Chapter 3:

z-Transform

Contents

3.1 The z-Transform

ZT of the sequence x(n) is defined as:

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Z, complex variable

Inverse ZT:

$$x(n) = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$$

3.1 The z-Transform

We have to ask if X(z) converge?

For any x(n), the region of converge (*ROC*) of ZT is the complex plane making X(z) converge.

i.e.: {z: X(z) exists}

Important:

Different x(n) with different ROC may have the same ZT.

So, the ROC of each X(z) should be defined.

z-Transform: Pole-zero Plot

System has ZT as:

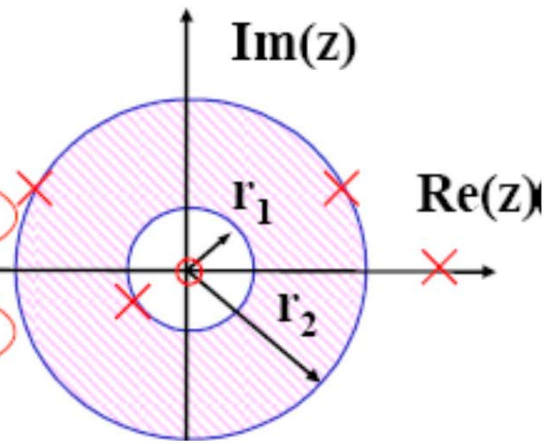
$$X(z) = \frac{P(z)}{D(z)} = \frac{p_0 + p_1 z^{-1} + p_2 z^{-2} \ldots + p_M z^{-M}}{d_0 + d_1 z^{-1} + d_2 z^{-2} \ldots + d_N z^{-N}}$$

$$X(z) = z^{N-M} \frac{p_0 z^M + p_1 z^{M-1} + \ldots + p_M}{d_0 z^N + d_1 z^{N-1} + \ldots + d_N}$$

$$X(z) = z^{N-M} \frac{p_0}{d_0} \frac{\prod_{i=1}^{M} (z - z_i)}{\prod_{j=1}^{N} (z - \lambda_j)}$$

zero point

pole point



z-Transform: Pole-zero Plot

ROC:

The *ROC* is determined by $|z|=r$, in terms of the theory of complex variable function, it can be a circular band:

$$r1<|z|<r2$$

In the ROC, *X(z)* is an analytic function, and the pole of *X(z)* is out of *ROC*, with the pole on the edge.

r1 can be zero, r2 can be ∞

If $r2<r1$， it means *ROC* is not exist, neither the *z-Transform*.

z-Transform

Right-side Sequence:

When n<0, x(n)=0;

Usually causal sequence;

X(z) only contains the negative indexes of *z*.

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

The ROC: |z|>r1, outside of radius r1.

z-Transform

Example 1:

Determine the ZT of x(n):

$$x_1(n) = a^n u(n)$$

z-Transform

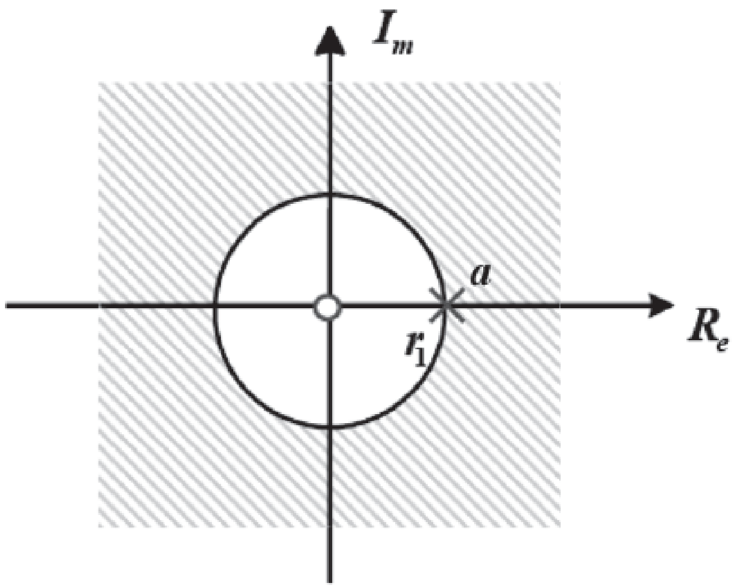
Solution:

