Chapter 1

Discrete Sequences and Systems

Outline

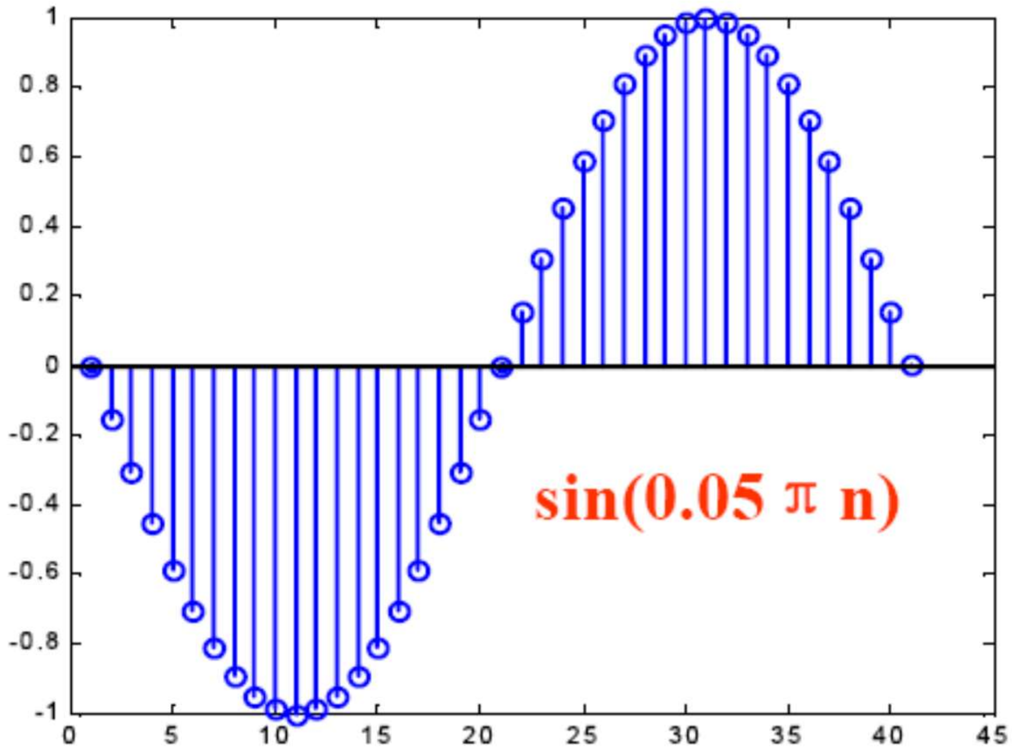
1.1.1 Discrete-time Signals

{x(n)}: n is integer in the range $-\infty$ and $+\infty$

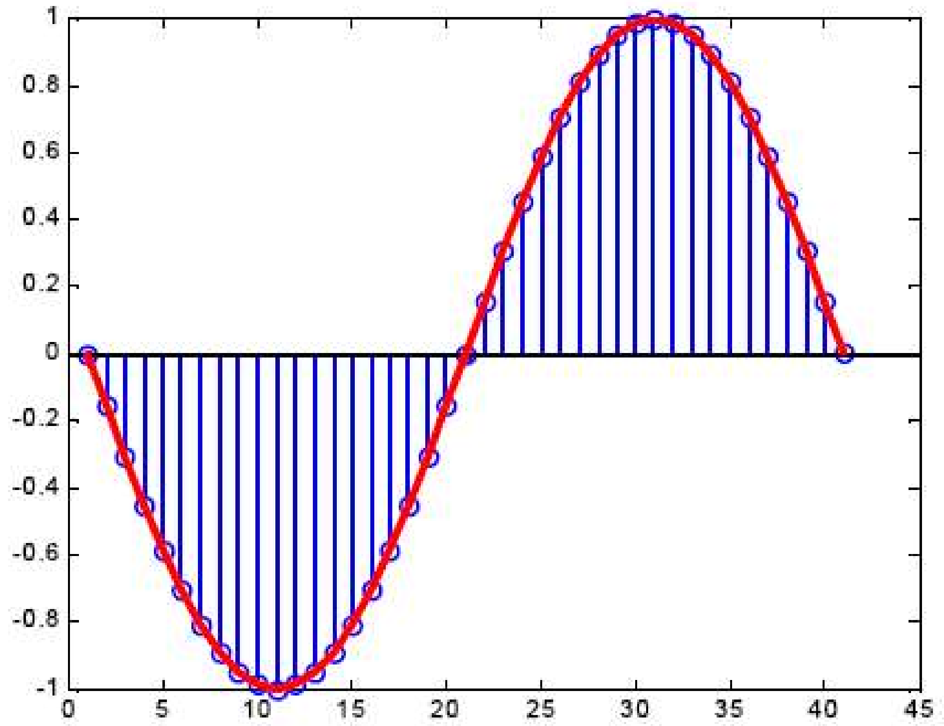
: x(n) is one sample

E.g.

