第一章　非完整基理论

1.1　完整基与非完整基的概念

在 ?? 节中，我们利用曲线坐标系 $\boldsymbol{X}(\boldsymbol{x})$ 构造了 $\mathbb{R}^m$ 上的一组（局部协变）基

$$\left\{ \boldsymbol{g}_i(\boldsymbol{x}) = \frac{\partial \boldsymbol{X}}{\partial x^i}(\boldsymbol{x}) \right\}_{i=1}^{m} \subset \mathbb{R}^m, \tag{1.1}$$

它们称为**完整基**．与之对应，不是由曲线坐标系诱导的基，称为**非完整基**．

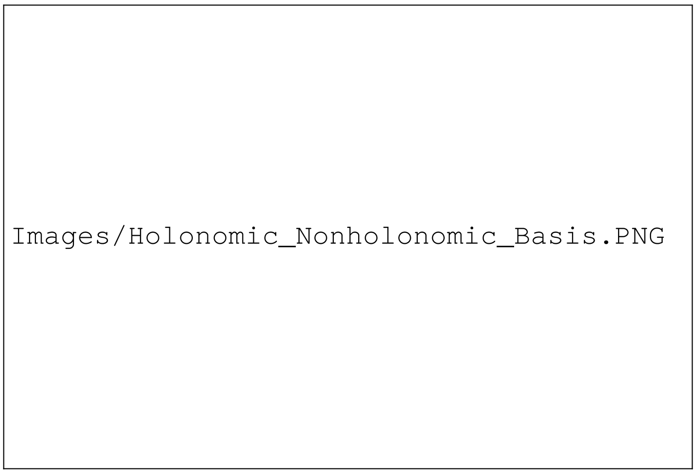

图 1.1: 完整基与非完整基

如图 1.1，x^i-线的切向量构成一组局部协变基 $\left\{\boldsymbol{g}_i(\boldsymbol{x})\right\}_{i=1}^m$，它和它的对偶 $\left\{\boldsymbol{g}^i(\boldsymbol{x})\right\}_{i=1}^m$ 都是完整基．除此以外，我们当然可以选取另外的基 $\left\{\boldsymbol{g}_{(i)}(\boldsymbol{x})\right\}_{i=1}^m$ 和 $\left\{\boldsymbol{g}^{(i)}(\boldsymbol{x})\right\}_{i=1}^m$，它们不是由曲线坐标系诱导，因而是非完整基．

1.2　非完整基下的张量梯度

下面我们来考察张量梯度在非完整基下的表达形式．在 ?? 节中，我们已经推导出了张量场的（右）梯度：

$$\left(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\right)(\boldsymbol{x}) \triangleq \frac{\partial \boldsymbol{\Phi}}{\partial x^\mu}(\boldsymbol{x}) \otimes \boldsymbol{g}^\mu(\boldsymbol{x}) = \nabla_\mu \Phi^{i\ k}_{\ j}(\boldsymbol{x})\, \boldsymbol{g}_i(\boldsymbol{x}) \otimes \boldsymbol{g}^j(\boldsymbol{x}) \otimes \boldsymbol{g}_k(\boldsymbol{x}) \otimes \boldsymbol{g}^\mu(\boldsymbol{x}). \tag{1.2}$$

这是一个四阶张量，对应的张量分量可记作

$$\left(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\right)^{i\ k}_{\ j\ \mu}(\boldsymbol{x}) := \nabla_\mu \Phi^{i\ k}_{\ j}(\boldsymbol{x}). \tag{1.3}$$

除此以外，其他的基当然也可以用来表示该张量，比如前文提到过的 $\left\{g_{(i)}(x)\right\}_{i=1}^{m}$ 和 $\left\{g^{(i)}(x)\right\}_{i=1}^{m}$，它们都是非完整基.

非完整基与完整基之间的关系，可以利用 **??** 小节中引入的坐标转换关系来获得：

$$\begin{cases} g_{(i)}(x) = c_{(i)}^{k}(x)\, g_k(x), & (1.4\text{-a}) \\ g^{(i)}(x) = c_{k}^{(i)}(x)\, g^k(x); & (1.4\text{-b}) \\ g_i(x) = c_i^{(k)}(x)\, g_{(k)}(x), & (1.4\text{-c}) \\ g^i(x) = c_{(k)}^{i}(x)\, g^{(k)}(x). & (1.4\text{-d}) \end{cases}$$

2017-01-20　坐标转换关系

其中的基转换系数都是已知量，它们的定义如下：[①]

$$\begin{cases} c_{(i)}^{j}(x) := \left\langle g_{(i)}(x), g^j(x) \right\rangle_{\mathbb{R}^m}, & (1.5\text{-a}) \\ c_j^{(i)}(x) := \left\langle g^{(i)}(x), g_j(x) \right\rangle_{\mathbb{R}^m}. & (1.5\text{-b}) \end{cases}$$

代入 (1.2) 式，可有[②]

$$\begin{aligned} \boldsymbol{\Phi} \otimes \boldsymbol{\nabla} &= \nabla_\mu \Phi^{i\ k}_{\ j} \left(g_i \otimes g^j \otimes g_k \otimes g^\mu \right) \\ &= \nabla_\mu \Phi^{i\ k}_{\ j}(x) \left[\left(c_i^{(p)} g_{(p)} \right) \otimes \left(c_{(q)}^{j} g^{(q)} \right) \otimes \left(c_k^{(r)} g_{(r)} \right) \otimes \left(c_{(\alpha)}^{\mu} g^{(\alpha)} \right) \right] \end{aligned}$$