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

$$R: |z| > a$$

The Pole:



z-Transform

Left-side Sequence:

When n ≥ 0, x(n)=0;

X(z) only contains the positive indexes of z.

$$X(z) = Z[x(n)] = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

<u>The ROC: |z|<r2, inside of radius r2.</u>

z-Transform

Example 2:

Determine the ZT of $x_1(n)$:

$$x_1(n) = -b^n u(-n-1)$$

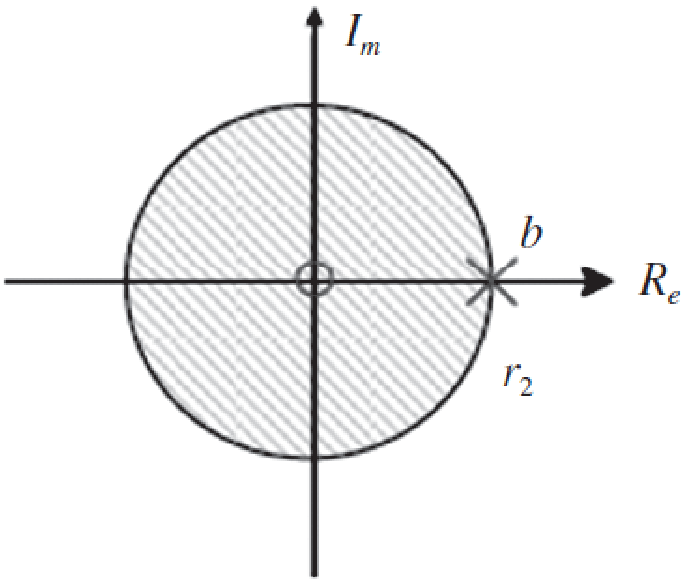
z-Transform

Solution:

$$X_1(z) = \sum_{n=-\infty}^{-1} -b^n z^{-n} = \sum_{n=1}^{\infty} -b^{-n} z^n$$

$$= 1 - \sum_{n=0}^{\infty} b^{-n} z^n = \frac{z}{z-b}$$

The Pole:



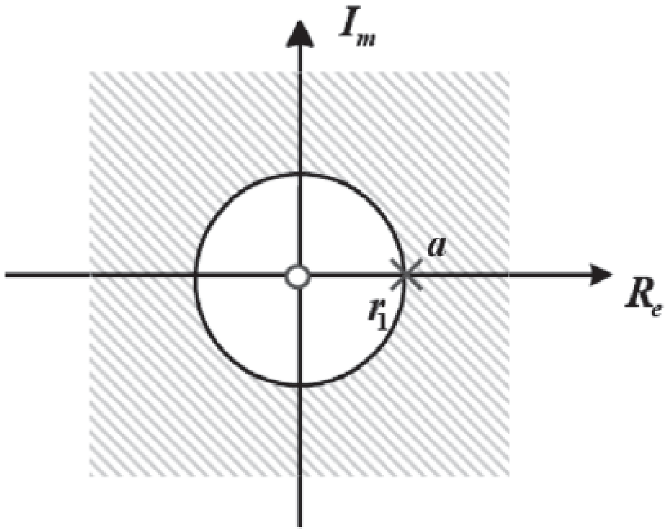
$$R : |Z| < b$$

z-Transform

About these two examples, if *b*=*a*:

$$X(z)=\frac{z}{z-a} \quad = \quad X_1(z)=\frac{z}{z-b}$$

But for *ROC*:



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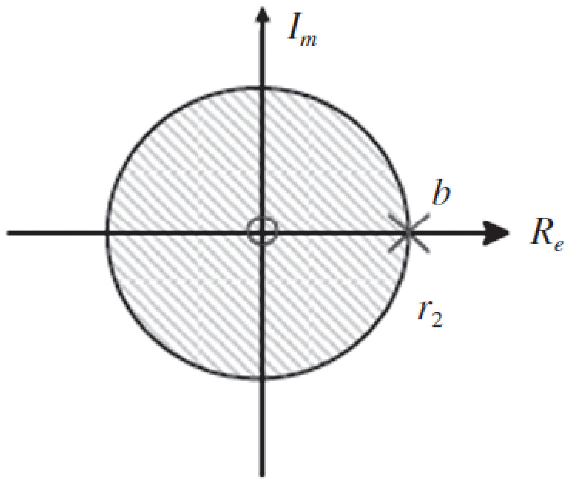


z-Transform

If *b=a*:

$X_1(z)$ has the same form with X(z), except for the ROC.