- **x(-1)=-0.1564;**
- **x(0)=0;**
- **x(1)=0.1564;**
- **x(2)=0.3090;**
- **…**



1.1.1 Discrete-time Signals



x(n)

comes from:

$x_a(t)=\sin(2\pi ft)$

Uniformed sampled as:

$x_a(nT)=\sin(2\pi f\underline{nT})$

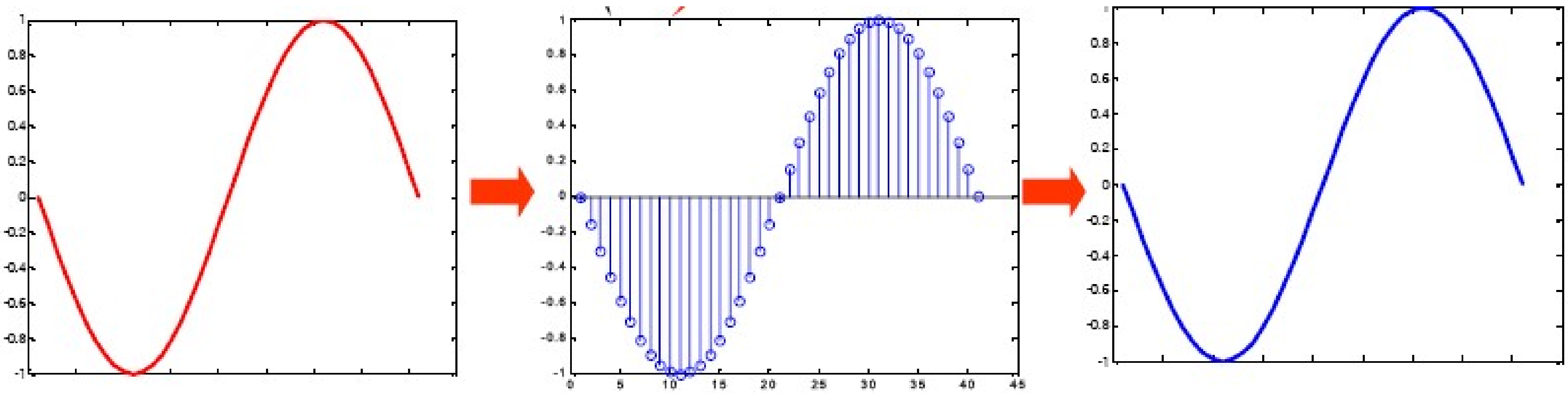
T **denotes sampling interval or period and its reciprocal is sampling frequency written as:**

sampling frequency $$f_s = \frac{1}{T}$$

1.1.1 Discrete-time Signals

Relationship between $x_a(nT)$ and $x_a(t)$:

- ✓ **Part and whole;**
- ✓ **Many curves connecting the two points $x(nT)$ and $x((n+1)T)$, but under *certain conditions*, $x_a(t)$ can be *exclusively reconstructed* based on $x_a(nT)$.**



1.1.1 Discrete-time Signals

About Sequence:

{x(n)} is a real sequence if x(n) is real for any n.

{x(n)} is a complex sequence if x(n) is complex.

$$\{x(n)\} = \{x_{re}(n)\} + j\{x_{im}(n)\}$$

x(n) is a finite length sequence if it is defined only for a finite time interval as: $N1 \le n \le N2$.

The *period of a finite length sequence* is:

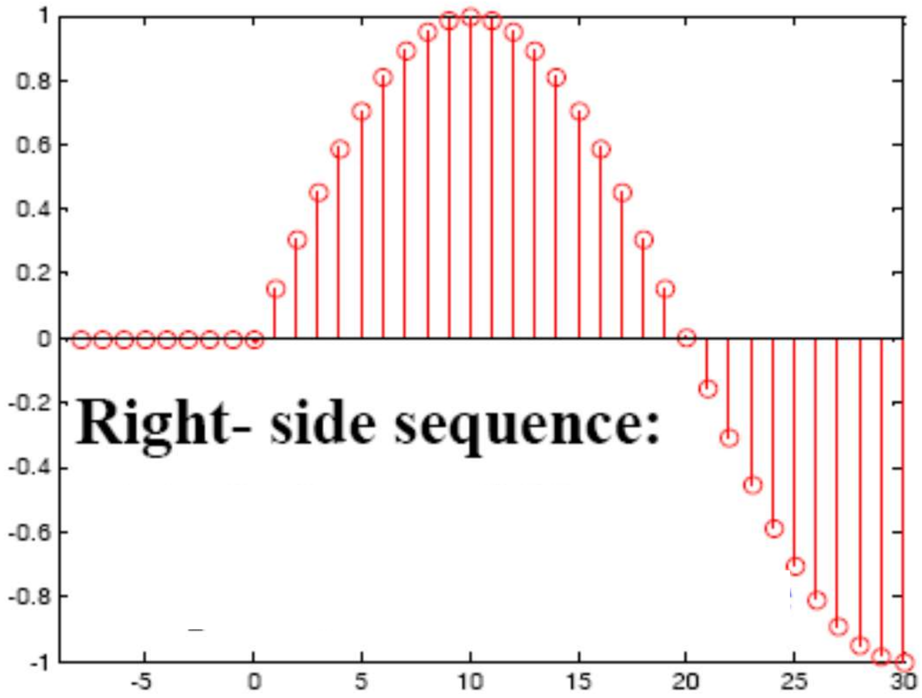
$N = N_2 - N_1 + 1$: N point sequence

1.1.1 Discrete-time Signals

About Sequence:

x(n) is a infinite length sequence if it is defined for infinite time interval.

(By *appending with zeros or zero-padding*, finite length sequence ⇨ infinite.)