根据线性性，提出系数：

$$= \left(c_i^{(p)} c_{(q)}^{j} c_k^{(r)} c_{(\alpha)}^{\mu} \nabla_\mu \Phi^{i\ k}_{\ j} \right) \left(g_{(p)} \otimes g^{(q)} \otimes g_{(r)} \otimes g^{(\alpha)} \right)$$

写成张量分量与简单张量“乘积”的形式，即为

$$=: \left(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla} \right)^{(p)\quad (r)}_{\quad (q)\quad (\alpha)} \left(g_{(p)} \otimes g^{(q)} \otimes g_{(r)} \otimes g^{(\alpha)} \right). \tag{1.6}$$

这样，我们就获得了非完整基下张量梯度的表示．再利用式 (1.3)，可知

$$\left(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla} \right)^{(p)\quad (r)}_{\quad (q)\quad (\alpha)} = c_i^{(p)} c_{(q)}^{j} c_k^{(r)} c_{(\alpha)}^{\mu} \left(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla} \right)^{i\ k}_{\ j\ \mu}. \tag{1.7}$$

以上结果与 **??** 小节中的推导是完全一致的.

1.3　非完整基的形式运算

在 1.2 节中，我们利用坐标转换关系获得了张量梯度在非完整基下的表示．而在本节，我们将通过定义，建立所谓“形式理论”，获得一套更统一、更连贯的表述.

2017-01-31　统一、连贯?

首先需要给出一些定义.

① 只有两个基转换系数的原因是内积具有交换律.

② 这里我们省略了“(x)”.

1. **形式偏导数**：

$$\frac{\partial}{\partial x^{(\mu)}} \triangleq c^l_{(\mu)} \frac{\partial}{\partial x^l}. \tag{1.8}$$

注意 $\partial/\partial x^{(\mu)}$ 本身是不能用极限形式来定义的，因为曲线坐标系中并不存在有 $x^{(\mu)}$ 坐标线.

2. **形式 Christoffel 符号**：

$$\Gamma^{(\gamma)}{}_{(\alpha)(\beta)} \triangleq c^i_{(\alpha)} c^j_{(\beta)} c^{(\gamma)}_k \Gamma^k{}_{ij} - c^i_{(\alpha)} c^j_{(\beta)} \frac{\partial c^{(\gamma)}_j}{\partial x^i} = c^i_{(\alpha)} c^j_{(\beta)} \left(c^{(\gamma)}_k \Gamma^k{}_{ij} - \frac{\partial c^{(\gamma)}_j}{\partial x^i} \right). \tag{1.9}$$

2017-01-31　第一类形式 Christoffel 符号

3. **形式协变导数**．我们以三阶张量 $\boldsymbol{\Phi}$ 为例给出定义．$\boldsymbol{\Phi}$ 在非完整基下可以用混合分量表示如下：

$$\Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)} := \boldsymbol{\Phi}\left(\boldsymbol{g}^{(\alpha)}, \boldsymbol{g}_{(\beta)}, \boldsymbol{g}^{(\gamma)}\right). \tag{1.10}$$

它相对 $x^{(\mu)}$ 分量的形式协变导数为

$$\nabla_{(\mu)} \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)} \triangleq \frac{\partial \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)}}{\partial x^{(\mu)}} + \Gamma^{(\alpha)}{}_{(\mu)(\sigma)} \Phi^{(\sigma)}{}_{(\beta)}{}^{(\gamma)} - \Gamma^{(\sigma)}{}_{(\mu)(\beta)} \Phi^{(\alpha)}{}_{(\sigma)}{}^{(\gamma)} + \Gamma^{(\gamma)}{}_{(\mu)(\sigma)} \Phi^{(\alpha)}{}_{(\beta)}{}^{(\sigma)}. \tag{1.11}$$

回顾 **??** 节，(**??**) 式给出了完整基下协变导数的定义：

$$\nabla_l \Phi^i{}_j{}^k \triangleq \frac{\partial \Phi^i{}_j{}^k}{\partial x^l} + \Gamma^i{}_{ls} \Phi^s{}_j{}^k - \Gamma^s{}_{lj} \Phi^i{}_s{}^k + \Gamma^k{}_{ls} \Phi^i{}_j{}^s. \tag{1.12}$$

可以看出形式协变导数的定义与它几乎一模一样.

接下来我们要证明

$$\nabla_{(\mu)} \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)} = c^l_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \nabla_l \Phi^i{}_j{}^k. \tag{1.13}$$

代入式 (1.3) 和 (1.7)，即得

$$\nabla_{(\mu)} \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)} = \left(\boldsymbol{\Phi} \otimes \boldsymbol{\nabla}\right)^{(p)}{}_{(q)}{}^{(r)}{}_{(\alpha)}. \tag{1.14}$$

换句话说，此处我们正是要验证这种“形式理论”与 1.2 节中坐标转换关系的一致性.

证明： 左边按照 (1.11) 式展开，第一项为

$$\frac{\partial \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)}}{\partial x^{(\mu)}} = c^l_{(\mu)} \frac{\partial \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)}}{\partial x^l}$$

这里用到了形式偏导数的定义 (1.8) 式．然后利用坐标转换关系展开张量分量：

$$= c^l_{(\mu)} \frac{\partial}{\partial x^l} \left(c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \Phi^i{}_j{}^k \right)$$