That implies that the *ROC* insures only one *ZT* of x(n).

Different ROC means different ZT.

ROC plays an important role in system analysis.

z-Transform

Two-side Sequence:

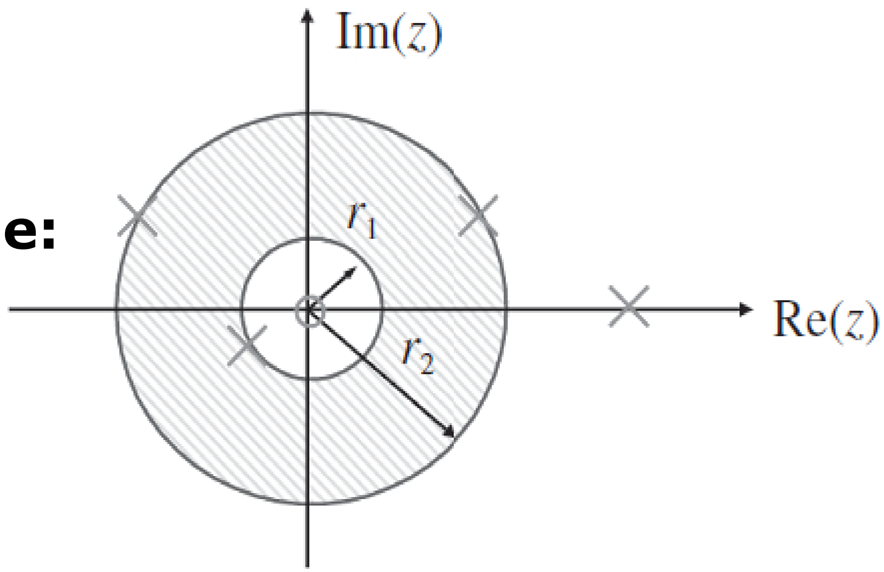
Contains *Right-side* sequence and *Left-side* sequence.

So, the *ROC* is defined as:

$$r1<|z|<r2$$

or *not exist* if r2<r1.

The Pole:



z-Transform

Example 3:

Define:

$$x_2(n) = x(n) + x_1(n) = a_n u(n) - b_n u(-n-1)$$

Determine the *ZT* of $x_2(n)$.

z-Transform

Solution:

$$X_2(z) = \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{-\infty}^{-1} b^n z^{-n}$$

$$= \left\{ \frac{z}{z-a}, ROC: |z| > |a| \right\} + \left\{ \frac{z}{z-b}, ROC1: |z| < |b| \right\}$$

$$= \frac{z}{z-a} + \frac{z}{z-b}; \qquad ROC2: ROC \bigcap ROC1$$

z-Transform

Conclusion:

$$X(Z)=\sum_{n=-\infty}^{\infty}x(n)Z^{-n}=\sum_{n=0}^{\infty}x(n)Z^{-n}+\sum_{n=-\infty}^{-1}x(n)Z^{-n}$$

<u>= Right-side sequence + Left-side sequence</u>

(1) The convergence condition is decided by the amplitude of |*z*|, so it converges in the boundary of a circle.

(2) <u>Right-side Sequence</u> (n≥0): the ROC is |*z*|>|a|, where *a* is the pole.

(3) <u>Left-side Sequence</u> (n<0): the ROC is |*z*|<|b|, where *b* is the pole.

