Chapter 4

Electrostatics

Contents

4.1 Maxwell's Equations

- The chapter opens as the first pure fields chapter
- The remaining chapters of the text focus on *Maxwell's equations* (4 total)

$$\nabla \cdot \mathbf{D} = \rho_v \tag{4.1}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.2}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4.3}$$

$$\nabla \times \mathbf{H} = \mathbf{J} = \frac{\partial \mathbf{D}}{\partial t}, \tag{4.4}$$

where

$$\begin{aligned} \mathbf{E} &= \text{electric field internsity} \\ \mathbf{D} &= \epsilon \mathbf{E} = \text{electric flux density} \\ \mathbf{H} &= \text{magnetic field intensity} \\ \mathbf{B} &= \mu \mathbf{H} = \text{magntic flux density} \\ \mathbf{J} &= \text{convection or conduction current density} \end{aligned}$$

- In Chapter 4 and 5 we will only consider *static* conditions, which means terms of the form $\partial/\partial t = 0$
- What remains of the four Maxwell's equations is two pairs of simplified equations:
  - **Electrostatics** (Chapter 4)

$$\nabla \cdot \mathbf{D} = \rho_v \tag{4.5}$$

$$\nabla \times \mathbf{E} = 0 \tag{4.6}$$

- **Magnetostatics** (Chapter 5)

$$\nabla \cdot \mathbf{B} = 0 \tag{4.7}$$

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{4.8}$$

- The above pairs of equations are said to be *decoupled*, which holds only for the static case

4.2 Charge and Current Distributions

With regard to electrostatics, working with charge current distributions is common place.

4.2.1 Charge Densities

- Charge densities are similar to probability densities studied in prob and stats and mass densities found in mechanics

- There are three basic forms:

  - Volume distribution

$$\rho_v = \lim_{\Delta\mathcal{V}\to 0} \frac{\Delta q}{\Delta\mathcal{V}} = \frac{dq}{d\mathcal{V}} \quad (\text{C/m}^3)$$

  Note:

$$Q = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} \quad (\text{C})$$

  - Surface distribution

$$\rho_s = \lim_{\Delta s\to 0} \frac{\Delta q}{\Delta s} = \frac{dq}{ds} \quad (\text{C/m}^2)$$

Note:

$$Q = \int_S \rho_s \, ds \quad \text{(C)}$$

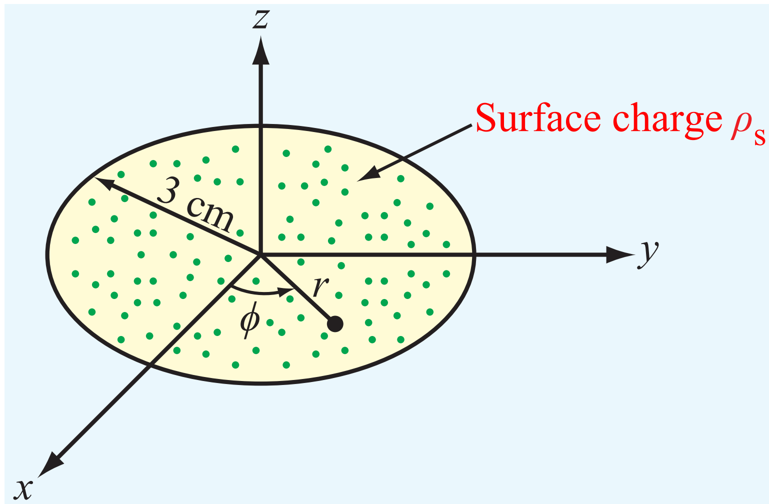

Figure 4.1: Circular surface charge ρ_s.

– Line distribution

$$\rho_\ell = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \quad \text{(C/m)}$$

Note:

$$Q = \int_l \rho_\ell \, dl \quad \text{(C)}$$

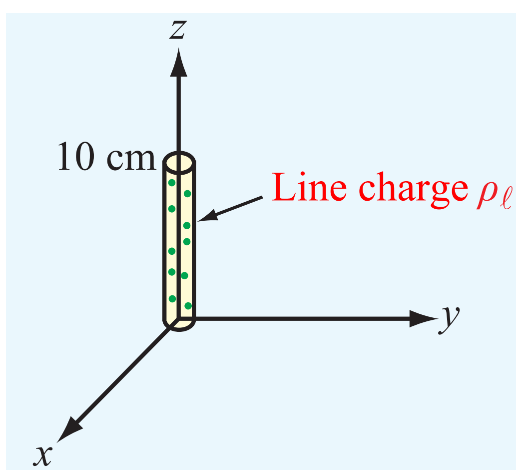

Figure 4.2: Linear line charge ρ_ℓ.

Example 4.1: **Nonuniform Surface Charge**

- Consider the surface charge density

$$\rho_s = \begin{cases} 4y^2 \ (\mu\text{C/m}^2), & -3 \le x, y \le 3 \text{ m} \\ 0, & \text{otherwise} \end{cases}$$

- Find the total charge

$$\begin{aligned} Q &= \int_{-3}^{3} \int_{-3}^{3} 4y^2 \, dy \, dx = \int_{-3}^{3} \left[\frac{4y^3}{3} \right]_{-3}^{3} dx \\ &= 72 \cdot [x]_{-3}^{3} = 72 \cdot \big(3 - (-3)\big) = 432 \quad (\mu\text{C}) \end{aligned}$$

4.2.2 Current Densities

- Current is related to charge density, except we have to put the charge into motion

- Consider charge in a tube having volume density ρ_v and moving from left to right with velocity $\mathbf{u}$

- In Δt s the charge moves $\Delta l = u \Delta t$, creating a charge flow across the tube's surface area, $\Delta s'$ of

$$\Delta q' = \rho_v \cdot \underbrace{(\Delta l \cdot \Delta s')}_{\mathcal{V} \cdot \Delta t} \stackrel{\text{also}}{=} \rho_v u \cdot \Delta s' \cdot \Delta t$$

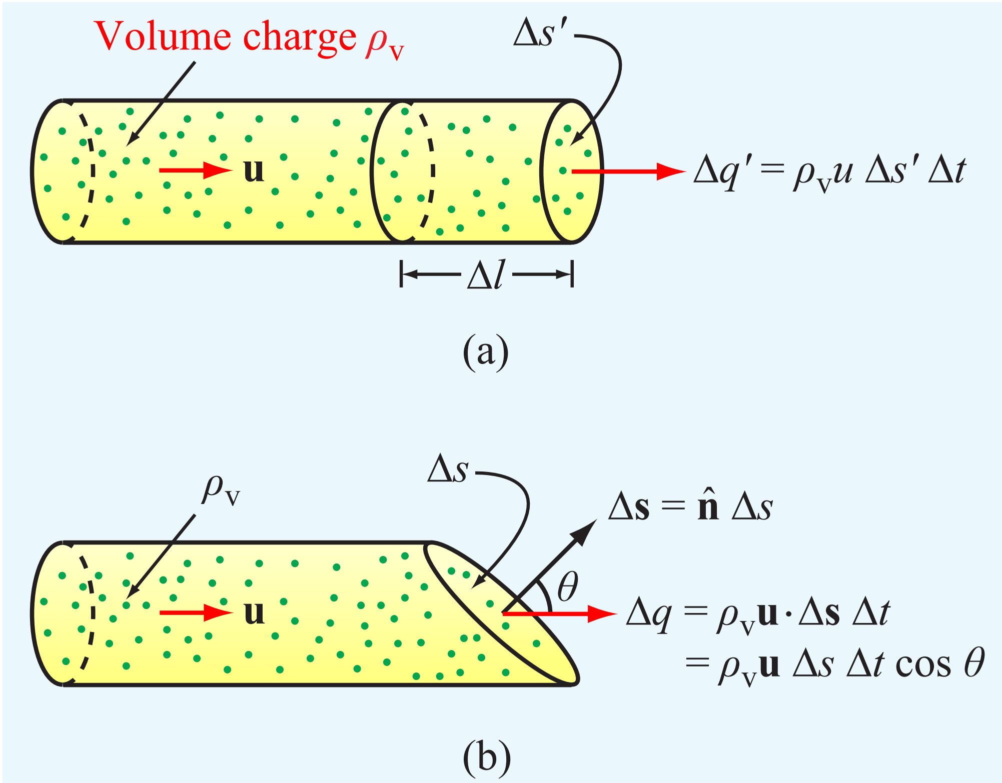

Figure 4.3: (a) Charges flowing in a tube with cross section $\Delta s'$ moving with velocity $\mathbf{u}$ m/s and (b) dealing with a surface normal differnt from the flow velocity.

- For the general case of charge flow across a surface (not parallel to the velocity $\mathbf{u}$) the normal to the surface $\Delta\mathbf{s}$, $\hat{\mathbf{n}}$, can be used to write $\Delta s = \hat{\mathbf{n}}\, \Delta s$, and then describe the general charge increment Δq (prime is dropped) as

$$\Delta q = \rho_v \mathbf{u} \cdot \Delta \mathbf{s} \, \Delta t$$

- Since current is charge flow per unit time, we have

$$\Delta I = \frac{\Delta q}{\Delta t} = \underbrace{\rho_v \mathbf{u}}_{(\mathrm{C/s})/\mathrm{m}^2} \cdot \Delta \mathbf{s} = \mathbf{J} \cdot \Delta \mathbf{s}$$

  where $\mathbf{J}$ is the *current density*

$$\boxed{\mathbf{J} = \rho_v \mathbf{u} \quad (\mathrm{A/m}^2)}$$

- Integrating over an arbitrary surface S yields the total current

$$I = \int_S \mathbf{J} \cdot d\mathbf{s} \quad \text{(A)}$$

  - For the movement of charged matter, $\mathbf{J}$ represents a *convection current*
  - For the movement of charged particles (e.g., electrons in a conductor), $\mathbf{J}$ represents a *conduction current*
  - **Note**: Conduction current obeys Ohm's law, while convection current does not!

4.3 Coulomb's Law

- First introduced in Chapter 1
- Now its time to get serious about studying it and working with it!
- **Review**: For an isolated charge q the induced electric field is

$$\mathbf{E} = \hat{\mathbf{R}} \frac{q}{4\pi\epsilon R^2} \quad \text{(V/m)},$$

  where $\hat{\mathbf{R}}$ points from q to the field point P

- **Review**: A test charge q' placed in electric field $\mathbf{E}$ at point P experiences force

$$\mathbf{F} = q'\mathbf{E} \quad \text{(N)}$$

  Note: It would appear that the units of $\mathbf{E}$ is also (N/C), i.e., (N/C) = (V/m)

- **Review**: When a material with permittivity $\epsilon = \epsilon_0\epsilon_r$ is present the electric flux density and electric field intensity are related by

$$\mathbf{D} = \epsilon\mathbf{E}$$

Note: The $\epsilon_0 = 8.85 \times 10^{-12}$ F/m

  - As long as ϵ is independent of the amplitude of $\mathbf{E}$, the material is *linear*
  - A material is said to be *isotropic* if ϵ is independent of the direction of $\mathbf{E}$; some PCB materials are anisotropic, meaning ϵ takes on one value in the (x, y) plane and another value in the z direction (sheet thickness)

4.3.1 Field Due to N Point Charges

- For N point charges $q_1, q_2, \ldots, q_N$ with corresponding position vectors $\mathbf{R}_i$, $i = 1, 2, \ldots, N$ connecting the charge location with the field point P, is the vector sum of the field due to the individual charges

$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon}\sum_{i=1}^{N}\frac{(\mathbf{R} - \mathbf{R}_i)}{|\mathbf{R} - \mathbf{R}_i|^3} \quad \text{(V/m)}}$$

Example 4.2: **Two Point Charges in Python and TI nspire**

- Consider two point charges as described in text Example 4-3
- In Cartesian coordinates we have

$$\mathbf{R}_1 = (1, 3, -1), \quad \mathbf{R}_2 = (-3, 1, -2), \quad \mathbf{R} = (3, 1, -2)$$

with

$$q_1 = 2 \times 10^{-5} \text{ (C)} \quad \text{and} \quad q_2 = -4 \times 10^{-5} \text{ (C)}$$

- We calculate **E** by plugging the 3D vector coefficients into

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\left[q_1 \frac{\mathbf{R}-\mathbf{R}_1}{|\mathbf{R}-\mathbf{R}_1|^3} + q_2 \frac{\mathbf{R}-\mathbf{R}_2}{|\mathbf{R}-\mathbf{R}_2|^3}\right] \quad \text{(V/m)}$$

- In Python the calculation is straight forward using numpy `ndarrays`

```
# Two charge example in Python
Ri = array([[1,3,-1],[-3,1,-2]])
R = array([3,1,-2])
E = 2*(R - Ri[0,:])/norm(R - Ri[0,:])**3 - \
    4*(R - Ri[1,:])/norm(R - Ri[1,:])**3
# Include the missing scale factors
E*1e-5/(4*pi*8.85e-12)
```

```
array([  3330.29803498, -13321.19213994,  -6660.59606997])
```

Figure 4.4: Python calculation of the two charge electric field at (3,1,-2).

- The nested list `[[1,2,3],[4,5,6]]` creates a 2D array having dimensions 2 by 3

- A 1D array is formed by indexing just one row, e.g., `Ri[i,:]`,; not the use of the colon operator to span all columns

- The `norm` function finds the length of an array in Cartesian coordinates

- Using the TI *n*spire calculator a symbolic solution can be obtained and then converted to a numerical form, no problem

Two Charge Calculation

On the nspire you can enter an array of three element lists using r_array:={{1,3,−1},{−3,1,−2}}. Once entered it looks like the following:

$$\textbf{r\_array}:=\begin{bmatrix}1 & 3 & -1\\ -3 & 1 & -2\end{bmatrix} \blacktriangleright \begin{bmatrix}1 & 3 & -1\\ -3 & 1 & -2\end{bmatrix}$$

$$\textbf{r\_point}:=\begin{bmatrix}3 & 1 & -2\end{bmatrix} \blacktriangleright \begin{bmatrix}3 & 1 & -2\end{bmatrix}$$

Now the calculation of the **E** field less the 10^{-5} factor and ε_o:

$$\frac{1}{4\cdot\pi\cdot\varepsilon 0}\cdot\left(2\cdot\frac{\textbf{r\_point}-\textbf{r\_array}[1]}{(\text{norm}(\textbf{r\_point}-\textbf{r\_array}[1]))^3}+{}^-4\cdot\frac{\textbf{r\_point}-\textbf{r\_array}[2]}{(\text{norm}(\textbf{r\_point}-\textbf{r\_array}[2]))^3}\right)$$

$$\blacktriangleright \begin{bmatrix}\frac{1}{108\cdot\varepsilon 0\cdot\pi} & \frac{-1}{27\cdot\varepsilon 0\cdot\pi} & \frac{-1}{54\cdot\varepsilon 0\cdot\pi}\end{bmatrix}$$

Now include the missing terms to get a pure numerical answer:

$$\frac{10^{-5}}{4\cdot\pi\cdot 8.85\cdot 10^{-12}}\cdot\left(2\cdot\frac{\textbf{r\_point}-\textbf{r\_array}[1]}{(\text{norm}(\textbf{r\_point}-\textbf{r\_array}[1]))^3}+{}^-4\cdot\frac{\textbf{r\_point}-\textbf{r\_array}[2]}{(\text{norm}(\textbf{r\_point}-\textbf{r\_array}[2]))^3}\right)$$

$$\blacktriangleright \begin{bmatrix}3330.3 & -13321.2 & -6660.6\end{bmatrix}$$

Figure 4.5: TI *n*spire calculation of the two charge electric field at (3,1,-2).