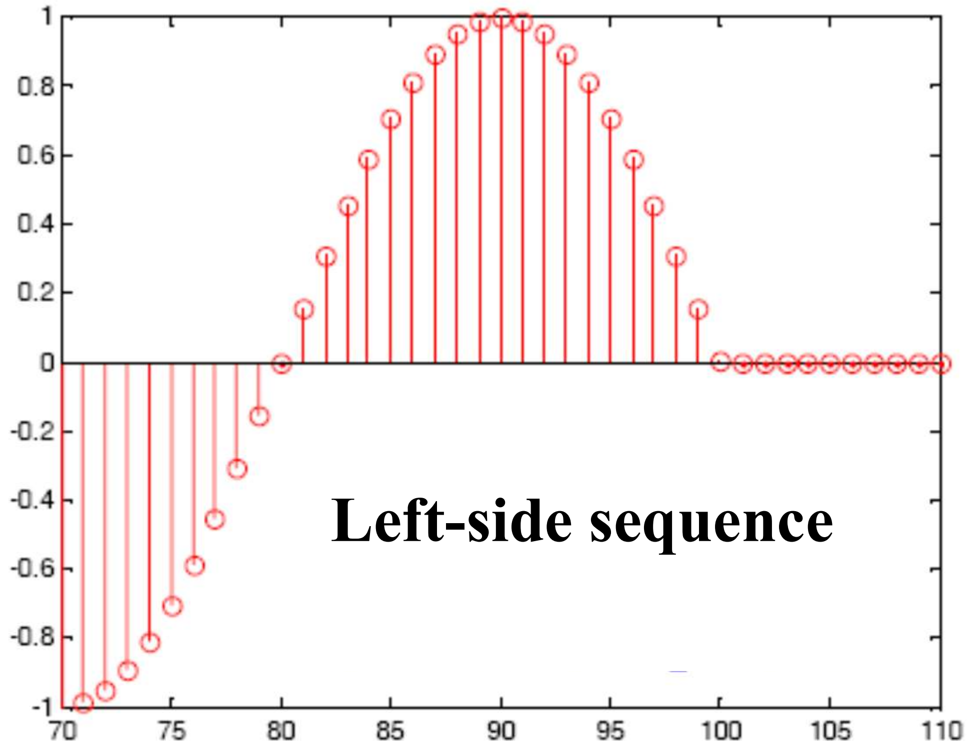
Three types of infinite length sequence, as:

<u>Right-side sequence</u>

1.1.1 Discrete-time Signals

About Sequence:

Left-side sequence



Two-side sequence: $-\infty<n<\infty$

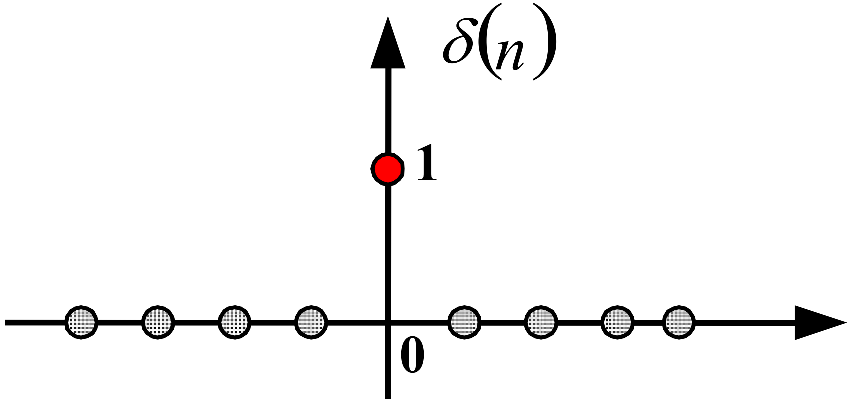
1.1.2 Frequently Used Discrete Sequences

(1) Unit sample sequence

discrete-time impulse

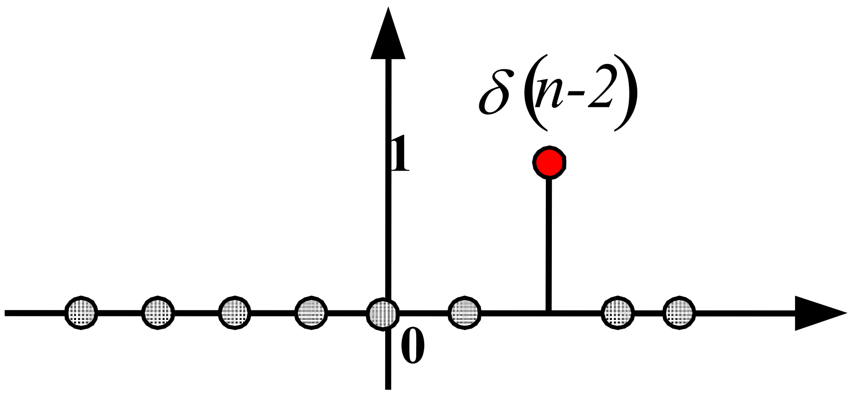
unit impulse

$$\delta(n)=\begin{cases}1 & ,n=0\\ 0 & ,n\neq 0\end{cases}$$



Shifted:

$$\delta(n-n_0)=\begin{cases}1 & ,n=n_0\\ 0 & ,n\neq n_0\end{cases}$$



1.1.2 Frequently Used Discrete Sequences

Any sequence could be expressed as the *<u>weighted sum</u>* of:

shifted unit sample sequences

$$x(n) = x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + \cdots$$

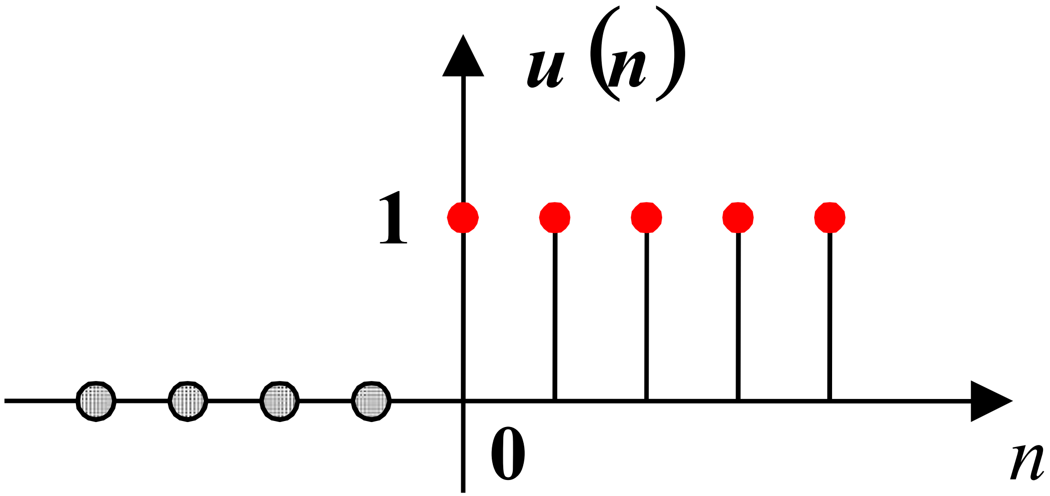
$$= \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)$$

➧ ***Important! Often used later***

1.1.2 Frequently Used Discrete Sequences

(2) Unit step sequence

$$u(n) = \begin{cases} 1 & , n \geq 0 \\ 0 & , n < 0 \end{cases}$$



1.1.2 Frequently Used Discrete Sequences

Relation between u(n) and δ(n)

- **δ(n) could be expressed as:**

$$\delta(n) = u(n) - u(n-1)$$

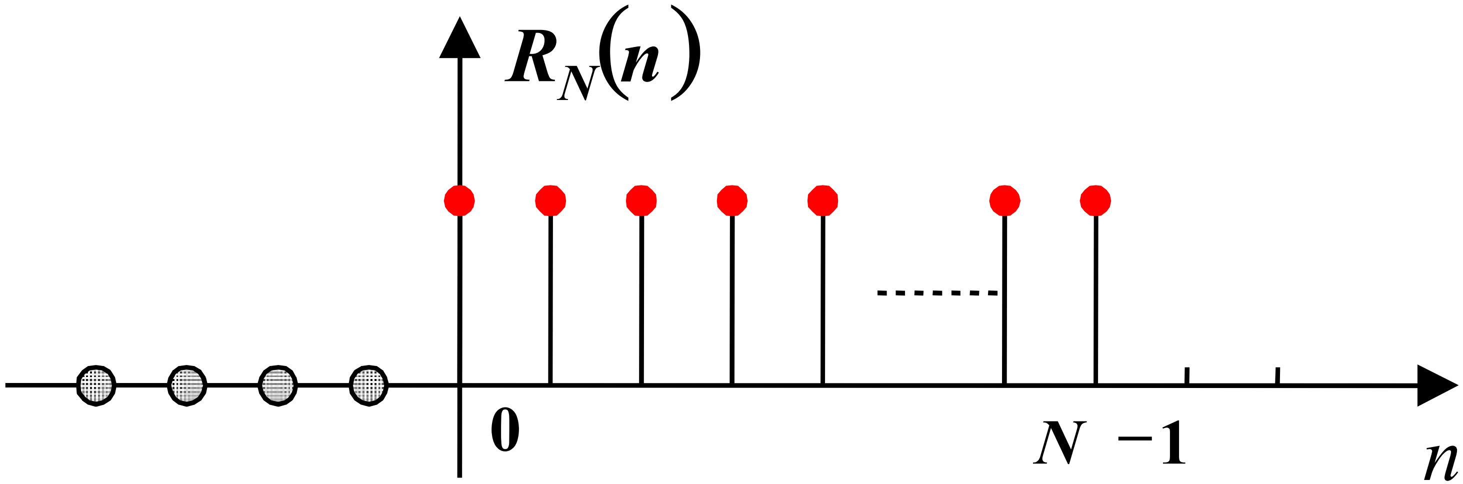
- **u(n) could be expressed as:**

$$u(n) = \sum_{k=0}^{+\infty} \delta(n-k) = \delta(n) + \delta(n-1) + \delta(n-2) + \cdots$$

1.1.2 Frequently Used Discrete Sequences

(3) Rectangular sequence

$$R_N(n)=\begin{cases}1, & 0\le n\le N-1\\ 0 & ,n<0,n\ge N\end{cases}$$



1.1.2 Frequently Used Discrete Sequences

Relation of $R_N(n)$, u(n) and δ(n)

