Exercise session notes - Week 11

Semidefinite Programs + Applications

1 Application: Markowitz portfolio optimization

In this problem we have a budget B that we want to invest in assets $1, \dots, n$. At time $t = 0$ we know the prices of the assets as

$$p_1^0, \dots, p_n^0$$

After some period of time we may know some predictions of the prices, and at time $t = T$ we know the distribution of the prices as

$$P_1^T, \dots, P_n^T$$

which are random variables. In order to model the problem as a mathematical problem we define the Rate of Return:

$$R_i = \frac{P_i^T}{p_i^0} - 1$$

which essentially gives us the amount that we receive at T by buying asset i. (if $R_i = 0$ then we receive the same amount, if $R_i = 1$ then we double the amount, if $R_i = -\frac{1}{2}$ then we lose half of the amount).
We then define our decision variables x_i = fraction of the budget B spent on asset i (the vector x is called portfolio). Then the final revenue/loss is the random variable

$$R^\top x = \sum_i R_i \cdot x_i$$

This RV has expected value

$$\mu^\top x := \sum_i x_i \cdot \mathbb{E}[R_i]$$

and variance

$$x^\top \Sigma x := \sum_{i,j} \mathrm{Cov}(R_i, R_j) x_i x_j$$

Finally we can write our mathematical program as follows:

- **Goal:** Minimize risk with lower bound on profit, as QP.

$$\begin{aligned} \min \quad & x^\top \Sigma x \\ \text{s.t} \quad & \mu^\top x \geq b \\ & \mathbf{1}^\top x = 1 \end{aligned}$$

- **Goal:** Maximize profit with upper bound on risk, as QCQP.

$$\begin{aligned} \max \quad & \mu^\top x \\ \text{s.t} \quad & x^\top \Sigma x \leq \gamma^2 \\ & \mathbf{1}^\top x = 1 \end{aligned}$$

- **Goal:** Maximize utility function (balance between profit and risk), as SOCP.

$$\max \quad \mu^\top x - \delta\sqrt{x^\top \Sigma x}$$
$$\text{s.t} \quad \mathbf{1}^\top x = 1$$

2 Application: Max-Cut Problem

We firstly recall the definition of a SemiDefinite Program:

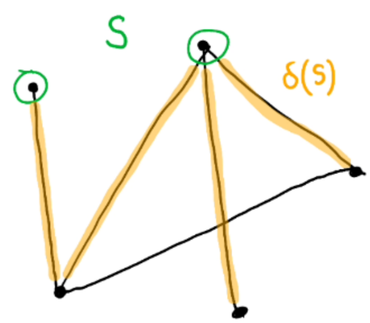
$$\begin{aligned} \min \quad & \operatorname{Tr}(CX) \\ \text{s.t.} \quad & \operatorname{Tr}(A_i X) = b \\ & X \succeq 0 \end{aligned}$$

As the objective function and the constraints are affine, and $X \succeq 0$ is equivalent to X being in the positive semidefinite cone, we get SDP $\subseteq$ CP. In the lecture notes it is also proven SOCP $\subseteq$ CP, and thus we concluded the inclusions related to convex programs.

In the Max-Cut Problem we are given a graph $G = (V, E)$, with edge weights w_e for $e \in E$ and $\sum_e w_e = 1$. The goal is to find a subset of vertices $S \subseteq V$ with cut value

$$c(S) := \sum_{e \in \delta(S)} w_e$$

maximized. For example the cut in the following graph has cut value 4 and it is the maximum that we can achieve



Note that for general graphs this problem is NP-hard, on the other hand for bipartite graphs and complete graphs this problem becomes trivial. There is also a theorem that gives us the existence of a cut with cut value $\geq \frac{1}{2}$ (see Ch. 7.5 in Lecture notes).

2.1 Semidefinite Formulation

We want to solve the mathematical problem

$$\max \quad \sum_{e \in \delta(S)} w_e \quad \text{s.t.} \quad S \subseteq V$$

To do that we first define the decision variables $x \in \{-1, 1\}^V$ with

$$x_v = 1 \iff v \in S$$

Therefore we can write an equivalent program as

$$\max \quad \sum_{uv \in E} w_{uv} \cdot \left(\frac{1 - x_u x_v}{2} \right) \quad \text{s.t.} \quad x \in \{-1, 1\}^V$$

Let us define the matrix $X = (x_u \cdot x_v)_{u,v \in V} \in \{-1, 1\}^n \times n$, then we can write the program as

$$\begin{aligned} \max \quad & \mathrm{Tr}(WX) = \sum_{uv} X_{u,v} W_{u,v} \\ \text{s.t.} \quad & X_{ii} = 1 \\ & X = xx^\top \\ & x \in \{-1, 1\}^n \end{aligned}$$

Finally we can drop the last two constraints and we get the SDP relaxation

$$\begin{aligned} \max \quad & \mathrm{Tr}(WX) = \sum_{uv} X_{u,v} W_{u,v} \\ \text{s.t.} \quad & X_{ii} = 1 \\ & X \succeq 0 \end{aligned}$$

By solving this problem we end up with an approximation of $0.87856\ldots$ of the value of the max-cut.