Chapter 2

Periodic Sampling

Outline

2.1 Aliasing: Signal Ambiguity in the Frequency Domain

There is a frequency-domain ambiguity associated with discrete-time signal samples that does not exist in the continuous signal world.

$$
\begin{aligned}
x(0) &= 0 \\
x(1) &= 0.866 \\
x(2) &= 0.866 \\
x(3) &= 0 \\
x(4) &= -0.866 \\
x(5) &= -0.866 \\
x(6) &= 0\ ,
\end{aligned}
$$

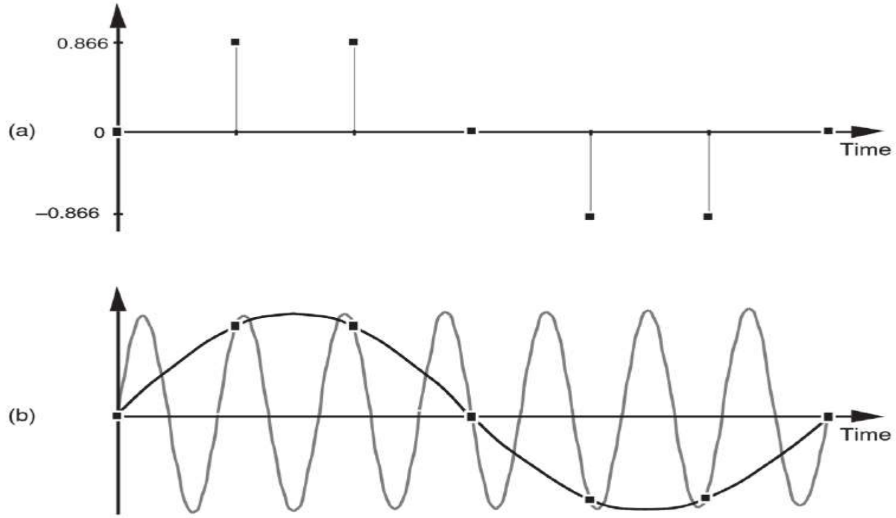
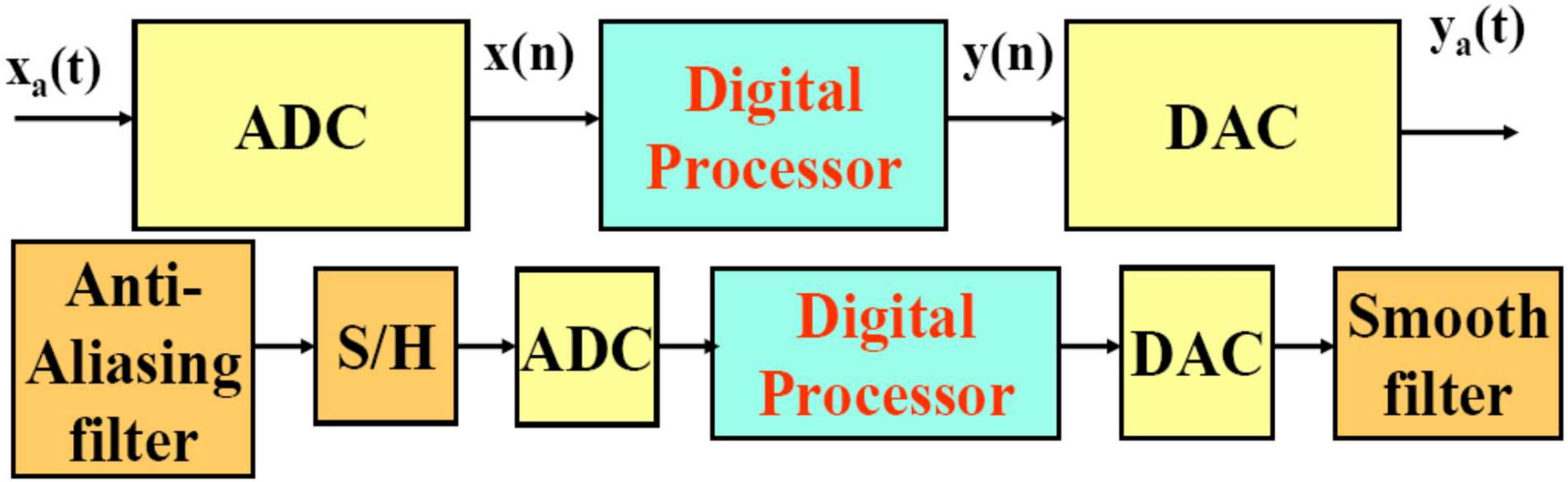


Figure 2-1 **Frequency ambiguity: (a) discrete-time sequence of values; (b) two different sinewaves that pass through the points of the discrete sequence.**

