Elecmag I - electromagnetics

electrical engineering principles (Technical University of Kenya)

Contents

1 VECTOR ANALYSIS

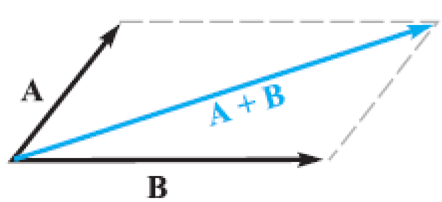
1.1 SCALAR AND VECTOR QUANTITIES

The term *scalar* refers to a quantity whose value may be represented by a single (positive or negative) real number. A *vector* quantity has both magnitude and some direction in space.

Of major concern in the world of Electromagnetics are scalar and vector *fields*. A field (scalar or vector) may be defined mathematically as some function that connects an arbitrary origin to a general point in space. We usually associate some physical effect with a field, such as the force on a compass needle in the earth's magnetic field, or the movement of smoke particles in the field defined by the vector velocity of air in some region of space. The field concept invariably is related to a region. Some quantity is defined at every point in a region. Both *scalar fields* and *vector fields* exist. The temperature throughout a bowl of soup and the density at any point in the earth are examples of scalar fields. The gravitational and magnetic fields of the earth, the voltage gradient in a cable, and the temperature gradient in a soldering-iron tip are examples of vector fields. The value of a field varies with both position and time.

1.2 VECTOR ALGEBRA

Addition of vectors follows the parallelogram law. Vector addition obeys the commutative law; $A + B = B + A$. Vector addition also obeys the associative law; $(A + B) + C = A + (B + C)$



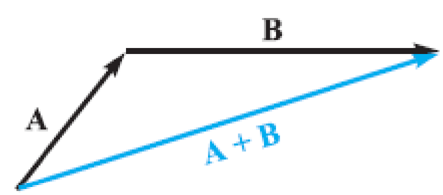


Figure 1.1: Vector addition

Two vectors may be added graphically either by drawing both vectors from a common origin and completing the parallelogram or by beginning the second vector from the head of the first and completing the triangle; either method is easily extended to three or more vectors.

When a vector is drawn as an arrow of finite length, its location is defined to be at the tail end of the arrow.

Coplanar vectors are vectors lying in a common plane, such as those shown in Figure 1.1 above. Both lie in the plane of the paper (printed copy of the notes!) and may be added by expressing each vector in terms of "horizontal" and "vertical" components and then adding the corresponding components. Vectors in three dimensions may likewise be added by expressing the vectors in terms of three components and adding the corresponding components.

The rule for the subtraction of vectors follows easily from that for addition, for we may always express $(\mathrm{A} - \mathrm{B})$ as$(\mathrm{A} + (-\mathrm{B}))$; the sign, or direction, of the second vector is reversed, and this vector is then added to the first by the rule for vector addition. Vectors may be multiplied by scalars. The magnitude of the vector changes, but its direction does not when the scalar is positive, although it reverses direction when multiplied by a negative scalar. Multiplication of a vector by a scalar also obeys the associative and distributive laws of algebra, leading to:

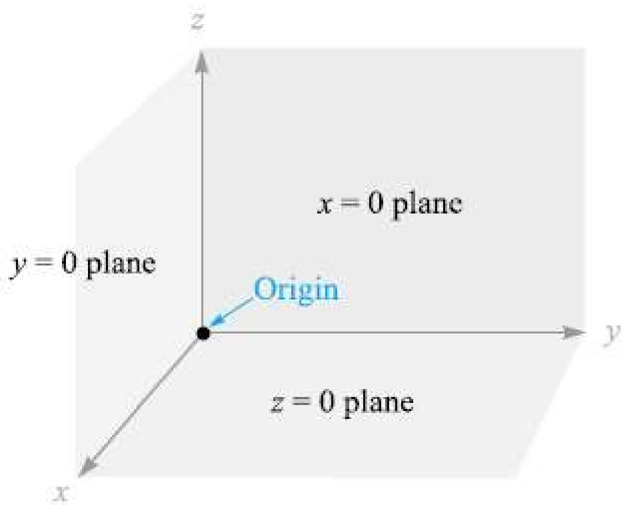
$$(r + s)(\mathbf{A} + \mathbf{B}) = r(\mathbf{A} + \mathbf{B}) + s(\mathbf{A} + \mathbf{B}) = r\mathbf{A} + r\mathbf{B} + s\mathbf{A} + s\mathbf{B}$$

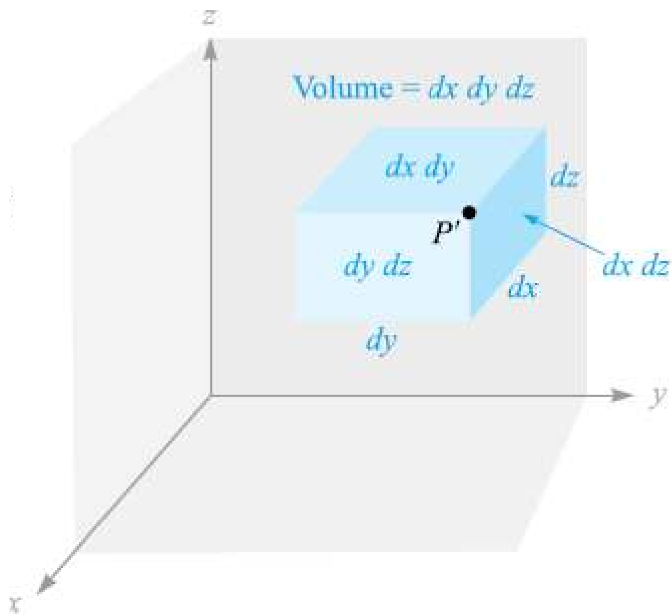
Division of a vector by a scalar is merely multiplication by the reciprocal of that scalar. Two vectors are said to be equal if their difference is zero, or $(\mathrm{A} = \mathrm{B})$ if$(\mathrm{A} - \mathrm{B}) = 0$.

1.3 THE RECTANGULAR COORDINATE SYSTEM

To describe a vector accurately, some specific lengths, directions, angles, projections, or components must be given. The rectangular (or rectangular Cartesian) coordinate system is the most basic vector description technique.

In the rectangular coordinate system we set up three coordinate axes mutually at right angles to each other and call them the x, y, and z axes. It is customary to choose a right-handed coordinate system, in which a rotation of the x axis into the y axis would cause a right-handed screw to progress in the direction of the z axis. If the right hand is used, then the thumb, forefinger, and middle finger may be identified, respectively, as the x, y, and z axes. Figure 1.2 below shows a right-handed rectangular coordinate system. A point is located by giving its x, y, and z coordinates. These are, respectively, the distances from the origin to the intersection of perpendicular lines dropped from the point to the x, y, and z axes.

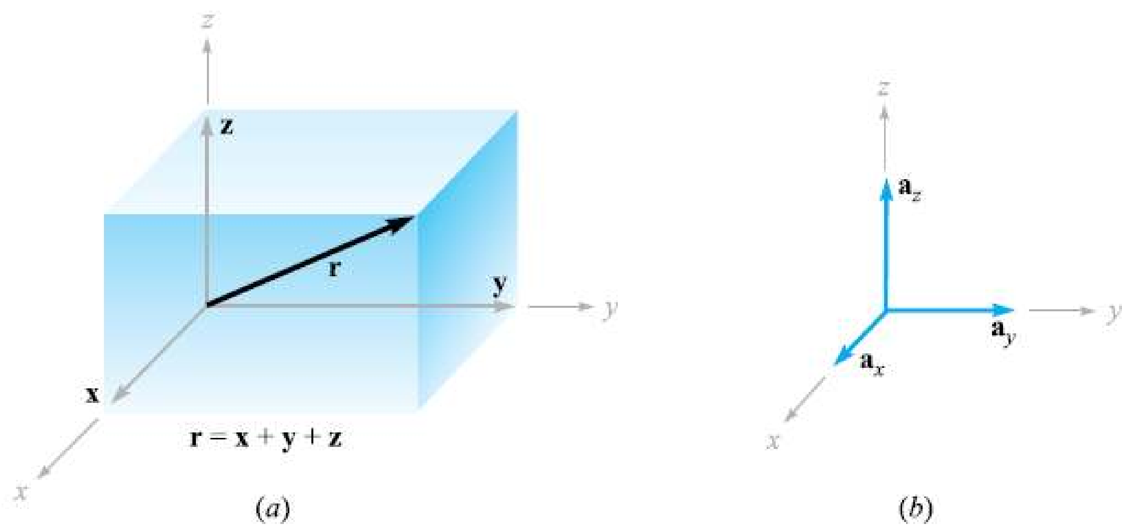




If we visualize three planes intersecting at the general point P, whose coordinates are x, y, and z, we may increase each coordinate value by a differential amount and obtain three slightly displaced planes intersecting at point P', whose coordinates are x +dx, y +dy, and z +dz. The six planes define a rectangular parallelepiped whose volume is dv = dxdydz; the surfaces have differential areas dS of dxdy, dydz, and dzdx. Finally, the distance dL from P to P' is the diagonal of the parallelepiped and has a length of $\sqrt{(dx)^2 + (dy)^2 + (dz)^2}$.