The Properties of ZT

1. Linear

$$Z[ax(n) + by(n)] = aX(z) + bY(z)$$

2. Time-domain shifting

$$Z[x(n + n_0)] = z^{n_0}X(z)$$

3. Frequency-domain shifting *(scaling in z-domain)*

$$Z[a^n x(n)] = X(\frac{z}{a}), \qquad |a| r_{x-} < |z| < |a| r_{x+}$$

The Properties of ZT

4. Differential

$$Z[nx(n)] = -z\frac{dX(z)}{dz}, \qquad r_{x-} < |z| < r_{x+}$$

5. Conjugation

$$Z[x^*(n)] = X^*(z^*), \qquad r_{x-} < |z| < r_{x+}$$

The Properties of ZT

6. Initial Value Theorem

If n<0, x(n)=0, then:

$$x(0) = \lim_{z \to \infty} X(z)$$

7. Convolution in z-domain

The convolution in the discrete time domain equals to the multiplication in z domain.

$$Z[x(n) * y(n)] = X(z)Y(Z)$$

$$r_{x-} < |z| < r_{x+}, \qquad r_{h-} < |z| < r_{h+}$$

Common ZT Pairs

Table of common ZT pairs:

Signal, x(n)	Z-transform	ROC
δ (n)	1	$\forall z$
μ (n)	$\frac{1}{1-z^{-1}}$	$\|z\|>1$
$-u(-n-1)$	$\frac{1}{1-z^{-1}}$	$\|z\|<1$
$a^n u(n)$	$\frac{1}{1-az^{-1}}$	$\|z\|>a$
$-b^n u(-n-1)$	$\frac{1}{1-bz^{-1}}$	$\|z\|<b$

Important!

3.2 The Inverse z-Transform

Definition of Inverse ZT:

$$X(z) \xrightarrow{Z^{-1}} x(n) \quad z \in R$$ <u>*(ROC)*</u>

$$x(n) = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$$

3.2 The Inverse z-Transform

Calculation of Inverse ZT:

The inverse ZT needs to calculate the integral in the complex contour C, usually it is complex and difficult.

Normal methods:

- ***Part fractional method***
- ***General Expression of Inverse z-Transform***
- ***Definition Method***

Inverse ZT - *Part fractional method*

We have already got some common sequences' ZT:

$$a^n u(n) \xleftrightarrow{Z} \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z| > |a|$$

$$-a^n u(-n-1) \xleftrightarrow{Z} \frac{1}{1-az^{-1}} = \frac{z}{z-a} \qquad |z| < |a|$$

Therefore, we could extract *X(z)* to the sum of many fractional parts.

Then, use the already got ZT form to get *x(n)*.

Note: ROC is very important!

Inverse ZT - ***Part fractional method***

Such as:

$$X(z) = \sum_i \frac{A_i z}{z - z_i}$$

For *ROC* (|z|>r1): $x(n) = \sum_i A_i {z_i}^n u(n)$

For *ROC* (|z|<r2): $x(n) = -\sum_i A_i {z_i}^n u(-n-1)$

Inverse ZT - ***<u>Part fractional method</u>***

For *ROC* (r1<|z|<r2):

$$X(z) = \sum_i \frac{B_i z}{z - p_i} + \sum_i \frac{C_i z}{z - s_i}$$

Two-side sequence:

$$x(n) = \sum_i B_i p_i{}^n u(n) - \sum_i C_i s_i{}^n u(-n-1)$$

Inverse ZT - ***Part fractional method***

Example:

If:

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

Please determine its time domain signal *x(n)*.

Inverse ZT - ***Part fractional method***

Solution:

$$X(z)=\frac{z}{3\left(z^2-\frac{4}{3}z+\frac{1}{3}\right)}=\frac{\frac{1}{3}z^{-1}}{1-\frac{4}{3}z^{-1}+\frac{1}{3}z^{-2}}$$

$$=\frac{\frac{1}{3}z^{-1}}{\left(1-z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}=\frac{1}{2}\left(\frac{1}{1-z^{-1}}\right)-\frac{1}{2}\left(\frac{1}{1-\frac{1}{3}z^{-1}}\right)$$

Inverse ZT - ***Part fractional method***

Solution:

X(z) has two poles, $z_1=1$ and $z_2=1/3$.

(1) For *ROC* |*z*|>1, x(n) is *Right-side* sequence.

$$x_1(n) = \frac{1}{2}u(n) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

(2) For *ROC* |*z*|<1/3, x(n) is *Left-side* sequence.

$$x_2(n) = -\frac{1}{2}u(-n-1) + \frac{1}{2}(\frac{1}{3})^n u(-n-1)$$

Inverse ZT - ***Part fractional method***

Solution:

X(z) has two poles, z_1=1 and z_2=1/3.