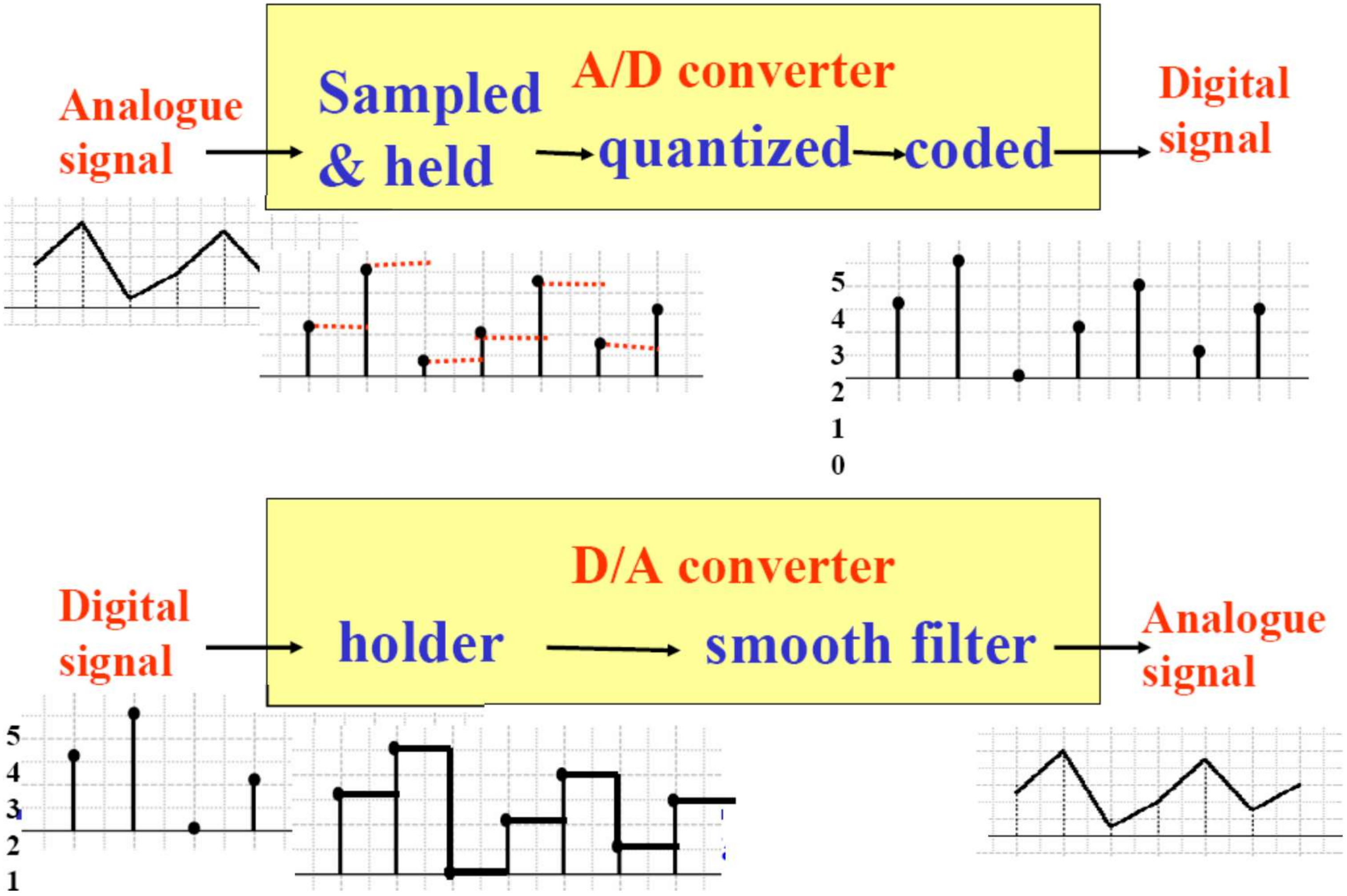
2.2 Sampling Low-Pass Signals

Introduction

The structure of DSP processing is composed of:



Introduction



Sampling of Continuous-Time Signal

Sampling & Interpolation:

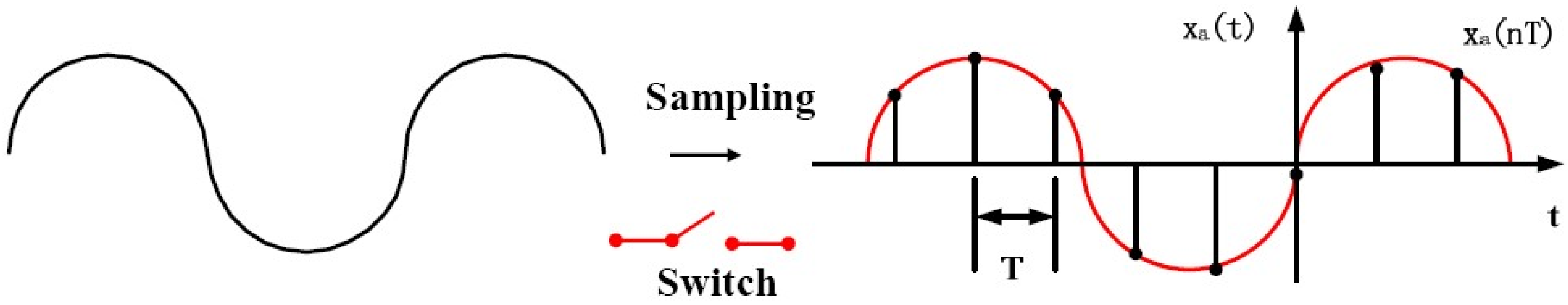
<u>Sampling</u>: Discretization process of analog signals.

The first matter of DSP is the discretization of signals (sampling).

<u>Interpolation</u>: The process in which the discrete-time signals becomes continuous-time signals.

Sampling of Continuous-Time Signal

The common method is uniform sampling, as:



T **denotes** ***sampling interval*** **and its reciprocal is** ***sampling frequency*****, written as:**

sampling frequency → $$f_s = \frac{1}{T}$$ $$\Omega_s = 2\pi f_s$$ ← analog angular frequency

Sampling Theorem

Question?

There're many curves exist connecting the two points $x(nT)$ and $x((n+1)T)$.

Do there exit any certain conditions that $x_a(t)$ can be reconstructed based on discrete-time signal $x_a(nT)$?

Sampling Theorem !

Sampling Theorem - Shannon Theorem

Shannon Theorem:

Assuming the highest frequency component of any given continuous-time signal $x_a(t)$ is f_m, $x_a(t)$ can be uniquely and accurately reconstructed from the sampling sequence $x_a(nT)$ as long as the sampling frequency is no less than $2f_m$.

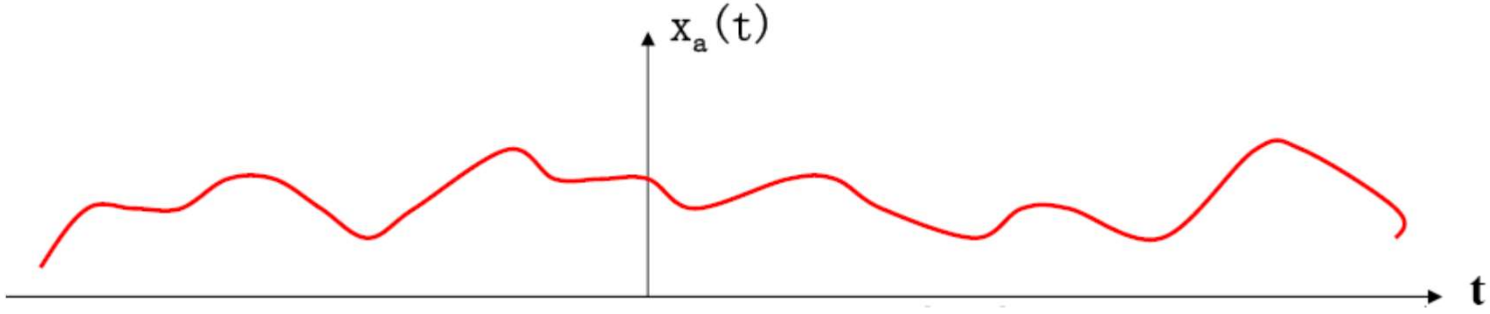
Two important points:

1. **The continuous function is *bandwidth limited*;**
2. **The *relationship* between sampling frequency and the highest frequency component is:**

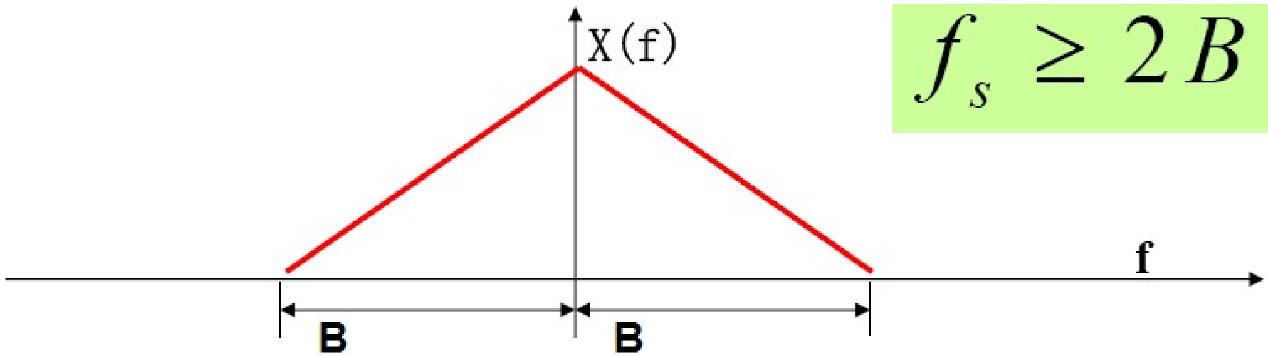
$$f_s \geq 2 f_m$$

Sampling Theorem - Shannon Theorem

A bandwidth limited signal is given:



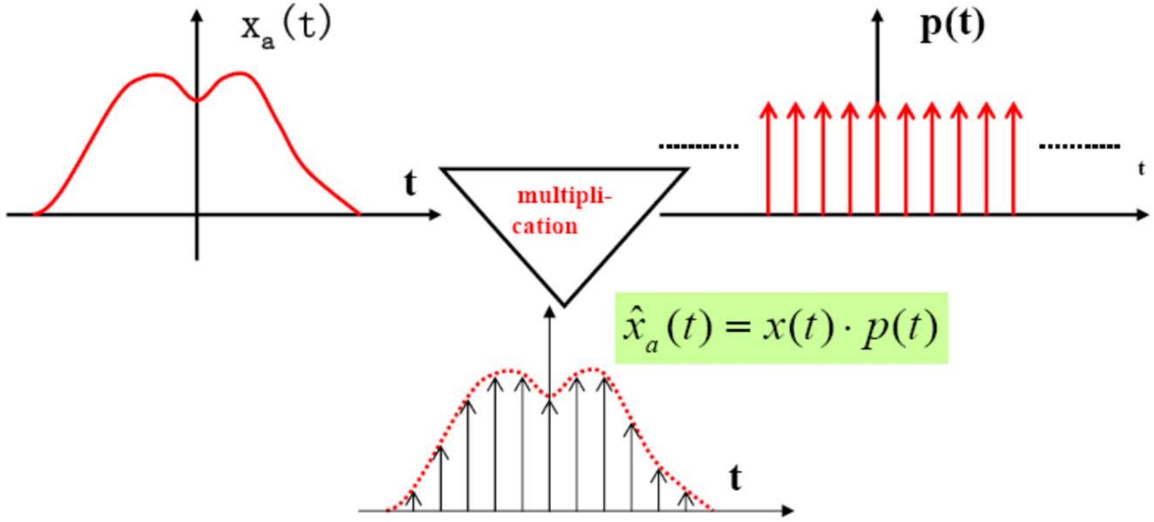
Its Fourier Transform *X(f)=0*, when │f│>=B is satisfied. Then the relationship between sampling frequency and the highest frequency component is:



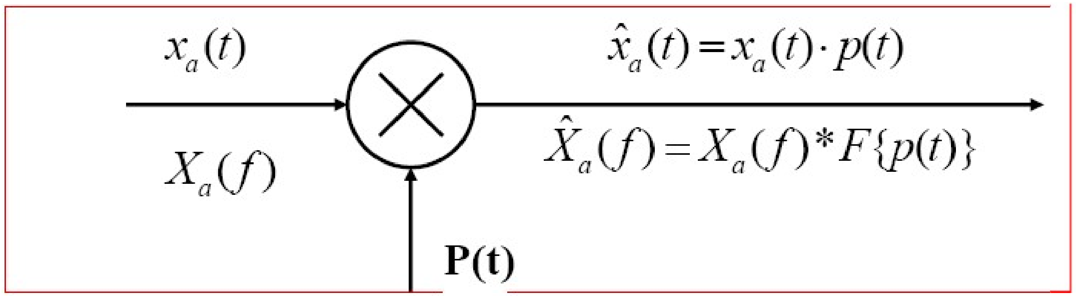
$$f_s \geq 2B$$

Time Domain Analysis

Time-domain analysis: ***uniform sampling***



The figure above can also be represented as:



Time Domain Analysis

The *Sampling Function* is defined as:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

\- **T : sampling interval**

Then:

$$\hat{x}_a(t) = x_a(t)p(t) = \sum_{n=\infty}^{\infty} x_a(t)\delta(t - nT) = \sum_{n=\infty}^{\infty} x_a(nT)\delta(t - nT)$$

It can be noticed that $\hat{x}_a(t)$ is the *weighted sum* of infinite uniformly spaced *Unit Sample sequence*.

Frequency Domain Analysis

Multiplication **in time-domain -> *Convolution* in frequency-domain, namely:**

$$F\left[p(t)x_a(t)\right] = P(f) * X_a(f)$$
$$= \frac{1}{2\pi} P(\Omega) * X_a(\Omega)$$

Therefore, the expression

$$\hat{x}_a(t) = x_a(t) \cdot p(t)$$

when mapped to frequency-domain:

$$\hat{X}_a(\Omega) = \frac{1}{2\pi}\left[X_a(\Omega) * P(\Omega)\right]$$

?