- The units (not shown in the figures) is of course (V/m)

4.3.2 Field Due to a Charge Distribution

- A practical extension to Coulomb's law is consider charge distributions: (1) volume, (2) surface, or (3) line distributions

Volume Distributon

- Consider a volume $\mathcal{V}'$ that contains charge density ρ_v

- The electric field at a point P due to a differential charge $dq = \rho_v \, d\mathcal{V}'$, is
$$d\mathbf{E} = \hat{\mathbf{R}}' \frac{dq}{4\pi\epsilon R'^2} = \hat{\mathbf{R}}' \frac{\rho_v d\mathcal{V}}{4\pi\epsilon R'^2},$$
where $\mathbf{R}'$ is the vector pointing from the differential charge to the field point P and $\hat{\mathbf{R}}'$ is the corresponding unit vector $\mathbf{R}'/|\mathbf{R}'|$

- Since superposition holds for discrete charges it holds here, so in integral form

$$\boxed{\mathbf{E} = \int_{\mathcal{V}'} d\mathbf{E} = \frac{1}{4\pi\epsilon} \int_{\mathcal{V}'} \hat{\mathbf{R}}' \frac{\rho_v \, d\mathcal{V}'}{R'^2} \quad \text{(V/m)}}$$

- This formula is nice and compact, but R' and $\mathbf{R}'$ are likely functions of the integration variables used to describe $\mathcal{V}'$

- Furthermore, there are no examples or homework problems in the book for a volume charge distribution

Surface Distribution

- For the case of a surface charge density $dq = \rho_s ds'$, we can write
$$\boxed{\mathbf{E} = \frac{1}{4\pi\epsilon} \int_{S'} \hat{\mathbf{R}}' \frac{\rho_s \, ds'}{R'^2} \quad \text{(V/m)}}$$

- With the charge distribution limited to just two dimensions, problem set-up and integration become easier

Line Distribution

- Finally for the case of a line charge density $dq = \rho_\ell dl'$, we can write

$$\mathbf{E} = \frac{1}{4\pi\epsilon}\int_{l'} \hat{\mathbf{R}}' \frac{\rho_\ell \, dl'}{R'^2} \quad \text{(V/m)}$$

- Here the charge distribution one dimensional, but may lie along a curve

- The math is manageable in many cases

Example 4.3: A Ring of Charge in Air

- Here we consider a circular line charge lying in the $x-y$ plane of radius b uniform positive density ρ_ℓ
  - Note: This is a classical, yet also important, example

- The field point for determining $\mathbf{E}$ is along the z-axis at $P = (0, 0, h)$

- The problem symmetry makes cylindrical coordinates the obvious choice

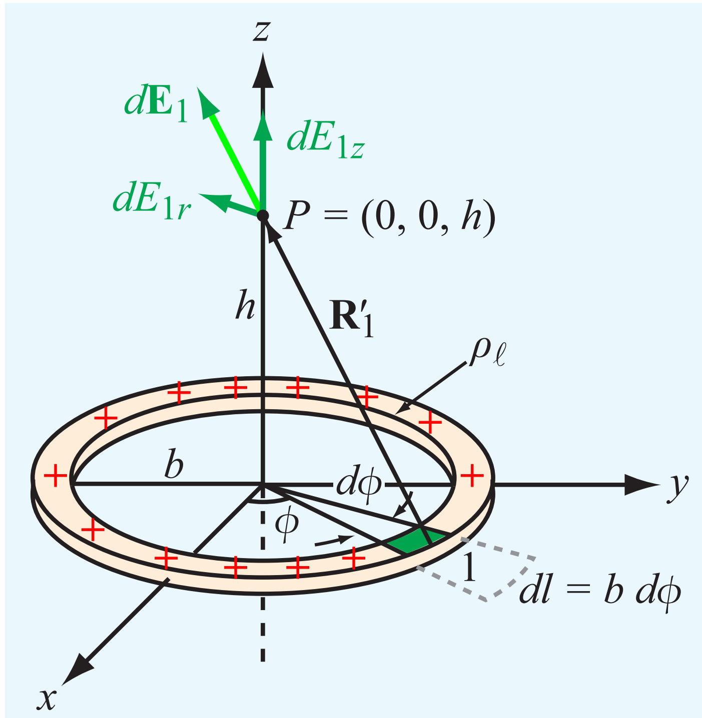

Figure 4.6: Setting up the ring of charge field calculation.

- The differential line charge segment length is $dl = bd\phi$ and the differential charge density is $dq = \rho_\ell dl = \rho_\ell b\, d\phi$
- From Figure 4.6 we see that

$$\mathbf{R}' = -\hat{\mathbf{r}}\, b + \hat{\mathbf{z}}\, h$$
$$R' = |\mathbf{R}'| = \sqrt{b^2 + h^2}$$
$$\hat{\mathbf{R}}' = \frac{\mathbf{R}'}{|\mathbf{R}'|} = \frac{-\hat{\mathbf{r}}\, b + \hat{\mathbf{z}}\, h}{\sqrt{b^2 + h^2}}$$

and

$$d\mathbf{E} = \frac{1}{4\pi\epsilon_0} \rho_\ell b \frac{-\hat{\mathbf{r}}\, b + \hat{\mathbf{z}}\, h}{\left(b^2 + h^2\right)^{3/2}} d\phi$$

- With the field point along the z-axis further simplification is possible, namely the radial ($\hat{\mathbf{r}}$) field contributions from charge segments on opposite sides of the ring cancel, leaving only the axial or $\hat{\mathbf{z}}$ component

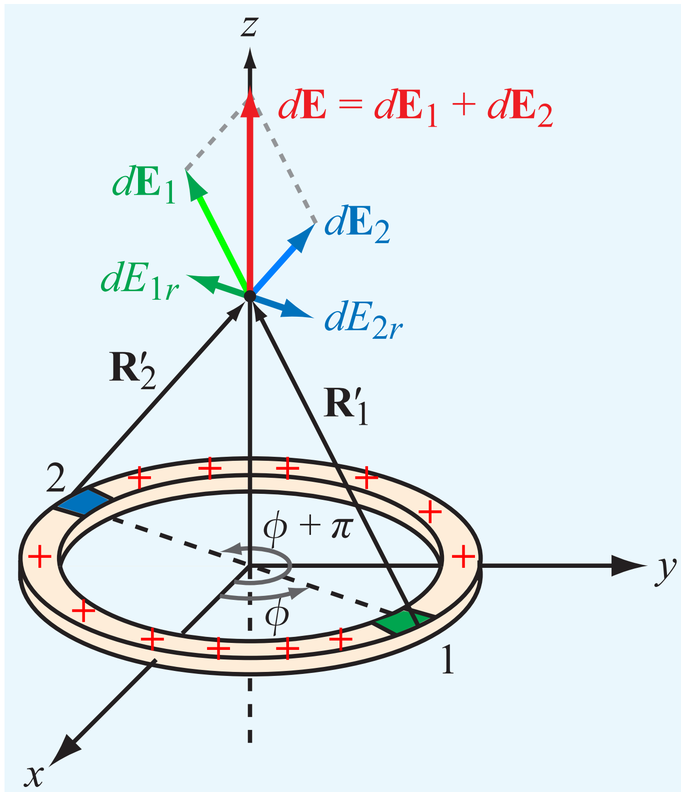

Figure 4.7: Opposing line charge segments result in radial component cancellation and constructive combining of the axial components due to the segment 1 and 2 semicircles.

- The charge ring is broken into two semicircles, each defined over $0 \leq \phi \leq \pi$
- We know the radial field components cancel and reason that the axial components constructively add, thus

$$\begin{aligned}\mathbf{E} &= \hat{\mathbf{z}}\, 2 \times \frac{\rho_\ell b h}{4\pi\epsilon_0 \big(b^2 + h^2\big)^{3/2}} \int_0^\pi d\phi \\ &= \hat{\mathbf{z}}\, \frac{\rho_\ell b h}{2\epsilon_0 \big(b^2 + h^2\big)^{3/2}} \overset{\text{also}}{=} \hat{\mathbf{z}}\, \frac{h}{4\pi\epsilon_0 \big(b^2 + h^2\big)^{3/2}}\, Q,\end{aligned}$$

where $Q = 2\pi b\rho_\ell$ is the total charge on the ring

- A curiosity is how does the diameter of the ring, relative to the field point distance h alter the field strength, and when does the ring look like a point charge?

- A simple plot (here using Python), helps explain

- In the code and corresponding plot shown below, the scale factors of Q and ϵ_0 are not included (normalized out)

- For comparison purposes the field due to a point charge of Q located at the origin is also included in the plot

```
# Axial distance h
h = arange(1,5,.01)
# Point charge Q at (x,y,z) = (0,0,0)
plot(h,1/h**2)
# Ring of charge with radius b
b = 1/8
plot(h,h/(b**2 + h**2)**(3/2))
b = 1/4
plot(h,h/(b**2 + h**2)**(3/2))
b = 1/2
plot(h,h/(b**2 + h**2)**(3/2))
b = 1
plot(h,h/(b**2 + h**2)**(3/2))
legend((r'Point charge',r'$b=1/8$ (m)',r'$b=1/4$ (m)',
        r'$b=1/2$ (m)',r'$b=1$ (m)'),loc='best')
title(r'Plot of $E_z \cdot 4\pi\epsilon_0/Q$ versus $h$ \
with $b$ a Parameter')
ylabel(r'Normalized Axial Field Intensity')
xlabel(r'Axial Field Point $h$ (m)')
grid();
```

Figure 4.8: Python code for calculating the charge ring axial field.

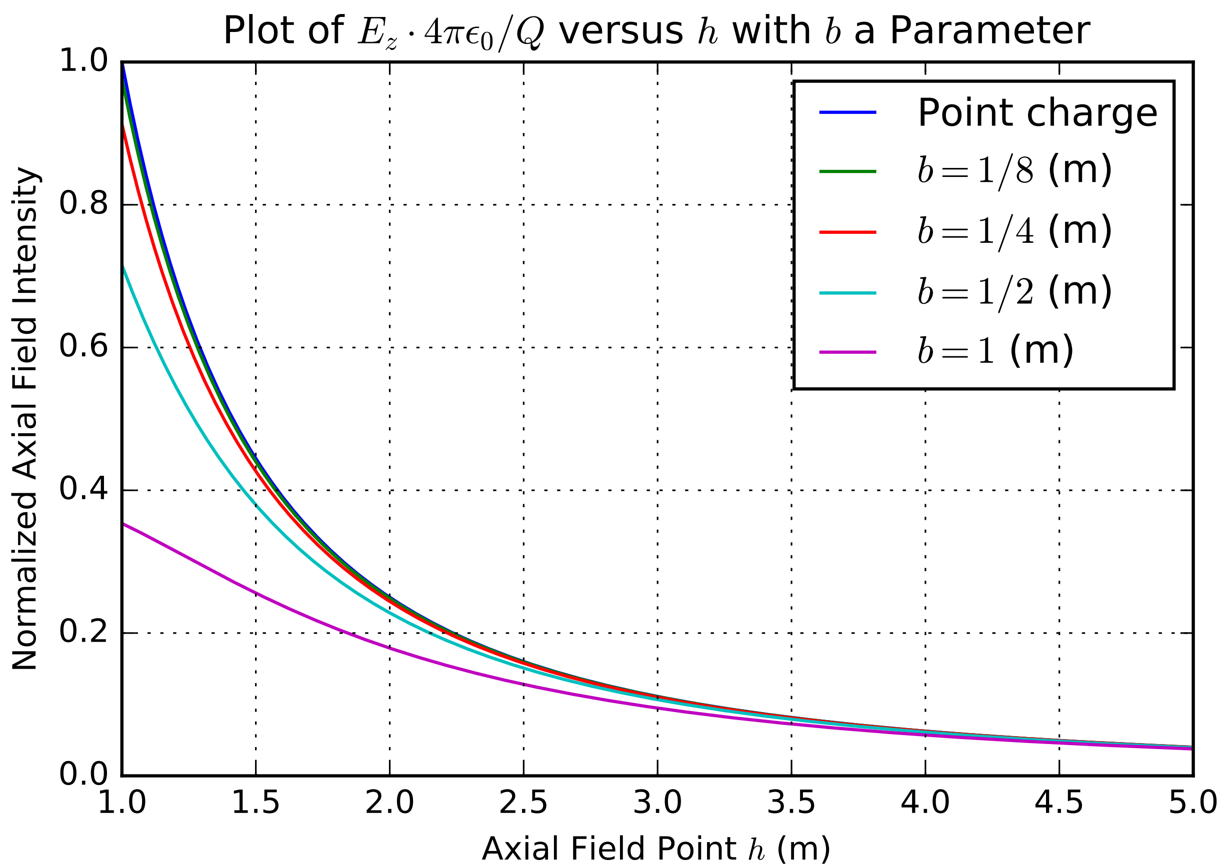

Figure 4.9: Axial field component of charge ring compared with equivalent point charge.

- The results are not too surprising:
  - The ring of charge looks like a point charge at not too greate a distance
  - For a larger ring radius the field stength on axis is reduced compared with the a smaller ring
  - **Note**: As h approaches zero the axial field component due to the ring is zero! why?

Example 4.4: Disk of Charge in the $x-y$-Plane

- Another classical example, that extends from the ring of charge, is a uniform circular disk of charge

$$\rho_s(r,\phi,z) = \begin{cases} \rho_s \text{ C/m}^2, & 0 \le r \le a, z = 0 \\ 0, & \text{otherwise} \end{cases}$$

- Field point is again $P = (0,0,h)$
- The elemental field contribution for the ring of charge can be extended to the case of the disk by adding in an integration over the charge differential $dq = 2\pi\rho_s r\, dr$ as shown in Figure 4.10

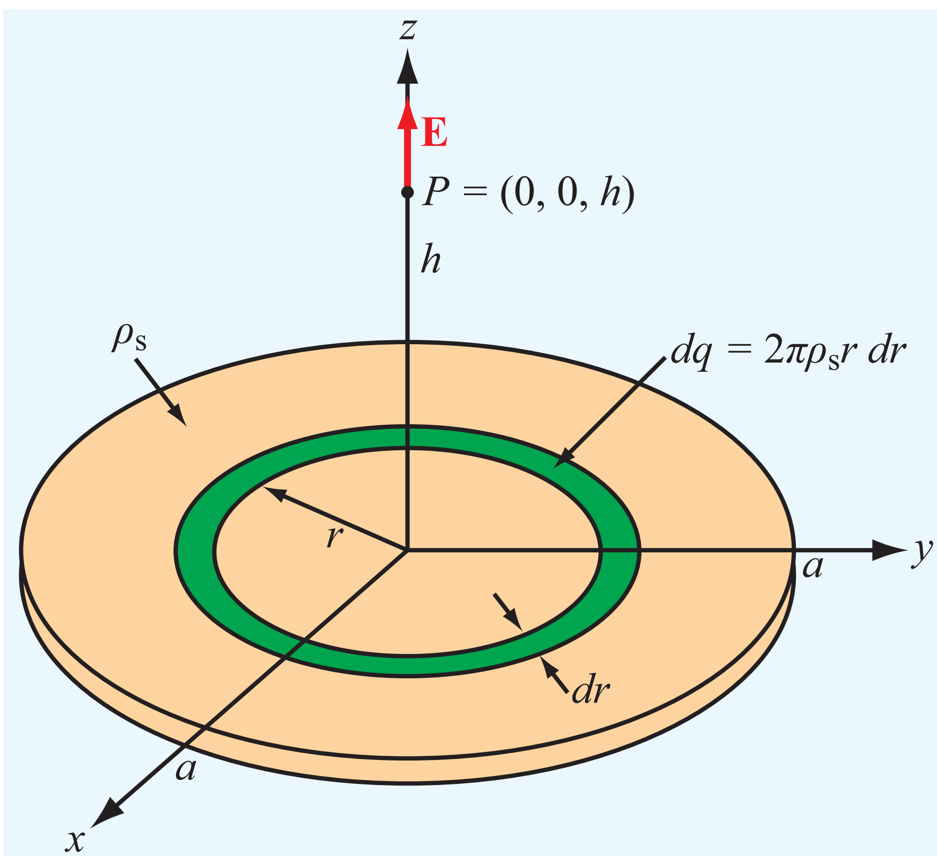

Figure 4.10: Set-up for calculating $\mathbf{E}$ due to a disk of charge in the $x-y$-plane.