- **$R_N(n)$ is expressed as:**

$$R_N(n) = u(n) - u(n-N)$$

- **$R_N(n)$ is expressed as:**

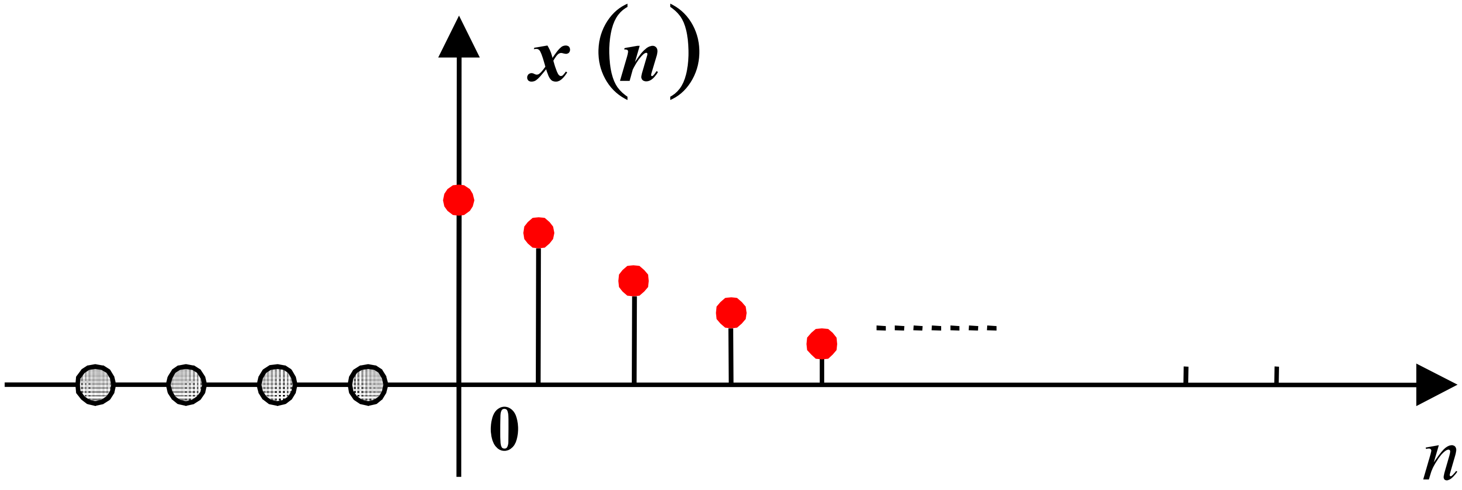
$$R_N(n) = \sum_{k=0}^{N-1} \delta(n-k) = \delta(n) + \delta(n-1) + \cdots + \delta(n-(N-1))$$

1.1.2 Frequently Used Discrete Sequences

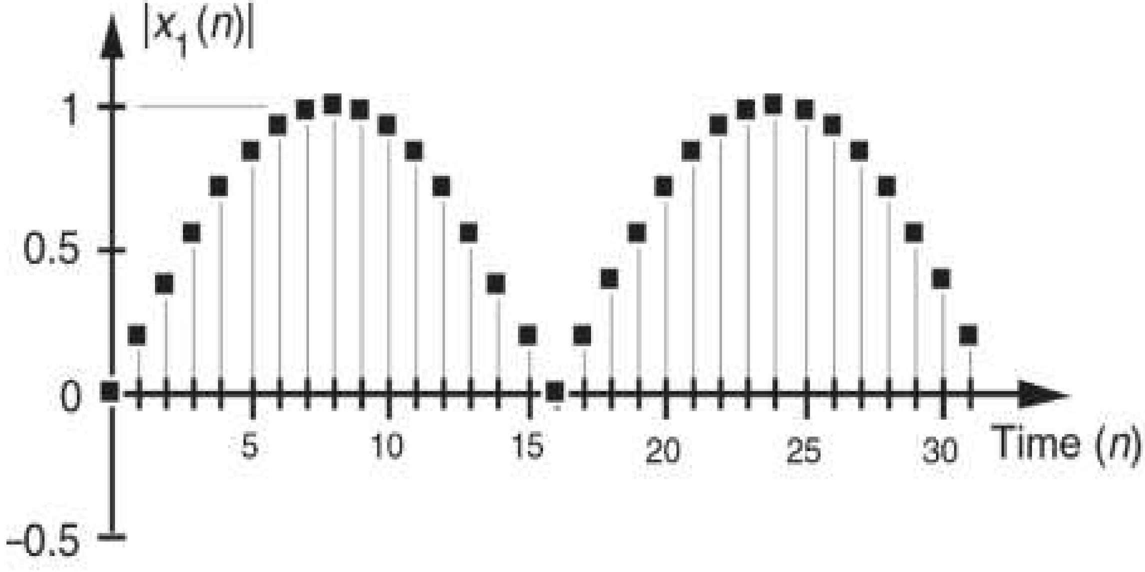
(4) Real exponential sequence

$$x(n) = a^n u(n), \qquad a \neq 0 \;\&\; a \in R$$

when 0<a<1



1.2 Signal Amplitude, Magnitude, Power



$$x_{\mathrm{pwr}}(n) = x(n)^2 = |x(n)|^2 ,$$

or

$$X_{\mathrm{pwr}}(m) = X(m)^2 = |X(m)|^2 .$$

1.3 Signal Processing Operational Symbols

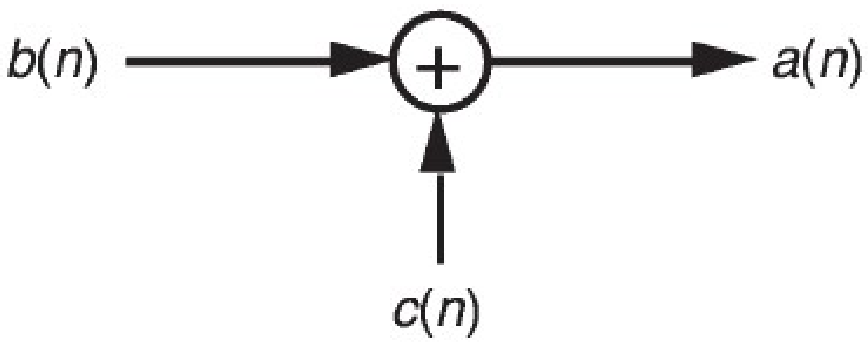
Operation on Sequences

- **Single Input-Single Output:**
  - ✓Input: corrupted signals
  - ✓Output: pure signals
- **M Input-N Output:**
  - ✓Several branches of signals are combined to output
- **But above system could be decomposed into simple operations, including:**
  - ✓modulator, scalar multiplication
  - ✓addition, unit advance

