# Exercise session notes - Week 5

## Strong Duality + KKT points

Let's consider a general mathematical program

$$\begin{aligned} \min \quad & f(x) \\ & g_i(x) \leq 0 \quad \forall i \\ & h_j(x) = 0 \quad \forall j \end{aligned}$$

Last week we constructed the Lagrange Dual Program as follows

$$\begin{aligned} \min \quad & \widehat{L}(\lambda, \nu) \\ & \lambda \geq 0 \end{aligned}$$

where the objective function is the Lagrange dual function, defined as

$$\widehat{L}(\lambda, \nu) = \inf_x \left\{ f(x) + \sum \lambda_i g_i(x) + \sum \nu_j h_j(x) \right\}$$

We now list some properties about duality.

**Proposition.** (Weak Duality) For any program we have

$$\max \widehat{L}(\lambda, \nu) \leq \min f(x)$$

**Definition.** (Strong Duality) We say strong duality holds if the duality gap is 0, i.e.

$$\max \widehat{L}(\lambda, \nu) = \min f(x)$$

**Proposition.** (Complementary Slackness) If strong duality holds, then for any optimal primal-dual solution $(x^*, \lambda^*, \nu^*)$ we have

$$\lambda_i^* \cdot g_i(x^*) = 0 \qquad \forall i$$

This means that for each $i$, either the $i$th primal constraint is tight ($g_i(x) = 0$) or the $i$th dual constraint is tight ($\lambda_i = 0$), or both of them are tight.

**Definition.** (KKT-conditions) A point $(x, \lambda, \nu)$ is a KKT-point if:

| | | | |
|---|---|---|---|
| 1 | Primal feasible | : | $g_i(x) \leq 0;\ h_j(x) = 0$ |
| 2 | Dual feasible | : | $\lambda_i \geq 0$ |
| 3 | Comp. slackness | : | $\lambda_i \cdot g_i(x) = 0$ |
| 4 | Gradient vanishes | : | $\nabla_x L(x, \lambda, \nu) = 0$ |

**Proposition.** For <u>any</u> mathematical program:

$$\text{Strong Duality holds} + (x, \lambda, \nu) \text{ is optimal} \implies (x, \lambda, \nu) \text{ is KKT point}$$

**Proposition.** For <u>convex</u> mathematical programs:

$$\text{Strong Duality holds} + (x, \lambda, \nu) \text{ is optimal} \iff (x, \lambda, \nu) \text{ is KKT point}$$

Now we show an example of a convex mathematical program for which Strong duality does not hold.

**Example.** Consider the following program (P)

$$\begin{aligned} \min \quad & e^{-x} \\ & \frac{x^2}{y} \leq 0 \end{aligned}$$

where we set the domain as $\mathcal{D} = \{(x, y) \mid y > 0\}$. Note that in this domain, the program is a convex program (Exercise). Moreover the optimal value is $f^* = 1$. We now compute the dual problem: Let $\lambda \geq 0$, then

$$\begin{aligned} L(x, y, \lambda) &= e^{-x} + \lambda \frac{x^2}{y} \\ \implies \widehat{L}(\lambda) &= \inf_{x,y} \left\{ e^{-x} + \lambda \frac{x^2}{y} \right\} \end{aligned}$$

Since $y$ is positive, we have $L(x, y, \lambda) \geq 0$. Moreover if we let $x = n$, $y = n^3$ for $n \in \mathbb{N}$ and tend $n \to \infty$ we obtain

$$L(n, n^3, \lambda) = e^{-n} + \lambda \frac{1}{n} \to 0$$

In particular we have $\widehat{L}(\lambda) = 0$. With this, we can write the dual program (D) of (P) by

$$\begin{aligned} \max \quad & \widehat{L}(\lambda) = 0 \\ & \lambda \geq 0 \end{aligned}$$

with optimal value $\widehat{L}^* = 0$. For this pair of programs we have a duality gap of 1, so strong duality does not hold. By the previous proposition there exists no KKT point, so let's see what goes wrong. Consider a point $(x, y, \lambda)$ in the domain of the primal/dual programs and assume it is a KKT-point.

1. We have $\frac{x^2}{y} = 0$ so $x = 0$. ✓
2. We have $\lambda \geq 0$. ✓
3. We have $\lambda \frac{x^2}{y} = \lambda \frac{0}{y} = 0$. ✓
4. We have $\nabla_{x,y} L = 0$ so

$$\frac{\partial}{\partial y} L = 0 - \lambda \frac{x^2}{y^2} = 0 \quad ✓$$

and

$$\frac{\partial}{\partial x} L = -e^{-x} + 2\lambda \frac{x}{y} = -e^0 = -1 \neq 0 \quad ✗$$

Therefore there are no KKT-points for this problem.