1.4 VECTOR COMPONENTSAND UNIT VECTORS

To describe a vector in the rectangular coordinate system, let us first consider a vector **r** extending outward from the origin. A logical way to identify this vector is by giving the three *component vectors*, lying along the three coordinate axes, whose vector sum must be the given vector. If the component vectors of the vector **r** are **x**, **y**, and **z**, then **r** = **x**+**y**+**z**. The component vectors are shown in Figure (a) below. Instead of one vector, we now have three, but this is a step forward because the three vectors are of a very simple nature; each is always directed along one of the coordinate axes. The component vectors have magnitudes that depend on the given vector (such as **r**), but they each have a known and constant direction. This suggests the use of *unit vectors* having unit magnitude by definition; these are parallel to the coordinate axes and they point in the direction of increasing coordinate values. We reserve the symbol **a** for a unit vector and identify its direction by an appropriate subscript. Thus $\mathbf{a}_x$, $\mathbf{a}_y$, and $\mathbf{a}_z$ are the unit vectors in the rectangular coordinate system. They are directed along the *x, y*, and *z* axes, respectively, as shown in Figure (b) below.



If the component vector y happens to be two units in magnitude, we should then write $\mathbf{y} = 2\mathbf{a}_y$. A vector $\mathbf{R}_P$ pointing from the origin to point P (1, 2, 3) is written $\mathbf{R}_P = \mathbf{a}_x + 2\mathbf{a}_y + 3\mathbf{a}_z$.

Any vector B then may be described by $B = B_x a_x + B_y a_y + B_z a_z$. The magnitude of B is given by:

$$\boxed{|\mathbf{B}| = \sqrt{B_x^2 + B_y^2 + B_z^2}}$$

A unit vector in a given direction is merely a vector in that direction divided by its magnitude. A unit vector in the direction of the vector B is

$$\mathbf{a}_B = \frac{\mathbf{B}}{\sqrt{B_x^2 + B_y^2 + B_z^2}} = \frac{\mathbf{B}}{|\mathbf{B}|}$$

1.5 THE VECTOR FIELD

A vector field is a vector function of a position vector. In general, the magnitude and direction of the function will change as we move throughout the region, and the value of the vector function must be determined using the coordinate values of the point in question. If we again represent the position vector as **r**, then a vector field **G** can be expressed in functional notation as **G**(**r**). If we inspect the velocity of the water in the ocean in some region near the surface where tides and currents are important, we might decide to represent it by a velocity vector that is in any direction, even up or down. If the z axis is taken as upward, the x axis in a northerly direction, the y axis to the west, and the origin at the surface, we have a right-handed coordinate system and may write the velocity vector as $\mathbf{v} = v_x a_x + v_y a_y + v_z a_z$, or $\mathbf{v}(\mathbf{r}) = v_x(\mathbf{r})a_x + v_y(\mathbf{r})a_y + v_z(\mathbf{r})a_z$; each of the components v_x , v_y , and v_z may be a function of the three variables x, y, and z. If we are in some portion of the Indian Ocean where the water is moving only to the north, then v_y and v_z are zero. Further simplifying assumptions might be made if the velocity falls off with depth and changes very slowly as we move north, south, east, or west. A suitable expression could be $v = 2e^{z/100}a_x$. We have a velocity of 2 m/s (meters per second) at the surface and a velocity of 0.736 m/s, at a depth of 100 m ($z = -100$). The velocity continues to decrease with depth, while maintaining a constant direction.

1.6 THE DOT PRODUCT

Given two vectors A and B, the dot product, or scalar product, is defined as the product of the magnitude of A, the magnitude of B, and the cosine of the smaller angle between them.

$$\boxed{\mathbf{A}\cdot\mathbf{B} = |\mathbf{A}|\,|\mathbf{B}|\cos\theta_{AB}}$$

Perhaps the most common application of the dot product is in mechanics, where a constant force **F** applied over a straight displacement **L** does an amount of work FL cos θ, which is more easily written **F** · **L**.

Another example might be taken from magnetic fields. The total flux crossing a surface of area S is given by BS if the magnetic flux density B is perpendicular to the surface and uniform over it. We define a vector surface **S** as having area for its magnitude and having a direction normal to the surface (avoiding for the moment the problem of which of the two possible normals to take). The flux crossing the surface is then **B** · **S**.

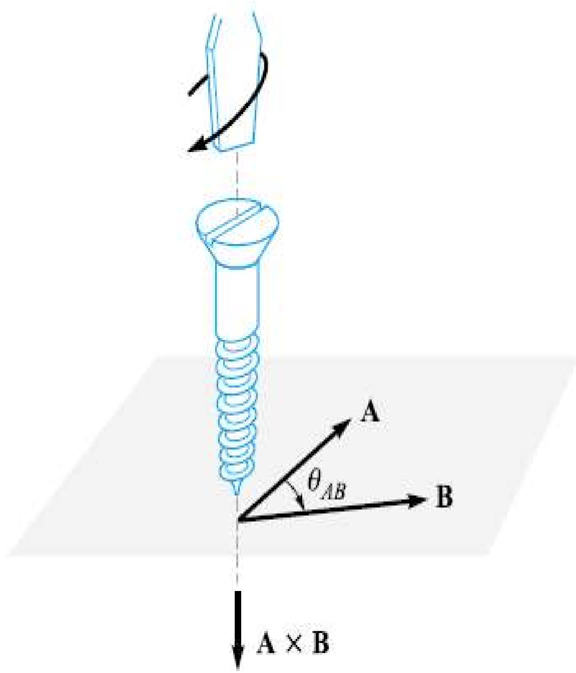
$$\mathbf{a}_x\cdot\mathbf{a}_y = \mathbf{a}_y\cdot\mathbf{a}_x = \mathbf{a}_x\cdot\mathbf{a}_z = \mathbf{a}_z\cdot\mathbf{a}_x = \mathbf{a}_y\cdot\mathbf{a}_z = \mathbf{a}_z\cdot\mathbf{a}_y = 0$$

$$\boxed{\mathbf{A}\cdot\mathbf{B} = A_xB_x + A_yB_y + A_zB_z}$$

$$\boxed{\mathbf{A}\cdot\mathbf{A} = A^2 = |\mathbf{A}|^2}$$

1.7 THE CROSS PRODUCT

Given two vectors **A** and **B**, we now define the cross product, or vector product, of A and B, written with a cross between the two vectors as **A** × **B** and read "A cross B." The cross product **A** × **B** is a vector; the magnitude of **A** × **B** is equal to the product of the magnitudes of A, B, and the sine of the smaller angle between **A** and **B**; the direction of **A**×**B** is perpendicular to the plane containing **A** and **B** and is along one of the two possible perpendiculars which is in the direction of advance of a right-handed screw as **A** is turned into **B**. This direction is illustrated in Figure below. Remember that either vector may be moved about at will, maintaining its direction constant, until the two vectors have a "common origin." This determines the plane containing both.



$$\mathbf{A} \times \mathbf{B} = \mathbf{a}_N |\mathbf{A}|\,|\mathbf{B}| \sin\theta_{AB}$$

B×**A** = − (**A**×**B**)

$\mathbf{a}_x \times \mathbf{a}_y = \mathbf{a}_z$ $\mathbf{a}_y \times \mathbf{a}_z = \mathbf{a}_x$ $\mathbf{a}_z \times \mathbf{a}_x = \mathbf{a}_y$

The cross product of any vector with itself is zero.

$$\begin{aligned}\mathbf{A} \times \mathbf{B} = {} & A_x B_x \mathbf{a}_x \times \mathbf{a}_x + A_x B_y \mathbf{a}_x \times \mathbf{a}_y + A_x B_z \mathbf{a}_x \times \mathbf{a}_z \\ & + A_y B_x \mathbf{a}_y \times \mathbf{a}_x + A_y B_y \mathbf{a}_y \times \mathbf{a}_y + A_y B_z \mathbf{a}_y \times \mathbf{a}_z \\ & + A_z B_x \mathbf{a}_z \times \mathbf{a}_x + A_z B_y \mathbf{a}_z \times \mathbf{a}_y + A_z B_z \mathbf{a}_z \times \mathbf{a}_z\end{aligned}$$

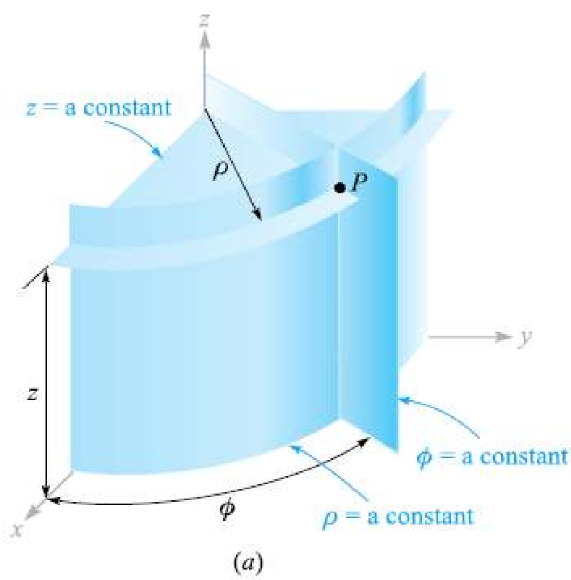
Results in:

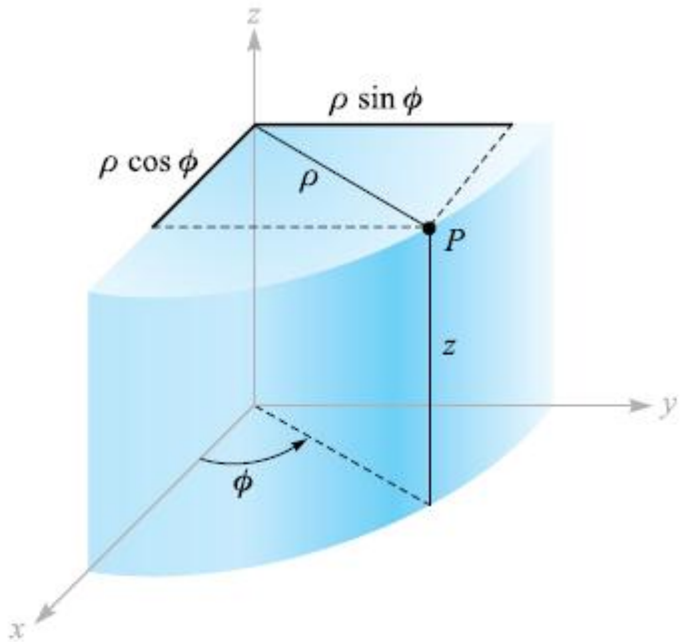
$$\mathbf{A} \times \mathbf{B} = (A_y B_z - A_z B_y)\mathbf{a}_x + (A_z B_x - A_x B_z)\mathbf{a}_y + (A_x B_y - A_y B_x)\mathbf{a}_z$$

