol{\Psi}_k{}^q + \boldsymbol{\Phi}^i{}_j{}^k\left(\Gamma^q{}_{\mu s}\boldsymbol{\Psi}_k{}^s\right). \tag{1.82}$$

很明显，式 (1.81) 中多了两项．不过稍作计算，就可知道

$$\Gamma^k{}_{\mu s}\boldsymbol{\Phi}^i{}_j{}^s\boldsymbol{\Psi}_k{}^q - \Gamma^s{}_{\mu k}\boldsymbol{\Phi}^i{}_j{}^k\boldsymbol{\Psi}_s{}^q$$

k 和 s 都是哑标，不妨在第二项中将二者交换：

$$= \Gamma^k{}_{\mu s}\boldsymbol{\Phi}^i{}_j{}^s\boldsymbol{\Psi}_k{}^q - \Gamma^k{}_{\mu s}\boldsymbol{\Phi}^i{}_j{}^s\boldsymbol{\Psi}_k{}^q = 0. \tag{1.83}$$

可见，Leibniz 法则经受住了考验．

把 Ricci 引理和 Leibniz 法则联合起来，便有

$$\begin{cases}\nabla_\mu\left(g_{ij}\boldsymbol{\Psi}^p{}_q\right) = \left(\nabla_\mu g_{ij}\right)\boldsymbol{\Psi}^p{}_q + g_{ij}\left(\nabla_\mu \boldsymbol{\Psi}^p{}_q\right) = g_{ij}\nabla_\mu \boldsymbol{\Psi}^p{}_q, & \text{(1.84-a)}\\ \nabla_\mu\left(\epsilon^i{}_j{}^k\boldsymbol{\Psi}^p{}_q\right) = \left(\nabla_\mu \epsilon^i{}_j{}^k\right)\boldsymbol{\Psi}^p{}_q + \epsilon^i{}_j{}^k\left(\nabla_\mu \boldsymbol{\Psi}^p{}_q\right) = \epsilon^i{}_j{}^k\nabla_\mu \boldsymbol{\Psi}^p{}_q. & \text{(1.84-b)}\end{cases}$$

这说明度量张量和 Eddington 张量类似常数，可以提到协变导数的外面．

1.4.3 混合协变导数

与混合偏导数定理类似，协变导数满足

$$\nabla_\nu\nabla_\mu \boldsymbol{\Phi}^i{}_j{}^k = \nabla_\mu\nabla_\nu \boldsymbol{\Phi}^i{}_j{}^k. \tag{1.85}$$

不必多说，张量分量依然可以任意选取．只是需要注意，该定理只在体积上张量场场论成立．

证明： 首先计算张量场整体的一阶偏导数：

$$\frac{\partial \boldsymbol{\Phi}}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}\left(\Phi^i{}_j{}^k \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right) = \nabla_\mu \Phi^i{}_j{}^k \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \in \mathscr{T}^3(\mathbb{R}^m). \tag{1.86}$$

再求一次偏导数，可有

$$\frac{\partial^2 \boldsymbol{\Phi}}{\partial x^\nu \partial x^\mu} := \frac{\partial}{\partial x^\nu}\left(\frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}\right) = \frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^i{}_j{}^k \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right). \tag{1.87}$$

请注意，括号里的张量带有一个独立指标 μ. 按照极限分析，有

$$\begin{aligned}
&\frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^i{}_j{}^k \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right) \\
&= \frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^i{}_j{}^k\right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \nabla_\mu \Phi^i{}_j{}^k \left(\frac{\partial \boldsymbol{g}_i}{\partial x^\nu} \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \boldsymbol{g}_i \otimes \frac{\partial \boldsymbol{g}^j}{\partial x^\nu} \otimes \boldsymbol{g}_k + \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \frac{\partial \boldsymbol{g}_k}{\partial x^\nu}\right) \\
&= \frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^i{}_j{}^k\right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k + \nabla_\mu \Phi^i{}_j{}^k \left(\Gamma^s{}_{\nu i} \boldsymbol{g}_s \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k - \Gamma^j{}_{\nu s} \boldsymbol{g}_i \otimes \boldsymbol{g}^s \otimes \boldsymbol{g}_k + \Gamma^s{}_{\nu k} \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_s\right) \\
&= \left[\frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^i{}_j{}^k\right) + \Gamma^i{}_{\nu s} \nabla_\mu \Phi^s{}_j{}^k - \Gamma^s{}_{\nu j} \nabla_\mu \Phi^i{}_s{}^k + \Gamma^k{}_{\nu s} \nabla_\mu \Phi^i{}_j{}^s\right] \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k.
\end{aligned} \tag{1.88}$$