- The charge disk is composed of concentric rings of charge
- The total charge on the disk is

$$Q = \int_0^{2\pi} \int_0^a \rho_s r \, dr \, d\phi = 2\pi\rho_s \frac{r^2}{2}\Big|_0^a = \pi\rho_s a^2$$

- Due to symmetry the radial component of **E** is again zero
- Putting the pieces together we have

$$\mathbf{E} = \hat{\mathbf{z}} \frac{\rho_s h}{2\epsilon_0} \int_0^a \frac{r \, dr}{\left(r^2 + h^2\right)^{3/2}}$$

- Recall/look up

$$\int \frac{r dr}{\left(r^2 + h^2\right)^{3/2}} = \frac{-1}{\sqrt{r^2 + h^2}},$$

so

$$\mathbf{E} = \begin{cases} \hat{\mathbf{z}} \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{|h|}{\sqrt{a^2+h^2}}\right], & h > 0 \\ -\hat{\mathbf{z}} \frac{\rho_s}{2\epsilon_0}\left[1 - \frac{|h|}{\sqrt{a^2+h^2}}\right], & h < 0 \\ 0, & h = 0 \end{cases}$$

- Writing in terms of the total charge Q amounts to replacing $\rho_s/(2\epsilon_0)$ with $Q/(\pi\epsilon_0 a^2)$, so

$$\mathbf{E} = \begin{cases} \hat{\mathbf{z}} \frac{Q}{2\pi\epsilon_0 a^2}\left[1 - \frac{|h|}{\sqrt{a^2+h^2}}\right], & h > 0 \\ -\hat{\mathbf{z}} \frac{Q}{2\pi\epsilon_0 a^2}\left[1 - \frac{|h|}{\sqrt{a^2+h^2}}\right], & h < 0 \\ 0, & h = 0 \end{cases}$$

- **Infinite Sheet of Charge**: By letting $a \to \infty$ we have the field due to an infinite sheet of charge being

$$\mathbf{E} = \begin{cases} \hat{\mathbf{z}}\,\frac{\rho_s}{2\epsilon_0}, & z > 0 \\ -\hat{\mathbf{z}}\,\frac{\rho_s}{2\epsilon_0}, & z < 0 \end{cases}$$

  - Notice no dependence on distance from the plane!

- As a final sanity check, we compare the axial electric field versus h while keeping the total charge, Q, the same in all cases

- Keeping the first two terms binomial in the expansion for $(1+x)^{-1/2}$ yields $1 - x/2$, so assuming $h \gg b$ or $h \gg a$ we have

$$E_{z,\text{ring}} \simeq Q\frac{(h^2 - b^2/2)^3}{4\pi\epsilon_0 h^8} \quad \text{and} \quad E_{z,\text{disk}} \simeq Q\frac{1}{4\pi\epsilon_0 h^2}$$

```
# Axial distance h
h = arange(1,5,.01)
# Point charge Q at (x,y,z) = (0,0,0)
plot(h,1/h**2)
# Ring of charge with radius b
b = 1/2
plot(h,h/(b**2 + h**2)**(3/2))
# Disk of charge with radius a
a = 1/2
plot(h,(2/a**2)*(1 - abs(h)/sqrt(a**2 + h**2)))
legend((r'Point charge',r'Ring $b=1/2$ (m)',r'Disk $a=1/2$ (m)'),
       loc='best')
title(r'Plot of $E_z \cdot 4\pi\epsilon_0/Q$ versus $h$ \
with $a$ and $b$ Parameters')
ylabel(r'Normalized Axial Field Intensity')
xlabel(r'Axial Field Point $h$ (m)')
grid();
```

Figure 4.11: Python code for calculating the charge disk axial field.

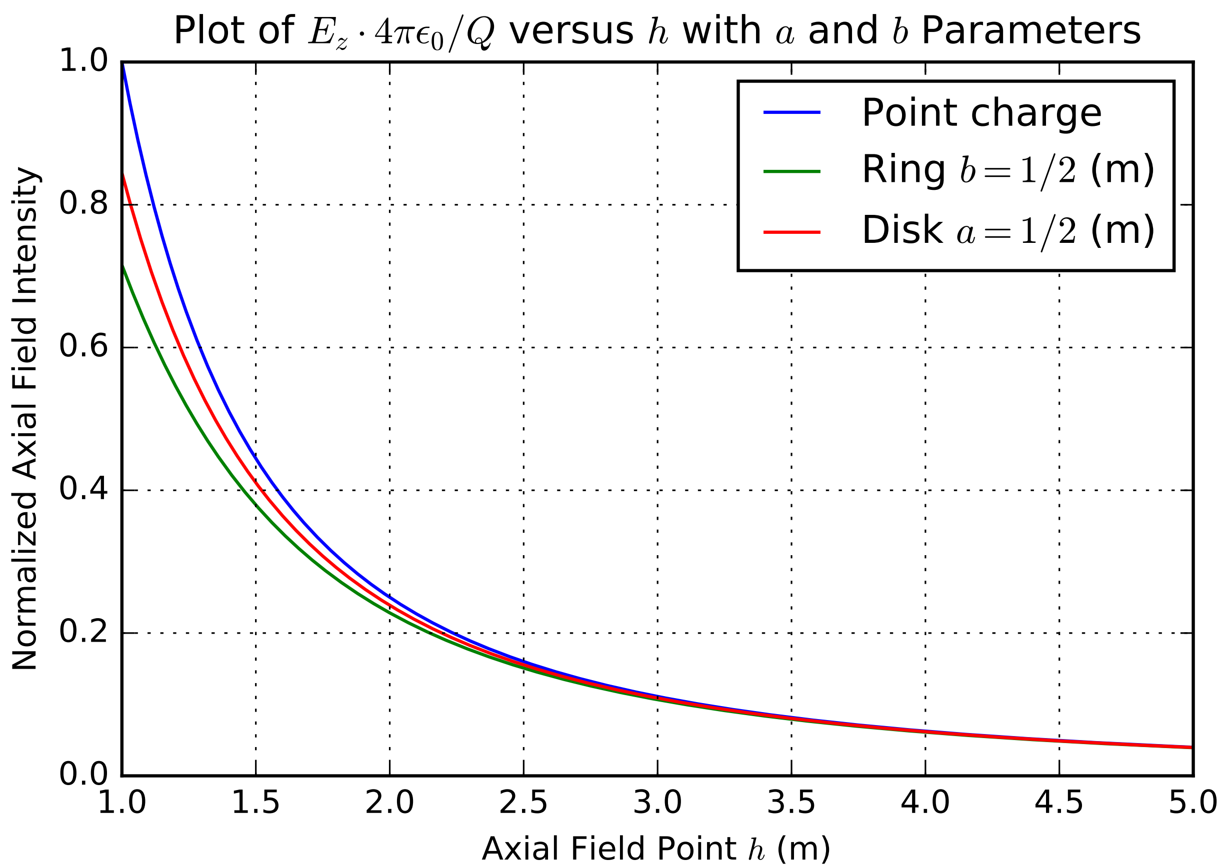

Figure 4.12: Axial field component of a charge disk compared with a ring and an equivalent point charge (total charge help constant for all three).

- The *asymptotic* behavior is as expected and is supported by the binomial expansion results too

4.4 Gauss's Law

- From physics you may remember this useful result
- Gauss's law is arrived at by starting from Maxwell's equation

$$\nabla \cdot \mathbf{D} = \rho_v$$

(*differential form* since partial derivatives are involved in the divergence calculation)

- The integral form of the above, which is obtained by integrating both sides over an arbitrary volme $\mathcal{V}$, is

$$\boxed{\int_{\mathcal{V}} \nabla \cdot \mathbf{D} \, d\mathcal{V} = \int_{\mathcal{V}} \rho_v \, d\mathcal{V} = Q}$$

- Now invoke the *divergence theorem* from Chapter 3 which says

$$\int_{\mathcal{V}} \nabla \cdot \mathbf{D} \, d\mathcal{V} = \oint_S \mathbf{D} \cdot d\mathbf{s},$$

where S is encloses $\mathcal{V}$ (S is known as a *Gaussian surface*)

- Finally, arrive at Gauss's Law:

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{s} = Q}$$

In words, the flux passing through S equals the enclosed charge Q

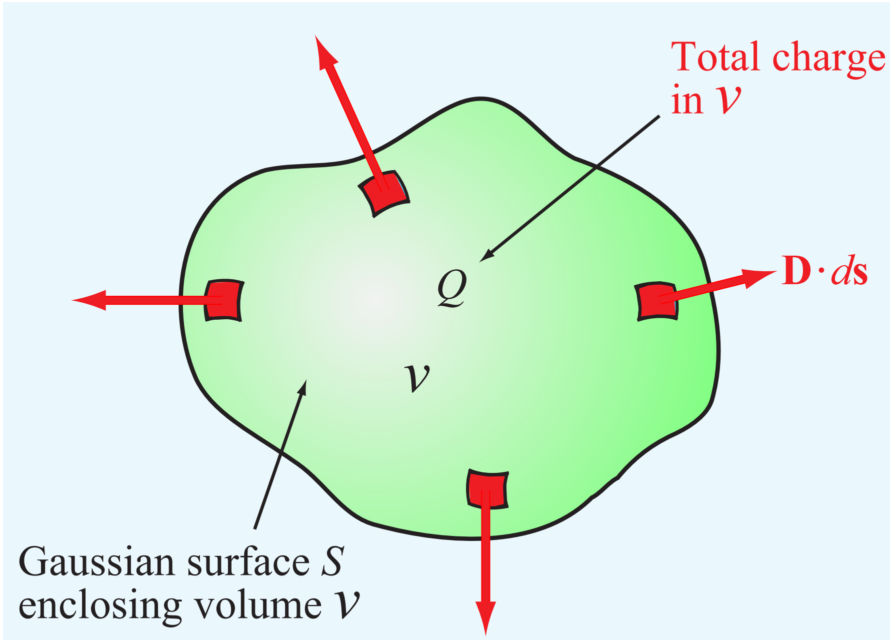

Figure 4.13: Gauss's law illustrated.

Example 4.5: **Classical Point Charge at the Origin**

- If you remember anything about Gauss's law, it is likely how you can calculate $\mathbf{E}$ by setting up a Gaussian surface at the origin that encloses a point charge q

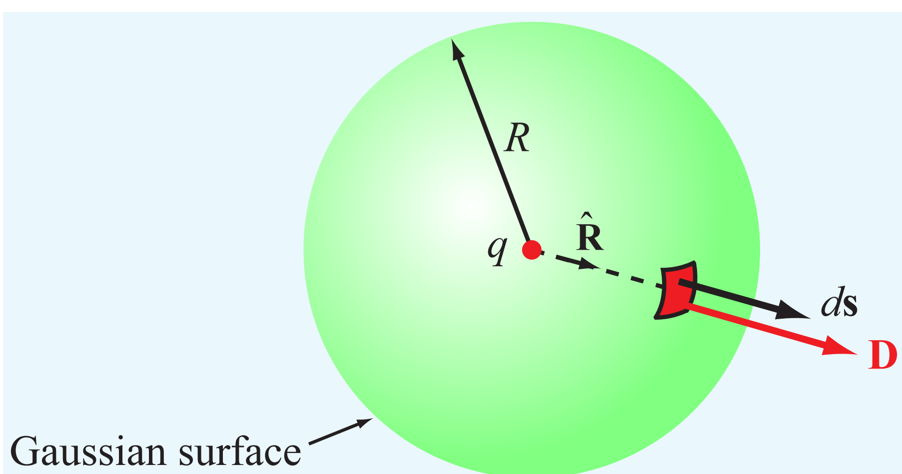

Figure 4.14: Point charge at the origin and the appropriate Gaussian surface.

- Suppose we do not know the exact form of $\mathbf{D}$ (or $\mathbf{E}$), but we know from symmetry that

$$\mathbf{D} = \hat{\mathbf{R}}\, D_R,$$

and a sphere centered at the origin has $d\mathbf{s} = \hat{\mathbf{R}}\, ds$, so

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_S \hat{\mathbf{R}} D_R \cdot \hat{\mathbf{R}}\, ds = \oint_S D_R\, ds = (4\pi R^2) D_R = q$$

- Clearly,

$$D_R = \frac{q}{4\pi R^2}$$

or

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon} = \hat{\mathbf{R}} \frac{q}{4\pi\epsilon R^2}$$

A familiar result!

Gaussian Surface Tips

- Picking the Gaussian surface is key to finding $\mathbf{D}$ using Gauss's law
  - Make use of symmetry so the form of $\mathbf{D}$ can be deduced (at least for each component)
  - Choose S so the form of $\mathbf{D}$ is normal <u>or</u> parallel to the surface, making integration trivial
  - Be clever in selecting S (OK, you need to practice at this)

Example 4.6: **Classical Infinite Line Charge**

- Given an infinite length line charge along the z axis (or parallel to it so you can shift the cylindrical coordinate frame) having uniform charge density ρ_ℓ
- The logical Gaussian surface is a cylinder axially centered over a portion of the line charge
- Suppose the cylinder has radius r and length/height h
- We deduce that an infinite line charge can **only** have a radial component $\mathbf{D} = \hat{\mathbf{r}} D_r$; Why?

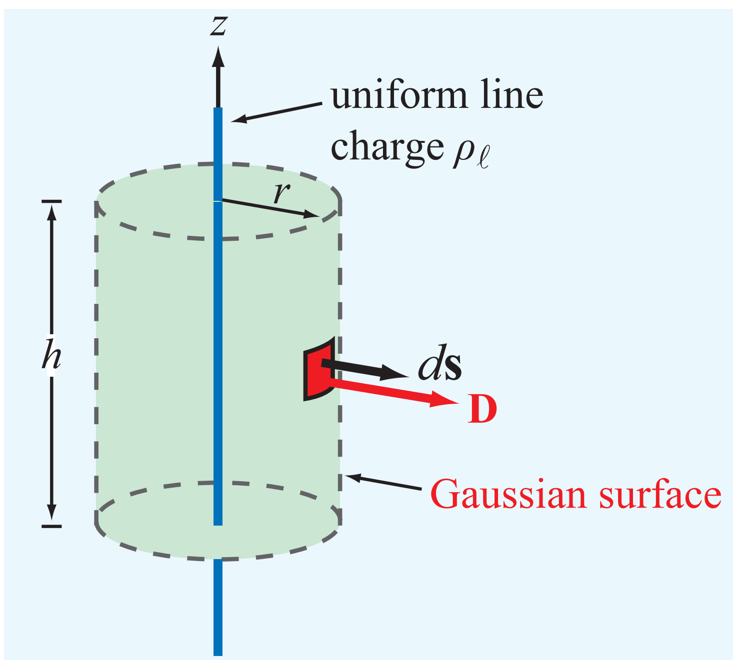

Figure 4.15: Gaussian surface for infinite line charge.

