DUBLIN INSTITUTE OF TECHNOLOGY

School of Mathematical Sciences

DT228 BSc (Honours) Degree in Computer Science

DT282 BSc (Honours) Degree in Computer Science (International)

Stage 2

WINTER EXAMINATIONS 2017/2018

CMPU 2012: MATHEMATICS 2

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9.30 – 11.30 am, Thursday, 11^{th} January 2018

Duration: 2 hours

ANSWER **QUESTION 1** AND **TWO** OF THE OTHER THREE QUESTIONS

Approved calculators may be used

Mathematical tables are provided

New Cambridge Statistical Tables are NOT permitted

1. a) Given that 499 is a prime number, use Fermat's Little Theorem (or otherwise) to calculate the residue of

$$5^{500} \bmod 499.$$

hence calculate residue of

$$5^{1001} \bmod 499.$$

(8)

b) Find **all** incongruent solutions to the following equations or say why no solution exists:

i) $3x \equiv 6 \pmod 9$

ii) $5x \equiv 6 \pmod{25}$

(8)

c) Based on her recent performances, Serena Williams' sponsors believe that her probability of winning any given match is $\frac{4}{5}$ in an upcoming Grand Slam tournament. She will play 9 matches in the first stage and must win all 9 matches in order to reach the semi-final. Assuming each match is an independent event, calculate the that probability that she:

i) Makes it to the semi-final.

ii) Loses at least one match.

In each case give your answer as a fraction.

(8)

d) Given the domain of discourse is the set of natural numbers $\mathbb{N}$ and the predicates:

$$\begin{aligned} G(x) &: x < 10 \\ L(x) &: x \geq 10 \\ H(x,y) &: x < y \end{aligned}$$

Write the following quantifications as English sentences and state whether they are true or false:

i) $\exists x\,(G(x) \wedge L(x))$

ii) $\forall x \exists y\, H(x,y)$

iii) $\forall x \forall y\,(G(x) \wedge L(y) \rightarrow H(x,y))$

(8)

e) Use *Proof by Induction* to prove that

$$1 + 5 + 5^2 + 5^3 + \cdots + 5^n = \frac{5^{n+1} - 1}{4} \quad \forall n \in \mathbb{Z}_+$$

Be sure to label all steps clearly.

(8)

[40]

2. **a)** A political cabinet of 14 ministers must be chosen from 11 women and 20 men.

i) How many combinations of cabinet ministers can be formed?

ii) How many possible combinations of cabinet ministers are there with exactly 3 women?

iii) What is the probability of having exactly 3 women in the cabinet?

(8)

b) A card player draws a card at random from a standard deck of 52 cards (without any jokers).

i) What is the probability that he draws a red card?

ii) What is the probability that he draws a red Jack?

iii) Are the events in i) and ii) independent? Explain why mathematically.

iv) Assuming he drew a red Jack from the deck, he now draws a second card (without replacing the red Jack), what is the probability that he draws a black Jack?

In each case give numerical answers as a fraction.

(10)

c) John's credit card details have just been hacked and now there is a probability of $\frac{1}{4}$ that the hackers will steal €300 from John every time he uses his card. Assuming that John uses his card 3 times and that each transaction is an independent event, calculate the following:

i) The probability that the hackers steal exactly €600 of John's money (i.e. they successfully steal from John exactly twice).

ii) The probability that the hackers steal at least €300 of John's money (i.e. they successfully steal from John at least once).

iii) If X is the amount of money that the hackers steal from John, then calculate the expected value of X, $E(X)$.

In each case give your answer as a fraction or whole number.

(12)

[30]

3. a) Use the *Chinese Remainder Theorem* (or otherwise) to solve the following:

$$\begin{aligned} x &\equiv 3 \pmod{4} \\ x &\equiv 5 \pmod{7} \\ x &\equiv 6 \pmod{9} \end{aligned}$$

(8)

b) The ciphertext "UFAA" was encrypted using the encryption matrix

$$E = \begin{pmatrix} 2 & 21 \\ 2 & 1 \end{pmatrix}$$

modulo 27, where:
☺ = 0, A = 1, B = 2, C = 3, D = 4, E = 5, F = 6, G = 7, H = 8, I = 9,
J = 10, K = 11, L = 12, M = 13, N = 14, O = 15, P = 16, Q = 17, R = 18,
S = 19, T = 20, U = 21, V = 22, W = 23, X = 24, Y = 25, Z = 26.

Find the decryption matrix (i.e. the inverse of E mod 27) and use it to decrypt the ciphertext above.

(10)

c) Use the *Extended Euclidean Algorithm* (or otherwise) to find the **general solution** to the Diophantine Equation:

$$56x + 138y = 100$$

(12)

[30]

4. a) Determine how many edges each of the following graphs have, giving reasons for your answers:

i) C_5
ii) K_4
iii) $K_{3,4}$

(6)

b) For the graphs G_1, G_2 and G_3 shown in Figure 1 state which of the following is:

i) A complete graph.

ii) A bipartite graph.

iii) A complete bipartite graph.

For those that are bipartite or complete bipartite, illustrate the partition of the graph.

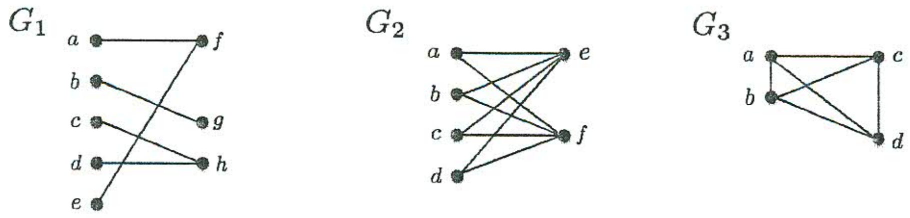

Figure 1: The graphs G_1, G_2 and G_3.

(12)

c) Use Kruskal's algorithm on the weighted graph H shown in Figure 2 in order to find its minimal spanning tree, being sure to clearly show the steps involved. Illustrate the resulting spanning tree and calculate its weight.

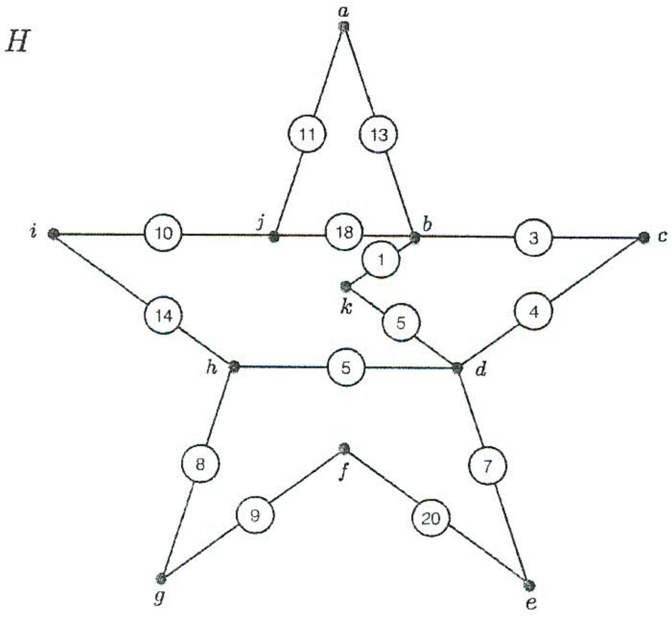

Figure 2: The weighted graph H.

(12)

[30]