但是 $\nabla_\mu \Phi^i{}_j{}^k$ 本身带有 4 个指标，因而

$$\nabla_\nu\left(\nabla_\mu \Phi^i{}_j{}^k\right) \triangleq \frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^i{}_j{}^k\right) + \Gamma^i{}_{\nu s} \nabla_\mu \Phi^s{}_j{}^k - \Gamma^s{}_{\nu j} \nabla_\mu \Phi^i{}_s{}^k + \Gamma^k{}_{\nu s} \nabla_\mu \Phi^i{}_j{}^s - \Gamma^s{}_{\nu\mu} \nabla_s \Phi^i{}_j{}^k. \tag{1.89}$$

代入 (1.88) 式，可得

$$\frac{\partial^2 \boldsymbol{\Phi}}{\partial x^\nu \partial x^\mu} = \frac{\partial}{\partial x^\nu}\left(\nabla_\mu \Phi^i{}_j{}^k \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right) = \left(\nabla_\nu \nabla_\mu \Phi^i{}_j{}^k + \Gamma^s{}_{\nu\mu} \nabla_s \Phi^i{}_j{}^k\right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k. \tag{1.90}$$

同理，

$$\frac{\partial^2 \boldsymbol{\Phi}}{\partial x^\mu \partial x^\nu} = \left(\nabla_\mu \nabla_\nu \Phi^i{}_j{}^k + \Gamma^s{}_{\mu\nu} \nabla_s \Phi^i{}_j{}^k\right) \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k. \tag{1.91}$$

根据 Christoffel 符号的性质，

$$\Gamma^s{}_{\nu\mu} = \Gamma^s{}_{\mu\nu}\,; \tag{1.92}$$

而按照一般赋范线性空间上的微分学, 当张量场具有足够正则性时, 成立

$$\frac{\partial^2 \boldsymbol{\Phi}}{\partial x^\nu \partial x^\mu} = \frac{\partial^2 \boldsymbol{\Phi}}{\partial x^\mu \partial x^\nu}. \tag{1.93}$$

这样就可得到

$$\nabla_\nu \nabla_\mu \Phi^i{}_j{}^k = \nabla_\mu \nabla_\nu \Phi^i{}_j{}^k. \tag{1.94}$$

□

1.5 场论恒等式（二）

本节将给出微分形式张量场场论中的若干恒等式，以及它们的推演过程.

1.5.1 微分算子

在 1.3.3 小节中，我们已经定义了**左梯度**

$$\left(\boldsymbol{\nabla}\otimes\boldsymbol{\Phi}\right)(\boldsymbol{x})\triangleq\left[\boldsymbol{g}^{\mu}\frac{\partial}{\partial x^{\mu}}(\boldsymbol{x})\right]\otimes\boldsymbol{\Phi}(\boldsymbol{x})\coloneqq\boldsymbol{g}^{\mu}(\boldsymbol{x})\otimes\frac{\partial\boldsymbol{\Phi}}{\partial x^{\mu}}(\boldsymbol{x}) \tag{1.95-a}$$

和（右）**梯度**

$$\left(\boldsymbol{\Phi}\otimes\boldsymbol{\nabla}\right)(\boldsymbol{x})\triangleq\boldsymbol{\Phi}(\boldsymbol{x})\otimes\left[\boldsymbol{g}^{\mu}\frac{\partial}{\partial x^{\mu}}(\boldsymbol{x})\right]\coloneqq\frac{\partial\boldsymbol{\Phi}}{\partial x^{\mu}}(\boldsymbol{x})\otimes\boldsymbol{g}^{\mu}(\boldsymbol{x}). \tag{1.95-b}$$

如果 $\boldsymbol{\Phi}(\boldsymbol{x})$ 是 r 阶张量，则左右梯度都是 $r+1$ 阶张量.