- Apply Gauss's law using the cylinder as S and notice that no flux passes through the ends of the cylinder, so the surface integral is just over the cylinder proper

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = \int_{z=0}^{h} \int_{\phi=0}^{2\pi} \hat{\mathbf{r}} D_r \cdot \hat{\mathbf{r}} r \, d\phi \, dz = 2\pi \, h \, D_r \, r \overset{\text{also}}{=} \rho_\ell h$$

- So, we can solve for D_r and hence $\mathbf{D}$ and/or $\mathbf{E}$

$$\mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \hat{\mathbf{r}} \frac{D_r}{\epsilon_0} = \hat{\mathbf{r}} \frac{\rho_\ell}{2\pi\epsilon_0 r}$$

  - The electric field from an infinite length line charge is inversely proportional to the radial (perpendicular) distance to the line

Example 4.7: Multiple Line Charges

- Find $\mathbf{E}$ when more than one line charge is parallel to the z axis

- As a special case consider a line charges at $(x, y) = (1, 0)$ and $(x, y) = (-1, 0)$ each having density ρ_ℓ as shown below (looking down frlom the $+z$ axis)

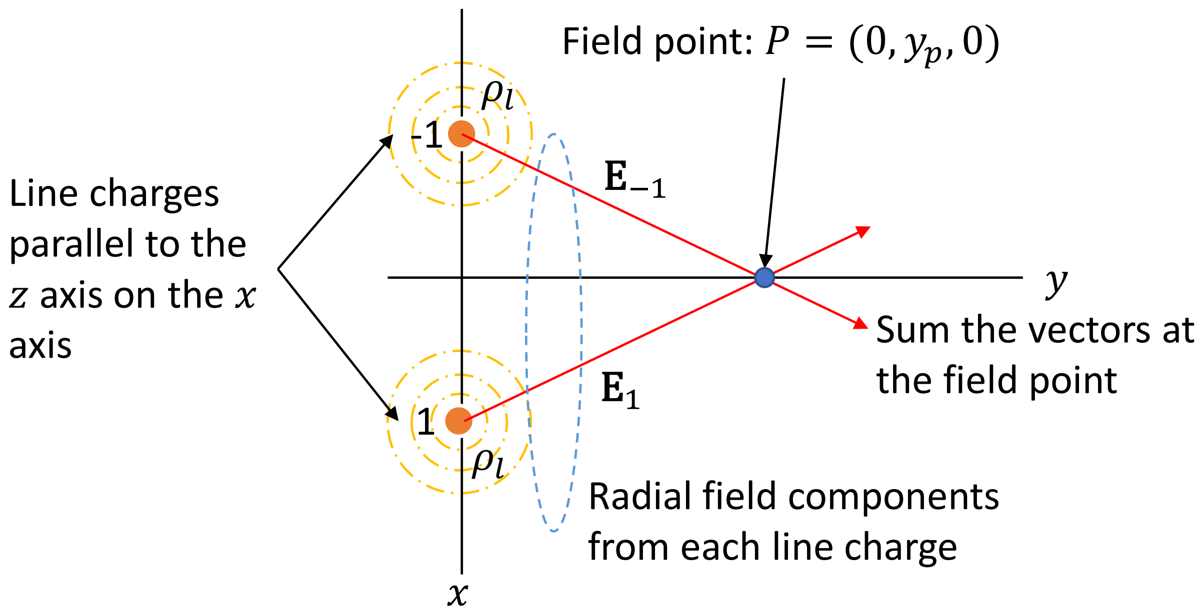

Figure 4.16: Configuration of two line charges and the vector addition of the fields.

- The field point we consider is $P = (0, y_p, 0)$
- A vector sum is needed to combine the offset radial components from each line charge and of course we need unit vectors at $(0, y_p, 0)$
- Denote the commponents $\mathbf{E}_1$ and $\mathbf{E}_{-1}$

$$\mathbf{E}_1 = \overbrace{\frac{-\hat{\mathbf{x}} + \hat{\mathbf{y}} y_p}{\sqrt{1 + y_p^2}}}^{\text{unit vector}} \overbrace{\frac{\rho_\ell}{2\pi\epsilon_0\sqrt{1 + y_p^2}}}^{|\mathbf{E}_1|}$$

$$\mathbf{E}_{-1} = \frac{\hat{\mathbf{x}} + \hat{\mathbf{y}} y_p}{\sqrt{1 + y_p^2}} \frac{\rho_\ell}{2\pi\epsilon_0\sqrt{1 + y_p^2}}$$

- Add the components

$$\mathbf{E} = \mathbf{E}_1 + \mathbf{E}_{-1} = \frac{\hat{\mathbf{y}}2y}{\sqrt{1+y_p^2}}\frac{\rho_\ell}{2\pi\epsilon_0\sqrt{1+y_p^2}} = \frac{\hat{\mathbf{y}}y_p\rho_\ell}{\pi\epsilon_0\big[1+y_p^2\big]}$$

- Note the $\hat{\mathbf{x}}$ component is zero due to symmetry
- Would moving the field point off the y axis make the $\hat{\mathbf{x}}$ component nonzero?

Example 4.8: A Uniform Surface Charge Density Sphere

- Given a thin shell of radius a contains a uniform surface charge of ρ_s, find $\mathbf{E}$ everywhere
- Due to symmetry the electric field will be of the form $\mathbf{E} = \hat{\mathbf{R}}\, E_R$

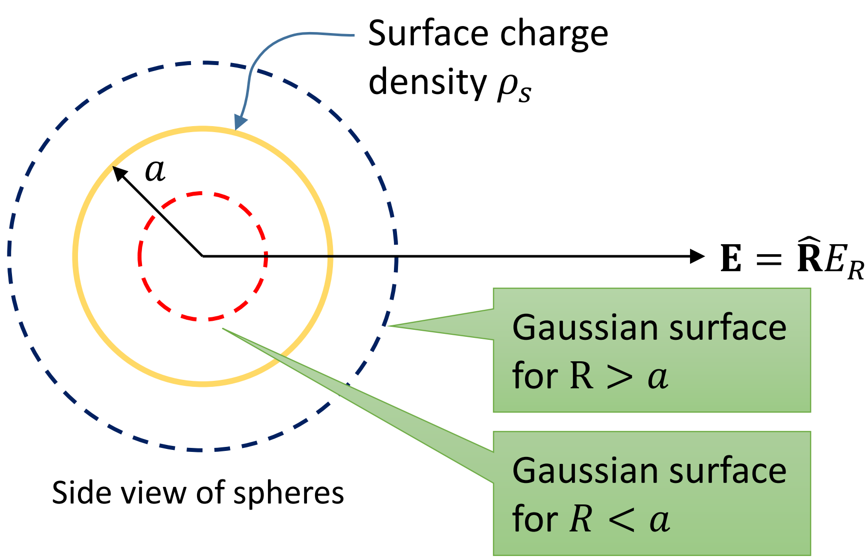

Figure 4.17: Gaussian surfaces for findng $\mathbf{E}$ inside and outside a sphere with surface charge density.

- From Figure 4.17 the appropriate Gaussian surface is a sphere centered at the origin having radius $R > a$ or $R < a$
- $\underline{R < a}$: The Gaussian surface does not enclose any charge, so $\mathbf{E} = 0$
- $\underline{R > a}$: Here the Gaussian surface encloses the surface charge of the thin sphere of radius a, allowing us to write

$$\oint_s \mathbf{D} \cdot d\mathbf{s} = D_R(4\pi R^2) = \int_s \rho_s \, ds = 4\pi a^2 \rho_s$$

$$\Rightarrow \quad \mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \hat{\mathbf{R}} \frac{\rho_s a^2}{\epsilon_0 R^2} \quad \text{(V/m)}$$

- In summary,

$$\mathbf{E} = \begin{cases} 0, & R < a \\ \hat{\mathbf{R}} \frac{\rho_s a^2}{\epsilon_0 R^2}, & R > a \end{cases} \quad \text{(V/m)}$$

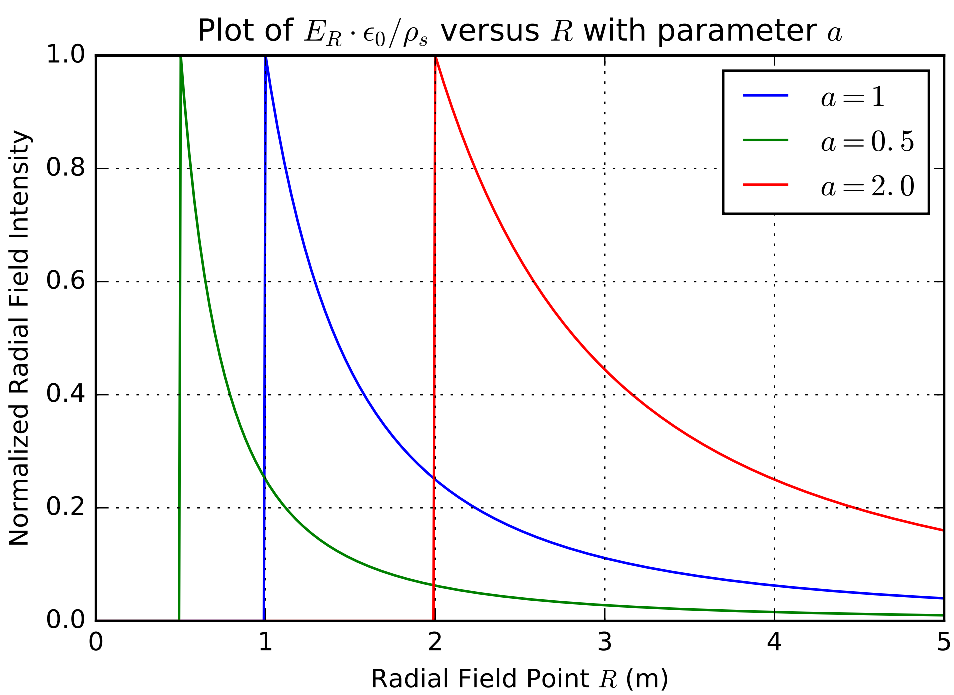

Figure 4.18: Normalized radial field component.

Example 4.9: A Uniform Volume Charge Density Sphere

- Given a spherical volume radius a contains a uniform volume charge of ρ_v, find $\mathbf{E}$ everywhere
- Due to symmetry the electric field will be of the form $\mathbf{E} = \hat{\mathbf{R}}\, E_R$

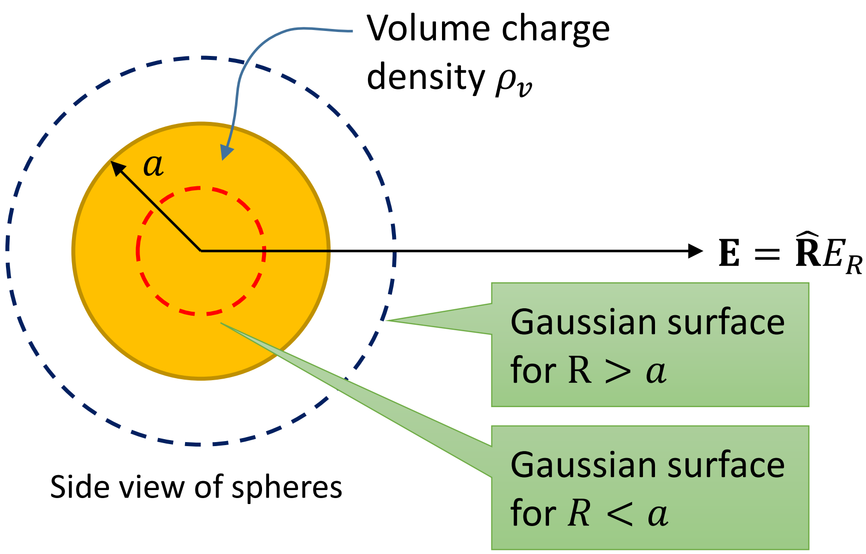

Figure 4.19: Gaussian surfaces for findng $\mathbf{E}$ inside and outside a sphere with volume charge density.

- From Figure 4.19 the appropriate Gaussian surface is a sphere centered at the origin having radius $R > a$ or $R < a$
- $\underline{R < a}$: The Gaussian surface encloses a portion of the total volume charge, so

$$\oint_s \mathbf{D} \cdot d\mathbf{s} = D_R\big(4\pi R^2\big) = \int_v \rho_v \, dv = \frac{4}{3}\pi R^3 \rho_v$$

$$\Rightarrow \quad \mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \hat{\mathbf{R}}\frac{\rho_v R}{3\epsilon_0} \quad \text{(V/m)}$$

- $\underline{R > a}$: Here the Gaussian surface encloses the surface charge of the thin sphere of radius a, allowing us to write

$$\oint_s \mathbf{D}\cdot d\mathbf{s} = D_R\big(4\pi R^2\big) = \int_v \rho_v\, dv = \frac{4}{3}\pi a^3\,\rho_v$$
$$\Rightarrow \quad \mathbf{E} = \frac{\mathbf{D}}{\epsilon_0} = \hat{\mathbf{R}}\frac{\rho_v a^3}{\epsilon_0 R^2} \quad \text{(V/m)}$$

- In summary,

$$\mathbf{E} = \begin{cases} \hat{\mathbf{R}}\frac{\rho_v R}{3\epsilon_0}, & R < a \\ \hat{\mathbf{R}}\frac{\rho_v a^3}{3\epsilon_0 R^2}, & R > a \end{cases} \quad \text{(V/m)}$$

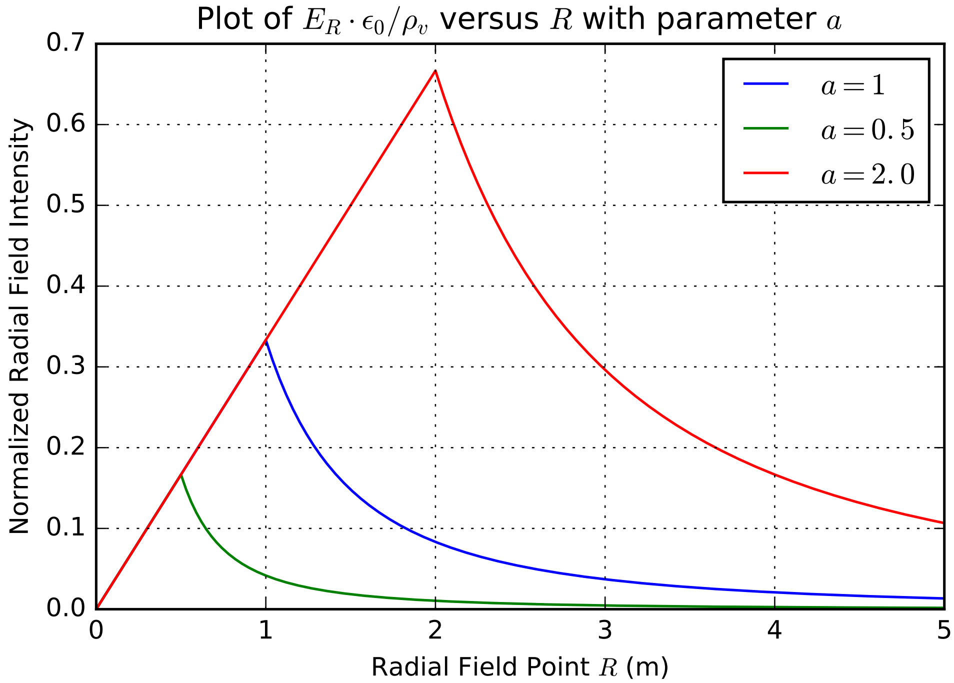

Figure 4.20: Normalized radial field component.

4.5 Electric Scalar Potential

- Associated with the electric field there is also an electric potential V or simply *voltage* V
- This is the same voltage you measure in a circuit
- Similar to the concept of voltage drop across an electrical component, in field theory we are interested in the *potential difference* between two points in space, e.g. $V_{21} = V_2 - V_1$ is the potential difference observed as you move from field point P_1 to P_2
- The definition of the potential difference lies in the *work* done (units of joules of (J)) in moving a charge from P_1 o P_2
- In physics work W is force times distance (N· m = J), i.e., to move a charge q a differential distance $d\mathbf{l}$ in the electric field $\mathbf{E}$ requires work
$$dW = -q\mathbf{E} \cdot d\mathbf{l},$$
where the minus sign comes from the fact that energy is expelled when we move the charge q in the opposite (or against) direction of the field
- The work per unit charge defines the potential difference, that is 1 V = 1 J/C, i.e.,
$$dV = \frac{dW}{q} = -\mathbf{E} \cdot d\mathbf{l}$$

- Finally, the potential difference in moving from field point P_1 to P_2 is defined to be

$$\boxed{V_{21} = V_2 - V_1 = -\int_{P_1}^{P_2} \mathbf{E} \cdot d\mathbf{l}}$$

- In an electrical circuit the sum of the voltage drops around a closed loop (Kirchoff's voltage law) is zero, so too in a static electric field

$$\boxed{\oint_C \mathbf{E} \cdot d\mathbf{l} = 0}$$

- This behavior means that the field is *conservative* or *irrotational*

- It is Maxwell's second equation, for $\partial/\partial t = 0$,

$$\nabla \times \mathbf{E} = 0$$

  that makes this conservation property concrete

- Furthermore, via Stokes theorem

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{s} = \oint_C \mathbf{E} \cdot d\mathbf{l} = 0$$

  where C surrounds the surface S

- What do we use as a voltage reference?

