Exercise session notes - Week 10

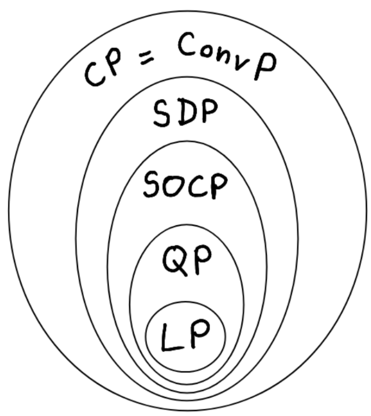
Second Order Cone Program + Robust Optimization

1 Convex Programs and Inclusions

Firstly we reviewed the types of convex programs that we studied so far.

$$\begin{aligned}
\text{LP:} &\quad \min\ f^\top x \quad \text{s.t} \quad Ax \le b &&; \quad Fx = g \\
\text{QP:} &\quad \min\ f^\top x \quad \text{s.t} \quad x^\top Ax + b\top x \le c &&; \quad Fx = g \\
\text{SOCP:} &\quad \min\ f^\top x \quad \text{s.t} \quad \|Ax+b\|_2 \le c^\top x + d &&; \quad Fx = g \\
\text{SDP:} &\quad ... \\
\text{CP:} &\quad \min\ f^\top x \quad \text{s.t} \quad x \in K &&; \quad Fx = g \\
\text{ConvP:} &\quad \min\ f(x) \quad \text{s.t} \quad x \in C \text{ convex set} &&; \quad Fx = g
\end{aligned}$$

We then pictured the diagram of all the inclusions between these sets.



The inclusions LP $\subseteq$ QP, SOCP $\subseteq$ CP and CP $\subseteq$ ConvP are trivial. Then we proved ConvP $\subseteq$ CP and QP $\subseteq$ SOCP, and next week we will consider also the Semi Definite Programs (SDP).

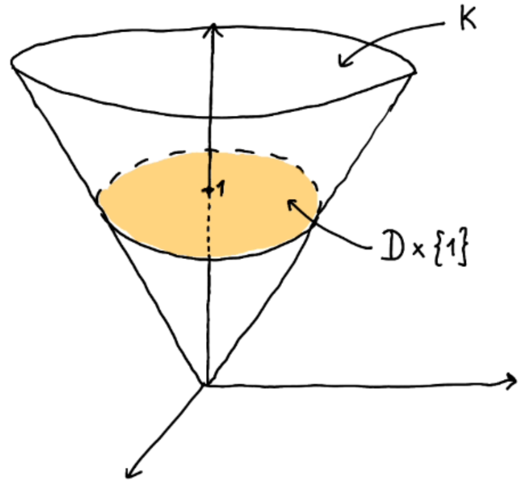
ConvP $\subseteq$ CP. Consider a general convex program

$$\min\ f(x) \quad \text{s.t} \quad x \in C \text{ convex set} ; \quad Fx = g$$

Then we can write an equivalent program as

$$\begin{aligned}
&\min\ t \quad \text{s.t} \quad t \ge f(x) ;\ x \in C \text{ convex set} ; \quad Fx = g \\
\iff &\min\ t \quad \text{s.t} \quad (x,t) \in \text{epi}(f) ;\ (x,t) \in C \times \mathbb{R} ; \quad Fx = g
\end{aligned}$$

Let $D := \text{epi}(f) \cap (C \times \mathbb{R})$ and note that this is a convex set. Then we pick the conic hull $K = \text{cone}(D \times \{1\})$ which is



Therefore $(x, t) \in D \iff (x, t, 1) \in E$. Finally we can write the equivalent conic program as

$$\min\ t \quad \text{s.t} \quad (x, t, s) \in K\ ; \quad s = 1\ ; \quad Fx = g$$

QP $\subseteq$ SOCP. Consider a general quadratic program with A being a positive definite matrix

$$\min\ f^\top x \quad \text{s.t} \quad x^\top A x + b^\top x + c \leq 0\ ; \quad Fx = g$$

Recall that by the Cholesky Decomposition there exists a matrix B with $A = B^\top B$. Then we can write an equivalent program as

$$\begin{aligned} \min\ f^\top x \quad \text{s.t} \quad x^\top B^\top B x \leq -b^\top x - c\ ; \quad Fx = g \\ \min\ f^\top x \quad \text{s.t} \quad \|Bx\|_2^2 \leq -b^\top x - c\ ; \quad Fx = g \end{aligned}$$

Then recall the definition of the rotated Lorentz cone:

$$\mathcal{L}^m_{rot} = \left\{ (x, y, z) \ \middle|\ \|x\|_2^2 \leq 2yz \right\}$$

Therefore we can model the program as the following SOCP

$$\min\ f^\top x \quad \text{s.t} \quad (Bx,\ -b^\top x - c,\ \tfrac{1}{2}) \in \mathcal{L}^m_{rot}\ ; \quad Fx = g$$

2 Robust Optimization

Assume we are working for an airline company and we want to assign people to fligths. We model this problem as an LP. In a perfect world we can write

$$\max \#\ \text{people served} \quad \text{s.t.} \quad \text{assignment is feasible}$$

Since flight disruptions may happen (flight cancellations, delays) we want to find an assignment that works for any possible scenario. Such a solution is called a Robust Solution. A non-example would be:

Rome $\to$ Zurich (wait 15 min)
Zurich $\to$ London

This solution is clearly feasible but a small delay could change entirely our assignment, therefore it is not Robust.
Let's formalize this problem (Ex. 6.6 from Lecture Notes). We want to solve an LP

$$\min \quad c^\top x \quad \text{s.t.} \quad a^\top x \leq b \;, \quad x \geq 0$$

but the vector $a \in \mathbb{R}^n$ is uncertain, it varies in some Ellipse

$$a \in E = \{a_0 + Pu \mid \|u\|_2 \leq 1\}$$

for a_0 the center and P the matrix corresponding to the stretch of the Ellipse. Our goal is to find a Robust solution, so a point x that is feasible for any choice of a and with minimal value. Therefore we may require

$$\begin{aligned} b &\geq \sup_{a \in E} a^\top x \\ &= a_0^\top x + \sup_{\|u\| \leq 1} u^\top P^\top x \\ &= a_0^\top x + \left\|P^\top x\right\|_2 \end{aligned}$$

Therefore we can model the robust LP as the following SOCP

$$\min \quad c^\top x \quad \text{s.t.} \quad a_0^\top x + \left\|P^\top x\right\|_2 \leq b \;, \quad x \geq 0$$