Frequency Domain Analysis

As it is a periodic function of *T*, it can be expanded to the form of *Fourier Series*:

$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) = \sum_{n=-\infty}^{\infty} a_n e^{jn\Omega_0 t}$$

$$\Omega_0 = 2\pi / T$$

In which:

$$\begin{aligned} a_n &= \frac{1}{T}\int_{-T/2}^{T/2} p(t) e^{-jn\Omega_0 t}\, dt \\ &= \frac{1}{T}\int_{-T/2}^{T/2} [\sum \delta(t - nT)] e^{-jn\Omega_0 t}\, dt] \end{aligned}$$

Frequency Domain Analysis

There's only one impulse within the integrating range, while others exist outside. Therefore:

$$a_n = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jn\Omega_0 t} dt = \frac{1}{T} e^{-jn\Omega_0 \cdot 0} = \frac{1}{T}$$

p(t) **is expanded to the form of** ***Fourier Series*****:**

$$\therefore \quad p(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jnt\Omega_0}$$

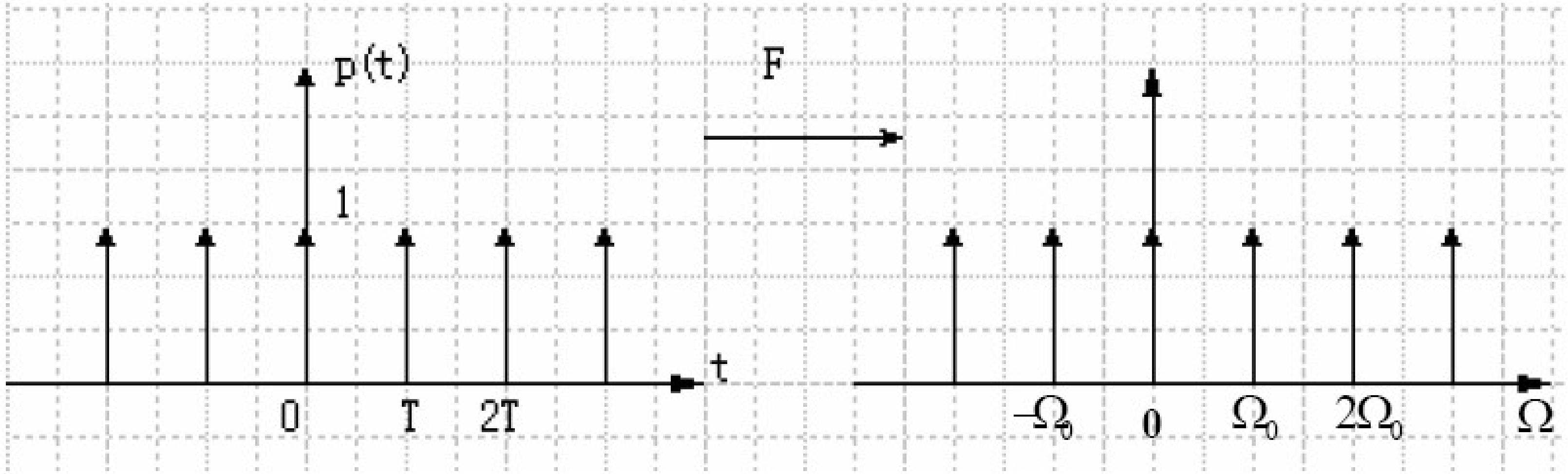
$$\because \quad e^{j\Omega_0 t} \xrightarrow{\mathcal{F}} 2\pi\delta(\Omega - \Omega_0)$$

Frequency Domain Analysis

Therefore:

$$p(t) = \frac{1}{T}\sum_{n=-\infty}^{\infty} e^{jn\Omega_0 t} \leftrightarrow P(\Omega) = \frac{2\pi}{T}\sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_0)$$

$$\Omega_0 = 2\pi / T$$



Frequency Domain Analysis

Finally:

$$\hat{X}_a(\Omega) = \frac{1}{2\pi} X_a(\Omega) * P(\Omega)$$

$$= \frac{1}{2\pi} X_a(\Omega) * \frac{2\pi}{T} \sum_{n=-\infty}^{\infty} \delta(\Omega - n\Omega_0)$$

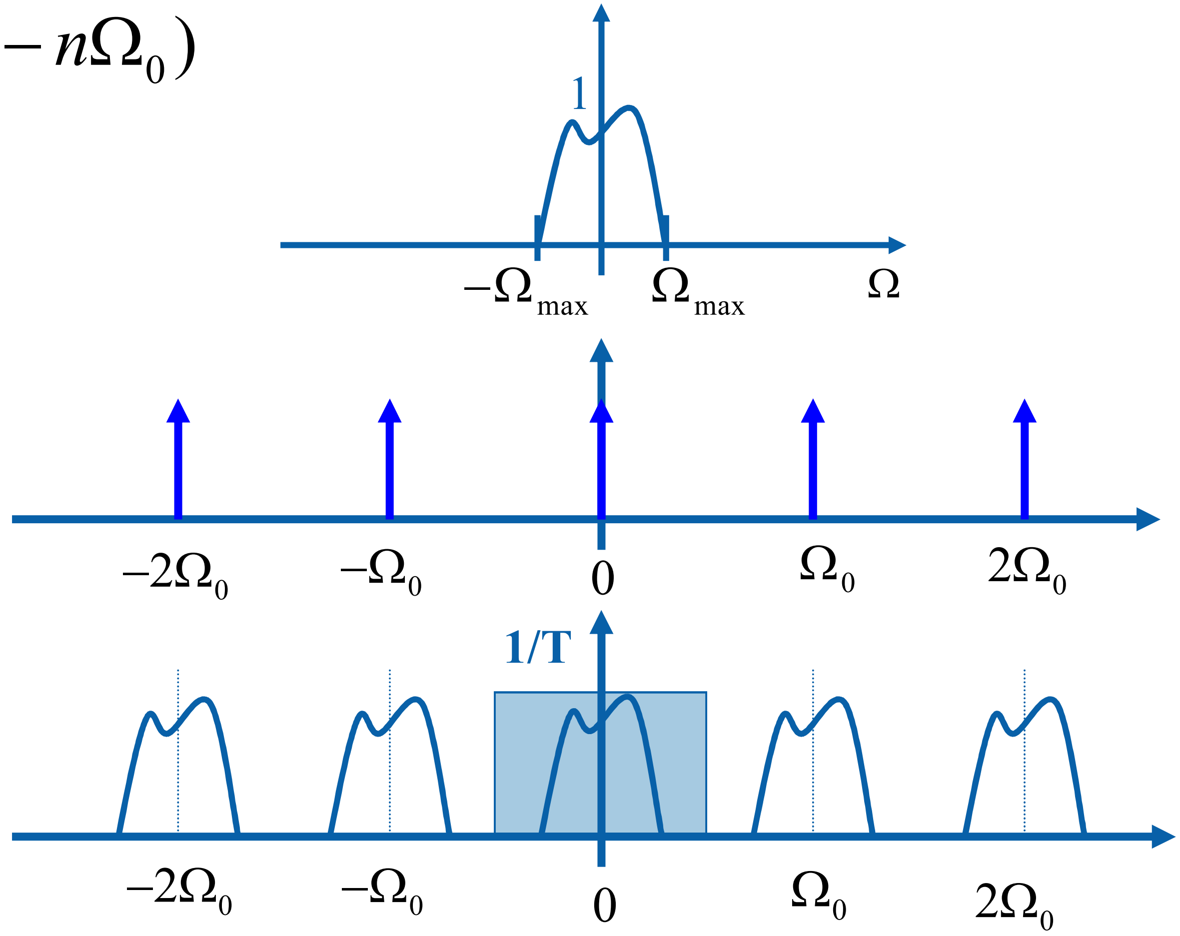
$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} X_a(\Omega) * \delta(\Omega - n\Omega_0)$$