Or written as a determinant in an easily remembered form;

$$\boxed{\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix}}$$

1.8 CYLINDRICAL COORDINATE SYSTEM





$$x = \rho \cos\phi$$
$$y = \rho \sin\phi$$
$$z = z$$

$$\rho = \sqrt{x^2 + y^2}$$
$$\phi = \tan^{-1}\frac{y}{x}$$
$$z = z$$

Given

$$\mathbf{A} = A_x\mathbf{a}_x + A_y\mathbf{a}_y + A_z\mathbf{a}_z$$

To obtain

$$\mathbf{A} = A_\rho\mathbf{a}_\rho + A_\phi\mathbf{a}_\phi + A_z\mathbf{a}_z$$

Recall

$$A_\rho = \mathbf{A}\cdot\mathbf{a}_\rho \qquad \text{and} \qquad A_\phi = \mathbf{A}\cdot\mathbf{a}_\phi$$

Then:

$$A_\rho = (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z) \cdot \mathbf{a}_\rho = A_x \mathbf{a}_x \cdot \mathbf{a}_\rho + A_y \mathbf{a}_y \cdot \mathbf{a}_\rho$$
$$A_\phi = (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z) \cdot \mathbf{a}_\phi = A_x \mathbf{a}_x \cdot \mathbf{a}_\phi + A_y \mathbf{a}_y \cdot \mathbf{a}_\phi$$

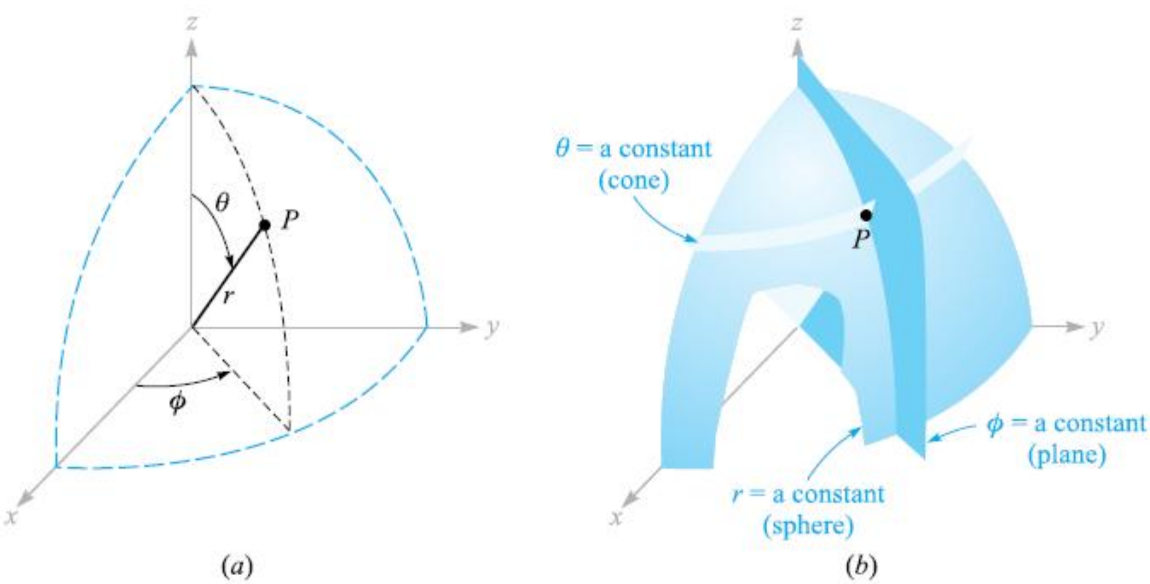
since $\mathbf{a}_z \cdot \mathbf{a}_\rho$ and $\mathbf{a}_z \cdot \mathbf{a}_\phi$ are zero

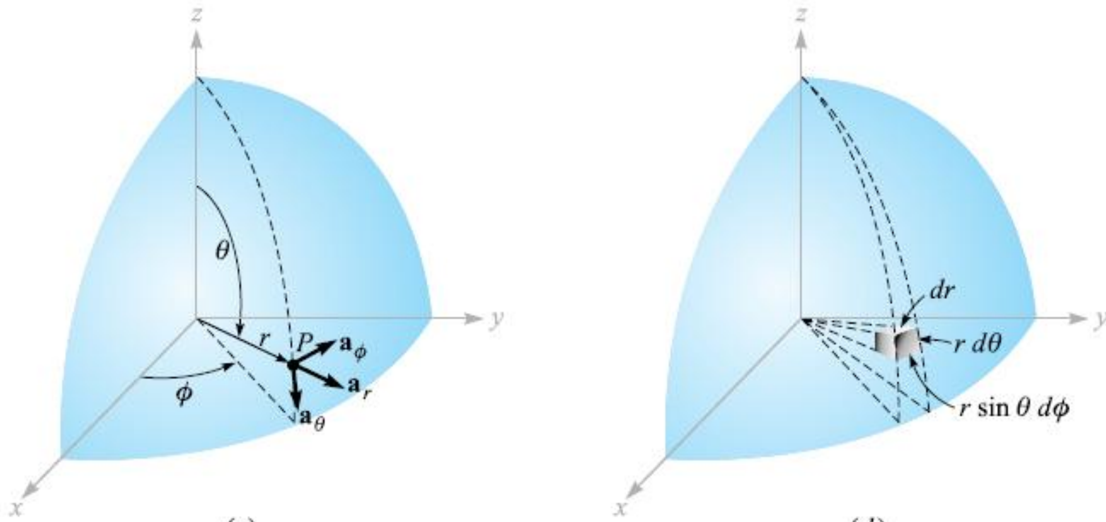
Also,

$$A_z = (A_x \mathbf{a}_x + A_y \mathbf{a}_y + A_z \mathbf{a}_z) \cdot \mathbf{a}_z = A_z \mathbf{a}_z \cdot \mathbf{a}_z = A_z$$

	$\mathbf{a}_\rho$	$\mathbf{a}_\phi$	$\mathbf{a}_z$
$\mathbf{a}_x \cdot$	$\cos\phi$	$-\sin\phi$	0
$\mathbf{a}_y \cdot$	$\sin\phi$	$\cos\phi$	0
$\mathbf{a}_z \cdot$	0	0	1

1.9 THE SPHERICAL COORDINATE SYSTEM





(c) (d)

$$x = r \sin\theta \cos\phi$$
$$y = r \sin\theta \sin\phi$$
$$z = r \cos\theta$$

$$r = \sqrt{x^2 + y^2 + z^2} \qquad (r \geq 0)$$
$$\theta = \cos^{-1} \frac{z}{\sqrt{x^2 + y^2 + z^2}} \qquad (0^\circ \leq \theta \leq 180^\circ)$$
$$\phi = \tan^{-1} \frac{y}{x}$$

	$\mathbf{a}_r$	$\mathbf{a}_\theta$	$\mathbf{a}_\phi$
$\mathbf{a}_x \cdot$	$\sin\theta \cos\phi$	$\cos\theta \cos\phi$	$-\sin\phi$
$\mathbf{a}_y \cdot$	$\sin\theta \sin\phi$	$\cos\theta \sin\phi$	$\cos\phi$
$\mathbf{a}_z \cdot$	$\cos\theta$	$-\sin\theta$	0

The dot products involving a_x and a_y require first the projection of the spherical unit vector on the x-y plane and then the projection onto the desired axis.

2 Coulomb's Law and Electric Field Intensity

2.1 THE EXPERIMENTAL LAW OF COULOMB

Records from at least 600 B.C. show evidence of the knowledge of static electricity. The Greeks were responsible for the term electricity, derived from their word for amber, and they spent many leisure hours rubbing a small piece of amber on their sleeves and observing how it would then attract pieces of fluff and stuff. However, their main interest lay in philosophy and logic, not in experimental science, and it was many centuries before the attracting effect was considered to be anything other than magic or a "life force." Dr. Gilbert, physician to Her Majesty the Queen of England, was the first to do any true experimental work with this effect, and in 1600 he stated that glass, sulfur, amber, and other materials, which he named, would "not only draw to themselves straws and chaff, but all metals, wood, leaves, stone, earths, even water and oil." Shortly thereafter, an officer in the French Army Engineers, Colonel Charles Coulomb, performed an elaborate series of experiments using a delicate torsion balance, invented by him, to determine quantitatively the force exerted between two objects, each having a static charge of electricity. His published result bears a great similarity to Newton's gravitational law (discovered about a hundred years earlier).