- In circuits it is *ground*

- In fields it is typical to assume $V_1 = 0$ when P_1 is at infinity, i.e., V at P is

$$\boxed{V = -\int_{\infty}^{P} \mathbf{E} \cdot d\mathbf{l} \quad \text{(V)}}$$

Electric Potential Due to Point Charges

- A point charge at the origin produces potential at radial distance R given by

$$V = \int_{\infty}^{R} \left(\hat{\mathbf{R}} \frac{q}{4\pi\epsilon R^2} \right) \cdot \hat{\mathbf{R}}\, d\, R = \frac{q}{4\pi\epsilon R} \quad \text{(V)}$$

- Generalizing to a point charge at location $\mathbf{R}_1$, we have

$$V = \frac{q}{4\pi\epsilon |\mathbf{R} - \mathbf{R}_1|} \quad \text{(V)}$$

- For an arbitrary configuration of point charges

$$\boxed{V = \frac{1}{4\pi\epsilon} \sum_{i=1}^{N} \frac{q_i}{|\mathbf{R} - \mathbf{R}_i|} \quad \text{(V)}}$$

Electric Potential Due to Continuous Distributions

- The electric potential can be solved with the three forms of charge distributions we have been using

- In particular

$$\begin{aligned} V &= \frac{1}{4\pi\epsilon} \int_{\mathcal{V}'} \frac{\rho_v}{R'}\, d\mathcal{V}' \\ V &= \frac{1}{4\pi\epsilon} \int_{S'} \frac{\rho_s}{R'}\, ds' \\ V &= \frac{1}{4\pi\epsilon} \int_{l'} \frac{\rho_\ell}{R'}\, dl' \end{aligned}$$

Electric Field from the Electric Potential

- The subject of this subsection is finding the electric field after first finding the electric potential

- The needed relationship comes about by first recalling from Chapter 3 that a scalar function V obeys

$$dV = \nabla V \cdot d\mathbf{l},$$

and for the just established scalar potential,

$$dV = -\mathbf{E} \cdot d\mathbf{l},$$

so putting the above equations together gives the key result

$$\boxed{\mathbf{E} = -\nabla V,}$$

which says the gradient of the potential function is the electric field!

- Finding $\mathbf{E}$ via V is a two-step process, but the integrals for the scalar potential are likely easier to evaluate

---

Example 4.10: Charge Ring Potential

- Consider a ring of charge in the $x - y$ plane having radius b and line charge density ρ_ℓ

- Calculate the potential along the z axis at the point $P(0, 0, h)$ and then $\mathbf{E}$

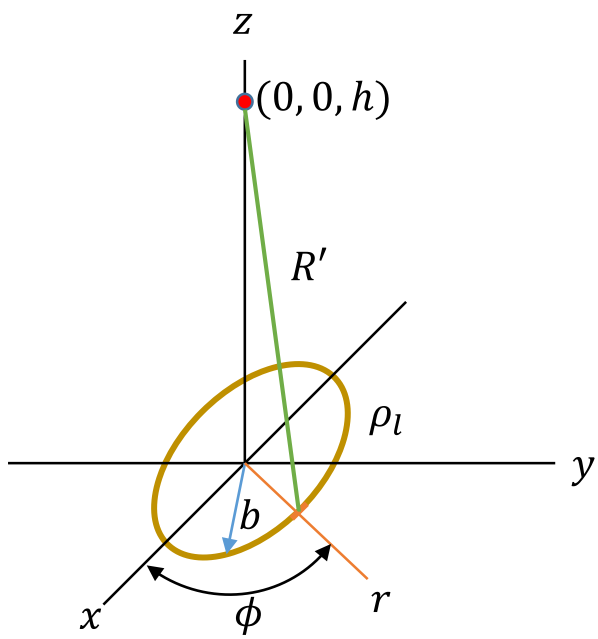

Figure 4.21: Charge ring configuration for potential along the z axis.

- The scalar potential follows easily from Figure 4.21

$$V = \frac{1}{4\pi\epsilon_0}\int_l \frac{\rho_\ell}{R'}\,dl' = \frac{\rho_\ell}{4\pi\epsilon_0}\int_0^{2\pi}\frac{1}{\sqrt{b^2+z^2}}\underbrace{b\,d\theta}_{dl'}$$
$$= \left.\frac{\rho_\ell b}{2\epsilon_0\sqrt{b^2+z^2}}\right|_{z=h} = \frac{\rho_\ell b}{2\epsilon_0\sqrt{b^2+h^2}} \quad \text{(V)}$$

- The electric field follows easily as well

$$\mathbf{E} = -\nabla V = \hat{\mathbf{z}}\frac{\partial}{\partial z}\frac{\rho_\ell b}{2\epsilon_0\sqrt{b^2+z^2}}$$
$$= \hat{\mathbf{z}}\frac{-\rho_\ell b}{2\epsilon_0}\left.\frac{2z(-1/2)}{\left(b^2+z^2\right)^{3/2}}\right|_{z=h} = \hat{\mathbf{z}}\frac{\rho_\ell b h}{2\epsilon_0\left(b^2+h^2\right)^{3/2}} \quad \text{(V/m)}$$

- This is consistent with the earlier Coulomb's law calculation

Example 4.11: Electric Dipole

- An *electric dipole* along the z axis is formed by placing pair of charges $\pm q$ at $z = \pm d/2$ respectively
  - In the study of antennas, structures for propagating electromagnetic energy, another type of dipole is studied
- Calculate V and $\mathbf{E}$ at the field point $P(R, \theta, \phi)$

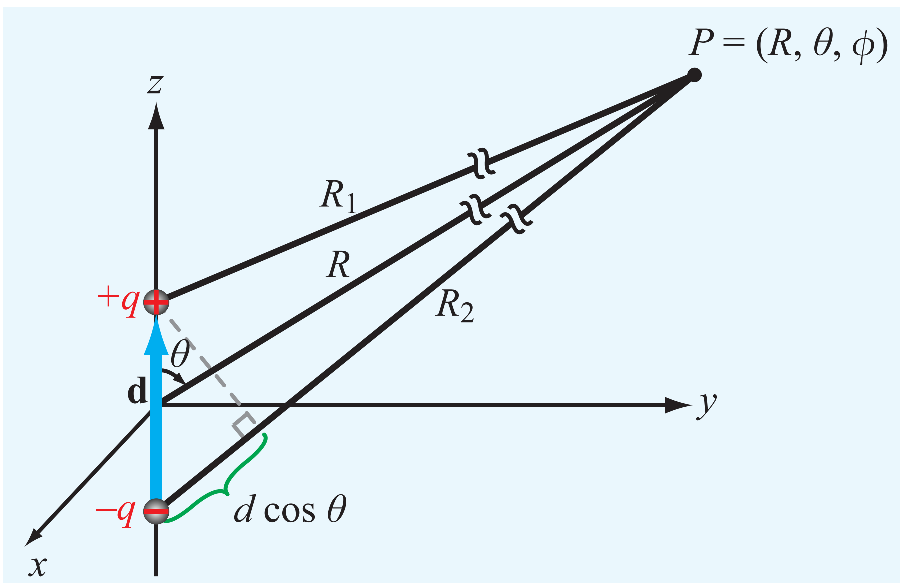

Figure 4.22: Electric dipole configuraton and far field approximation.

- The potential is

$$V = \frac{q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) = \frac{q}{4\pi\epsilon_0}\left(\frac{R_2 - R_1}{R_1 R_2}\right) \quad \text{(V)}$$

- The intent of this example to obtain a *far field* approximation, which means for $R \gg d$, so

- $R_2 - R_1 \approx d\cos\theta$
- $R_1 R_2 \approx R^2$

- Using these approximations V becomes

$$V \approx \frac{qd\cos\theta}{4\pi\epsilon_0} = \frac{\mathbf{p}\cdot\hat{\mathbf{R}}}{4\pi\epsilon_0 R^2}$$

where $\mathbf{p} = q\mathbf{d}$ is the *dipole moment* with $\mathbf{d}$ and also $\mathbf{p}$ pointing in the direction from $-q$ to $+q$, and $\hat{\mathbf{R}}$ is the unit vector from the dipole center to the field point

- The electric field follows from $-\nabla V$ using the spherical coordinates form to calculate the gradient

$$\mathbf{E} = -\nabla V \approx \frac{qd}{4\pi\epsilon_0 R^3}\left(\hat{\mathbf{R}}2\cos\theta + \hat{\boldsymbol{\theta}}\sin\theta\right) \quad \text{(V/m)}$$

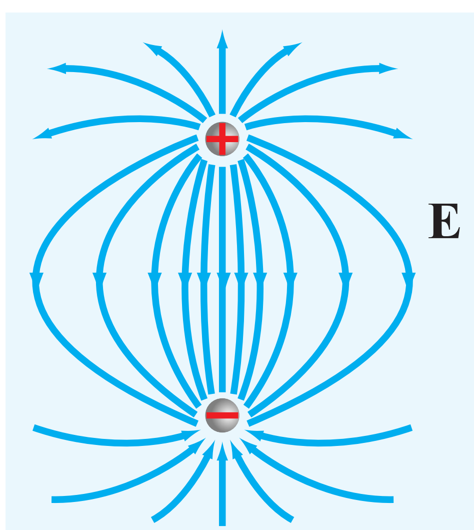

Figure 4.23: Exact electric field pattern for the electric dipole.

Poisson and Laplace's Equation in Electrostatics

- Gauss's law was originally presented in differential form as

$$\nabla \cdot \mathbf{D} = \rho_v \quad \text{or} \quad \nabla \cdot \mathbf{E} = \frac{\rho_v}{\epsilon}$$

- If we set $\mathbf{E} = -\nabla V$ we arrive at

$$\underbrace{\nabla \cdot \overbrace{(\nabla V)}^{\text{a vector}}}_{\text{a scalar}} = \nabla^2 V = -\frac{\rho_v}{\epsilon} \qquad \textbf{(Poisson's equation)}$$

- Recall from Chapter 3 that in Cartesian coordinates

$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

- If there is no charge present in the medium, Poisson's equation reduces to Laplace's equation

$$\nabla^2 V = 0 \qquad \textbf{(Laplace's equation)}$$

- Laplace's equation in particular pops up when we want to solve for the electrostatic potential when boundaries (boundary conditions), such as the plates of a capacitor, have a known potential

- A course in *partial differential equations* considers problems of this sort

Example 4.12: Potential Inside A Spherical Shell

- Consider a spherical shell of radius a with uniform surface charge density ρ_s
- Find the potential and electric field at $P(0, 0, 0)$

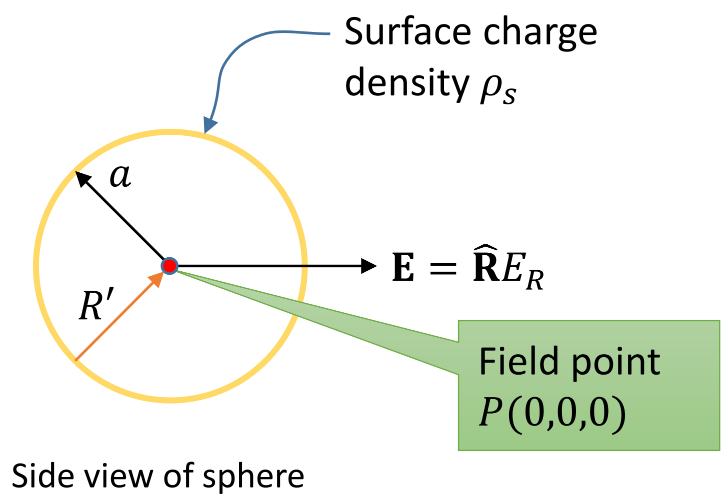

Figure 4.24: Set up for finding the potential inside a spherical shell of radius a.

- From Figure 4.24 we see that

$$
\begin{aligned}
V &= \frac{1}{4\pi\epsilon}\int_{s'}\frac{\rho_s}{a}\,ds' = \frac{\rho_s}{4\pi\epsilon}\int_0^{2\pi}\int_0^{\pi}\frac{1}{a}a^2\sin\theta\,d\theta\,d\phi \\
&= \frac{\rho_s}{4\pi\epsilon}4\pi a = \frac{\rho_s a}{\epsilon} \quad \text{(V)}
\end{aligned}
$$

- To find $\mathbf{E}$ we form

$$\mathbf{E} = -\nabla V = 0,$$

as there is no variation with R, θ, or ϕ

4.6 Conductors

- In the section the focus is conductors and the conduction current $\mathbf{J}$, introduced earlier

- Again the material *constitutive parameters* of permittivity, ϵ, permeability, μ, and conductivity, σ are of interest

- From a materials consideration, we have *conductors* (metals) or *dielectrics* (insulators)

- What an electric field is applied to a conductor *conduction current* flows in the same direction as the electric field:

$$\mathbf{J} = \sigma \mathbf{E} \quad (\text{A/m}^2),$$

  where σ is the material conductivity in (S/m)

- This relationship is known in this context as **Ohm's law**

- We expect:

  – A perfect dielectric to have $\sigma = 0$ (for good insulators $10^{-17} \leq \sigma \leq 10^{-10}$ S/m)

  – A perfect conductor to have $\sigma = \infty$ (for good conductors $10^{6} \leq \sigma \leq 10^{7}$ S/m)

Perfect dielectric: $\mathbf{J} = 0$ since $\mathbf{J} = \sigma \mathbf{E}$ and $\sigma = 0$
Perfect conductor: $\mathbf{E} = 0$ since $\mathbf{E} = \mathbf{J}/\sigma$ and $\sigma = \infty$

4.6.1 Drift Velocity

- In a conductor electrons have a *drift velocity* of $\mathbf{u}_e$ such that

$$\mathbf{u}_e = -\mu_e \mathbf{E},$$

where μ_e is the *electron mobility* in (m^2/V· s)

- In semiconductors there is also hole velocity and hole mobility (positive charge carriers)

$$\mathbf{u}_h = \mu_h \mathbf{E}$$

- The total conduction current is

$$\mathbf{J} = \mathbf{J}_e + \mathbf{J}_h = \rho_{\text{ve}} \mathbf{u}_e + \rho_{\text{vh}} \mathbf{u}_h \quad (\text{A/m}^2),$$

where ρ_{ve} and ρ_{vh} are volume charge densities

- In particular, $\rho_{\text{ve}} = -N_e e$ and $\rho_{\text{vh}} = N_h e$, where N_e and N_h are the number of free electrons and holes respectively, per unit volume, and $e = 1.6 \times 10^{-19}$ C

- Because $\mathbf{u}_{e/h}$ is related to $\mathbf{E}$ via $\mu_{e/h}$,

$$\mathbf{J} = \underbrace{(-\rho_{\text{ve}} \mu_e + \rho_{\text{vh}} \mu_h)}_{\sigma} \mathbf{E}$$

- For a good conductor

$$\boxed{\sigma = -\rho_{ve} \mu_e = N_e \mu_e e \quad (\text{S/m}),}$$

4.6.2 Resistance

- Consider the resistance, R, of a conductor of length l and cross section A
- We assume the conductor has uniform cross section and lies along $\hat{\mathbf{x}}$, making $\mathbf{E} = \hat{\mathbf{x}}E_x$
- The voltage applied across the terminals is V, so relative to reference points x_1 and x_2

$$V = -\int_{x_2}^{x_1} \mathbf{E} \cdot d\mathbf{l} = E_x l \quad \text{(V)}$$

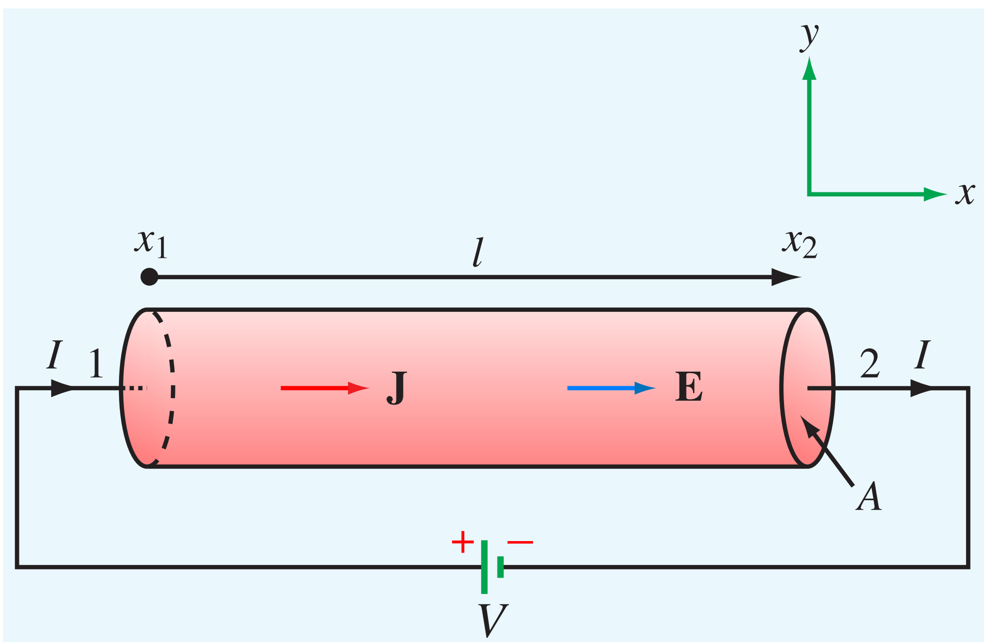

Figure 4.25: A resistor from a materials view point.