$$= \frac{1}{T} \sum_{n=-\infty}^{\infty} X_a(\Omega - n\Omega_0)$$

(The spectrum of discrete-time signals is the periodic continuation of the original one.)

Frequency Domain Analysis

$$\hat{X}_a(\Omega) = \frac{1}{T}\sum_{n=-\infty}^{\infty} X_a(\Omega - n\Omega_0)$$



Frequency Domain Analysis

Thus, if $x_a(t)$ is a bandwidth limited function:

$$X_a(\Omega) = \begin{cases} X_a(\Omega), & |\Omega| < \frac{\Omega_0}{2} = \frac{\pi}{T} \\ 0, & |\Omega| >= \frac{\Omega_0}{2} = \frac{\pi}{T} \end{cases}$$

As long as the sampling frequency is high enough that satisfy the condition below:

$$|\Omega|_{max} \leq \frac{\Omega_0}{2} = \frac{\pi}{T}$$

The spectrum of $\hat{X}_a(\Omega)$ won't overlap and it can be reconstructed using ideal low-pass filter.

-> Shannon Theorem

Frequency Domain Analysis

Nyquist Frequency **(Folding Frequency):**

$$\frac{\Omega_0}{2} = \frac{\pi}{T}$$

Related definitions:

- **Oversampling:**

$$\Omega_0 > 2\Omega_{max}$$

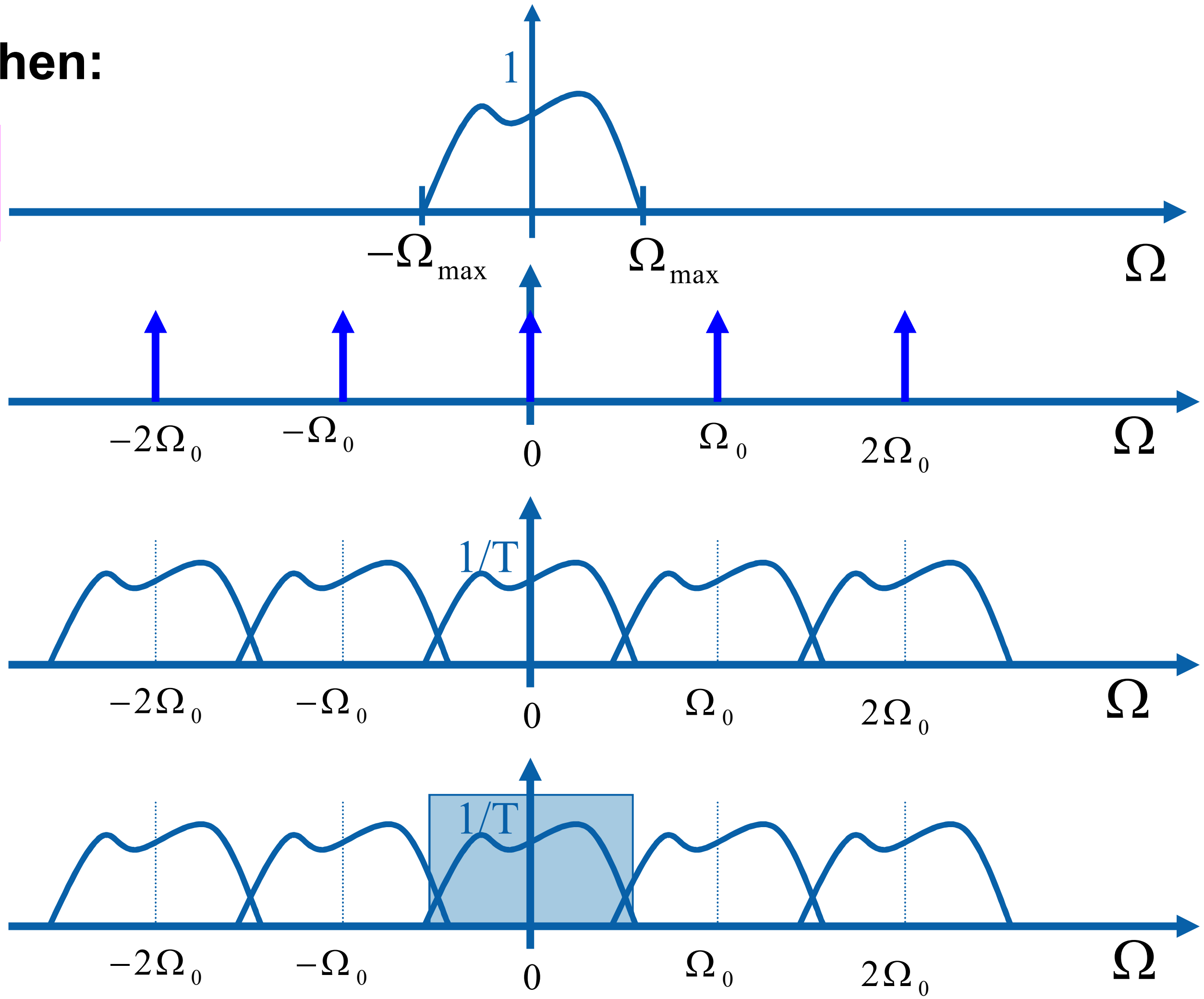
Undersampling:

$$\Omega_0 < 2\Omega_{max}$$

Frequency Domain Analysis

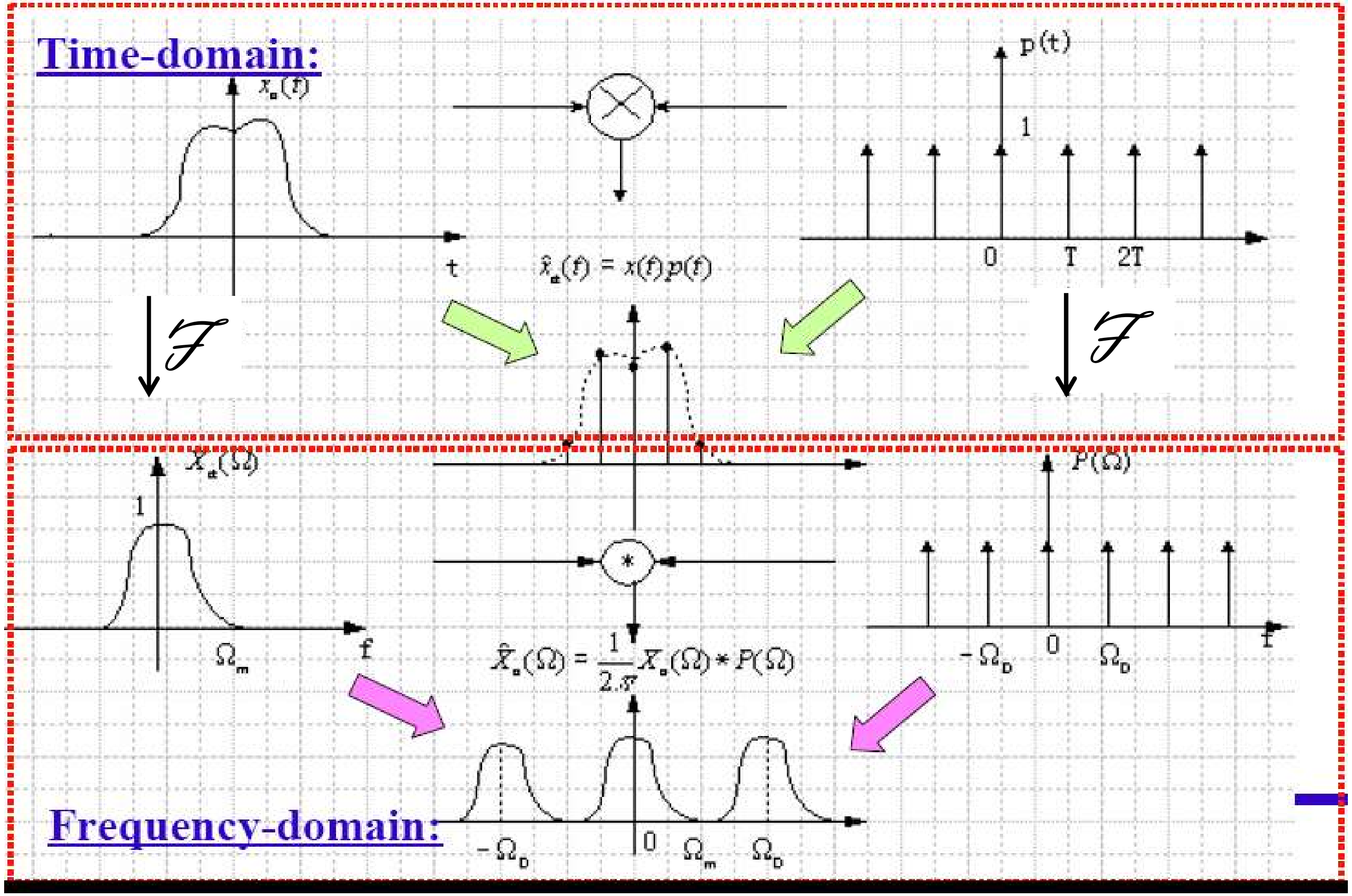
Overlapping occurs when:

$$|\Omega|_{max} > \frac{\Omega_0}{2} = \frac{\pi}{T}$$



Summary of the Sampling Theorem

In summary, sampling process is depicted in this figure:



Examples in applications

Digital Phone System:

3.4kHz could satisfy the conversation requirement.

So, 8kHz is enough for digital systems.

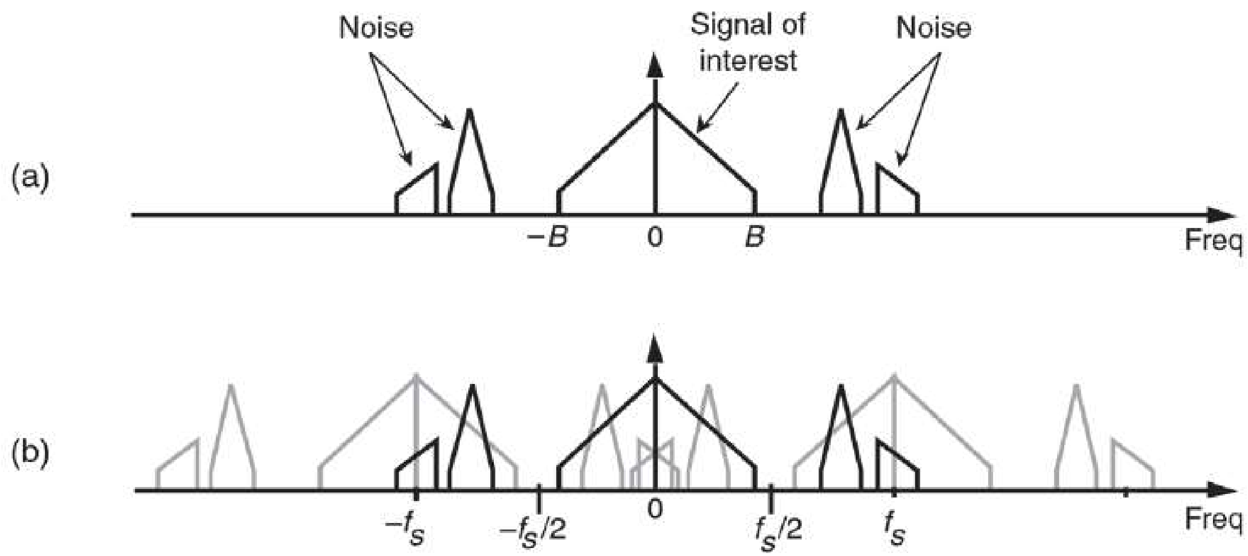
CD Music:

20kHz could satisfy high-quality analogue music requirement.

So, 44.1kHz is enough for digital systems.

Applications - Sampling Low-Pass Signals

A more advanced sampling topic that's proven so useful in practice.