Coulomb stated that the force between two very small objects separated in a vacuum or free space by a distance, which is large compared to their size, is proportional to the charge on each and inversely proportional to the square of the distance between them, or

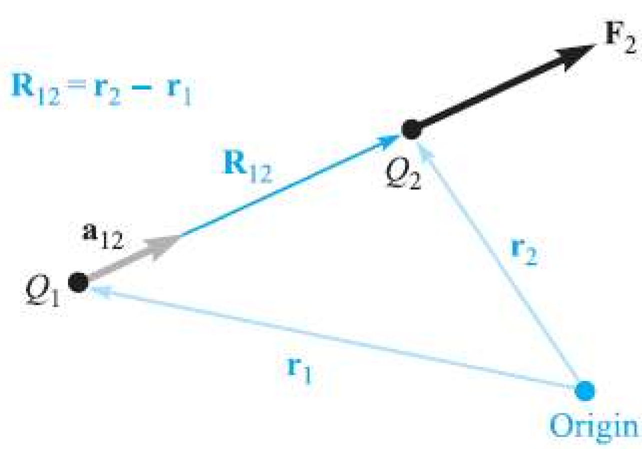
$$F = k\frac{Q_1 Q_2}{R^2} \qquad k = \frac{1}{4\pi\epsilon_0} \qquad \boxed{\epsilon_0 = 8.854 \times 10^{-12} \doteq \frac{1}{36\pi}10^{-9}\ \text{F/m}}$$

where Q_1 and Q_2 are the positive or negative quantities of charge and R is the separation distance.

The coulomb is an extremely large unit of charge, for the smallest known quantity of charge is that of the electron (negative) or proton (positive), given in SI units as 1.602×10^{-19} C; hence a negative charge of one coulomb represents about 6×10^{18} electrons. Coulomb's law shows that the force between two charges of one coulomb each, separated by one meter, is 9×10^{9} N, or about one million tons. The electron has a rest mass of 9.109×10^{-31} kg and has a radius of the order of magnitude of 3.8×10^{-15} m. This does not mean that the electron is spherical in shape, but merely describes the size of the region in which a slowly moving electron has the greatest probability of being found. All other known charged particles, including the proton, have larger masses and larger radii, and occupy a probabilistic volume larger than does the electron.

In order to write the vector form of Coulomb's law, we need the additional fact (furnished also by Colonel Coulomb) that the force acts along the line joining the two charges and is repulsive if the charges are alike in sign or attractive if they are of opposite sign.

Let the vector $\mathbf{r}1$ locate $Q1$, whereas $\mathbf{r}2$ locates $Q2$. Then the vector $\mathbf{R}12 = \mathbf{r}2 - \mathbf{r}1$ represents the directed line segment from $Q1$ to $Q2$, as shown in Figure below. The vector $\mathbf{F}2$ is the force on $Q2$ and is shown for the case where $Q1$ and $Q2$ have the same sign.



The vector form of Coulomb's law is

$$\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12}$$

where $\mathbf{a}_{12}$ = a unit vector in the direction of R_{12}, or

$$\mathbf{a}_{12} = \frac{\mathbf{R}_{12}}{|\mathbf{R}_{12}|} = \frac{\mathbf{R}_{12}}{R_{12}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{|\mathbf{r}_2 - \mathbf{r}_1|}$$

EXAMPLE:

We illustrate the use of the vector form of Coulomb's law by locating a charge of $Q_1 = 3 \times 10^{-4}$ C at $M(1, 2, 3)$ and a charge of $Q_2 = -10^{-4}$ C at $N(2, 0, 5)$ in a vacuum. We desire the force exerted on Q_2 by Q_1.

Solution. We use (3) and (4) to obtain the vector force. The vector $\mathbf{R}_{12}$ is

$$\mathbf{R}_{12} = \mathbf{r}_2 - \mathbf{r}_1 = (2-1)\mathbf{a}_x + (0-2)\mathbf{a}_y + (5-3)\mathbf{a}_z = \mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z$$

leading to $|\mathbf{R}_{12}| = 3$, and the unit vector, $\mathbf{a}_{12} = \frac{1}{3}(\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z)$. Thus,

$$\mathbf{F}_2 = \frac{3 \times 10^{-4}(-10^{-4})}{4\pi(1/36\pi)10^{-9} \times 3^2}\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right)$$
$$= -30\left(\frac{\mathbf{a}_x - 2\mathbf{a}_y + 2\mathbf{a}_z}{3}\right) \text{ N}$$

The magnitude of the force is 30 N, and the direction is specified by the unit vector, which has been left in parentheses to display the magnitude of the force. The force on Q_2 may also be considered as three component forces,

$$\mathbf{F}_2 = -10\mathbf{a}_x + 20\mathbf{a}_y - 20\mathbf{a}_z$$

The force expressed by Coulomb's law is a mutual force, for each of the two charges experiences a force of the same magnitude, although of opposite direction.

We might equally well have written

$$\boxed{\mathbf{F}_1 = -\mathbf{F}_2 = \frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{21} = -\frac{Q_1 Q_2}{4\pi\epsilon_0 R_{12}^2}\mathbf{a}_{12}}$$

Coulomb's law is linear, for if we multiply $Q1$ by a factor n, the force on $Q2$ is also multiplied by the same factor n. It is also true that the force on a charge in the presence of several other charges is the sum of the forces on that charge due to each of the other charges acting alone.

D2.1. A charge $Q_A = -20\,\mu$C is located at $A(-6, 4, 7)$, and a charge $Q_B = 50\,\mu$C is at $B(5, 8, -2)$ in free space. If distances are given in meters, find: (*a*) $\mathbf{R}_{AB}$; (*b*) R_{AB}. Determine the vector force exerted on Q_A by Q_B if $\epsilon_0 =$ (*c*) $10^{-9}/(36\pi)$ F/m; (*d*) 8.854×10^{-12} F/m.

Ans. $11\mathbf{a}_x + 4\mathbf{a}_y - 9\mathbf{a}_z$ m; 14.76 m; $30.76\mathbf{a}_x + 11.184\mathbf{a}_y - 25.16\mathbf{a}_z$ mN; $30.72\mathbf{a}_x + 11.169\mathbf{a}_y - 25.13\mathbf{a}_z$ mN

2.2 ELECTRIC FIELD INTENSITY

If we now consider one charge fixed in position, say Q_1, and move a second charge slowly around, we note that there exists everywhere a force on this second charge; in other words, this second charge is displaying the existence of a force field that is associated with charge, Q_1. Call this second charge a test charge Q_t. The force on it is given by Coulomb's law,

$$\mathbf{F}_t = \frac{Q_1 Q_t}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

Writing this force as a force per unit charge gives the electric field intensity, $\mathbf{E}_1$ arising from Q_1:

$$\mathbf{E}_1 = \frac{\mathbf{F}_t}{Q_1} = \frac{Q_1}{4\pi\epsilon_0 R_{1t}^2}\mathbf{a}_{1t}$$

$\mathbf{E}_1$ is interpreted as the vector force, arising from charge Q_1 that acts on a unit positive test charge. More generally, we write the defining expression:

$$\boxed{\mathbf{E} = \frac{\mathbf{F}_t}{Q_t}} \qquad \boxed{\mathbf{E} = \frac{Q}{4\pi\epsilon_0 R^2}\mathbf{a}_R}$$

We remember that R is the magnitude of the vector **R**, the directed line segment from the point at which the point charge Q is located to the point at which **E** is desired, and **a**R is a unit vector in the **R** direction.

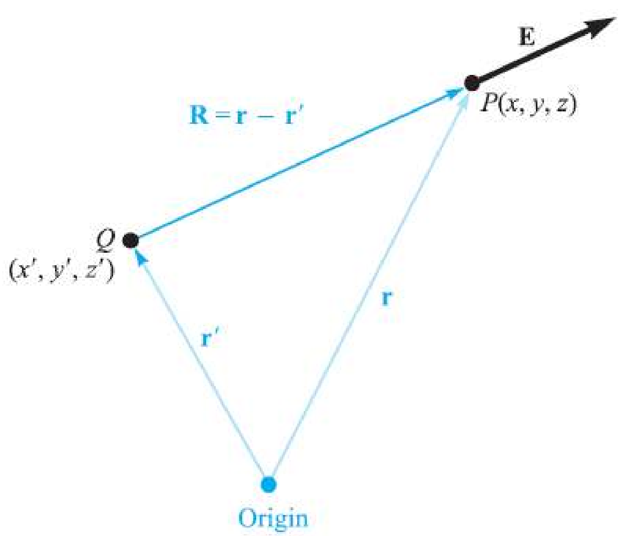
We arbitrarily locate $Q1$ at the center of a spherical coordinate system. The unit vector **a**R then becomes the radial unit vector **a**r , and R is r . Hence

$$\mathbf{E} = \frac{Q_1}{4\pi\epsilon_0 r^2}\mathbf{a}_r$$

The field has a single radial component, and its inverse-square-law relationship is quite obvious.