- The current flowing is

$$I = \int_A \mathbf{J} \cdot d\mathbf{s} = \int_A \sigma \mathbf{E} \cdot d\mathbf{s} = \sigma E_x A \quad \text{(A)}$$

- From Ohm's law it follows that

$$R = \frac{V}{I} = \frac{l}{\sigma A}$$

- For any conductor shape R can be found as

$$\boxed{R = \frac{V}{I} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_s \mathbf{J} \cdot d\mathbf{s}} = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_s \sigma \mathbf{E} \cdot d\mathbf{s}}}$$

Example 4.13: Coax Cable with Finite σ

- The coax cable was studied in Chapter 2 and equations for tline parameters were given

- Here we establish the conductance per unit length using field theory and the equation for $R = 1/G$

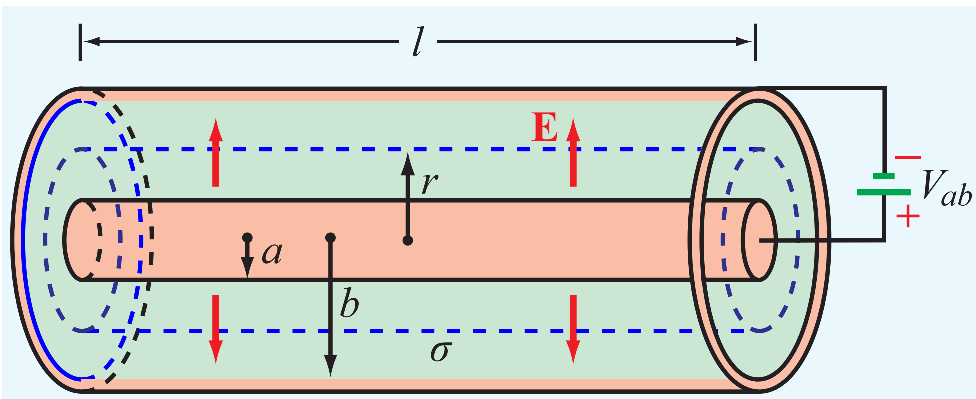

Figure 4.26: Finding the shunt conductance of a coax cable.

- The dielectric filling is assumed to have conductivity σ

- Given V_{ab} is connected from the center conductor to the outer shield, we need to find the corresponding current flow I

- For a length l section of line, the surface area at some $a < r < b$ through which current flows is $A = 2\pi r l$

- The current density $\mathbf{J} = \sigma \mathbf{E}$ is outward radially i.e., $\hat{\mathbf{r}}$, as the potential is higher on the center conductor

$$\mathbf{J} = \hat{\mathbf{r}}\frac{I}{A} = \hat{\mathbf{r}}\frac{I}{2\pi r l} \quad \text{or} \quad \mathbf{E} = \hat{\mathbf{r}}\frac{I}{2\pi \sigma r l}$$

- The voltage between the conductors, V_{ab}, must be

$$V_{ab} = -\int_b^a \mathbf{E} \cdot d\mathbf{l} = -\frac{I}{2\pi\sigma l} \int_b^a \frac{\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dr}{r}$$
$$= -\frac{I}{2\pi\sigma l} \ln\left(r\right)\Big|_b^a = \frac{I}{2\pi\sigma l} \ln\left(\frac{b}{a}\right)$$

- Finally,

$$R = \frac{V_{ab}}{I} = \frac{1}{2\pi\sigma l} \ln\left(\frac{b}{a}\right) \quad (\Omega)$$
$$G' = \frac{1}{Rl} = \frac{2\pi\sigma}{\ln(b/a)} \quad \text{(S/m)}$$

4.6.3 Joule's Law

- The power dissipated in a conducting medium is

$$P = \int_{\mathcal{V}} \mathbf{E} \cdot \mathbf{J}\, d\mathcal{V} = \int_{\mathcal{V}} \sigma |\mathbf{E}|^2\, d\mathcal{V} \quad \text{(W)}$$

- For the case of a simple resistor as cylindrical conductor, Joule's law reduces to

$$P = I^2 R \quad \text{(W)}$$

- **Power Dissipated in Coax Dielectric**: For a length l line section

$$P = I^2 R = I^2 \ln(b/a)/(2\pi\sigma l)$$

4.7 Dielectrics

- When a dielectric material is subject to an electric field, the atoms or molecules of the material become polarized

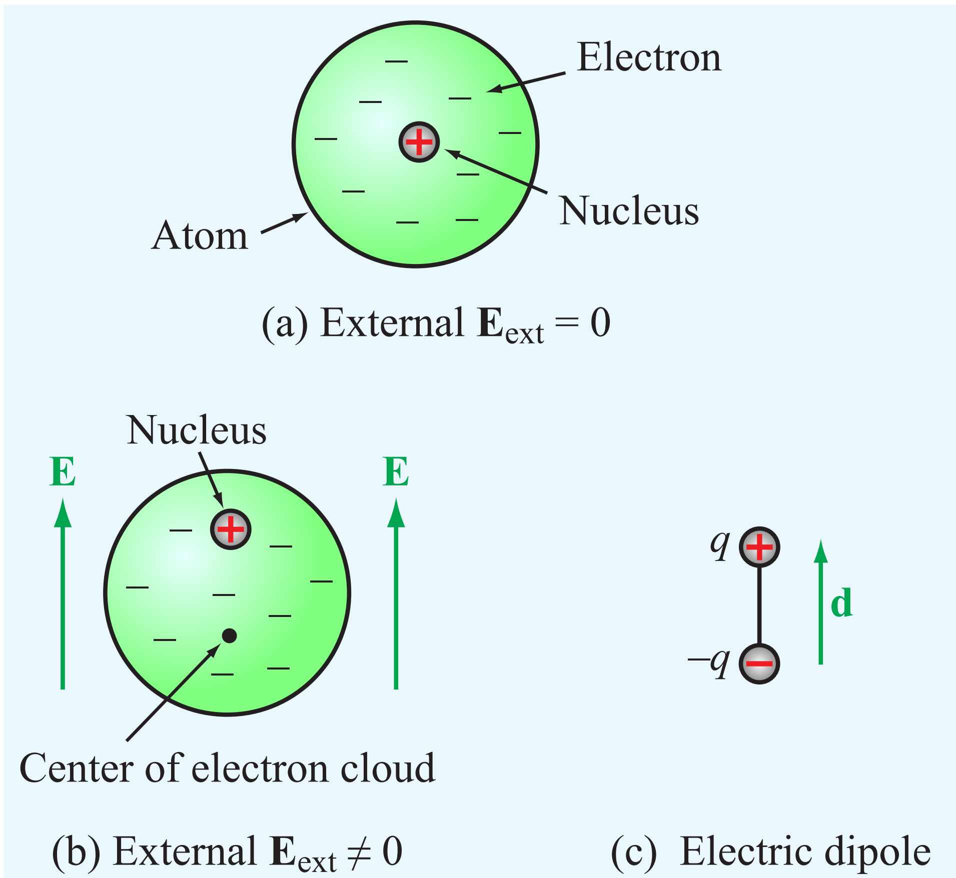

Figure 4.27: Impact of $\mathbf{E}$ on material atoms and the creation of a dipole moment.

- When no field is present the electron cloud is symmetrical about the nucleus (a) in Figure 4.27

- In a dielectric when the field is applied (b) in Figure 4.27, a shift occurs and $\mathbf{E}$ is said to *polarize* the atoms and create a dipole

  - The dipole creates its own electric field known as the polarization field, **P**

- Molecules such as water have a permanent dipole moment, but the dipoles are randomly aligned until an applied field is applied

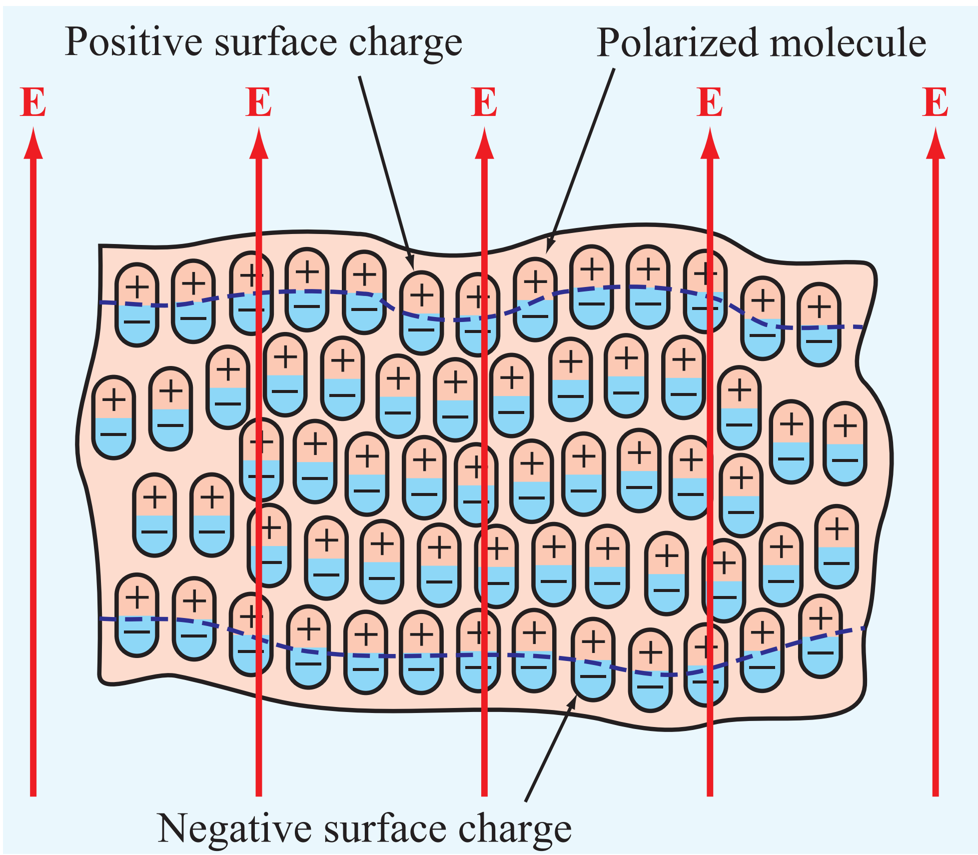

Figure 4.28: Applying an electric field polarizes the molecules creating an effective surface charge.

4.7.1 Polarization Field

- In a dielectric material the total flux density under the influence of an external **E** is

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$$

where **P** is the *electric polarization field*

- In a linear, isotropic, and homogeneous medium, **P** is proportional to the applied **E** via

$$\mathbf{P} = \epsilon_0 \chi_e \mathbf{E}$$

where χ_e is the *electric susceptibility* of the material

- In the end it is χ_e that defines the material permittivity, as

$$\mathbf{D} = \epsilon_0 \mathbf{E} + \epsilon_0 \chi_e \mathbf{E} = \epsilon_0 (1 + \chi_e) \mathbf{E} = \epsilon \mathbf{E}$$

so it must be that $\epsilon_r = \epsilon/\epsilon_0 = 1 + \chi_e$

4.7.2 Dielectric Breakdown

- Real materials are subject to *dielectric breakdown*
- The electric field magnitude E_{ds}, known as the *dielectric strength*, is the largest field strength a material can handle without breakdown
- For air E_{ds} is about 3 (MV/m)
- For mica E_{ds} is about 200 (MV/m), hence mica is a strong dielectric

4.8 Electric Boundary Conditions

- A vector field does not experience abrupt changes in it magnitude or direction unless is passes from one medium to another, e.g., a dielectric interface or a metal conductor

- Boundary conditions among **E**, **D**, and **J** dielectric and conductor interfaces are now established with the aid of Figure 4.29

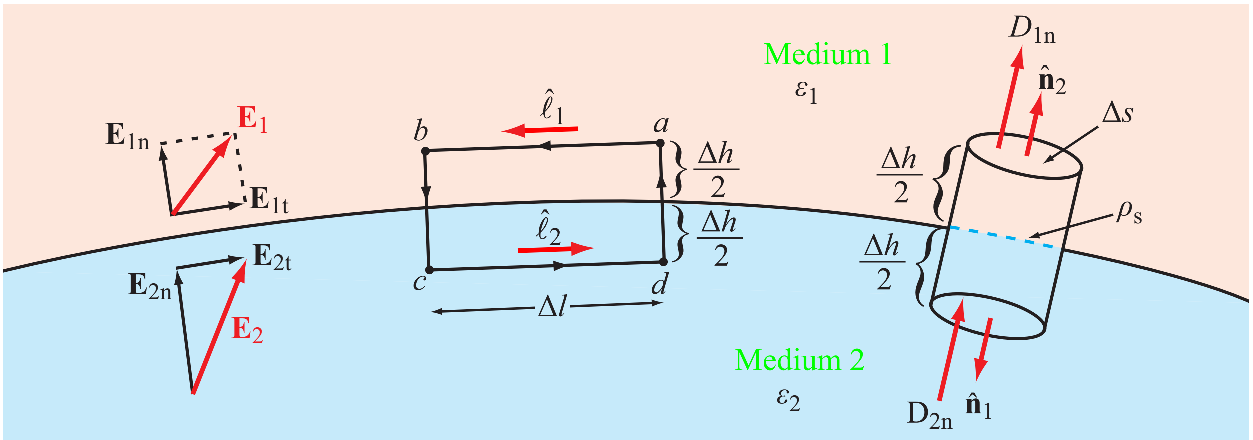

Figure 4.29: Establishing electric field/flux boundary conditions between two mediums.