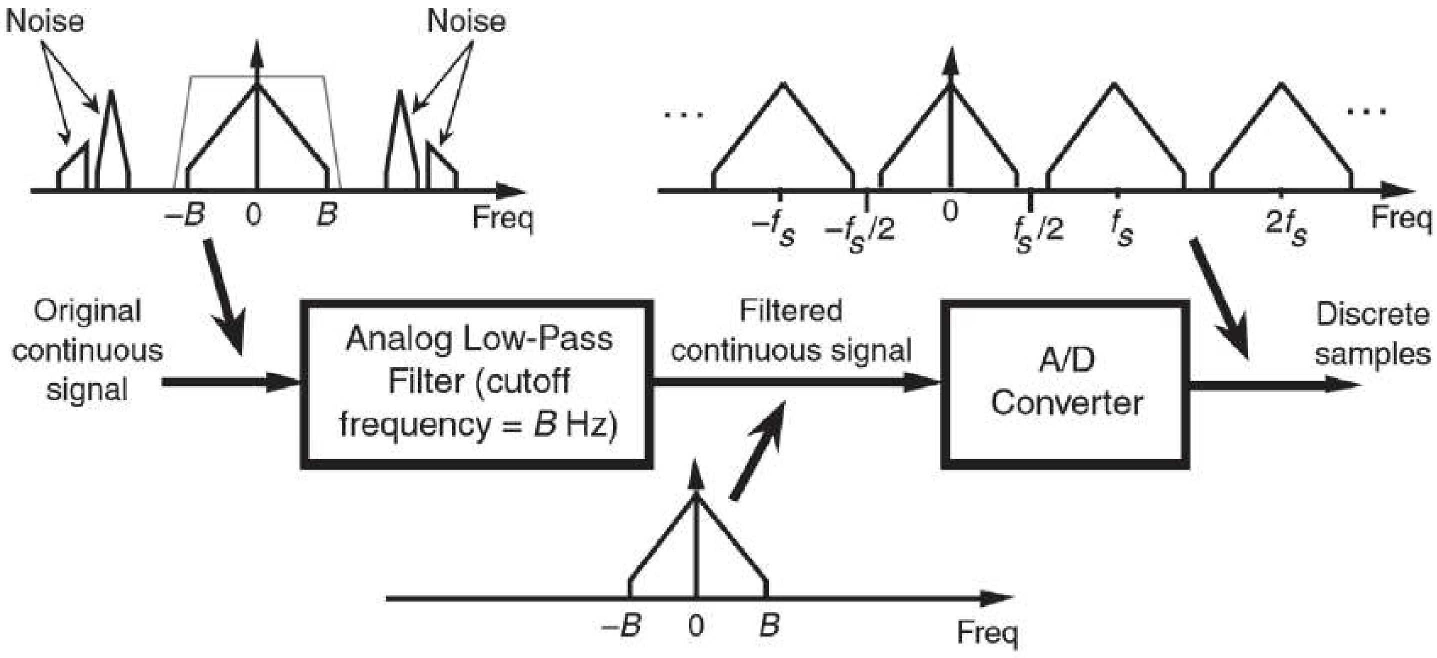


Spectral replications:
(a) original continuous signal plus noise spectrum;
(b) discrete spectrum with noise contaminating the signal of interest.

Applications - Sampling Low-Pass Signals

Solution:

Low-pass analog filtering prior to sampling at a rate of f_s Hz.



2.3 A Generic Description of Discrete Convolution

Discrete Convolution in the Time Domain:

We have two input sequences h(k) of length P and x(k) of length Q in the time domain. The output sequence y(n) of the convolution of the two inputs is defined mathematically,

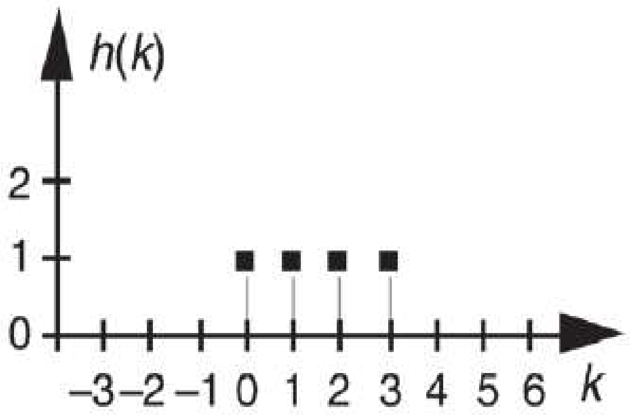
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

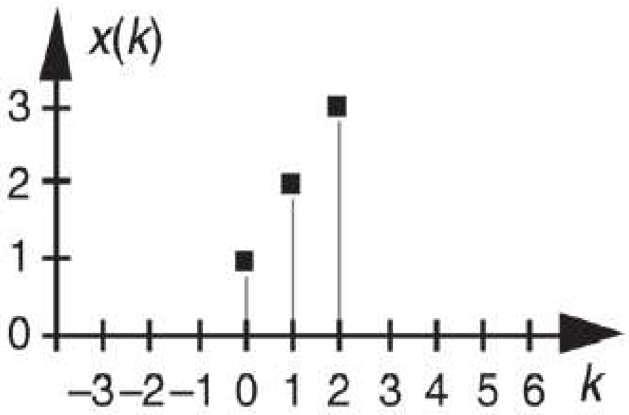
$$= \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

Discrete Convolution in the Time Domain

Example:

h(k) and x(k) is shown in the following:





Discrete Convolution in the Time Domain

Solution:

$$
\begin{aligned}
y(0) &= h(0)x(0) + h(1)x(-1) + h(2)x(-2) + h(3)x(-3) \ , \\
y(1) &= h(0)x(1) + h(1)x(0) + h(2)x(-1) + h(3)x(-2) \ , \\
y(2) &= h(0)x(2) + h(1)x(1) + h(2)x(0) + h(3)x(-1) \ , \\
y(3) &= h(0)x(3) + h(1)x(2) + h(2)x(1) + h(3)x(0) \ , \\
y(4) &= h(0)x(4) + h(1)x(3) + h(2)x(2) + h(3)x(1) \ , \\
y(5) &= h(0)x(5) + h(1)x(4) + h(2)x(3) + h(3)x(2) \ .
\end{aligned}
$$

Discrete Convolution in the Time Domain

Calculation steps:

<u>***(1) Time-reversed***</u>

h(-k) ←h(k)

<u>***(2) Right shift n***</u>

h(n-k)←h(-k)