If we consider a charge that is *not* at the origin of our coordinate system, the field no longer possesses spherical symmetry, and we might as well use rectangular coordinates. For a charge Q located at the source point **r**', as illustrated in Figure below, we find the field at a general field point **r** by expressing **R** as **r** – **r**', and then

$$\mathbf{E}(\mathbf{r}) = \frac{Q}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^2}\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|} = \frac{Q(\mathbf{r}-\mathbf{r}')}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}'|^3}$$
$$= \frac{Q[(x-x')\mathbf{a}_x + (y-y')\mathbf{a}_y + (z-z')\mathbf{a}_z]}{4\pi\epsilon_0[(x-x')^2+(y-y')^2+(z-z')^2]^{3/2}}$$

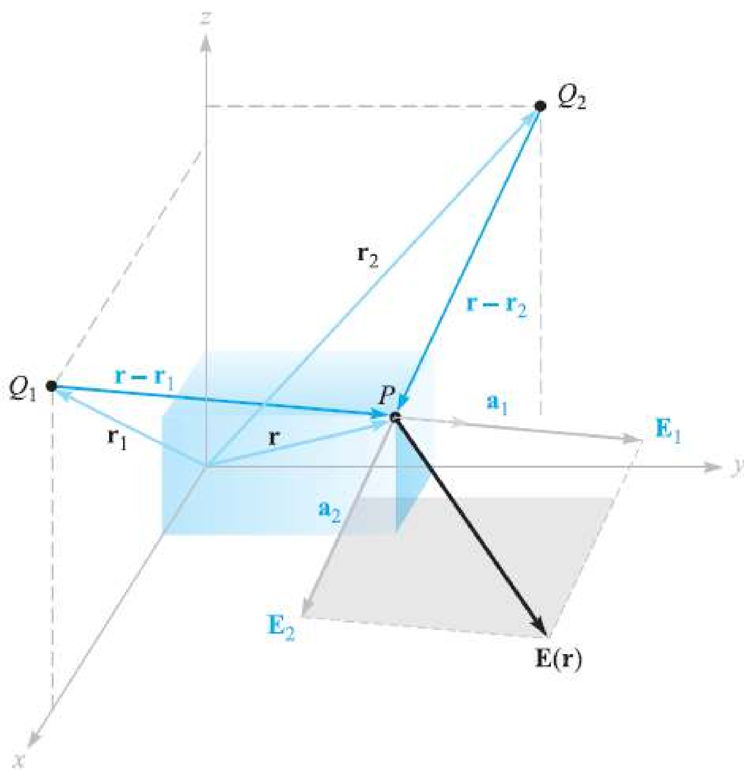


Earlier, we defined a vector field as a vector function of a position vector, and this is emphasized by letting **E** be symbolized in functional notation by **E**(**r**).

Because the coulomb forces are linear, the electric field intensity arising from two point charges, *Q*1 at **r**1 and *Q*2 at **r**2, is the sum of the forces on *Qt* caused by *Q*1 and *Q*2 acting alone, or

$$\mathbf{E}(\mathbf{r}) = \frac{Q_1}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_1|^2}\mathbf{a}_1 + \frac{Q_2}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_2|^2}\mathbf{a}_2$$

where **a**1 and **a**2 are unit vectors in the direction of (**r**–**r**1) and (**r**–**r**2), respectively.



If we add more charges at other positions, the field due to *n* point charges is:

$$\mathbf{E}(\mathbf{r}) = \sum_{m=1}^{n} \frac{Q_m}{4\pi\epsilon_0|\mathbf{r}-\mathbf{r}_m|^2}\mathbf{a}_m$$

EXAMPLE

In order to illustrate the application of (11), we find $\mathbf{E}$ at $P(1, 1, 1)$ caused by four identical 3-nC (nanocoulomb) charges located at $P_1(1, 1, 0)$, $P_2(-1, 1, 0)$, $P_3(-1, -1, 0)$, and $P_4(1, -1, 0)$, as shown in Figure 2.4.

Solution. We find that $\mathbf{r} = \mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z$, $\mathbf{r}_1 = \mathbf{a}_x + \mathbf{a}_y$, and thus $\mathbf{r} - \mathbf{r}_1 = \mathbf{a}_z$. The magnitudes are: $|\mathbf{r} - \mathbf{r}_1| = 1$, $|\mathbf{r} - \mathbf{r}_2| = \sqrt{5}$, $|\mathbf{r} - \mathbf{r}_3| = 3$, and $|\mathbf{r} - \mathbf{r}_4| = \sqrt{5}$. Because $Q/4\pi\epsilon_0 = 3 \times 10^{-9}/(4\pi \times 8.854 \times 10^{-12}) = 26.96$ V · m, we may now use (11) to obtain

$$\mathbf{E} = 26.96 \left[\frac{\mathbf{a}_z}{1}\frac{1}{1^2} + \frac{2\mathbf{a}_x + \mathbf{a}_z}{\sqrt{5}} \frac{1}{(\sqrt{5})^2} + \frac{2\mathbf{a}_x + 2\mathbf{a}_y + \mathbf{a}_z}{3}\frac{1}{3^2} + \frac{2\mathbf{a}_y + \mathbf{a}_z}{\sqrt{5}}\frac{1}{(\sqrt{5})^2} \right]$$

or

$$\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z \text{ V/m}$$

D2.2. A charge of $-0.3\,\mu$C is located at $A(25, -30, 15)$ (in cm), and a second charge of $0.5\,\mu$C is at $B(-10, 8, 12)$ cm. Find $\mathbf{E}$ at: (*a*) the origin; (*b*) $P(15, 20, 50)$ cm.

Ans. $92.3\mathbf{a}_x - 77.6\mathbf{a}_y - 94.2\mathbf{a}_z$ kV/m; $11.9\mathbf{a}_x - 0.519\mathbf{a}_y + 12.4\mathbf{a}_z$ kV/m

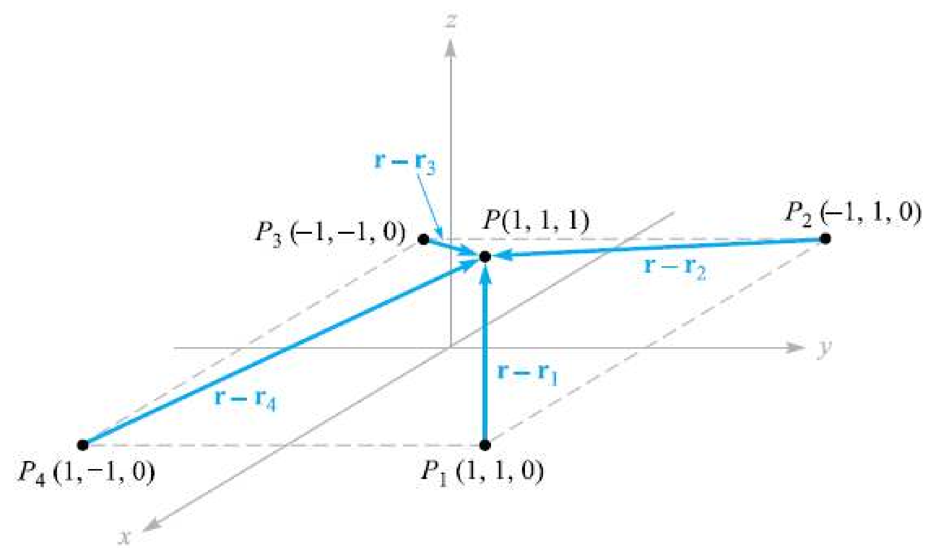


Figure 2.4 A symmetrical distribution of four identical 3-nC point charges produces a field at P, $\mathbf{E} = 6.82\mathbf{a}_x + 6.82\mathbf{a}_y + 32.8\mathbf{a}_z$ V/m.

D2.3. Evaluate the sums: (*a*) $\sum_{m=0}^{5} \frac{1+(-1)^m}{m^2+1}$; (*b*) $\sum_{m=1}^{4} \frac{(0.1)^m+1}{(4+m^2)^{1.5}}$

Ans. 2.52; 0.176

2.3 FIELD ARISING FROM A CONTINUOUS VOLUME CHARGE DISTRIBUTION

If we now visualize a region of space filled with a tremendous number of charges separated by minute distances, we see that we can replace this distribution of very small particles with a smooth continuous distribution described by a *volume charge density*, just as we describe water as having a density of 1 g/cm^3 (gram per cubic centimeter) even though it consists of atomic- and molecular-sized particles. We can do this only if we are uninterested in the small irregularities (or ripples) in the field as we move from electron to electron or if we care little that the mass of the water actually increases in small but finite steps as each new molecule is added.

This is really no limitation at all, because the end results for electrical engineers are almost always in terms of a current in a receiving antenna, a voltage in an electronic circuit, or a charge on a capacitor, or in general in terms of some large-scale *macroscopic* phenomenon. It is very seldom that we must know a current electron by electron.

We denote volume charge density by ρ_v, having the units of coulombs per cubic meter (C/m^3).

The small amount of charge ΔQ in a small volume Δv is

$$\Delta Q = \rho_v \Delta v$$

$$\boxed{\rho_v = \lim_{\Delta v \to 0} \frac{\Delta Q}{\Delta v}}$$

The total charge within some finite volume is obtained by integrating the corresponding *volume charge density* throughout that volume,

$$\boxed{Q = \int_{\text{vol}} \rho_v \, dv}$$

Only one integral sign is customarily indicated, but the differential *dv* signifies integration throughout a volume, and hence a triple integration.

As an example of the evaluation of a volume integral, we find the total charge contained in a 2-cm length of the electron beam shown in Figure 2.5.

Solution. From the illustration, we see that the charge density is

$$\rho_v = -5 \times 10^{-6} e^{-10^5 \rho z} \ \mathrm{C/m^2}$$

The volume differential in cylindrical coordinates is given in Section 1.8; therefore,

$$Q = \int_{0.02}^{0.04} \int_{0}^{2\pi} \int_{0}^{0.01} -5 \times 10^{-6} e^{-10^5 \rho z} \rho \, d\rho \, d\phi \, dz$$

We integrate first with respect to ϕ because it is so easy,

$$Q = \int_{0.02}^{0.04} \int_{0}^{0.01} -10^{-5} \pi e^{-10^5 \rho z} \rho \, d\rho \, dz$$

and then with respect to z, because this will simplify the last integration with respect to ρ,

$$Q = \int_{0}^{0.01} \left(\frac{-10^{-5}\pi}{-10^5 \rho} e^{-10^5 \rho z} \rho \, d\rho \right)_{z=0.02}^{z=0.04}$$

$$= \int_{0}^{0.01} -10^{-5}\pi (e^{-2000\rho} - e^{-4000\rho}) d\rho$$

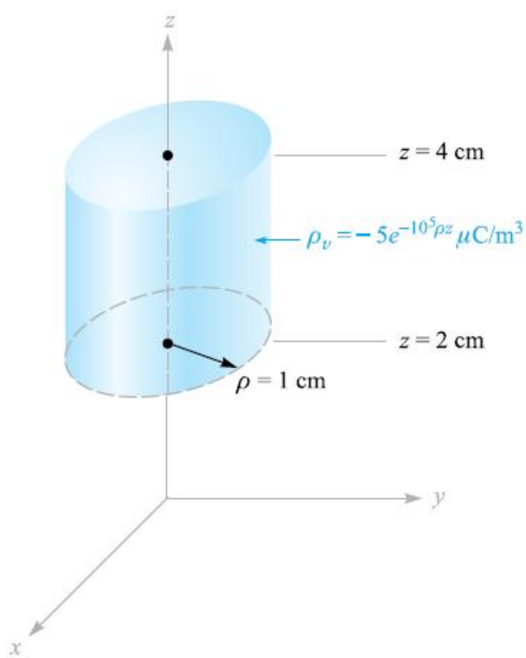


Figure 2.5 The total charge contained within the right circular cylinder may be obtained by evaluating $Q = \int_{\mathrm{vol}} \rho_v dv$.

