Exercise session notes - Week 7

Theorem of Alternatives

In this week we viewed how to apply duality theory for the study of feasibility of a system involving inequalities and equalities. Consider the following system (S1)

$$\begin{aligned} g_i(x) \leq 0 \quad \forall i \qquad (S1) \\ Ax = b \end{aligned}$$

then we can check its feasibility by solving the following mathematical program (P1)

$$\begin{aligned} \min \quad & 0 & \qquad (P1) \\ \text{s.t.} \quad & g_i(x) \leq 0 \quad \forall i \\ & Ax = b \end{aligned}$$

We obtain

$$(S1) \text{ is feasible } \iff (P1) \text{ has optimal value } 0$$

Let's look at the dual program (P2) of (P1)

$$\begin{aligned} \max \quad & \widehat{L}(\lambda, \nu) = \inf_x \left(\sum_i \lambda_i g_i(x) + \nu^\top (Ax - b) \right) \qquad (P2) \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

If we just consider the system (S2)

$$\begin{aligned} & \widehat{L}(\lambda, \nu) > 0 \qquad (S2) \\ \text{s.t.} \quad & \lambda \geq 0 \end{aligned}$$

by weak duality we obtain

$$\begin{aligned} (S1) \text{ is feasible } & \iff (P1) \text{ has optimal value } 0 \\ & \implies (P2) \text{ has optimal value } 0 \iff (S2) \text{ is infeasible} \end{aligned}$$

and

$$\begin{aligned} (S2) \text{ is feasible } & \iff (P2) \text{ has optimal value ¿}0 \\ & \implies (P1) \text{ has optimal value } \infty \iff (S1) \text{ is infeasible} \end{aligned}$$

Therefore the two system of inequalities (S1) and (S2) are <u>weak alternatives</u> (at most one is feasible).
If we swap inequalities with strictly inequalities we get the following systems:

$$\begin{aligned} g_i(x) < 0 \quad \forall i \qquad (S3) \\ Ax = b \end{aligned}$$

and

$$
\begin{aligned}
&\widehat{L}(\lambda,\nu) \geq 0 \qquad\qquad (S4) \\
&\text{s.t.} \quad \lambda \in \mathbb{R}^n \geq 0,\ \lambda \neq 0
\end{aligned}
$$

Then, if there exists a point x in the relative interiour of the program (P1), then the two systems are strong alternatives (exactly one is feasible). The proof uses Slater's condition applied on an auxiliary program and convexity of g, and you can find it on the script (Proposition 3.7.5).