再按照通常的偏导数法则直接打开：

$$\begin{aligned} &= c^l_{(\mu)} c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial c^{(\alpha)}_i}{\partial x^l} \Phi^i{}_j{}^k + c^l_{(\mu)} c^{(\alpha)}_i c^{(\gamma)}_k \frac{\partial c^j_{(\beta)}}{\partial x^l} \Phi^i{}_j{}^k + c^l_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} \frac{\partial c^{(\gamma)}_k}{\partial x^l} \Phi^i{}_j{}^k \\ &\quad + c^l_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial \Phi^i{}_j{}^k}{\partial x^l} \end{aligned}$$

$$
\begin{aligned}
&= c^l_{(\mu)} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \left(c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial c^{(\alpha)}_i}{\partial x^l} + c^{(\alpha)}_i c^{(\gamma)}_k \frac{\partial c^j_{(\beta)}}{\partial x^l} + c^{(\alpha)}_i c^j_{(\beta)} \frac{\partial c^{(\gamma)}_k}{\partial x^l} \right) \\
&\quad + c^l_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial \boldsymbol{\Phi}^{i}{}_{j}{}^{k}}{\partial x^l} .
\end{aligned} \tag{1.15}
$$

接下来处理含有形式 Christoffel 符号的三项，分别是

$$
\begin{aligned}
\Gamma^{(\alpha)}{}_{(\mu)(\sigma)} \boldsymbol{\Phi}^{(\sigma)}{}_{(\beta)}{}^{(\gamma)} &= c^p_{(\mu)} c^q_{(\sigma)} \left(c^{(\alpha)}_s \Gamma^s{}_{pq} - \frac{\partial c^{(\alpha)}_q}{\partial x^p} \right) \cdot \boldsymbol{\Phi}^{(\sigma)}{}_{(\beta)}{}^{(\gamma)} \\
&= c^p_{(\mu)} c^q_{(\sigma)} \left(c^{(\alpha)}_s \Gamma^s{}_{pq} - \frac{\partial c^{(\alpha)}_q}{\partial x^p} \right) \cdot c^{(\sigma)}_i c^j_{(\beta)} c^{(\gamma)}_k \boldsymbol{\Phi}^{i}{}_{j}{}^{k}
\end{aligned}
$$

根据式 (**??**)，我们有 $c^q_{(\sigma)} c^{(\sigma)}_i = \delta^q_i$，于是

$$
= c^p_{(\mu)} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \left(c^{(\alpha)}_s c^j_{(\beta)} c^{(\gamma)}_k \Gamma^s{}_{pi} - c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial c^{(\alpha)}_i}{\partial x^p} \right); \tag{1.16-a}
$$

$$
\begin{aligned}
-\Gamma^{(\sigma)}{}_{(\mu)(\beta)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\sigma)}{}^{(\gamma)} &= -c^p_{(\mu)} c^q_{(\beta)} \left(c^{(\sigma)}_s \Gamma^s{}_{pq} - \frac{\partial c^{(\sigma)}_q}{\partial x^p} \right) \cdot \boldsymbol{\Phi}^{(\alpha)}{}_{(\sigma)}{}^{(\gamma)} \\
&= -c^p_{(\mu)} c^q_{(\beta)} \left(c^{(\sigma)}_s \Gamma^s{}_{pq} - \frac{\partial c^{(\sigma)}_q}{\partial x^p} \right) \cdot c^{(\alpha)}_i c^j_{(\sigma)} c^{(\gamma)}_k \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \\
&= c^p_{(\mu)} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \left(-c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k \Gamma^j{}_{pq} + c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k c^j_{(\sigma)} \frac{\partial c^{(\sigma)}_q}{\partial x^p} \right);
\end{aligned} \tag{1.16-b}
$$

$$
\begin{aligned}
\Gamma^{(\gamma)}{}_{(\mu)(\sigma)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\sigma)} &= c^p_{(\mu)} c^q_{(\sigma)} \left(c^{(\gamma)}_s \Gamma^s{}_{pq} - \frac{\partial c^{(\gamma)}_q}{\partial x^p} \right) \cdot \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\sigma)} \\
&= c^p_{(\mu)} c^q_{(\sigma)} \left(c^{(\gamma)}_s \Gamma^s{}_{pq} - \frac{\partial c^{(\gamma)}_q}{\partial x^p} \right) \cdot c^{(\alpha)}_i c^j_{(\beta)} c^{(\sigma)}_k \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \\
&= c^p_{(\mu)} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \left(c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_s \Gamma^s{}_{pk} - c^{(\alpha)}_i c^j_{(\beta)} \frac{\partial c^{(\gamma)}_k}{\partial x^p} \right).
\end{aligned} \tag{1.16-c}
$$

以上三式都有公因子 $c^p_{(\mu)} \boldsymbol{\Phi}^{i}{}_{j}{}^{k}$．为了进一步化简，不妨将哑标 p 换为 l．这样可有

$$
\begin{aligned}
&\Gamma^{(\alpha)}{}_{(\mu)(\sigma)} \boldsymbol{\Phi}^{(\sigma)}{}_{(\beta)}{}^{(\gamma)} - \Gamma^{(\sigma)}{}_{(\mu)(\beta)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\sigma)}{}^{(\gamma)} + \Gamma^{(\gamma)}{}_{(\mu)(\sigma)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\sigma)} \\
&= c^l_{(\mu)} \boldsymbol{\Phi}^{i}{}_{j}{}^{k} \Bigg[\left(c^{(\alpha)}_s c^j_{(\beta)} c^{(\gamma)}_k \Gamma^s{}_{li} - c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k \Gamma^j{}_{lq} + c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_s \Gamma^s{}_{lk} \right) \\
&\qquad - c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial c^{(\alpha)}_i}{\partial x^l} + c^{(\alpha)}_i c^q_{(\beta)} c^{(\gamma)}_k c^j_{(\sigma)} \frac{\partial c^{(\sigma)}_q}{\partial x^l} - c^{(\alpha)}_i c^j_{(\beta)} \frac{\partial c^{(\gamma)}_k}{\partial x^l} \Bigg] .
\end{aligned} \tag{1.17}
$$