Finally,

$$Q = -10^{-10}\pi \left(\frac{e^{-2000\rho}}{-2000} - \frac{e^{-4000\rho}}{-4000} \right)_0^{0.01}$$

$$Q = -10^{-10}\pi \left(\frac{1}{2000} - \frac{1}{4000} \right) = \frac{-\pi}{40} = 0.0785 \text{ pC}$$

where pC indicates picocoulombs.

The incremental contribution to the electric field intensity at $\mathbf{r}$ produced by an incremental charge ΔQ at $\mathbf{r}'$ is

$$\Delta \mathbf{E}(\mathbf{r}) = \frac{\Delta Q}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} = \frac{\rho_v \Delta v}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|}$$

If we sum the contributions of all the volume charge in a given region and let the volume element Δv approach zero as the number of these elements becomes infinite, the summation becomes an integral,

$$\mathbf{E}(\mathbf{r}) = \int_{\text{vol}} \frac{\rho_v(\mathbf{r}')\, dv'}{4\pi\epsilon_0 |\mathbf{r} - \mathbf{r}'|^2} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|} \tag{15}$$

This is again a triple integral, and (except in Drill Problem 2.4) we shall do our best to avoid actually performing the integration.

The significance of the various quantities under the integral sign of (15) might stand a little review. The vector $\mathbf{r}$ from the origin locates the field point where $\mathbf{E}$ is being determined, whereas the vector $\mathbf{r}'$ extends from the origin to the source point where $\rho_v(\mathbf{r}')dv'$ is located. The scalar distance between the source point and the field point is $|\mathbf{r} - \mathbf{r}'|$, and the fraction $(\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|$ is a unit vector directed from source point to field point. The variables of integration are x', y', and z' in rectangular coordinates.

D2.4. Calculate the total charge within each of the indicated volumes: $(a)\, 0.1 \le |x|, |y|, |z| \le 0.2$: $\rho_v = \dfrac{1}{x^3y^3z^3}$; $(b)\, 0 \le \rho \le 0.1, 0 \le \phi \le \pi, 2 \le z \le 4$; $\rho_v = \rho^2 z^2 \sin 0.6\phi$; (c) universe: $\rho_v = e^{-2r}/r^2$.

Ans. 0; 1.018 mC; 6.28 C

3 Electric Flux Density, Gauss's Law, and Divergence

Mmmh… the concept of having drawn lines depicting the direction of the force on a test charge at every point…….. it is difficult to avoid giving these lines a physical significance and thinking of them as *flux* lines. No physical particle is projected radially outward from the point charge, and there are no steel tentacles reaching out to attract or repel an unwary test charge, but as soon as the streamlines are drawn on paper there seems to be a picture showing “something” is present.

It is very helpful to invent an *electric flux* that streams away symmetrically from a point charge and is coincident with the streamlines and to visualize this flux wherever an electric field is present.

3.1 ELECTRIC FLUX DENSITY

About 1837, the director of the Royal Society in London, Michael Faraday, became very interested in static electric fields and the effect of various insulating materials on these fields. This problem had been bothering him during the past ten years when he was experimenting in his now-famous work on induced electromotive force. With that subject completed, he had a pair of concentric metallic spheres constructed, the outer one consisting of two hemispheres that could be firmly clamped together. He also prepared shells of insulating material (or dielectric material or simply *dielectric*) that would occupy the entire volume between the concentric spheres. We will immediately use his findings about dielectric materials.

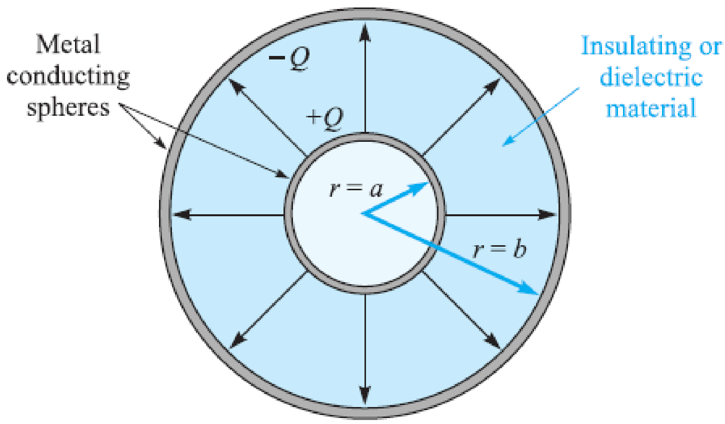
His experiment, then, consisted essentially of the following steps:

1. With the equipment dismantled, the inner sphere was given a known positive charge.

2. The hemispheres were then clamped together around the charged sphere with about 2 cm of dielectric material between them.

3. The outer sphere was discharged by connecting it momentarily to ground.

4. The outer space was separated carefully, using tools made of insulating material in order not to disturb the induced charge on it, and the negative induced charge on each hemisphere was measured.



Faraday found that the total charge on the outer sphere was equal in *magnitude* to the original charge placed on the inner sphere and that this was true regardless of the dielectric material separating the two spheres. He concluded that there was some sort of "displacement" from the inner sphere to the outer which was independent of the medium, and we now refer to this flux as *displacement*, *displacement flux,* or simply *electric flux*.

Faraday's experiments also showed, of course, that a larger positive charge on the inner sphere induced a correspondingly larger negative charge on the outer sphere, leading to a direct proportionality between the electric flux and the charge on the inner sphere.

If electric flux is denoted by Ψ (psi) and the total charge on the inner sphere by Q, then for Faraday's experiment

$$\boxed{\Psi = Q}$$

Electric flux density **D** (sometimes described as "flux lines per square meter," for each line is due to one coulomb) at a point r meters from a point charge Q is given by:

$$\boxed{\mathbf{D} = \frac{Q}{4\pi r^2}\mathbf{a}_r}$$

The electric flux density **D** is a vector field and is a member of the "flux density" class of vector fields, as opposed to the "force fields" class, which includes the electric field intensity **E**. The direction of **D** at a point is the direction of the flux lines at that point, and the magnitude is given by the number of flux lines crossing a surface normal to the lines divided by the surface area.

Comparing **D** and **E**:

$$\boxed{\mathbf{E} = \frac{Q}{4\pi \epsilon_0 r^2}\mathbf{a}_r}$$

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E}} \qquad \text{(free space only)}$$

$$\boxed{\mathbf{D} = \int_{\text{vol}} \frac{\rho_v dv}{4\pi R^2}\mathbf{a}_R}$$

D3.1. Given a 60-µC point charge located at the origin, find the total electric flux passing through: (*a*) that portion of the sphere $r = 26$ cm bounded by $0 < \theta < \frac{\pi}{2}$ and $0 < \phi < \frac{\pi}{2}$; (*b*) the closed surface defined by $\rho = 26$ cm and $z = \pm 26$ cm; (*c*) the plane $z = 26$ cm.

Ans. 7.5 µC; 60 µC; 30 µC

D3.2. Calculate **D** in rectangular coordinates at point $P(2, -3, 6)$ produced by: (*a*) a point charge $Q_A = 55$ mC at $Q(-2, 3, -6)$; (*b*) a uniform line charge $\rho_{LB} = 20$ mC/m on the x axis; (*c*) a uniform surface charge density $\rho_{SC} = 120\,\mu\text{C/m}^2$ on the plane $z = -5$ m.

Ans. $6.38\mathbf{a}_x - 9.57\mathbf{a}_y + 19.14\mathbf{a}_z\ \mu\text{C/m}^2$; $-212\mathbf{a}_y + 424\mathbf{a}_z\ \mu\text{C/m}^2$; $60\mathbf{a}_z\ \mu\text{C/m}^2$

3.2 GAUSS'S LAW

The results of Faraday's experiments with the concentric spheres could be summed up as an experimental law by stating that the electric flux passing through any imaginary spherical surface lying between the two conducting spheres is equal to the charge enclosed within that imaginary surface. This enclosed charge is distributed on the surface of the inner sphere, or it might be concentrated as a point charge at the center of the imaginary sphere. However, because one coulomb of electric flux is produced by one coulomb of charge, the inner conductor might just as well have been a cube or a brass door key and the total induced charge on the outer sphere would still be the same.