- To establish these relationships, we rely on:

$$\oint_C \mathbf{E} \cdot d\mathbf{l} = 0 \Leftrightarrow \nabla \times \mathbf{E} = 0 \quad \text{(conservation prop.)}$$

$$\oint_S \mathbf{D} \cdot d\mathbf{s} = Q \Leftrightarrow \nabla \cdot \mathbf{D} = \rho_v \quad \text{(divergence prop.)}$$

- Working through the details sketched out in Figure 4.29, we establish from the conservation of **E** that at an interface the tangential electric field component is continuous, i.e.,

$$\boxed{\mathbf{E}_{1\mathbf{t}} = \mathbf{E}_{2\mathrm{t}} \quad \text{(V/m)}}$$

- Similarly working from the divergence property the flux density normal to the interface is continuous, subject any added charge density that may be present, i.e.,

$$\boxed{\hat{\mathbf{n}}_2 \cdot (\mathbf{D}_1 - \mathbf{D}_2) = \rho_s \quad (\text{C/m}^2)}$$

or

$$D_{1n} - D_{2n} = \rho_s \quad (\text{C/m}^2)$$

- Any abrupt change in the flux density, **D**, normal to the interface, is due to a surface charge density being present at the interface

- The above results in Table 4.1, including a specialization for a dielectric conductor interface, yet to be explained

Table 4.1: Electric field/flux boundary conditions.

Field Component	Any Two Media	Medium 1 Dielectric ε_1	Medium 2 Conductor
Tangential E	$\mathbf{E}_{1t} = \mathbf{E}_{2t}$	$\mathbf{E}_{1t} = \mathbf{E}_{2t} = 0$	
Tangential D	$\mathbf{D}_{1t}/\varepsilon_1 = \mathbf{D}_{2t}/\varepsilon_2$	$\mathbf{D}_{1t} = \mathbf{D}_{2t} = 0$	
Normal E	$\varepsilon_1 E_{1n} - \varepsilon_2 E_{2n} = \rho_s$	$E_{1n} = \rho_s/\varepsilon_1$	$E_{2n} = 0$
Normal D	$D_{1n} - D_{2n} = \rho_s$	$D_{1n} = \rho_s$	$D_{2n} = 0$
Notes: (1) ρ_s is the surface charge density at the boundary; (2) normal components of $\mathbf{E}_1$, $\mathbf{D}_1$, $\mathbf{E}_2$, and $\mathbf{D}_2$ are along $\hat{\mathbf{n}}_2$, the outward normal unit vector of medium 2.			

Example 4.14: A Dielectric–Dielctric Interface

- An important special case to consider is when **E** travels from a material having permittivity ϵ_1 to ϵ_2

- From physics you may recall that when light rays pass from one medium to the next, there is a direction change that takes place due to the change in the *index of refraction*

- The same concept applies here and is in fact related
- Consider now the scenario of Figure 4.30

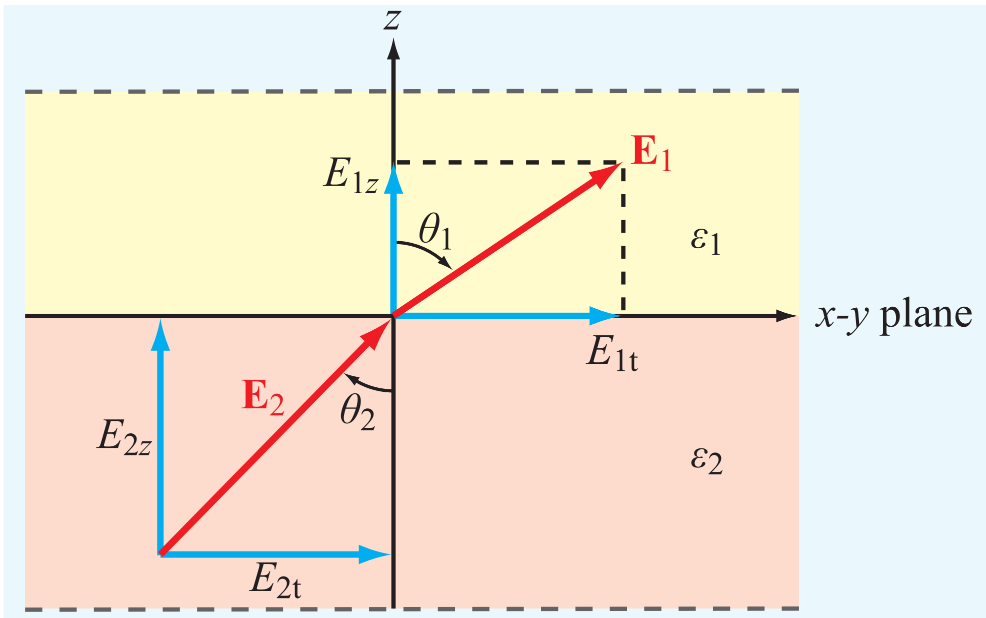

Figure 4.30: Electric field angle change at an ϵ_1 to ϵ_2 interface.

- Assume that $\mathbf{E}_1 = \hat{\mathbf{x}}E_{1x} + \hat{\mathbf{y}}E_{1y} + \hat{\mathbf{z}}E_{1z}$ and find $\mathbf{E}_2 = \hat{\mathbf{x}}E_{2x} + \hat{\mathbf{y}}E_{2y} + \hat{\mathbf{z}}E_{2z}$ in terms of $\mathbf{E}_1$, also find the relationship between the angles θ_1 and θ_2 (assume $\rho_s = 0$ at the interface)
- The continuity of $\mathbf{E}_t$ means that

$$E_{1t} = E_{2t} \Rightarrow E_{1x} = E_{2x} \textbf{ and } E1y = E_{2y}$$

- Similarly for the normal components, which by construction of the problem lie along the z axis,

$$\epsilon_1 E_{1z} = \epsilon_2 E_{2z}$$

- So we can write that

$$\begin{aligned}\mathbf{E}_2 &= \hat{\mathbf{x}}E_{2x} + \hat{\mathbf{y}}E_{2y} + \hat{\mathbf{z}}E_{2z} \\ &= \hat{\mathbf{x}}E_{1x} + \hat{\mathbf{y}}E_{1y} + \frac{\epsilon_1}{\epsilon_2}\hat{\mathbf{z}}E_{1z}\end{aligned}$$

- The angle reationships are

$$\tan\theta_1 = \frac{E_{1t}}{E_{1n}} = \frac{E_{1t}}{E_{1z}} = \frac{\sqrt{E_{1x}^2 + E_{1y}^2}}{E_{1z}}$$

$$\tan\theta_2 = \frac{E_{2t}}{E_{2n}} = \frac{E_{2t}}{E_{2z}} = \frac{\sqrt{E_{2x}^2 + E_{2y}^2}}{E_{2z}} \overset{\text{also}}{=} \frac{\sqrt{E_{2x}^2 + E_{2y}^2}}{(\epsilon_1/\epsilon_2)E_{1z}}$$

- In particular

$$\frac{\tan\theta_2}{\tan\theta_1} = \frac{\epsilon_2}{\epsilon_1}$$

4.8.1 Dielectric-Conductor Boundary

- Consider the special case of medium 1 a dielectric and medium 2 a conductor

- In a perfect conductor there are no fields or fluxes, i.e., $\mathbf{E} = \mathbf{D} = 0$, so for the tangential and normal boundary conditions we have

$$E_{1\mathrm{t}} = D_{1\mathrm{t}} = 0$$
$$D_{1\mathrm{n}} = \epsilon_1 E_{1\mathrm{n}} = \rho_s$$

- The key result from the above is that a charge density at the conductor surface is induced by the normal component of the electric field

$$\mathbf{D}_1 = \epsilon_1\mathbf{E}_1 = \hat{\mathbf{n}}\rho_s$$

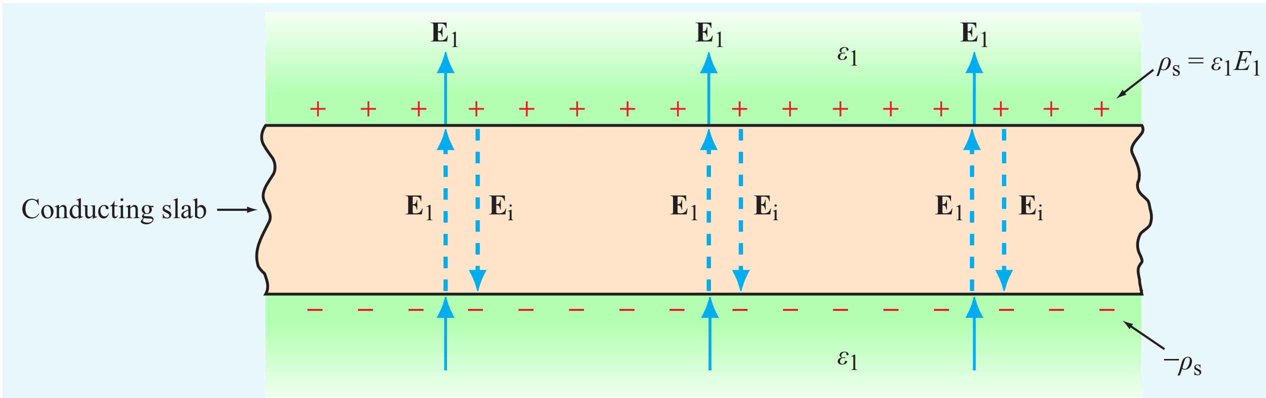

Figure 4.31: The interaction of an electric field in a sandwidge of dielectric-conductor-dielectric.

- Also, the flux lines at the conductor interface are *always normal* ($\hat{\mathbf{n}}$) with a positive charge density when $\mathbf{E}_1$ is away from the interface and a negative charge density when $\mathbf{E}_1$ is toward the interface (see Figure 4.31)

Example 4.15: **Metallic Sphere in a Uniform E Field**

- Consider a metallic sphere placed in a uniform electric field

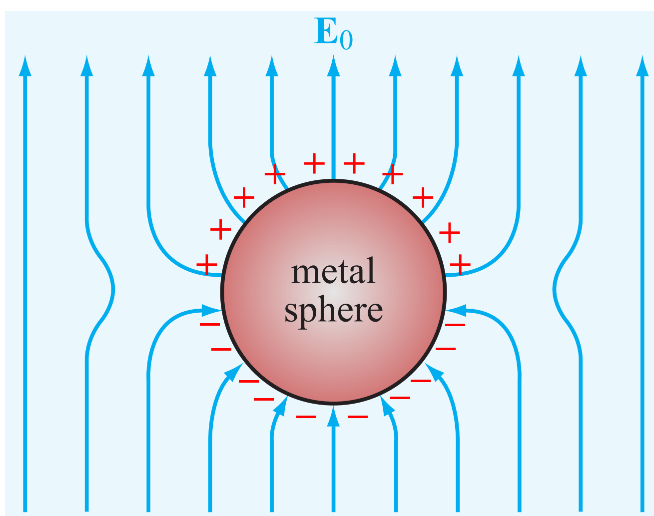

Figure 4.32: The flux lines bend at the surface of a metallic sphere to insure they remain normal everywhere.

4.8.2 Conductor-Conductor Boundary

- Taking the special case one step further, suppose we have two conductors of different conductivity interfaced
  - Note, neither conductor is perfect in this scenario

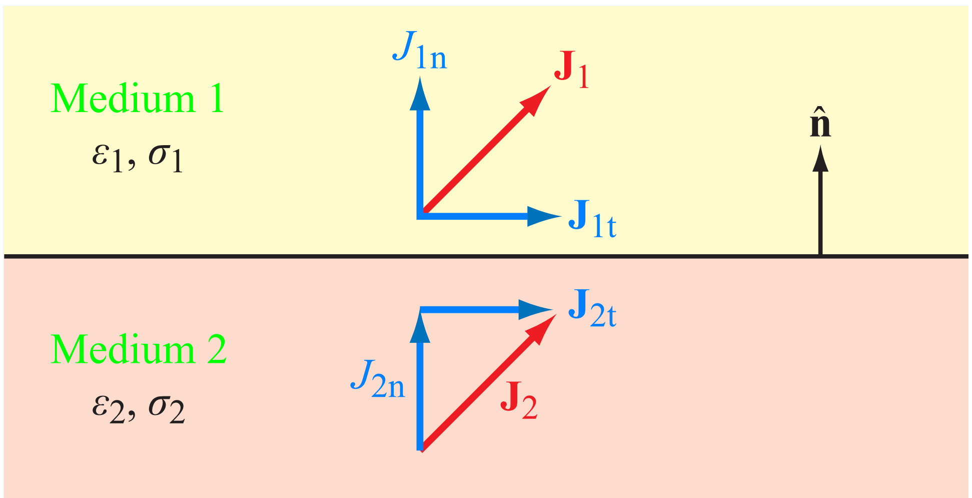

Figure 4.33: A finite conductivity conductor–conductor interface scenario.

- The boundary conditions require that

$$\mathbf{E}_{1t} = \mathbf{E}_{2t} \quad \text{and} \quad \epsilon_1 E_{1n} - \epsilon_2 E_{2n} = \rho_s$$

- Since for conductors $\mathbf{J_i} = \sigma_i \mathbf{E}_i$ we can also write that

$$\frac{\mathbf{J}_{1t}}{\sigma_1} = \frac{\mathbf{J}_{2t}}{\sigma_2} \quad \text{and} \quad \epsilon_1 \frac{E_{1n}}{\sigma_1} - \epsilon_2 \frac{E_{2\mathbf{n}}}{\sigma_2} = \rho_s$$

- The tangential components **can** coexist as parallel current flow

- The normal components cannot be different, since this requires a ρ_s that is not constant with time and hence not a static condition

- To resolve this dilemma, we force $J_{1n} = J_{2n}$, so

$$\boxed{J_{1n}\left(\frac{\epsilon_1}{\sigma_1} - \frac{\epsilon_2}{\sigma_2}\right) = \rho_s}$$

4.9 Capacitance

- Any two conductors in space, separated by a dielectric (air is valid), form a capacitor
- A voltage placed across the two conductors allows $+Q$ and $-Q$ charges to accumulate
- The ratio of charge to voltage defines the *capacitance* in farads

$$C = \frac{Q}{V} \quad \text{(C/V or F)}$$

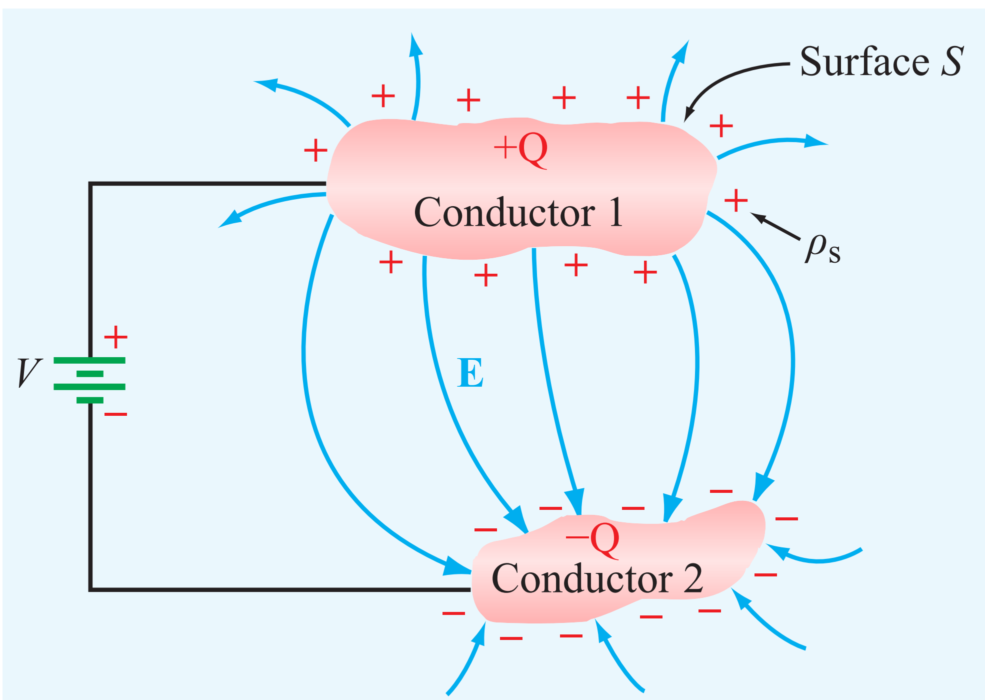

Figure 4.34: Establishing the capacitance between two conductors by applying voltage V.