该式与 (1.15) 式相加，得

$$
\nabla_{(\mu)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\gamma)} \triangleq \frac{\partial \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\gamma)}}{\partial x^{(\mu)}} + \Gamma^{(\alpha)}{}_{(\mu)(\sigma)} \boldsymbol{\Phi}^{(\sigma)}{}_{(\beta)}{}^{(\gamma)} - \Gamma^{(\sigma)}{}_{(\mu)(\beta)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\sigma)}{}^{(\gamma)} + \Gamma^{(\gamma)}{}_{(\mu)(\sigma)} \boldsymbol{\Phi}^{(\alpha)}{}_{(\beta)}{}^{(\sigma)}
$$

$$
\begin{aligned}
&= c^l_{(\mu)} c_i^{(\alpha)} c^j_{(\beta)} c_k^{(\gamma)} \frac{\partial \Phi^i{}_j{}^k}{\partial x^l} + c^l_{(\mu)} \Phi^i{}_j{}^k \Bigg[c^j_{(\beta)} c_k^{(\gamma)} \frac{\partial c_i^{(\alpha)}}{\partial x^l} + c_i^{(\alpha)} c_k^{(\gamma)} \frac{\partial c^j_{(\beta)}}{\partial x^l} + c_i^{(\alpha)} c^j_{(\beta)} \frac{\partial c_k^{(\gamma)}}{\partial x^l} \\
&\qquad + \left(c_s^{(\alpha)} c^j_{(\beta)} c_k^{(\gamma)} \Gamma^s{}_{li} - c_i^{(\alpha)} c^q_{(\beta)} c_k^{(\gamma)} \Gamma^j{}_{lq} + c_i^{(\alpha)} c^j_{(\beta)} c_s^{(\gamma)} \Gamma^s{}_{lk} \right) \\
&\qquad - c^j_{(\beta)} c_k^{(\gamma)} \frac{\partial c_i^{(\alpha)}}{\partial x^l} + c_i^{(\alpha)} c^q_{(\beta)} c_k^{(\gamma)} c^j_{(\sigma)} \frac{\partial c_q^{(\sigma)}}{\partial x^l} - c_i^{(\alpha)} c^j_{(\beta)} \frac{\partial c_k^{(\gamma)}}{\partial x^l} \Bigg]
\end{aligned}
$$

高亮部分相互抵消：

$$
\begin{aligned}
&= c^l_{(\mu)} c_i^{(\alpha)} c^j_{(\beta)} c_k^{(\gamma)} \frac{\partial \Phi^i{}_j{}^k}{\partial x^l} + c^l_{(\mu)} \Phi^i{}_j{}^k \Bigg[\left(c_s^{(\alpha)} c^j_{(\beta)} c_k^{(\gamma)} \Gamma^s{}_{li} - c_i^{(\alpha)} c^q_{(\beta)} c_k^{(\gamma)} \Gamma^j{}_{lq} + c_i^{(\alpha)} c^j_{(\beta)} c_s^{(\gamma)} \Gamma^s{}_{lk} \right) \\
&\qquad + c_i^{(\alpha)} c_k^{(\gamma)} \frac{\partial c^j_{(\beta)}}{\partial x^l} + c_i^{(\alpha)} c^q_{(\beta)} c_k^{(\gamma)} c^j_{(\sigma)} \frac{\partial c_q^{(\sigma)}}{\partial x^l} \Bigg]
\end{aligned} \tag{1.18}
$$

注意到 $c^j_{(\beta)} = c^q_{(\beta)} \delta^j_q = c^q_{(\beta)} c^j_{(\sigma)} c_q^{(\sigma)}$，因此

$$
\frac{\partial c^j_{(\beta)}}{\partial x^l} = \frac{\partial}{\partial x^l} \left(c^q_{(\beta)} c^j_{(\sigma)} c_q^{(\sigma)} \right) = c^j_{(\sigma)} c_q^{(\sigma)} \frac{\partial c^q_{(\beta)}}{\partial x^l} + c^q_{(\beta)} c_q^{(\sigma)} \frac{\partial c^j_{(\sigma)}}{\partial x^l} + c^q_{(\beta)} c^j_{(\sigma)} \frac{\partial c_q^{(\sigma)}}{\partial x^l}. \tag{1.19}
$$

所以 (1.18) 式中最后一步的第二行就能够写成

$$
\begin{aligned}
& c_i^{(\alpha)} c_k^{(\gamma)} \frac{\partial c^j_{(\beta)}}{\partial x^l} + c_i^{(\alpha)} c^q_{(\beta)} c_k^{(\gamma)} c^j_{(\sigma)} \frac{\partial c_q^{(\sigma)}}{\partial x^l} \\
&= c_i^{(\alpha)} c_k^{(\gamma)} \left(\frac{\partial c^j_{(\beta)}}{\partial x^l} + c^q_{(\beta)} c^j_{(\sigma)} \frac{\partial c_q^{(\sigma)}}{\partial x^l} \right) \\
&= c_i^{(\alpha)} c_k^{(\gamma)} \left(c^j_{(\sigma)} c_q^{(\sigma)} \frac{\partial c^q_{(\beta)}}{\partial x^l} + c^q_{(\beta)} c_q^{(\sigma)} \frac{\partial c^j_{(\sigma)}}{\partial x^l} + c^q_{(\beta)} c^j_{(\sigma)} \frac{\partial c_q^{(\sigma)}}{\partial x^l} + c^q_{(\beta)} c^j_{(\sigma)} \frac{\partial c_q^{(\sigma)}}{\partial x^l} \right)
\end{aligned}
$$

