Exercise session notes - Week 3

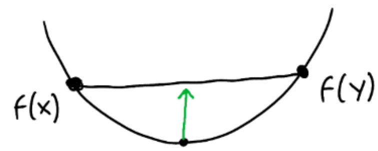
Convex Functions

Definitions. A function $f : X \to \mathbb{R}$ is convex if X is convex and one of the following (equivalent) conditions hold:

- 0th-order characterization: for any $x, y \in X$ and $\lambda \in [0,1]$ we have

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda) f(y)$$

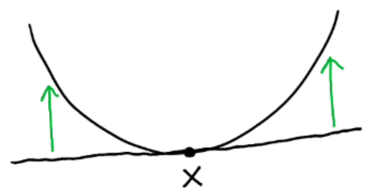
Meaning: the line between any two points in the graph of f lies above the function f.



- 1st-order characterization: for any $x, y \in X$ we have

$$f(y) \geq f(x) + \nabla f(x)^\top (y - x)$$

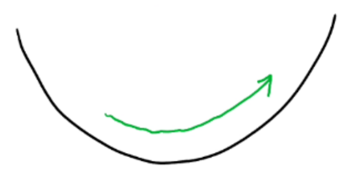
Meaning: the function f entirely lies above the first order taylor approximation at each point x.



- 2nd-order characterization: for any $x \in X$ we have

$$\begin{aligned} &\nabla^2 f(x) \text{ is Positive Semi Definite} \\ \iff\ & v^\top \cdot \nabla^2 f(x) \cdot v \geq 0 \qquad \forall v \in \mathbb{R}^n \end{aligned}$$

Meaning: the curvature of the function f is positive (the function f is "trying to go upwards").



Examples. Here are some examples of important convex/concave functions.

Convex Functions.	$\longleftrightarrow$	Concave Functions
Affine $ax + b$ for $a, b \in \mathbb{R}$		Affine $ax + b$ for $a, b \in \mathbb{R}$
Exponentials e^{ax} for $a \in \mathbb{R}$		Logarithms $\log_a(x)$ for $a \geq 1$
Powers x^a in $\mathbb{R}_{>0}$ for $a \leq 0$ or $a \geq 1$		Powers x^a in $\mathbb{R}_{>0}$ for $a \in [0, 1]$
Any norm $\|x\|_p = \left(\sum \lvert x_i\rvert^p\right)^{1/p}$ for $p \in [1, \infty]$		
LogSumExp $\log\left(e^{x_1} + \cdots + e^{x_n}\right)$. $\hookrightarrow$ This is a smooth approximator of the max funcion.		

Properties. Here are some operations that preserve convexity.

- Sums (conic combinations) $\sum a_i f_i$ for $a_i \geq 0$
  In particular $f_1 + f_2$ is convex.
- Affine Precompositions. If f convex then $f(Ax + b)$ is convex.
  E.g. $\|Ax + b\|$ is convex (Norm approximation error)
- If $f(x, y)$ convex in x then $g(x) := \sup\limits_y f(x, y)$ is convex.
  E.g. Max eigenvalue of symmetric matrix $\lambda_{max}(A) = \sup\limits_{\|y\|=1} y^\top Ay$
- If f convex + non-decreasing and g convex then $f \circ g$ convex.
  E.g. If g convex then $e^{g(x)}$ convex
- If f convex + non-increasing and g concave then $f \circ g$ convex.
  E.g. If g concave+positive then $1/g(x)$ convex.