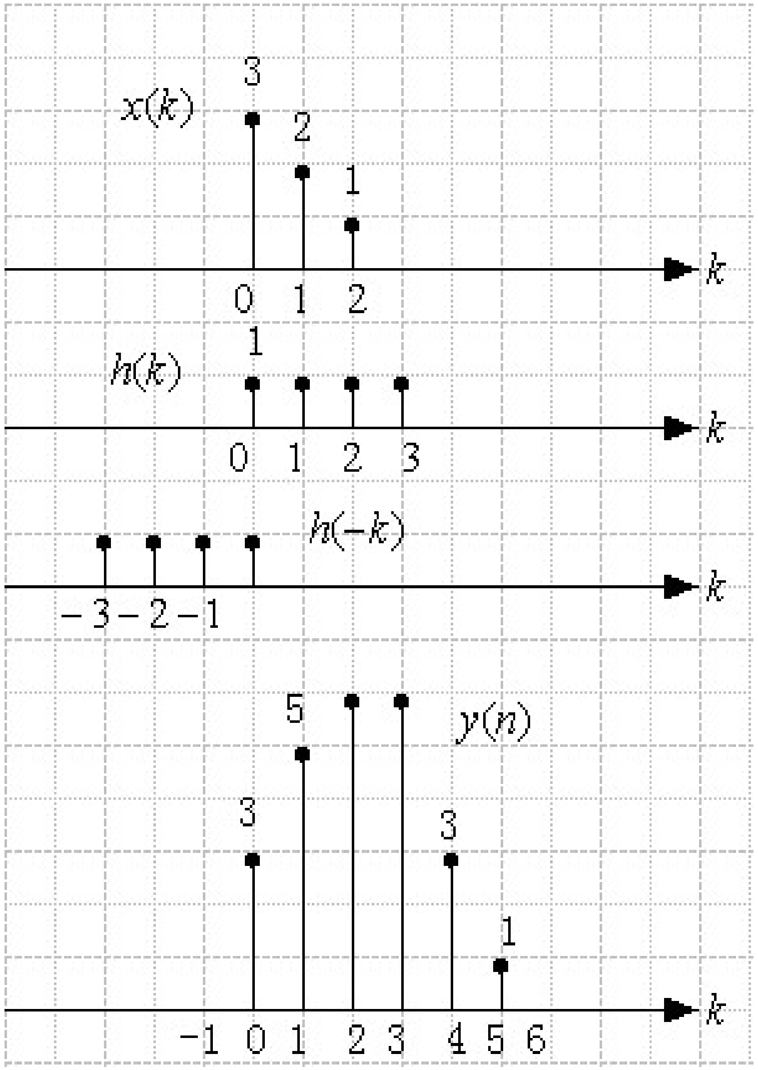
<u>***(3) Multiplication***</u>

x(n)·h(n-k)

<u>***(4) Sum***</u>

∑[x(n)·h(n-k)]

Note: the resulted length is <u>*L=N+M-1*</u>.



Discrete Convolution in the Time Domain

Conclusion of convolution:

Step 1: plotting both h(k) and x(k),

Step 2: flipping the x(k) sequence about the k = 0 value to get x(-k),

Step 3: summing the products of h(k) and x(0-k) for all k to get y(0),

Step 4: shifting x(-k) one sample to the right,

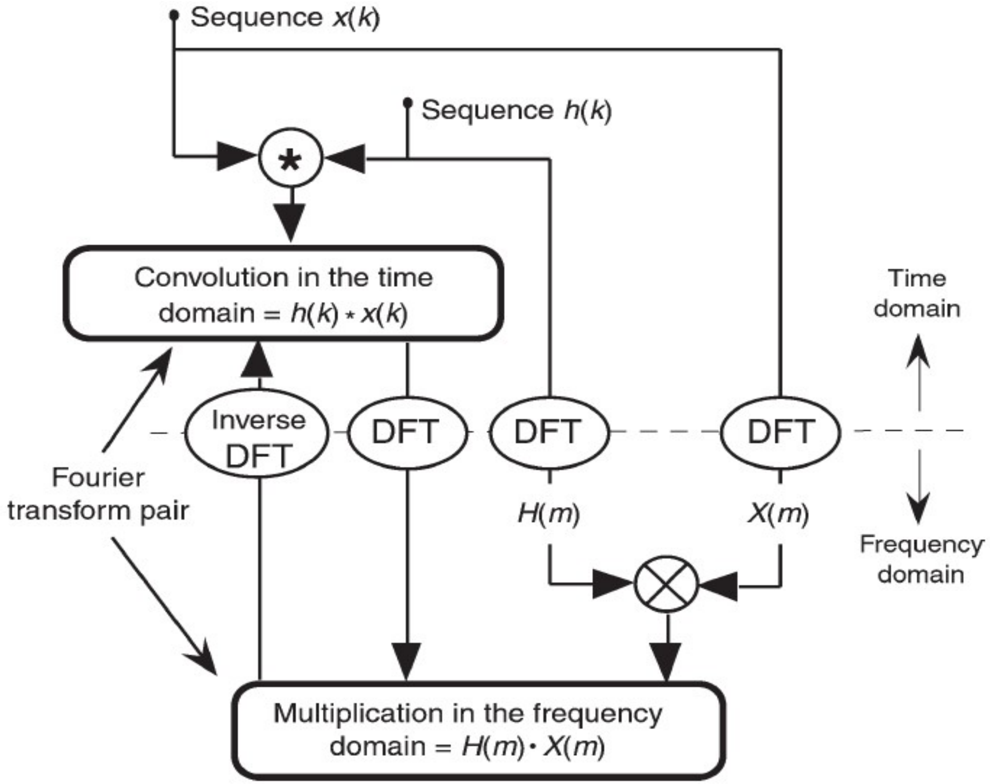
Step 5: summing the products of h(k) and x(1-k) for all k to get y(1)

Step 6: shifting and summing products until there's no overlap of h(k) and the shifted x(n-k), in which case all further y(n)s are zero and we're done.

The Convolution Theorem

$$y(n) = h(k) * x(k) \qquad h(k) * x(k) \underset{\text{IDFT}}{\overset{\text{DFT}}{\rightleftarrows}} H(m) \cdot X(m).$$

The following is the relationships of the convolution theorem:



Summary

2.1 Aliasing: Signal Ambiguity in the Frequency Domain

2.2 Sampling Low-Pass Signals

2.3 A Generic Description of Discrete Convolution

- 2.3.1 Discrete Convolution in the Time Domain
- 2.3.2 The Convolution Theorem
- 2.3.3 Applying the Convolution Theorem