类似地，我们还可以定义**左散度**

$$\left(\boldsymbol{\nabla}\cdot\boldsymbol{\Phi}\right)(\boldsymbol{x})\triangleq\left[\boldsymbol{g}^{\mu}\frac{\partial}{\partial x^{\mu}}(\boldsymbol{x})\right]\cdot\boldsymbol{\Phi}(\boldsymbol{x})\coloneqq\boldsymbol{g}^{\mu}(\boldsymbol{x})\cdot\frac{\partial\boldsymbol{\Phi}}{\partial x^{\mu}}(\boldsymbol{x}) \tag{1.96-a}$$

和**右散度**

$$\left(\boldsymbol{\Phi}\cdot\boldsymbol{\nabla}\right)(\boldsymbol{x})\triangleq\boldsymbol{\Phi}(\boldsymbol{x})\cdot\left[\boldsymbol{g}^{\mu}\frac{\partial}{\partial x^{\mu}}(\boldsymbol{x})\right]\coloneqq\frac{\partial\boldsymbol{\Phi}}{\partial x^{\mu}}(\boldsymbol{x})\cdot\boldsymbol{g}^{\mu}(\boldsymbol{x}). \tag{1.96-b}$$

如果 $\boldsymbol{\Phi}(\boldsymbol{x})$ 是 r 阶张量，则左右散度都是 $r-1$ 阶张量.

当然，如果底空间是 $\mathbb{R}^3$，即 $\boldsymbol{\Phi}(\boldsymbol{x})\in\mathscr{T}^{r}\left(\mathbb{R}^3\right)$，还不能忘了定义**左旋度**

$$\left(\boldsymbol{\nabla}\times\boldsymbol{\Phi}\right)(\boldsymbol{x})\triangleq\left[\boldsymbol{g}^{\mu}\frac{\partial}{\partial x^{\mu}}(\boldsymbol{x})\right]\times\boldsymbol{\Phi}(\boldsymbol{x})\coloneqq\boldsymbol{g}^{\mu}(\boldsymbol{x})\times\frac{\partial\boldsymbol{\Phi}}{\partial x^{\mu}}(\boldsymbol{x}) \tag{1.97-a}$$

和**右旋度**

$$\left(\boldsymbol{\Phi}\times\boldsymbol{\nabla}\right)(\boldsymbol{x})\triangleq\boldsymbol{\Phi}(\boldsymbol{x})\times\left[\boldsymbol{g}^{\mu}\frac{\partial}{\partial x^{\mu}}(\boldsymbol{x})\right]\coloneqq\frac{\partial\boldsymbol{\Phi}}{\partial x^{\mu}}(\boldsymbol{x})\times\boldsymbol{g}^{\mu}(\boldsymbol{x}). \tag{1.97-b}$$

旋度不改变张量的阶数，即左右旋度仍属于 $\mathscr{T}^{r}\left(\mathbb{R}^3\right)$.

左右梯度、散度和梯度都是张量场中常用的微分算子．向量微积分中的梯度、散度和梯度，其实就是一阶张量的特殊情况.

1.5.2 推演举例

首先是为人熟知的“梯度场无旋，旋度场无源”：

$$\forall\boldsymbol{\Phi}\in\mathscr{T}^{r}\left(\mathbb{R}^3\right),\quad 有\begin{cases}\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\otimes\boldsymbol{\Phi}\right)=\boldsymbol{0}\in\mathscr{T}^{r+1}\left(\mathbb{R}^3\right), & (1.98\text{-a})\\ \boldsymbol{\nabla}\cdot\left(\boldsymbol{\nabla}\times\boldsymbol{\Phi}\right)=\boldsymbol{0}\in\mathscr{T}^{r-1}\left(\mathbb{R}^3\right). & (1.98\text{-b})\end{cases}$$

证明： 不失一般性，我们设 $\boldsymbol{\Phi}$ 是一个三阶张量．代入上一小节中梯度和旋度的定义，可有

$$\begin{aligned}\boldsymbol{\nabla}\times\left(\boldsymbol{\nabla}\otimes\boldsymbol{\Phi}\right)&=\left(\boldsymbol{g}^{\nu}\frac{\partial}{\partial x^{\nu}}\right)\times\left(\boldsymbol{g}^{\mu}\otimes\frac{\partial\boldsymbol{\Phi}}{\partial x^{\mu}}\right)\\&=\left(\boldsymbol{g}^{\nu}\frac{\partial}{\partial x^{\nu}}\right)\times\left(\nabla_{\mu}\Phi^{i}{}_{j}{}^{k}\boldsymbol{g}^{\mu}\otimes\boldsymbol{g}_{i}\otimes\boldsymbol{g}^{j}\otimes\boldsymbol{g}_{k}\right)\\&=\boldsymbol{g}^{\nu}\times\frac{\partial}{\partial x^{\nu}}\left(\nabla_{\mu}\Phi^{i}{}_{j}{}^{k}\boldsymbol{g}^{\mu}\otimes\boldsymbol{g}_{i}\otimes\boldsymbol{g}^{j}\otimes\boldsymbol{g}_{k}\right)\end{aligned}$$