合并同类项：

$$
\begin{aligned}
&= c_i^{(\alpha)} c_k^{(\gamma)} \left[c^j_{(\sigma)} \left(c_q^{(\sigma)} \frac{\partial c^q_{(\beta)}}{\partial x^l} + c^q_{(\beta)} \frac{\partial c_q^{(\sigma)}}{\partial x^l} \right) + c^q_{(\beta)} \left(c_q^{(\sigma)} \frac{\partial c^j_{(\sigma)}}{\partial x^l} + c^j_{(\sigma)} \frac{\partial c_q^{(\sigma)}}{\partial x^l} \right) \right] \\
&= c_i^{(\alpha)} c_k^{(\gamma)} \left[c^j_{(\sigma)} \frac{\partial}{\partial x^l} \left(c_q^{(\sigma)} c^q_{(\beta)} \right) + c^q_{(\beta)} \frac{\partial}{\partial x^l} \left(c_q^{(\sigma)} c^j_{(\sigma)} \right) \right]
\end{aligned}
$$

再次利用式 (**??**)，可得

$$
= c_i^{(\alpha)} c_k^{(\gamma)} \left(c^j_{(\sigma)} \frac{\partial \delta^\sigma_\beta}{\partial x^l} + c^q_{(\beta)} \frac{\partial \delta^j_q}{\partial x^l} \right) = 0. \tag{1.20}
$$

代回式 (1.18)，有

$$
\begin{aligned}
\nabla_{(\mu)} \Phi^{(\alpha)}{}_{(\beta)}{}^{(\gamma)} &= c^l_{(\mu)} c_i^{(\alpha)} c^j_{(\beta)} c_k^{(\gamma)} \frac{\partial \Phi^i{}_j{}^k}{\partial x^l} + c^l_{(\mu)} \Phi^i{}_j{}^k \left(c_s^{(\alpha)} c^j_{(\beta)} c_k^{(\gamma)} \Gamma^s{}_{li} - c_i^{(\alpha)} c^q_{(\beta)} c_k^{(\gamma)} \Gamma^j{}_{lq} + c_i^{(\alpha)} c^j_{(\beta)} c_s^{(\gamma)} \Gamma^s{}_{lk} \right) \\
&= c^l_{(\mu)} c_i^{(\alpha)} c^j_{(\beta)} c_k^{(\gamma)} \frac{\partial \Phi^i{}_j{}^k}{\partial x^l} + c^l_{(\mu)} \left(c_s^{(\alpha)} c^j_{(\beta)} c_k^{(\gamma)} \Gamma^s{}_{li} \Phi^i{}_j{}^k - c_i^{(\alpha)} c^q_{(\beta)} c_k^{(\gamma)} \Gamma^j{}_{lq} \Phi^i{}_j{}^k + c_i^{(\alpha)} c^j_{(\beta)} c_s^{(\gamma)} \Gamma^s{}_{lk} \Phi^i{}_j{}^k \right)
\end{aligned}
$$

下面要对哑标进行重排. 括号里的第一项: $s \leftrightarrow i$; 第二项: $j \to s, q \to j$; 第三项: $s \leftrightarrow k$. 于是

$$
\begin{aligned}
&= c^l_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \frac{\partial \boldsymbol{\Phi}^{i\ k}_{\ j}}{\partial x^l} + c^l_{(\mu)} \left(c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \boldsymbol{\Gamma}^i_{ls} \boldsymbol{\Phi}^{s\ k}_{\ j} - c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \boldsymbol{\Gamma}^s_{lj} \boldsymbol{\Phi}^{i\ k}_{\ s} + c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \boldsymbol{\Gamma}^k_{ls} \boldsymbol{\Phi}^{i\ s}_{\ j} \right) \\
&= c^l_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \left(\frac{\partial \boldsymbol{\Phi}^{i\ k}_{\ j}}{\partial x^l} + \boldsymbol{\Gamma}^i_{ls} \boldsymbol{\Phi}^{s\ k}_{\ j} - \boldsymbol{\Gamma}^s_{lj} \boldsymbol{\Phi}^{i\ k}_{\ s} + \boldsymbol{\Gamma}^k_{ls} \boldsymbol{\Phi}^{i\ s}_{\ j} \right) \\
&= c^l_{(\mu)} c^{(\alpha)}_i c^j_{(\beta)} c^{(\gamma)}_k \nabla_l \boldsymbol{\Phi}^{i\ k}_{\ j} .
\end{aligned} \tag{1.21}
$$

这就完成了证明. □

如前文所言, 此种形式理论与我们在 1.2 节中所使用的方法 (坐标转换) 并无二致, 但它在某些特定情况下将会十分有用, 这就是下一节要介绍的内容.

1.4 单位正交基

1.4.1 选取非完整基

实际情况下, 为了计算的方便, 我们通常会取一组**正交基**作为完整基, 它们满足

$$
\left\langle \boldsymbol{g}_i, \boldsymbol{g}_j \right\rangle_{\mathbb{R}^m} = 0, \quad i \neq j . \tag{1.22}
$$

注意此处的 $\boldsymbol{g}_i$ 和 $\boldsymbol{g}_j$ 都是协变基.

2017-02-01　为什么不直接取单位正交基

这样, 度量 g_{ij} 就可以用矩阵形式写成

$$
\left[g_{ij} \right] = \begin{bmatrix} g_{11} & & \\ & \ddots & \\ & & g_{mm} \end{bmatrix}, \tag{1.23}
$$

它是一个对角矩阵. 根据式 (??), 我们有

$$
\left[g_{ik} \right] \left[g^{kj} \right] = \left[\delta^j_i \right] = \boldsymbol{I}_m ; \tag{1.24}
$$

而根据线性代数的知识, 对角矩阵的逆同样是对角阵, 因此 $\left[g^{ij} \right]$ 也是一个对角矩阵. 换句话说, 逆变基同样是一组正交基.