1.3 Signal Processing Operational Symbols

Addition (Adder):

- **a(n)=b(n)+c(n)**
- **sum of samples at the same instant.**



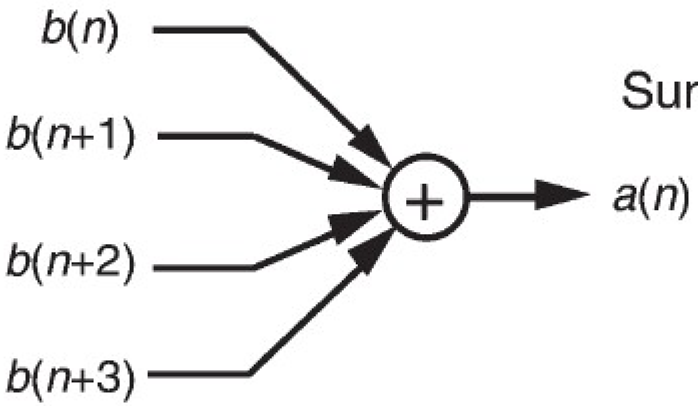
Adder:

$$a(n) = b(n) + c(n)$$

1.3 Signal Processing Operational Symbols

Addition (Adder):

- **a(n)** $= \sum_{k=n}^{n+3} b(k)$
- **summation of samples.**



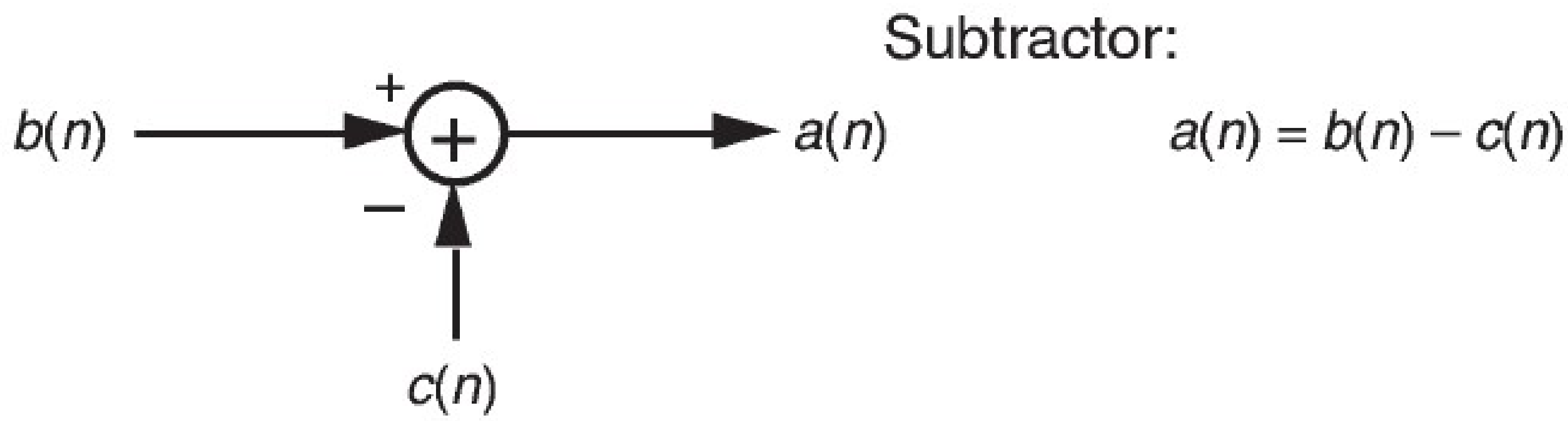
Summation:

$$a(n) = \sum_{k=n}^{n+3} b(k) = b(n) + b(n+1) + b(n+2) + b(n+3)$$

1.3 Signal Processing Operational Symbols

Subtract :

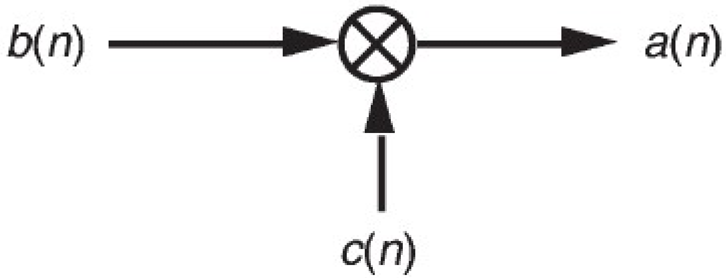
- **a(n)=b(n)-c(n)**
- **difference of samples at the same instant.**



1.3 Signal Processing Operational Symbols

Multiplication:

- **a(n)=b(n)·c(n)**
- **product of samples at the same instant.**
- **Like: *<u>Windowing operation</u>*.**



Multiplication:

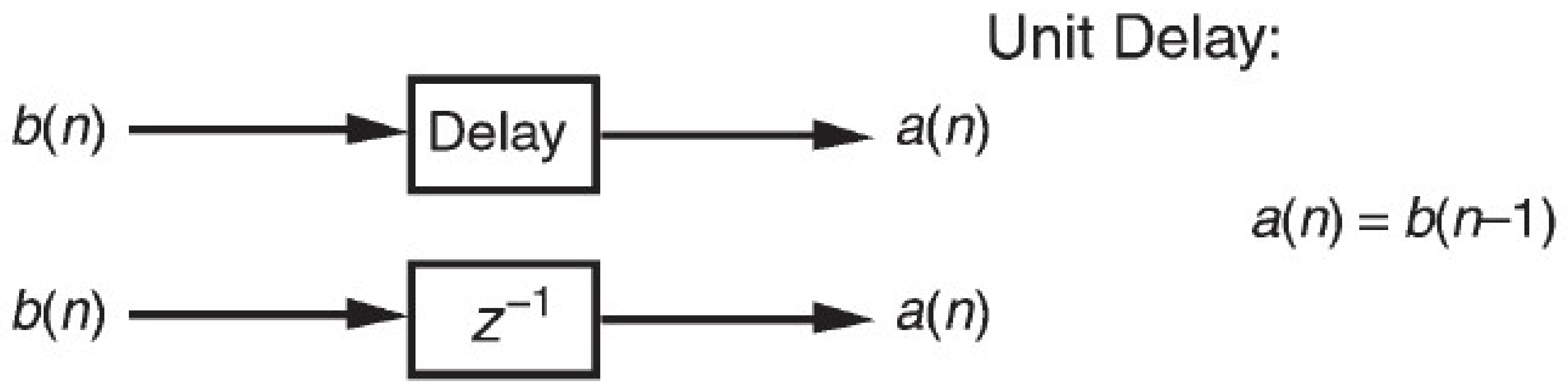
$a(n) = b(n)c(n) = b(n) \cdot c(n)$

[Sometimes we use a "·" to signify multiplication.]

1.3 Signal Processing Operational Symbols

Unit Delay:

- $a(n) = b(n{-}1)$
- **A delayed version of *sample***



1.4 Introduction to Discrete Linear Time-Invariant Systems

- ***Linearity*** **and** ***Time-Invariance (LTI)*** **are two important system characteristics having very special properties.**

- **We need to recognize and understand the notions of** ***Linearity*** **and** ***Time-Invariance*** **not just because the vast majority of discrete systems used in practice are LTI systems, but also LTI systems are very accommodating when it comes to their analysis.**

1.5 Discrete Linear Systems

- **The term *Linear* defines a special class of systems where the output is the superposition, or sum, of the individual outputs had the individual inputs been applied separately to the system.**

$$x_1(n) \xrightarrow{\text{results in}} y_1(n)$$

$$x_2(n) \xrightarrow{\text{results in}} y_2(n)$$

$$x_1(n)+x_2(n) \xrightarrow{\text{results in}} y_1(n)+y_2(n)$$

$$c_1x_1(n)+c_2x_2(n) \xrightarrow{\text{results in}} c_1y_1(n)+c_2y_2(n)$$

Example

E.g：

- **Given** $y(n) = 3x(n) + 4$ **:**
- **Is this sequence linear ?**

Example

E.g：

- **Given** $y(n) = 3x(n) + 4$ **:**
- **Is this sequence linear ?**

$$y_1(n) = T[x_1(n)] = 3x_1(n) + 4$$

$$y_2(n) = T[x_2(n)] = 3x_2(n) + 4$$

$$ay_1(n) + by_2(n) = 3ax_1(n) + 3bx_2(n) + 4(a+b)$$

$$T[ax_1(n) + bx_2(n)] = 3[ax_1(n) + bx_2(n)] + 4$$

$$\because \quad T[ax_1(n) + bx_2(n)] \neq ay_1(n) + by_2(n)$$

$$\therefore \quad Non-linear$$

1.6 Time-Invariant Systems

- **A time-invariant system is one where a time delay (or shift) in the input sequence causes a equivalent time delay in the system's output sequence.**

$$x(n) \xrightarrow{\text{results in}} y(n)$$

$$x'(n) = x(n+k) \xrightarrow{\text{results in}} y'(n) = y(n+k)$$

Example

E.g：

- **Given** $y(n) = 3x(n) + 4$ **:**
- **Is this sequence time-invariant ?**

Example

Time-shift Invariant ?