(3) For *ROC* 1/3<|*z*|<1, the x(n) is *Two-side* sequence.

$$x_3(n) = -\frac{1}{2}u(-n-1) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

Inverse ZT - *General Expression*

The general expression for the inverse ZT is given by :

$$x(n) = \frac{1}{2\pi j} \oint_c X(z) z^{n-1} dz$$

$\oint_c$ **is a closed curve in the *ROC*, containing the origin.**

Inverse ZT - *General Expression*

General Expression of Inverse ZT:

Using the Residue method, the integration becomes:

$$\begin{cases} (1)\, x(n) = \sum [X(z) z^{n-1} \text{ the residue number of the pole in } C] \\ \qquad\qquad\qquad\qquad n \geq m, \quad \text{causal} \\ (2)\, x(n) = -\sum [X(z) z^{n-1} \text{ the residue number of the pole out of } C] \\ \qquad\qquad\qquad\qquad n < m, \quad \text{anticausal} \end{cases}$$

Inverse ZT - *General Expression*

The calculation of the Residue:

For: $$X(z)z^{n-1} = \frac{\psi(z)}{(z-z_0)^s}$$

The Residue of the pole is:

$$\mathrm{Res}[X(z)z^{n-1}, z = z_0] = \frac{1}{(s-1)!}\frac{d^{s-1}\psi(z)}{dz^{s-1}}\Big|_{z=z_0}$$

For *1-order* Pole:

$$\mathrm{Res}[X(z)z^{n-1}, z = z_0] = \psi(z_0)$$

Inverse ZT - ***General Expression***

Note:

If *ROC* is out of a circle, we usually use Formula No.1:

$$x(n) = \sum [X(z) z^{n-1} \text{the residue of the pole in C}]$$

If *ROC* is inside of a circle, we usually use Formula No.2:

$$x(n) = -\sum [X(z) z^{n-1} \text{the residue of the pole out of C}]$$

If *ROC* is a ring, use both of them.

Inverse ZT - ***General Expression***

Note:

The poles of $X(z)z^{n-1}$ include two parts:

- ***Poles from X(z): usually have limited numbers and orders.***
- ***Poles from z^{n-1} : usually exist at $z=0$ and $z=\infty$.***

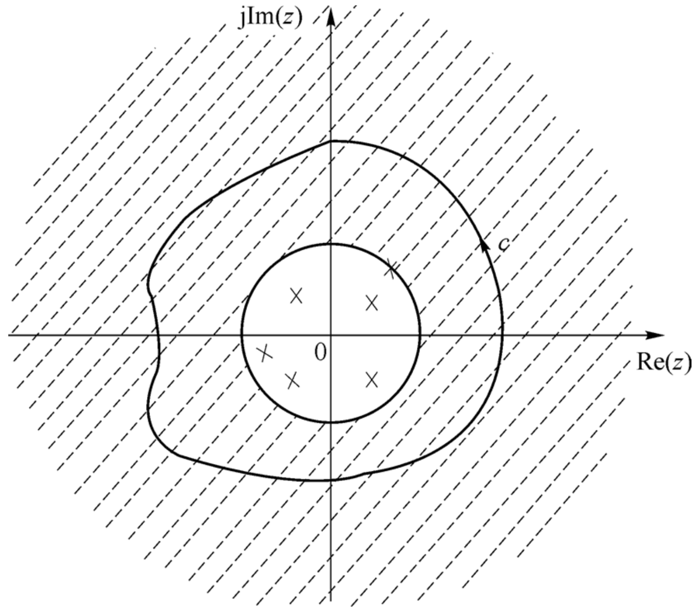
Usually, we want to choose the region that $X(z)z^{n-1}$ have limited numbers and orders' poles to calculate the residue easily and try to avoid the residue at $z=\infty$.

Inverse ZT - *General Expression*

Note:

If the *ROC* is outside of a circle, usually we calculate x(n) at n>0 (Right-side Sequence) and choose Formula No.1.

Because X(z) have limited poles inside of *C* and z^{n-1} is analytic at $z=0$, but there are high order poles at $z=\infty$ for z^{n-1} when *n* is large.