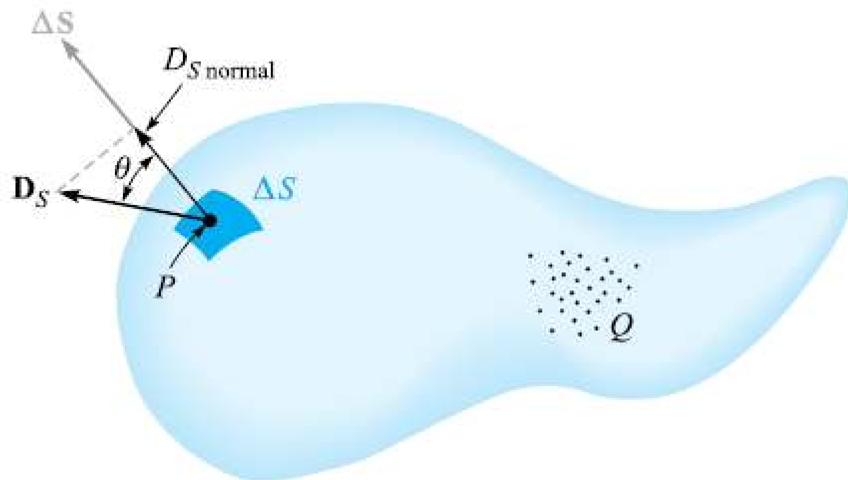
Certainly the flux density would change from its previous symmetrical distribution to some unknown configuration, but +Q coulombs on any inner conductor would produce an induced charge of −Q coulombs on the surrounding sphere. Going one step further, we could now replace the two outer hemispheres by an empty (but completely closed) soup can. Q coulombs on the brass door key would produce $\Psi = Q$ lines of electric flux and would induce −Q coulombs on the tin can. These generalizations of Faraday's experiment lead to the following statement, which is known as Gauss's law:

The electric flux passing through any closed surface is equal to the total charge enclosed by that surface.

The contribution of Gauss, one of the greatest mathematicians the world has ever produced, was actually not in stating the law as we have, but in providing a mathematical form for this statement, which we will now obtain.

Let us imagine a distribution of charge, shown as a cloud of point charges in Figure below, surrounded by a closed surface of any shape. The closed surface may be the surface of some real material, but more generally it is any closed surface we wish to visualize. If the total charge is Q, then Q coulombs of electric flux will pass through the enclosing surface. At every point on the surface the electric-flux-density vector **D** will have some value $\mathbf{D}_S$, where the subscript S merely reminds us that **D** must be evaluated at the surface, and $\mathbf{D}_S$ will in general vary in magnitude and direction from one point on the surface to another.

We must now consider the nature of an incremental element of the surface. An incremental element of area ΔS is very nearly a portion of a plane surface, and the complete description of this surface element requires not only a statement of its magnitude ΔS but also of its orientation in space. In other words, the incremental surface element is a vector quantity. The only unique direction that may be associated with $\Delta\mathbf{S}$ is the direction of the normal to that plane which is tangent to the surface at the point in question. There are, of course, two such normals, and the ambiguity is removed by specifying the outward normal whenever the surface is closed and "outward" has a specific meaning.



At any point P, consider an incremental element of surface ΔS and let $\mathbf{D}S$ make an angle θ with $\Delta\mathbf{S}$, as shown in Figure above. The flux crossing ΔS is then the product of the normal component of $\mathbf{D}_S$ and $\Delta\mathbf{S}$,

$$\Delta\Psi = \text{flux crossing } \Delta S = D_{S,\text{norm}}\Delta S = D_S \cos\theta \Delta S = \mathbf{D}_S \cdot \Delta\mathbf{S}$$

The *total* flux passing through the closed surface is obtained by adding the differential contributions crossing each surface element $\Delta\mathbf{S}$,

$$\Psi = \int d\Psi = \oint_{\substack{\text{closed}\\ \text{surface}}} \mathbf{D}_S \cdot d\mathbf{S}$$

The resultant integral is a *closed surface integral*, and since the surface element $d\mathbf{S}$ always involves the differentials of two coordinates, such as $dx\, dy$, $\rho\, d\varphi\, d\rho$, or $r^2 \sin\theta\, d\theta\, d\varphi$, the integral is a double integral. Usually only one integral sign is used for brevity, and we will always place an S below the integral sign to indicate a surface integral, although this is not actually necessary, as the differential $d\mathbf{S}$ is automatically the signal for a surface integral. One last convention is to place a small circle on the integral sign itself to indicate that the integration is to be performed over a *closed* surface. Such a surface is often called a *gaussian surface*. We then have the mathematical formulation of Gauss's law,

$$\boxed{\Psi = \oint_S \mathbf{D}_S \cdot d\mathbf{S} = \text{charge enclosed} = Q}$$

Also, Gauss's law may be written in terms of the charge distribution as

$$\boxed{\oint_S \mathbf{D}_S \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v \, dv}$$

D3.3. Given the electric flux density, $\mathbf{D} = 0.3r^2\mathbf{a}_r$ nC/m^2 in free space: (*a*) find **E** at point $P(r = 2, \theta = 25°, \phi = 90°)$; (*b*) find the total charge within the sphere $r = 3$; (*c*) find the total electric flux leaving the sphere $r = 4$.

Ans. $135.5\mathbf{a}_r$ V/m; 305 nC; 965 nC

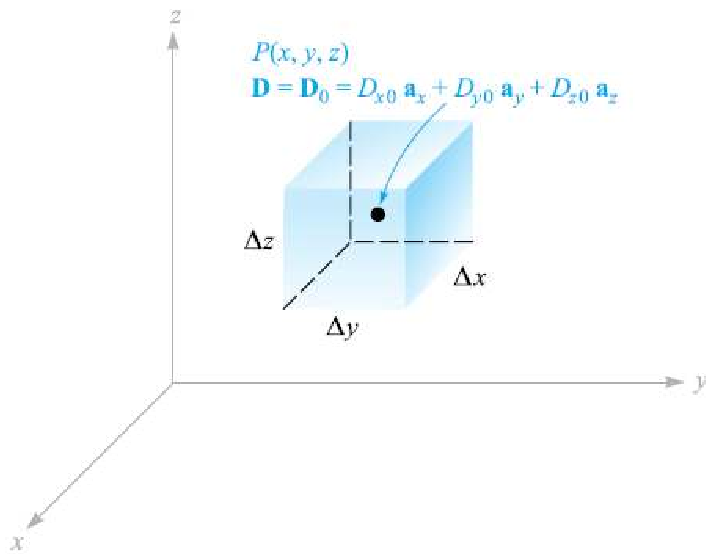
D3.4. Calculate the total electric flux leaving the cubical surface formed by the six planes $x, y, z = \pm 5$ if the charge distribution is: (*a*) two point charges, 0.1 μC at $(1, -2, 3)$ and $\frac{1}{7}$ μC at $(-1, 2, -2)$; (*b*) a uniform line charge of π μC/m at $x = -2$, $y = 3$; (*c*) a uniform surface charge of 0.1 μC/m^2 on the plane $y = 3x$.

Ans. 0.243 μC; 31.4 μC; 10.54 μC

3.3 AN APPLICATION OF GAUSS'S LAW: DIFFERENTIAL VOLUME ELEMENT

We are now going to apply the methods of Gauss's law to a type of problem that does not possess any symmetry at all. At first glance, it might seem that our case is hopeless, for without symmetry, a simple gaussian surface cannot be chosen such that the normal component of **D** is constant or zero everywhere on the surface. Without such a surface, the integral cannot be evaluated. There is only one way to circumvent these difficulties and that is to choose such a very small closed surface that **D** is *almost* constant over the surface, and the small change in **D** may be adequately represented by using the first two terms of the Taylor's-series expansion for **D**. The result will become more nearly correct as the volume enclosed by the gaussian surface decreases, and we intend eventually to allow this volume to approach zero.

In this example we will receive some extremely valuable information about the way **D** varies in the region of our small surface. This leads directly to one of Maxwell's four equations, which are basic to all electromagnetic theory.



Let us consider any point *P*, shown in Figure above, located by a rectangular coordinate system. The value of **D** at the point *P* may be expressed in rectangular components. We choose as our closed surface the small rectangular box, centered at *P*, having sides of lengths Δx, Δy, and Δz, and apply Gauss's law,

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = Q}$$

In order to evaluate the integral over the closed surface, the integral must be broken up into six integrals, one over each face,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{front}} + \int_{\text{back}} + \int_{\text{left}} + \int_{\text{right}} + \int_{\text{top}} + \int_{\text{bottom}}$$

Consider the first of these in detail. Because the surface element is very small, **D** is essentially constant (over *this* portion of the entire closed surface) and

$$\begin{aligned} \int_{\text{front}} &\doteq \mathbf{D}_{\text{front}} \cdot \Delta\mathbf{S}_{\text{front}} \\ &\doteq \mathbf{D}_{\text{front}} \cdot \Delta y \, \Delta z \, \mathbf{a}_x \\ &\doteq D_{\text{x,front}} \Delta y \, \Delta z \end{aligned}$$

where we have only to approximate the value of D_x at this front face. The front face is at a distance of $\Delta x/2$ from P, and hence

$$\begin{aligned} D_{x,\text{front}} &\doteq D_{x0} + \frac{\Delta x}{2} \times \text{rate of change of } D_x \text{with } x \\ &\doteq D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \end{aligned}$$

where D_{x0} is the value of D_x at P, and where a partial derivative must be used to express the rate of change of D_x with x, as D_x in general also varies with y and z. This expression could have been obtained more formally by using the constant term and the term involving the first derivative in the Taylor's-series expansion for D_x in the neighborhood of P.