出于量纲一致等因素的考虑, 我们常常需要将正交基单位化, 使其成为**单位正交基**. 当然, 之前的完整基也就成了非完整基:

$$
\boldsymbol{g}_{(\alpha)} \triangleq c^i_{(\alpha)} \boldsymbol{g}_i , \tag{1.25}
$$

式中,

$$
c^i_{(\alpha)} = \begin{cases} \dfrac{1}{\sqrt{g_{ii}}}, & i = \alpha ; \\ 0, & i \neq \alpha . \end{cases} \tag{1.26}
$$

这里 g_{ii} 中的指标 i 不求和. [①] 对于逆变基，也是同样的：

$$\boldsymbol{g}^{(\alpha)} \triangleq c_i^{(\alpha)} \boldsymbol{g}^i, \tag{1.27}$$

其中的

$$c_i^{(\alpha)} = \begin{cases} \sqrt{g_{ii}}, & i = \alpha; \\ 0, & i \neq \alpha. \end{cases} \tag{1.28}$$

1.4.2　形式运算

首先考虑形式偏导数：

$$\frac{\partial}{\partial x^{(\mu)}} \triangleq c^l_{(\mu)} \frac{\partial}{\partial x^l}. \tag{1.29}$$

一般来说，$\boldsymbol{x} \in \mathbb{R}^m$，因而该式包含着 m 项的求和. 但在完整基是正交基、非完整基是单位正交基的情况下，系数 $c^l_{(\mu)}$ 仅在 $l = \mu$ 的时候才有非零值. 所以

$$\frac{\partial}{\partial x^{(\mu)}} = c^\mu_{(\mu)} \frac{\partial}{\partial x^\mu} = \frac{1}{\sqrt{g_{\mu\mu}}} \frac{\partial}{\partial x^\mu}. \tag{1.30}$$

指标 μ 不求和，此式便只剩下了一项.

下面处理形式 Christoffel 符号：

$$\Gamma^{(\gamma)}{}_{(\alpha)(\beta)} \triangleq c^i_{(\alpha)} c^j_{(\beta)} c^{(\gamma)}_k \Gamma^k{}_{ij} - c^i_{(\alpha)} c^j_{(\beta)} \frac{\partial c^{(\gamma)}_j}{\partial x^i}$$

同样，特殊情况下只需要考虑非零值：

$$= c^\alpha_{(\alpha)} c^\beta_{(\beta)} c^{(\gamma)}_\gamma \Gamma^\gamma{}_{\alpha\beta} - c^\alpha_{(\alpha)} c^\beta_{(\beta)} \frac{\partial c^{(\gamma)}_\beta}{\partial x^\alpha}$$

代入式 (1.26) 和 (1.28)，可得

$$= \frac{1}{\sqrt{g_{\alpha\alpha}}} \frac{1}{\sqrt{g_{\beta\beta}}} \sqrt{g_{\gamma\gamma}} \cdot \Gamma^\gamma{}_{\alpha\beta} - \frac{1}{\sqrt{g_{\alpha\alpha}}} \frac{1}{\sqrt{g_{\beta\beta}}} \cdot \frac{\partial c^{(\gamma)}_\beta}{\partial x^\alpha}. \tag{1.31}$$

此处的指标 α、β、γ 均不表示求和.

上式包含一个完整基下的 Christoffel 符号 $\Gamma^\gamma{}_{\alpha\beta}$. 根据 **??** 小节中的 (**??**) 式和 (**??**) 式，很容易利用度量把它计算出来：

$$\Gamma^\gamma{}_{\alpha\beta} = g^{\gamma s} \Gamma_{\alpha\beta,s} = g^{\gamma s} \cdot \frac{1}{2} \left(\frac{\partial g_{\beta s}}{\partial x^\alpha} + \frac{\partial g_{\alpha s}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^s} \right)$$

注意指标 s 需要求和！但是由于度量的非对角元均为零，所以可以直接写成

$$= g^{\gamma\gamma} \Gamma_{\alpha\beta,\gamma} = \frac{1}{g_{\gamma\gamma}} \cdot \frac{1}{2} \left(\frac{\partial g_{\beta\gamma}}{\partial x^\alpha} + \frac{\partial g_{\alpha\gamma}}{\partial x^\beta} - \frac{\partial g_{\alpha\beta}}{\partial x^\gamma} \right). \tag{1.32}$$

同样，指标都不表示求和.

现在我们来分 4 种情况，进一步化简 $\Gamma^\gamma{}_{\alpha\beta}$.

① 在 $i = \alpha$ 的情况下，该系数常被称作 **Lamé** 系数.