与 (1.88) 式不同，第二个括号中的 μ、i、j、k 都是哑标，所以偏导数可以直接用协变导数表示，而不会出现多余的 Christoffel 符号：

$$= \boldsymbol{g}^{\nu} \times \left[\left(\nabla_{\nu} \nabla_{\mu} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \right) \boldsymbol{g}^{\mu} \otimes \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \right]$$

按照叉乘的定义（见 **??** 节），$\boldsymbol{g}^{\nu}$ 将与构成简单张量的第一个基向量相乘，即

$$= \left(\nabla_{\nu} \nabla_{\mu} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \right) (\boldsymbol{g}^{\nu} \times \boldsymbol{g}^{\mu}) \otimes \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k}$$

利用 Levi-Civita 记号展开叉乘项，有

$$= \epsilon^{\mu\nu s} \left(\nabla_{\nu} \nabla_{\mu} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \right) \boldsymbol{g}_{s} \otimes \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k}. \tag{1.99}$$

考虑交换哑标 μ、ν，结果必然保持不变．但是 Eddington 张量关于指标 $\mu\nu$ 反对称[①]，即 $\epsilon^{\mu\nu s} = -\epsilon^{\nu\mu s}$；另一方面，混合协变导数却又满足 $\nabla_{\nu} \nabla_{\mu} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} = \nabla_{\mu} \nabla_{\nu} \boldsymbol{\Phi}^{i}{}_{j}{}^{k}$，因此总的结果将变为其相反数．这样，结果只可能为零，即

$$\boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \otimes \boldsymbol{\Phi} \right) = \mathbf{0}. \tag{1.100}$$

同理，也可证明“旋度场无源”：

$$\begin{aligned}
\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \boldsymbol{\Phi} \right) &= \left(\boldsymbol{g}^{\nu} \frac{\partial}{\partial x^{\nu}} \right) \cdot \left(\boldsymbol{g}^{\mu} \times \frac{\partial \boldsymbol{\Phi}}{\partial x^{\mu}} \right) \\
&= \left(\boldsymbol{g}^{\nu} \frac{\partial}{\partial x^{\nu}} \right) \cdot \left[\boldsymbol{g}^{\mu} \times \left(\nabla_{\mu} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \, \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \right) \right] \\
&= \boldsymbol{g}^{\nu} \cdot \frac{\partial}{\partial x^{\nu}} \left[\boldsymbol{g}^{\mu} \times \left(\nabla_{\mu} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \, \boldsymbol{g}_{i} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \right) \right] \\
&= \boldsymbol{g}^{\nu} \cdot \frac{\partial}{\partial x^{\nu}} \left[\nabla_{\mu} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \left(\boldsymbol{g}^{\mu} \times \boldsymbol{g}_{i} \right) \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \right] \\
&= \boldsymbol{g}^{\nu} \cdot \frac{\partial}{\partial x^{\nu}} \left(\nabla_{\mu} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \epsilon^{\mu}{}_{i}{}^{s} \, \boldsymbol{g}_{s} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \right)
\end{aligned}$$

μ、s、i、j、k 全部是哑标，因此

$$= \boldsymbol{g}^{\nu} \cdot \left[\left(\nabla_{\nu} \nabla_{\mu} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \epsilon^{\mu}{}_{i}{}^{s} \right) \boldsymbol{g}_{s} \otimes \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \right]$$

点乘之后出来 Kronecker δ：

$$\begin{aligned}
&= \left(\nabla_{\nu} \nabla_{\mu} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \epsilon^{\mu}{}_{i}{}^{s} \right) \delta^{\nu}_{s} \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k} \\
&= \epsilon^{\mu}{}_{i}{}^{\nu} \left(\nabla_{\nu} \nabla_{\mu} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \right) \boldsymbol{g}^{j} \otimes \boldsymbol{g}_{k}.
\end{aligned} \tag{1.101}$$

交换哑标 μ、ν，仿上，便有

$$\boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \times \boldsymbol{\Phi} \right) = \mathbf{0}. \tag{1.102}$$