- Note: The charge on each conductor is distributed to insure that $\mathbf{E} = 0$ within the conductor and the potential is the same at all points

- We know from an earlier discussion that only the normal component of $\mathbf{E}$ exists, so

$$\boxed{E_n = \hat{\mathbf{n}} \cdot \mathbf{E} = \frac{\rho_s}{\epsilon}}$$

- The total charge over the positively charged conductor is

$$Q = \int_S \rho_s ds = \int_S \epsilon \mathbf{E} \cdot d\mathbf{s}$$

- The voltage V is formally

$$V = V_{12} = -\int_{\text{Cond. 2}}^{\text{Cond. 1}} \mathbf{E} \cdot d\mathbf{l}$$

  Putting the pieces together we have,

$$\boxed{C = \frac{\int_S \epsilon \mathbf{E} \cdot d\mathbf{s}}{-\int_l \mathbf{E} \cdot d\mathbf{l}} \quad \text{(F)}}$$

  Note: C is positive and a function of only the geometry and the permittivity

- If the material has loss, i.e., a small conductivity, then there is also a resistance between the two conductors given by

$$R = \frac{-\int_l \mathbf{E} \cdot d\mathbf{l}}{\int_S \sigma \mathbf{E} \cdot d\mathbf{s}}$$

  as we have seen earlier

- **Interesting Observation**: For materials having uniform σ and ϵ, it follows that
$$\boxed{RC = \frac{\epsilon}{\sigma},}$$
so given C then R is known and likewise given R, C is known

Example 4.16: Classical Parallel Plate Capacitor

- From physics you likely recall the parallel plate capacitor and $\epsilon A/d$, where A is the plate area and d is the plate separation

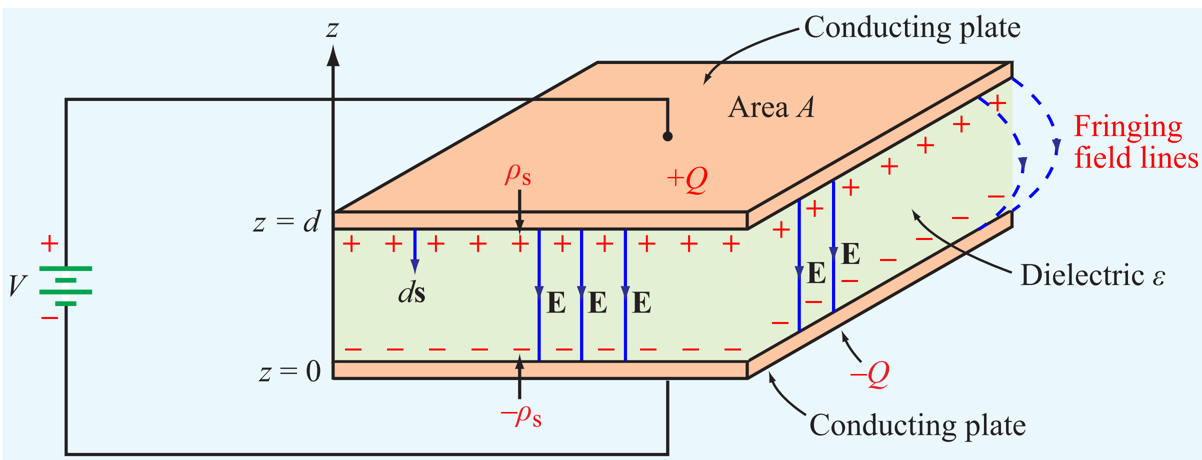

Figure 4.35: Parallel plate capacitor analysis model using applied voltage V.

- With respect to the top plate $\mathbf{E} = -\hat{\mathbf{z}}E$ and from the boundary conditions $E = \rho_s/\epsilon$

- Also the neglecting fringing fields, the charge density ρ_s is uniform, so $Q = \rho_s A \rightarrow E = Q/(\epsilon A)$

- The voltage across the plates is
$$V = -\int_0^d \mathbf{E} \cdot d\mathbf{l} = -\int_0^d (-\hat{\mathbf{z}}E) \cdot \hat{\mathbf{z}}\, dz = Ed,$$

so

$$C = \frac{Q}{V} = \frac{E\epsilon A}{Ed} = \frac{\epsilon A}{d} \quad \text{(F)}$$

Example 4.17: Coax Capacitance

- Another classical structure to analyze is the coax capacitor

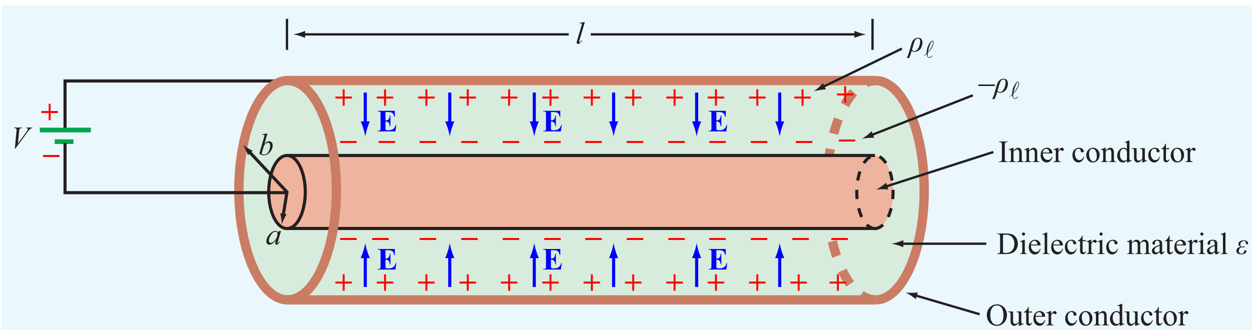

Figure 4.36: Coax capacitor analysis model using applied voltage V.

- Using Gauss's law and knowing the form of the **E** field for an infinite line charge (pure radial and inverse proportional to r), we have

$$\mathbf{E} = -\hat{\mathbf{r}}\frac{Q}{2\pi\epsilon r l}$$

- The potential can be calculated as

$$V = -\int_a^b \mathbf{E} \cdot d\mathbf{l} = -\int_a^b \left(-\hat{\mathbf{r}}\frac{Q}{2\pi\epsilon r l}\right) \cdot \hat{\mathbf{r}}\, dr$$

$$= \frac{Q}{2\pi\epsilon l} \ln(r)\Big|_a^b = \frac{Q}{2\pi\epsilon l} \ln\left(\frac{b}{a}\right)$$

- Finally,

$$C = \frac{Q}{V} = \frac{2\pi\epsilon l}{\ln(b/a)} \quad \text{(F)}$$

- In terms of capacitance per unit length we have

$$C' = \frac{C}{l} = \frac{2\pi\epsilon}{\ln(b/a)} \quad \text{(F/m)}$$

4.10 Electrostatic Potential Energy

- When a voltage is applied to a lossless capacitor energy goes into the structure and is stored in the electric field

- What work is done in charging up this capacitor?

- When the voltage is applied we are moving charge from one plate to another

- A voltage increment υ corresponds to charge q/C, so the differential electrostatic work, W_e, is

$$dW_e = \upsilon\, dq = \frac{q}{C}\, dq$$

- Building up a total charge Q accumulates total work

$$W_e = \int_0^Q \frac{q}{C}\, dq = \frac{1}{2}\frac{Q^2}{C} \quad \text{(J)},$$

but since $C = Q/V$, $Q^2 = C^2V^2$ and substituting yields

$$\boxed{W_e = \frac{1}{2}CV^2 \quad \text{(J)}}$$

- When an electric field is present in a region it can be viewed as an *electrostatic energy density via*

$$w_e = \frac{W_e}{\mathcal{V}} = \frac{1}{2}\epsilon E^2 \quad (\text{J/m}^3)$$

- From the energy density it follows that the potential energy stored in volume $\mathcal{V}$ is,

$$W_e = \frac{1}{2}\int_{\mathcal{V}} \epsilon E^2 \, d\mathcal{V} \quad (\text{J})$$

Example 4.18: **Energy Stored in Coax**

- In the coax capacitance example we found that

$$C = \frac{2\pi\epsilon l}{\ln(b/a)}$$

and

$$V = \frac{Q}{2\pi\epsilon l}\ln(b/a) = \frac{l\rho_\ell}{2\pi\epsilon l}\ln(b/a),$$

where $Q = l\rho\ell$

- Using $W_e = (1/2)CV^2$ we have

$$W_e = \frac{1}{2}\left(\frac{2\pi\epsilon l}{\ln(b/a)}\right)\left(\frac{\rho_\ell}{2\pi\epsilon}\ln(b/a)\right)^2$$
$$= \frac{1}{2}\frac{l\rho_\ell}{2\pi\epsilon}\ln\left(\frac{b}{a}\right) \quad (\text{J})$$

- Starting from energy density and integrating over the volume we should get the same answer

$$\begin{aligned} W_e &= \frac{1}{2}\int_0^{2\pi}\int_0^{l}\int_a^{b} \epsilon\left(\frac{\rho_\ell}{2\pi\epsilon r}\right)^2 r dr\, dz\, d\phi \\ &= \frac{2\pi\epsilon l}{2}\,\frac{\rho_\ell^2}{(2\pi)^2\epsilon^2}\ln\left(\frac{b}{a}\right) = \frac{1}{2}\,\frac{\rho_\ell^2}{2\pi\epsilon}\ln\left(\frac{b}{a}\right) \quad \text{(J)} \end{aligned}$$

The same result!

- Suppose that $a = 2$ cm, $b = 5$ cm, $\epsilon_r = 4$, $\rho_\ell = 10^{-4}$ C/m, and $l = 20$ cm

- Plugging the numbers into either of the above equations yields

$$W_e = 4.12 \quad \text{(J)}$$

Coax Stored Energy

$$W_e = \frac{1}{2}\cdot\frac{0.2\cdot\left(10^{-4}\right)^2}{2\cdot\pi\cdot 4\cdot 8.85\cdot 10^{-12}}\cdot\ln\left(\frac{5}{2}\right) \blacktriangleright 4.11955 \text{ (J)}$$

Figure 4.37: TI Nspire calculation of W_e.

4.11 Image Method

- When a charge distribution is placed over an infinite ground plane Coulomb's law and Gauss's law **cannot** be readily applied

- Solving Poisson's or Laplace's equation is an option, but this is also mathematically challenging, likely to require a numerical solution

- It turns out that an electrically equivalent problem can be created using the *image distribution* with the ground plane removed

- A simple example of a single point charge Q distance d above a ground plane is shown if Figure 4.38

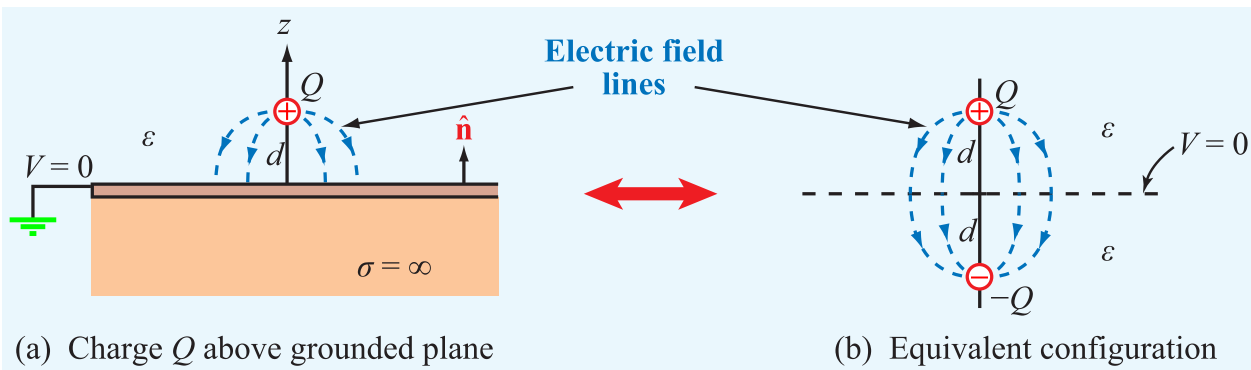

Figure 4.38: The image theory concept in solving electrostatics problems with charge over a uniform ground plane.

Example 4.19: Ulaby 4.71 – A Corner Reflector Charge

- Construct image distribution for a corner reflector charge

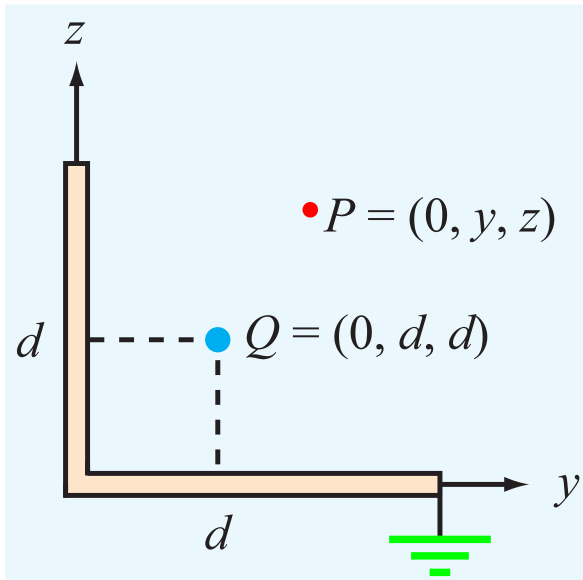

Figure 4.39: Ulaby 4.61.

Example 4.20: Ulaby 4.63 – Conducting Cylinder Over a Ground Plane

- An infinite length charged cylinder over a ground plane is an assigned homework problem

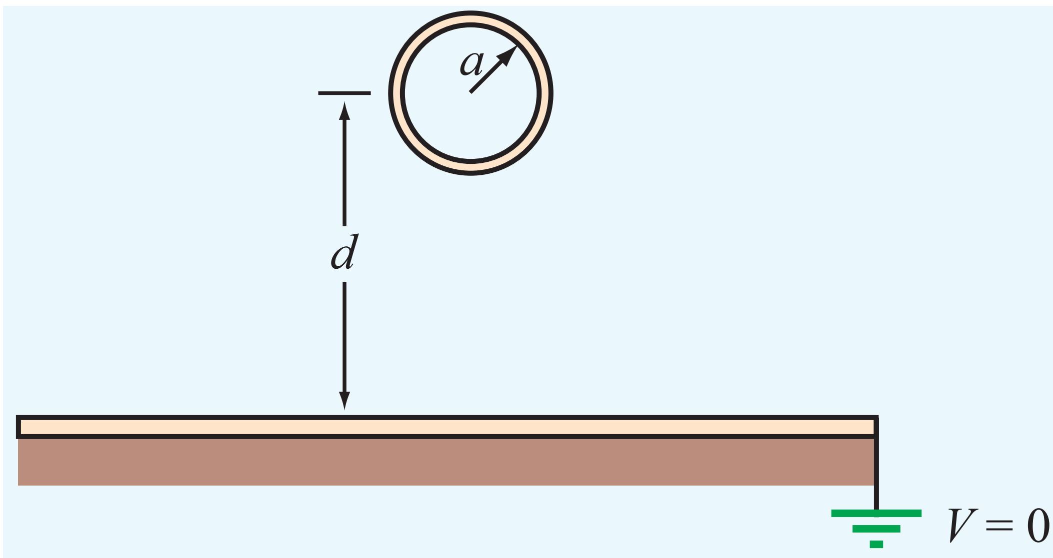

Figure 4.40: Capacitance of an Infinite Cylinder Over a Ground Plane.