$$T[x(n-m)] = 3x(n-m) + 4$$

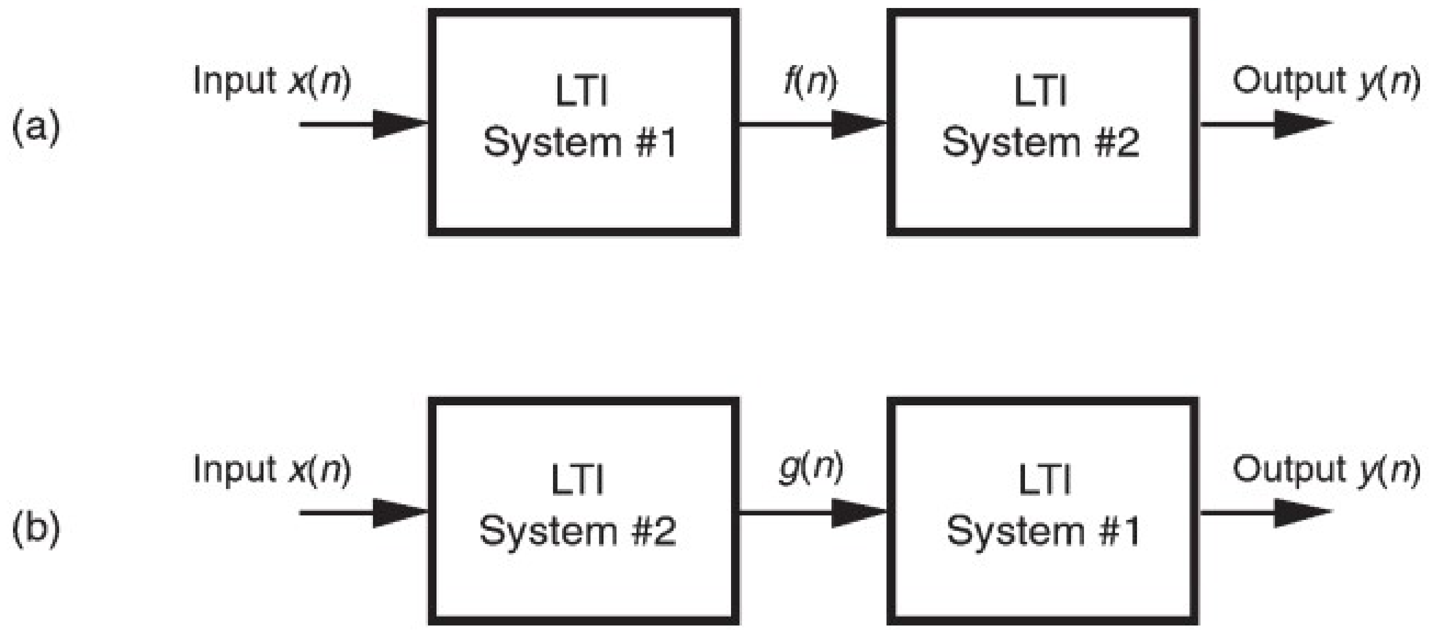
$$y(n-m) = 3x(n-m) + 4$$

- **So, it's Time-shift Invariant .**

- ***<u>-> This system is not a LTI system.</u>***

1.7 The Commutative Property of Linear Time-Invariant Systems

- **Swapping the order of two cascaded systems does not alter the final output.**



1.8 The Causality Property of Linear Time-Invariant Systems

Causality definition:

$$y_1[n] = y_2[n] \quad \text{for } n < N$$

This implies that:

For the causal system, if $x_1(n)=x_2(n)$ for $n<n_0$, then $y_1(n)=y_2(n)$ for $n<n_0$.

1.8 The Causality Property of Linear Time-Invariant Systems

Causality:

If a LTI system is a causal system, it satisfies:

$$h(n) = 0, \qquad n < 0$$

——Realizable System, it is used to prove the causality of the system (*Important*).

1.9 The Stability Property of Linear Time-Invariant Systems

Stability:

If and only if for every bounded input, the output is also bounded (*BIBO*).

- **i.e.**

$$\left| x(n) \right| < \infty, \qquad every \quad n$$

- **Then**

$$\left| y(n) \right| < \infty, \qquad every \quad n$$

In the later discussion, the involved discrete system is the LTI system. General practical systems are causal and stable.

1.9 The Stability Property of Linear Time-Invariant Systems

Significant Conclusion:

•**For a LTI system, the *sufficient* and *necessary* condition to stability is:**

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

1.9 The Stability Property of Linear Time-Invariant Systems

Proof:

- **Assuming |x(n)|≤M.**
- **Then,**

$$\begin{aligned}
|y(n)| &\le \sum_{k=-\infty}^{\infty} |h(k)x(n-k)| \\
&\le \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)| \\
&\le M \sum_{k=-\infty}^{\infty} |h(k)| < \infty
\end{aligned}$$

Stability and Causality

We focus on:

Stable and Causal LTI System

simultaneously

stable + **causal**

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

$$h(n) = 0, \qquad n < 0$$

Stability and Causality

E.g.:

Impulse response of a LTI system is given by

$$h(n) = a^n u(n)$$

Question:

Is it a causal system?

Is it a stable system?

Stability and Causality

Solution:

$$n<0 \quad u(n)=0$$

$$\therefore n>0 \quad h(n)=a^n u(n)$$

causal

$$\sum_{n=-\infty}^{\infty}|h(n)|=\sum_{n=-\infty}^{\infty}|a^n u(n)|=\begin{cases} |a|<1 & \dfrac{1}{1-a} \\ |a|>1 & \dfrac{1-a^{n+1}}{1-a} \end{cases}$$

bounded stable

not stable

1.10 Analyzing Linear Time-Invariant Systems

- **Knowing the *(unit) impulse response* of an LTI system, we can determine the system's output sequence for any input sequence because *the output is equal to the convolution of the input sequence and the system's impulse response*.**

$$y(n) = T[x(n)] = x(n) * h(n)$$

$$h(n) = T[\delta(n)]$$

Summary

- Discrete Sequences
- Signal Amplitude, Magnitude, Power
- Signal Processing Operational Symbols
- Introduction to Discrete Linear Time-Invariant Systems
- Discrete Linear Systems
- Time-Invariant Systems
- The Commutative Property of Linear Time-Invariant Systems
- The Causality Property of Linear Time-Invariant Systems
- The Stability Property of Linear Time-Invariant Systems
- Analyzing Linear Time-Invariant Systems