Exercise session notes - Week 14

Interiour Point Method

As always we want to solve a convex program

$$\begin{aligned} \min \quad & f(x) \\ \text{s.t} \quad & g(x) \leq 0 \\ & Ax = b \end{aligned}$$

Assume the problem satisfies Slater's condition, then strong duality holds and the dual optimum is attained, i.e. there exists a point (x^*, λ^*, μ^*) satisfying the KKT conditions:

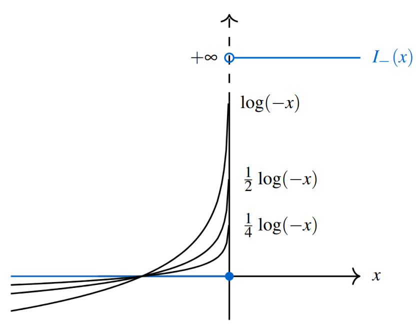
1. Primal feasibility: $Ax = b$, $g(x) \leq 0$
2. Dual feasibility: $\lambda \geq 0$
3. Comp. Slackness: $\mu \cdot g(x) = 0$
4. Optimal Lagrangian: $\nabla f(x) + \mu \cdot \nabla g(x) + A^\top \lambda = 0$

As in the duality chapter, we can find an equivalent program by introducing the function $I_{\text{-}}(\cdot)$:

$$\begin{aligned} \min \quad & f(x) + I_{\text{-}}(g(x)) \\ \text{s.t} \quad & Ax = b \end{aligned}$$

In the Interiour point method we approximate this function with the logarithmic barrier

$$x \mapsto -\frac{1}{t} \log(-x)$$



Note that by increasing t we get a better approximation of the function $I_-(\cdot)$, so the new program (P) is

$$\begin{aligned} \min \quad & f(x) - \tfrac{1}{t} \log(-g(x)) \\ \text{s.t} \quad & Ax = b \end{aligned}$$

The set of tuples $\{(x^*(t), t) \mid t > 0\}$, for $x^*(t)$ being the minimizer in the previous program, is called central path. In the Barrier Method we try to solve the original convex problem by picking solutions in the central path for larger and larger t.

Algorithm 1 Barrier Method (Phase II)

Given a strictly feasible solution x, $t > 0$, tolerance ε and incrementing factor $\mu > 1$
while gap of solution $m/t > \varepsilon$ **do**
  Step: solve (P) with given t using Newton method (starting at point x) and find solution $x^*(t)$
  Update: $x \leftarrow x^*(t)$
  Increment: $t \leftarrow \mu \cdot t$
end while

Algorithm 2 Barrier Method (Phase I)

Solve the convex program $\min \{s \mid g(x) \leq s,\ Ax = b\}$ using Barrier Method (Phase II) with any starting point x in the domain of the problem and any $s > g(x)$ large enough.

We can see the implementation of the Barrier Method applied to the program

$$\begin{aligned} \min \quad & x^2 + y^2 \\ \text{s.t} \quad & (x-1)^2 + (y-1)^2 \leq 1 \end{aligned}$$