□

接下来回忆一下向量场旋度的复合

$$\forall \boldsymbol{A} \in \mathbb{R}^{3}, \quad \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \boldsymbol{A}) = \boldsymbol{\nabla} \left(\boldsymbol{\nabla} \cdot \boldsymbol{A} \right) - \boldsymbol{\nabla}^{2} \boldsymbol{A} \in \mathbb{R}^{3}, \tag{1.103}$$

① 根据式 (**??**)，Levi-Civita 记号由行列式定义，而行列式交换两列将改变符号．

式中，$\boldsymbol{\nabla}^2$ 称为 **Laplace 算子**，其定义为

$$\boldsymbol{\nabla}^2 \boldsymbol{A} \triangleq \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \otimes \boldsymbol{A}\right). \tag{1.104}$$

推广到张量场上，我们有如下两式：

$$\forall \boldsymbol{\Phi} \in \mathscr{T}^r\left(\mathbb{R}^3\right), \quad \begin{cases} \boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \boldsymbol{\Phi}\right) = \boldsymbol{\nabla} \otimes \left(\boldsymbol{\nabla} \cdot \boldsymbol{\Phi}\right) - \boldsymbol{\nabla}^2 \boldsymbol{\Phi}, & \text{(1.105-a)} \\ \left(\boldsymbol{\Phi} \times \boldsymbol{\nabla}\right) \times \boldsymbol{\nabla} = \left(\boldsymbol{\Phi} \cdot \boldsymbol{\nabla}\right) \otimes \boldsymbol{\nabla} - \boldsymbol{\Phi} \boldsymbol{\nabla}^2, & \text{(1.105-b)} \end{cases}$$

其中，Laplace 算子的定义为

$$\begin{cases} \boldsymbol{\nabla}^2 \boldsymbol{\Phi} \triangleq \boldsymbol{\nabla} \cdot \left(\boldsymbol{\nabla} \otimes \boldsymbol{\Phi}\right), & \text{(1.106-a)} \\ \boldsymbol{\Phi} \boldsymbol{\nabla}^2 \triangleq \left(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\right) \cdot \boldsymbol{\nabla}. & \text{(1.106-b)} \end{cases}$$

证明： 同样，我们以三阶张量为例进行计算．代入旋度的定义，有

$$\begin{aligned} \boldsymbol{\nabla} \times \left(\boldsymbol{\nabla} \times \boldsymbol{\Phi}\right) &= \left(\boldsymbol{g}^\nu \frac{\partial}{\partial x^\nu}\right) \times \left(\boldsymbol{g}^\mu \times \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}\right) \\ &= \boldsymbol{g}^\nu \times \frac{\partial}{\partial x^\nu} \left[\boldsymbol{g}^\mu \times \left(\nabla_\mu \Phi^i{}_j{}^k \, \boldsymbol{g}_i \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right)\right] \\ &= \boldsymbol{g}^\nu \times \frac{\partial}{\partial x^\nu} \left(\nabla_\mu \Phi^i{}_j{}^k \epsilon^\mu{}_i{}^s \, \boldsymbol{g}_s \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right) \\ &= \boldsymbol{g}^\nu \times \left[\left(\nabla_\nu \nabla_\mu \Phi^i{}_j{}^k \epsilon^\mu{}_i{}^s\right) \boldsymbol{g}_s \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k\right] \\ &= \epsilon^\mu{}_i{}^s \epsilon^\nu{}_s{}^t \left(\nabla_\nu \nabla_\mu \Phi^i{}_j{}^k\right) \boldsymbol{g}_t \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k \end{aligned}$$

为了使用“前前后后，里里外外”之法则（见 **??** 小节），需要交换第二个 Eddington 张量的指标 s 挪到最后，不要忘了添上负号：

$$= -\epsilon^\mu{}_i{}^s \epsilon^{\nu t}{}_s \left(\nabla_\nu \nabla_\mu \Phi^i{}_j{}^k\right) \boldsymbol{g}_t \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k$$

这样就可以顺利用上口诀：

$$= -\left(g^{\mu\nu} \delta_i^t - \delta_i^\nu g^{\mu t}\right) \left(\nabla_\nu \nabla_\mu \Phi^i{}_j{}^k\right) \boldsymbol{g}_t \otimes \boldsymbol{g}^j \otimes \boldsymbol{g}_k .$$

接下来，形式上引入**逆变导数**

$$\nabla^\mu \coloneqq g^{\mu\nu} \nabla_\nu , \tag{1.107}$$

则上式可化为 □