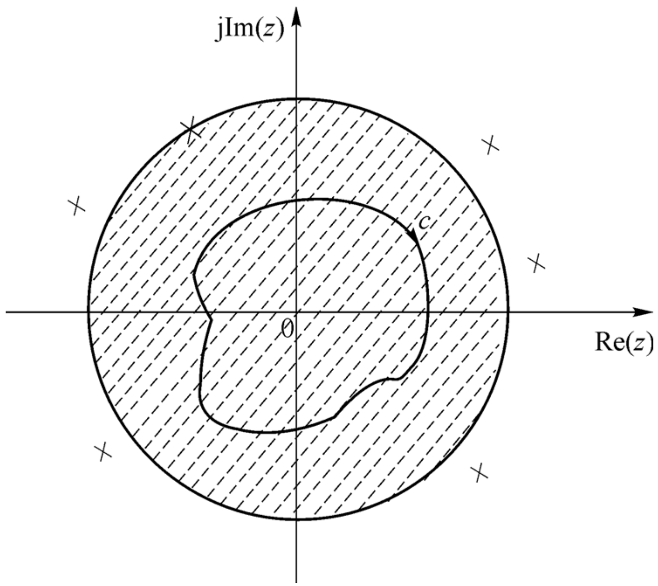


Inverse ZT - *General Expression*

Note:

If the *ROC* is inside of a circle, usually we calculate x(n) at n<0 (Left Sequence) and choose Formula No.2.

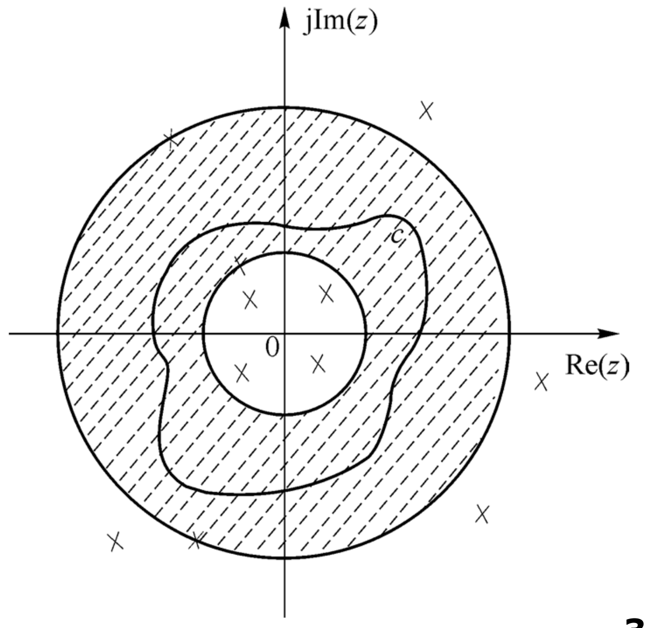
Because X(z) have limited poles outside of *C* and z^{n-1} is analytic at $z=\infty$, but there are high order poles at $z=0$ for z^{n-1} when *n* is large.



Inverse ZT - ***General Expression***

Note:

If the *ROC* is a ring, usually we use both of Formula No.1 and No.2.



Inverse ZT - *General Expression*

Note:

Actually, *n=0* is not the only edge for the residue method. We can reach one more general method:

Extract $X(z)=X_0(z)z^m$, m is an integer. Therefore, $X_0(z)$ is analytic at both $n=0$ and $n=\infty$. Then:

$$X(z)z^{n-1} = X_0(z)z^m z^{n-1} = X_0(z)z^{n+m-1} = X_1(z)$$

After determining the *ROC* of $X(z)$ and *C*:

We can get $x(n)$ at $n \geq 1-m$ by calculate the residue of $X_1(z)$ inside of C and then get $-x(n)$ at $n<1-m$ outside of *C*.

Inverse ZT - *General Expression*

Example:

If:

$$X(z) = \frac{z}{3z^2 - 4z + 1}$$

Please determine its time domain signal *x(n)* with the residue method.