1. $\alpha \neq \beta \neq \gamma$. 前文已经提到，度量的非对角元均为零，即

$$g_{\beta\gamma} = g_{\alpha\gamma} = g_{\alpha\beta} = 0, \tag{1.33}$$

因此结果非常简单：

$$\Gamma^{\gamma}{}_{\alpha\beta} = 0. \tag{1.34}$$

2. $\alpha = \beta \neq \gamma$，即 $\Gamma^{\gamma}{}_{\alpha\alpha}$. 直接计算，可有

$$\Gamma^{\gamma}{}_{\alpha\alpha} = g^{\gamma\gamma}\,\Gamma_{\alpha\alpha,\,\gamma} = \frac{1}{g_{\gamma\gamma}} \cdot \frac{1}{2}\left(-\frac{\partial g_{\alpha\alpha}}{\partial x^{\gamma}}\right) = -\frac{1}{2}\frac{1}{g_{\gamma\gamma}}\frac{\partial g_{\alpha\alpha}}{\partial x^{\gamma}}. \tag{1.35}$$

3. $\alpha = \gamma \neq \beta$，即 $\Gamma^{\alpha}{}_{\alpha\beta}$. 根据式 (**??**)，它又等于 $\Gamma^{\alpha}{}_{\beta\alpha}$. 同样，直接来进行计算：

$$\Gamma^{\alpha}{}_{\alpha\beta} = g^{\alpha\alpha}\,\Gamma_{\alpha\beta,\,\alpha} = \frac{1}{g_{\alpha\alpha}} \cdot \frac{1}{2}\left(\frac{\partial g_{\alpha\alpha}}{\partial x^{\beta}}\right) = \frac{1}{2}\frac{1}{g_{\alpha\alpha}}\frac{\partial g_{\alpha\alpha}}{\partial x^{\beta}}. \tag{1.36}$$

4. $\alpha = \beta = \gamma$，即 $\Gamma^{\alpha}{}_{\alpha\alpha}$. 指标只剩下了一个，喜闻乐见.

$$\Gamma^{\alpha}{}_{\alpha\alpha} = g^{\alpha\alpha}\,\Gamma_{\alpha\alpha,\,\alpha} = \frac{1}{g_{\alpha\alpha}} \cdot \frac{1}{2}\left(\frac{\partial g_{\alpha\alpha}}{\partial x^{\alpha}}\right) = \frac{1}{2}\frac{1}{g_{\alpha\alpha}}\frac{\partial g_{\alpha\alpha}}{\partial x^{\alpha}}. \tag{1.37}$$

算好了完整基（正交基）下的 Christoffel 符号，就可以考虑非完整基（单位正交基）下的情况了. 我们在式 (1.32) 中已经计算出了非完整基下的 Christoffel 符号，现在只要把以上四种情况逐一代入即可.

1. $\alpha \neq \beta \neq \gamma$. 已经知道 $\Gamma^{\gamma}{}_{\alpha\beta} = 0$，而根据 (1.28) 式，又有 $c^{(\gamma)}_{\beta} = 0$，于是

$$\Gamma^{(\gamma)}{}_{(\alpha)(\beta)} = 0. \tag{1.38}$$

2. $\alpha = \beta \neq \gamma$. 此时有

$$\begin{aligned}
\Gamma^{(\gamma)}{}_{(\alpha)(\alpha)} &= \frac{1}{g_{\alpha\alpha}}\sqrt{g_{\gamma\gamma}} \cdot \Gamma^{\gamma}{}_{\alpha\alpha} - \frac{1}{g_{\alpha\alpha}} \cdot \frac{\partial c^{(\gamma)}_{\alpha}}{\partial x^{\alpha}} \\
&= \frac{1}{g_{\alpha\alpha}}\sqrt{g_{\gamma\gamma}} \cdot \left(-\frac{1}{2}\frac{1}{g_{\gamma\gamma}}\frac{\partial g_{\alpha\alpha}}{\partial x^{\gamma}}\right) - 0 \\
&= -\frac{1}{\sqrt{g_{\gamma\gamma}}} \cdot \left(\frac{1}{2g_{\alpha\alpha}}\frac{\partial g_{\alpha\alpha}}{\partial x^{\gamma}}\right).
\end{aligned} \tag{1.39}$$

考虑到

$$\frac{\partial}{\partial x}\ln\sqrt{f(x)} = \frac{\partial}{\partial x}\left[\frac{1}{2}\ln f(x)\right] = \frac{1}{2}\frac{1}{f(x)}\frac{\partial f(x)}{\partial x}, \tag{1.40}$$

于是

$$\Gamma^{(\gamma)}{}_{(\alpha)(\alpha)} = -\frac{1}{\sqrt{g_{\gamma\gamma}}}\frac{\partial}{\partial x^{\gamma}}\left(\ln\sqrt{g_{\alpha\alpha}}\right). \tag{1.41}$$

3. $\alpha = \gamma \neq \beta$. 此时

$$\begin{aligned}
\Gamma^{(\alpha)}{}_{(\alpha)(\beta)} &= \frac{1}{\sqrt{g_{\beta\beta}}} \cdot \Gamma^{\alpha}{}_{\alpha\beta} - \frac{1}{\sqrt{g_{\alpha\alpha}}}\frac{1}{\sqrt{g_{\beta\beta}}} \cdot \frac{\partial c^{(\alpha)}_{\beta}}{\partial x^{\alpha}} \\
&= \frac{1}{\sqrt{g_{\beta\beta}}} \cdot \left(\frac{1}{2}\frac{1}{g_{\alpha\alpha}}\frac{\partial g_{\alpha\alpha}}{\partial x^{\beta}}\right) - 0
\end{aligned}$$

同理，利用对数，可得

$$= \frac{1}{\sqrt{g_{\beta\beta}}} \frac{\partial}{\partial x^{\beta}} \left(\ln \sqrt{g_{\alpha\alpha}} \right). \tag{1.42}$$

注意这里没有负号.

3*. $\beta = \gamma \neq \alpha$，即 ${\Gamma^{(\beta)}}_{(\alpha)(\beta)}$. 不过我们暂时先从 ${\Gamma^{(\alpha)}}_{(\beta)(\alpha)}$ 开始.

