# Exercise session notes - Week 13

## Log-Log convexity + Geometric Programs

First we reviewed the theory behind log- and log-log-convexity.

**Definition.** A function $f : X \to \mathbb{R}$ is called **log-convex** if $f(x) > 0$ and $\log f$ is convex.
A function $f : X \to \mathbb{R}$ is called **log-concave** if $1/f$ is log-convex.

**Proposition.** *A function $f$ is log-convex iff $f > 0$ and*

$$f\left(tx + (1-t)y\right) \leq f(x)^t \cdot f(y)^{1-t} \qquad \forall x, y, t$$

**Remark.**

- If $f$ is log-convex then $f$ is convex
- Sum $f_1 + f_2$, product $\alpha f$, product $f_1 \cdot f_2$ of log-convex functions are log-convex
- Affine $a^\top x + b$ is log-concave
- Powers $x^a$ is log-convex for $a \leq 0$ and log-concave for $a \geq 0$
- Exponentials $e^{ax}$ is log-affine
- Determinant $\det(X)$ is log-concave

**Definition.** A function $f : X \to \mathbb{R}$ is **log-log-convex** if $f > 0$ and $\log f(e^{x_1}, \ldots, e^{x_n})$ is convex.

**Proposition.** *A function $f$ is log-log-convex iff $f > 0$ and*

$$f\left(x^t \circ y^{1-t}\right) \leq f(x)^t \cdot f(y)^{1-t}$$

*where the product and the powers on the left are pointwise, i.e.*

$$x^t \circ y^{1-t} = \begin{pmatrix} x_1^t \cdot y_1^{1-t} \\ \vdots \\ x_n^t \cdot y_n^{1-t} \end{pmatrix}$$

**Proposition.** *A function $f$ is log-log-convex iff the log-log epigraph*

$$\log \operatorname{epi} f = \{(\log x, \log t) \mid f(x) \leq t\}$$

*is a convex set.*

**Remark.**

- Posynomials are log-log-convex
- Maximum $\max\{x_i\}$ is log-log-convex
- $L_p$-Norms are log-log-convex

Now we are able to define a Geometric Program. A **monomial** is a function defined as

$$\mathbf{x} \mapsto c \cdot x_1^{\alpha_1} \cdots x_n^{\alpha_n}$$

with $c > 0$ and $\alpha_i \in \mathbb{R}$. A **posynomial** is a function of the form

$$\mathbf{x} \mapsto \sum_j c_j \cdot x_1^{\alpha_1^j} \cdots x_n^{\alpha_n^j}$$

with $c_j > 0$ and $\alpha_i^j \in \mathbb{R}$, so a posynomial is just any sum of monomials. Finally a Geometric program has the form

$$\begin{aligned} \min \quad & f(x) & \\ \text{s.t.} \quad & g_i(x) \leq 1 & \forall i \\ & h_j(x) = 1 & \forall j \end{aligned}$$

where $f, g_i$ are posynomials and $h_j$ are monomials. An important property of GPs is the following

**Proposition.** *Any Geometric Program can be casted into an equivalent convex program by a change of variables.*

### Exercise 11.2 (Model Wheel Chair Ramp)

Finally we solved together Ex.11.2 (Model Wheel Chair Ramp) from the Exercise Sheet. We formulated the problem as a the following GP.

$$\begin{aligned} \min \quad & C_m w\ell + \frac{C_w}{2} wdh \\ \text{s.t.} \quad & 5hd^{-1} \leq 1 \\ & \ell w^{-1} \leq 1 \\ & d^2\ell^{-2} + h^2\ell^{-2} \leq 1 \\ & h \in [0.15, 1.5] \\ & w \in (0, 1.5] \\ & d \in (0, 1.5] \end{aligned}$$

Recall that we relaxed the last constraint with the following
**Claim.** Any optimal solution of (GP) is an optimal solution for the problem (it satisfies the constraint with equality)

*Proof of Claim.* Assume we have a optimal solution $(w, h, d, \ell)$. If $d^2 + h^2 = \ell^2$ then we are done. Otherwise we let

$$\tilde{\ell}^2 := d^2 + h^2 < \ell^2$$

Then $(w, h, d, \tilde{\ell})$ is still a feasible solution with cost

$$C_m w\tilde{\ell} + \frac{C_w}{2} wdh \ < \ C_m w\ell + \frac{C_w}{2} wdh$$

Therefore $(w, h, d, \ell)$ is not <u>optimal</u>. ↯ ■

Finally we can apply the log-log transformation to the objective and constraints and we obtain the following equivalent convex problem

$$
\begin{aligned}
\min \quad & \log(C_m e^{W+L} + \frac{C_w}{2} e^{W+D+H}) \\
\text{s.t.} \quad & \log(5e^{H-D}) \leq 0 \\
& \log(e^{L-W}) \leq 0 \\
& \log(e^{2D-2L} + e^{2H-2L}) \leq 0 \\
& H \in [\log(0.15), \log(1.5)] \\
& W \in (-\infty, \log(1.5)] \\
& D \in (-\infty, \log(1.5)]
\end{aligned}
$$

(Recall that log-sum-exp is a convex function). After that we can return to the original variables by

$$w = e^W, \quad d = e^D, \quad \ell = e^L, \quad h = e^H$$