Inverse ZT - ***General Expression***

Solution:

$$X(z) = \frac{z}{3\left(z^2 - \frac{4}{3}z + \frac{1}{3}\right)} = \frac{z}{3(z-1)(z-\frac{1}{3})}$$

$$X_1(z) = X(z)z^{n-1} = \frac{z^n}{3(z-1)(z-\frac{1}{3})} = X_0(z)z^n$$

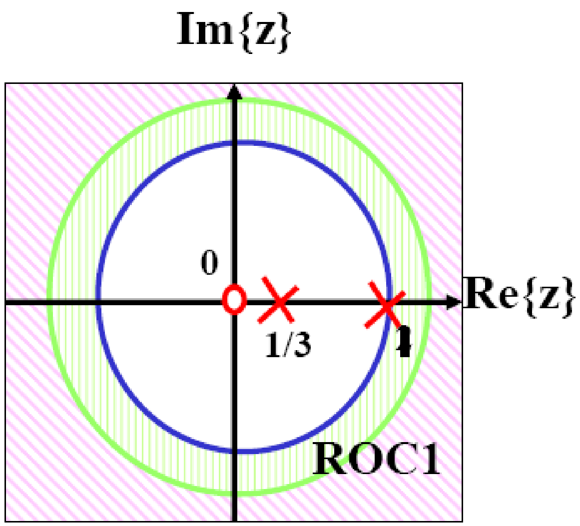
$$= X_0(z)z^{m+n-1}$$

Inverse ZT - ***General Expression***

Solution:

$$X_0(z) = \frac{1}{3(z-1)(z-\frac{1}{3})}$$ **and** $m = 1$.

Evidently, the function has two poles at $z=1/3$ **and** $z=1$**, and it is analytic out of the poles.**



Inverse ZT - ***General Expression***

Solution:

(1) If the *ROC* > 1:

Both of the two poles are inside of *C*, use Formula No.1.

When *n<1-m=0*, *x(n)=0*. When *n≥1-m=0*:

$$x_1(n) = \mathrm{Re}\,s[X_1(z), z=1] + \mathrm{Re}\,s[X_1(z), z=\frac{1}{3}]$$

$$= (z-1)X_1(z)\Big|_{z=1} + (z-\frac{1}{3})X_1(z)\Big|_{z=\frac{1}{3}} = \frac{z^n}{3(z-\frac{1}{3})}\Big|_{z=1} + \frac{z^n}{3(z-1)}\Big|_{z=\frac{1}{3}}$$

$$= \frac{1}{2} - \frac{1}{2}\left(\frac{1}{3}\right)^n$$

Inverse ZT - *General Expression*

Solution:

(2) If the *ROC* < 1/3:

Both of the two poles are outside of *C*, use Formula No.2.

When $n \geq 1-m=0$, $x(n)=0$. When $n<1-m=0$:

$$x_2(n) = -\mathrm{Re}\,s[X_1(z), z=1] - \mathrm{Re}\,s[X_1(z), z=\frac{1}{3}]$$

$$= -\frac{1}{2} + \frac{1}{2}(\frac{1}{3})^n$$

Inverse ZT - *General Expression*

Solution:

(3) If 1/3<*ROC*<1:

Pole *z=1/3* is inside of *C*, use Formula No.1. When *n≥0*:

$$x_3(n) = \mathrm{Re}\,s[X_1(z), z = \frac{1}{3}] = -\frac{1}{2}(\frac{1}{3})^n$$

Pole *z=1* is outside of *C*, use Formula No.2. When *n<0*:

$$x_3(n) = -\mathrm{Re}\,s[X_1(z), z = 1] = -\frac{1}{2}$$

Inverse ZT - ***General Expression***

Solution:

(3) If 1/3<*ROC*<1:

$$x_3(n) = -\frac{1}{2}u(-n-1) - \frac{1}{2}\left(\frac{1}{3}\right)^n u(n)$$

Inverse ZT - *General Expression*

Conclusion of the residue method:

1) Draw the zero-pole plot, find the ROC, and draw the closed curve C, containing the origin.

2) Calculate the residue in and out of C, and get the x(n).

Summary

3.1 The z-Transform

3.1.1 Poles and Zeros on the z-Plane and Stability

3.1.2 The ROC of z-Transform

3.1.3 The Properties of z-Transform

3.2 The Inverse z-Transform

3.2.1 General Expression of Inverse z-Transform

3.2.2 Inverse z-Transform by Partial-Fraction Expansion