之前虽然已经计算了 ${\Gamma^{(\alpha)}}_{(\alpha)(\beta)}$，但由于我们并未证明形式 Christoffel 符号的下标可以交换①，因而仍要从头来算：

$$ {\Gamma^{(\alpha)}}_{(\beta)(\alpha)} = \frac{1}{\sqrt{g_{\beta\beta}}} \cdot {\Gamma^{\alpha}}_{\beta\alpha} - \frac{1}{\sqrt{g_{\beta\beta}}} \frac{1}{\sqrt{g_{\alpha\alpha}}} \cdot \frac{\partial c_{\alpha}^{(\alpha)}}{\partial x^{\beta}} $$

交换 Christoffel 符号的下标，同时代入(1.28) 式，可有

$$\begin{aligned} &= \frac{1}{\sqrt{g_{\beta\beta}}} \cdot {\Gamma^{\alpha}}_{\alpha\beta} - \frac{1}{\sqrt{g_{\beta\beta}}} \frac{1}{\sqrt{g_{\alpha\alpha}}} \cdot \frac{\partial}{\partial x^{\beta}} \sqrt{g_{\alpha\alpha}} \\ &= \frac{1}{\sqrt{g_{\beta\beta}}} \cdot \left(\frac{1}{2} \frac{1}{g_{\alpha\alpha}} \frac{\partial g_{\alpha\alpha}}{\partial x^{\beta}} \right) - \frac{1}{\sqrt{g_{\beta\beta}}} \frac{1}{\sqrt{g_{\alpha\alpha}}} \cdot \frac{1}{2\sqrt{g_{\alpha\alpha}}} \frac{\partial g_{\alpha\alpha}}{\partial x^{\beta}} \\ &= 0. \end{aligned} \tag{1.43}$$

回过头来，若要得到 ${\Gamma^{(\beta)}}_{(\alpha)(\beta)}$，只需交换 α、β，结果当然不变：

$$ {\Gamma^{(\beta)}}_{(\alpha)(\beta)} = 0. \tag{1.44}$$

4. $\alpha = \beta = \gamma$. 指标全部相同，有

$$\begin{aligned} {\Gamma^{(\alpha)}}_{(\alpha)(\alpha)} &= \frac{1}{\sqrt{g_{\alpha\alpha}}} \cdot {\Gamma^{\alpha}}_{\alpha\alpha} - \frac{1}{g_{\alpha\alpha}} \cdot \frac{\partial c_{\alpha}^{(\alpha)}}{\partial x^{\alpha}} \\ &= \frac{1}{\sqrt{g_{\alpha\alpha}}} \cdot \left(\frac{1}{2} \frac{1}{g_{\alpha\alpha}} \frac{\partial g_{\alpha\alpha}}{\partial x^{\alpha}} \right) - \frac{1}{g_{\alpha\alpha}} \cdot \frac{\partial}{\partial x^{\alpha}} \sqrt{g_{\alpha\alpha}} \\ &= \frac{1}{2g_{\alpha\alpha}\sqrt{g_{\alpha\alpha}}} \frac{\partial g_{\alpha\alpha}}{\partial x^{\alpha}} - \frac{1}{g_{\alpha\alpha}} \cdot \frac{1}{2\sqrt{g_{\alpha\alpha}}} \frac{\partial g_{\alpha\alpha}}{\partial x^{\alpha}} \\ &= 0, \end{aligned} \tag{1.45}$$

依然是个很漂亮的结果.

到此，我们可以看到，只有两种形式的 Christoffel 符号非零：

$$\begin{cases} {\Gamma^{(\beta)}}_{(\alpha)(\alpha)} = -\dfrac{1}{\sqrt{g_{\beta\beta}}} \dfrac{\partial}{\partial x^{\beta}} \left(\ln \sqrt{g_{\alpha\alpha}} \right) = -\dfrac{\partial}{\partial x^{(\beta)}} \ln \sqrt{g_{\alpha\alpha}}, & \text{(1.46-a)} \\ {\Gamma^{(\alpha)}}_{(\alpha)(\beta)} = \dfrac{1}{\sqrt{g_{\beta\beta}}} \dfrac{\partial}{\partial x^{\beta}} \left(\ln \sqrt{g_{\alpha\alpha}} \right) = \dfrac{\partial}{\partial x^{(\beta)}} \ln \sqrt{g_{\alpha\alpha}}. & \text{(1.46-b)} \end{cases}$$

后一个等号利用了形式偏导数 (1.30) 式.

对于单位正交基，其度量满足

$$g^{(\alpha)(\beta)} = g_{(\alpha)(\beta)} = \delta_{\alpha}\beta, \tag{1.47}$$

① 所以这里也多了一种情况需要讨论.

因此协变基与逆变基只好相同．这样一来，协变分量与逆变分量也就没有了差别，我们统一用尖括号标出．于是上面的 Christoffel 符号就可以写成

$$
\begin{cases}
\Gamma_{\langle\alpha\alpha\beta\rangle} = \Gamma_{(\alpha)(\alpha),\,(\beta)} = \Gamma^{(\beta)}{}_{(\alpha)(\alpha)} = -\dfrac{\partial}{\partial x^{(\beta)}} \ln \sqrt{g_{\alpha\alpha}}, & \text{(1.48-a)} \\
\Gamma_{\langle\alpha\beta\alpha\rangle} = \Gamma_{(\alpha)(\beta),\,(\alpha)} = \Gamma^{(\alpha)}{}_{(\alpha)(\beta)} = \dfrac{\partial}{\partial x^{(\beta)}} \ln \sqrt{g_{\alpha\alpha}}. & \text{(1.48-b)}
\end{cases}
$$

显然，它们的第二、第三指标具有反对称性：

$$
\Gamma_{\langle\alpha\alpha\beta\rangle} = -\Gamma_{\langle\alpha\beta\alpha\rangle}. \tag{1.49}
$$