We now have

$$\int_{\text{front}} \doteq \left(D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \, \Delta z$$

Consider now the integral over the back surface,

$$\begin{aligned} \int_{\text{back}} &\doteq \mathbf{D}_{\text{back}} \cdot \Delta \mathbf{S}_{\text{back}} \\ &\doteq \mathbf{D}_{\text{back}} \cdot (-\Delta y \, \Delta z \, \mathbf{a}_x) \\ &\doteq -D_{x,\text{back}} \Delta y \, \Delta z \end{aligned}$$

and

$$D_{x,\text{back}} \doteq D_{x0} - \frac{\Delta x}{2} \frac{\partial D_x}{\partial x}$$

giving

$$\int_{\text{back}} \doteq \left(-D_{x0} + \frac{\Delta x}{2} \frac{\partial D_x}{\partial x} \right) \Delta y \, \Delta z$$

If we combine these two integrals, we have

$$\int_{\text{front}} + \int_{\text{back}} \doteq \frac{\partial D_x}{\partial x} \Delta x \, \Delta y \, \Delta z$$

By exactly the same process we find that

$$\int_{\text{right}} + \int_{\text{left}} \doteq \frac{\partial D_y}{\partial y} \Delta x \, \Delta y \, \Delta z$$

and

$$\int_{\text{top}} + \int_{\text{bottom}} \doteq \frac{\partial D_z}{\partial z} \Delta x \, \Delta y \, \Delta z$$

and these results may be collected to yield

$$\oint_S \mathbf{D} \cdot d\mathbf{S} \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta x \, \Delta y \, \Delta z$$

or

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \Delta \nu \qquad (7)$$

The expression is an approximation which becomes better as $\Delta \nu$ becomes smaller, and in the following section we shall let the volume $\Delta \nu$ approach zero. For the moment, we have applied Gauss's law to the closed surface surrounding the volume element $\Delta \nu$ and have as a result the approximation (7) stating that

$$\boxed{\text{Charge enclosed in volume } \Delta \nu \doteq \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} \right) \times \text{volume } \Delta \nu} \qquad (8)$$

EXAMPLE

Find an approximate value for the total charge enclosed in an incremental volume of 10^{-9} m^3 located at the origin, if $\mathbf{D} = e^{-x} \sin y\, \mathbf{a}_x - e^{-x} \cos y\, \mathbf{a}_y + 2z\mathbf{a}_z$ C/m^2.

Solution. We first evaluate the three partial derivatives in (8):

$$\frac{\partial D_x}{\partial x} = -e^{-x} \sin y$$

$$\frac{\partial D_y}{\partial y} = e^{-x} \sin y$$

$$\frac{\partial D_z}{\partial z} = 2$$

At the origin, the first two expressions are zero, and the last is 2. Thus, we find that the charge enclosed in a small volume element there must be approximately $2\Delta\nu$. If $\Delta\nu$ is 10^{-9} m^3, then we have enclosed about 2 nC.

D3.6. In free space, let $\mathbf{D} = 8xyz^4\mathbf{a}_x + 4x^2z^4\mathbf{a}_y + 16x^2yz^3\mathbf{a}_z$ pC/m^2. (*a*) Find the total electric flux passing through the rectangular surface $z = 2$, $0 < x < 2$, $1 < y < 3$, in the $\mathbf{a}_z$ direction. (*b*) Find $\mathbf{E}$ at $P(2, -1, 3)$. (*c*) Find an approximate value for the total charge contained in an incremental sphere located at $P(2, -1, 3)$ and having a volume of 10^{-12} m^3.

Ans. 1365 pC; $-146.4\mathbf{a}_x + 146.4\mathbf{a}_y - 195.2\mathbf{a}_z$ V/m; -2.38×10^{-21} C

3.4 THE VECTOR OPERATOR ∇ AND THE DIVERGENCE THEOREM

If we remind ourselves again that divergence is an operation on a vector yielding a scalar result, just as the dot product of two vectors gives a scalar result, it seems possible that we can find something that may be dotted formally with **D** to yield the scalar

$$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}$$

Obviously, this cannot be accomplished by using a dot *product*; the process must be a dot *operation*.

With this in mind, we define the *del operator* ∇ as a *vector operator*,

$$\boxed{\nabla = \frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z}$$

Consider $\nabla \cdot \mathbf{D}$, signifying

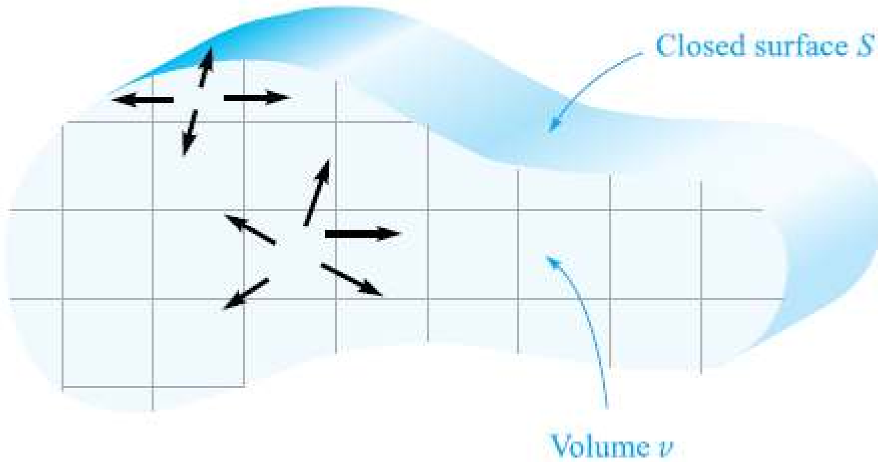
$$\nabla \cdot \mathbf{D} = \left(\frac{\partial}{\partial x}\mathbf{a}_x + \frac{\partial}{\partial y}\mathbf{a}_y + \frac{\partial}{\partial z}\mathbf{a}_z\right) \cdot (D_x\mathbf{a}_x + D_y\mathbf{a}_y + D_z\mathbf{a}_z)$$

$$\boxed{\nabla \cdot \mathbf{D} = \frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \mathrm{div}(\mathbf{D})}$$

Divergence theorem: *The integral of the normal component of any vector field over a closed surface is equal to the integral of the divergence of this vector field throughout the volume enclosed by the closed surface.*

This theorem applies to any vector field for which the appropriate partial derivatives exist, although it is easiest for us to develop it for the electric flux density. We have actually obtained it already and now have little more to do than point it out and name it, for starting from Gauss's law, we have

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\mathrm{vol}} \nabla \cdot \mathbf{D}\, dv}$$



Its benefits derive from the fact that it relates a triple integration throughout some volume to a double integration over the surface of that volume. For example, it is much easier to look for leaks in a bottle full of some agitated liquid by inspecting the surface than by calculating the velocity at every internal point.

The divergence theorem becomes obvious physically if we consider a volume v, shown in cross section in Figure above, which is surrounded by a closed surface S. Division of the volume into a number of small compartments of differential size and consideration of one cell show that the flux diverging from such a cell enters, or converges on, the adjacent cells unless the cell contains a portion of the outer surface.

In summary, the divergence of the flux density throughout a volume leads, then, to the same result as determining the net flux crossing the enclosing surface.

Evaluate both sides of the divergence theorem for the field $\mathbf{D} = 2xy\mathbf{a}_x + x^2\mathbf{a}_y$ C/m^2 and the rectangular parellelepiped formed by the planes $x = 0$ and 1, $y = 0$ and 2, and $z = 0$ and 3.

Solution. Evaluating the surface integral first, we note that $\mathbf{D}$ is parallel to the surfaces at $z = 0$ and $z = 3$, so $\mathbf{D} \cdot d\mathbf{S} = 0$ there. For the remaining four surfaces we have

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (\mathbf{D})_{x=0} \cdot (-dy\,dz\,\mathbf{a}_x) + \int_0^3 \int_0^2 (\mathbf{D})_{x=1} \cdot (dy\,dz\,\mathbf{a}_x)$$
$$+ \int_0^3 \int_0^1 (\mathbf{D})_{y=0} \cdot (-dx\,dz\,\mathbf{a}_y) + \int_0^3 \int_0^1 (\mathbf{D})_{y=2} \cdot (dx\,dz\,\mathbf{a}_y)$$

$$= -\int_0^3 \int_0^2 (D_x)_{x=0} dy\,dz + \int_0^3 \int_0^2 (D_x)_{x=1} dy\,dz$$
$$- \int_0^3 \int_0^1 (D_y)_{y=0} dx\,dz + \int_0^3 \int_0^1 (D_y)_{y=2} dx\,dz$$

However, $(D_x)_{x=0} = 0$, and $(D_y)_{y=0} = (D_y)_{y=2}$, which leaves only

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_0^3 \int_0^2 (D_x)_{x=1} dy\,dz = \int_0^3 \int_0^2 2y\,dy\,dz$$
$$= \int_0^3 4\,dz = 12$$

Since

$$\nabla \cdot \mathbf{D} = \frac{\partial}{\partial x}(2xy) + \frac{\partial}{\partial y}(x^2) = 2y$$

the volume integral becomes

$$\int_{\text{vol}} \nabla \cdot \mathbf{D}\,dv = \int_0^3 \int_0^2 \int_0^1 2y\,dx\,dy\,dz = \int_0^3 \int_0^2 2y\,dy\,dz$$
$$= \int_0^3 4\,dz = 12$$

and the check is accomplished. Remembering Gauss's law, we see that we have also determined that a total charge of 12 C lies within this parallelepiped.

D3.9. Given the field $\mathbf{D} = 6\rho \sin \frac{1}{2}\phi \, \mathbf{a}_\rho + 1.5\rho \cos \frac{1}{2}\phi \, \mathbf{a}_\phi$ C/m^2, evaluate both sides of the divergence theorem for the region bounded by $\rho = 2$, $\phi = 0$, $\phi = \pi$, $z = 0$, and $z = 5$.

Ans. 225; 225

3.5 POISSON'S AND LAPLACE'S EQUATIONS

One way of evaluating **capacitance** is by first assuming a known charge distribution on the conductors and then finding the potential difference in terms of the assumed charge. An alternate approach would be to start with known potentials on each conductor, and then work backward to find the charge in terms of the known potential difference. The capacitance in either case is found by the ratio *Q/V*.

The first objective in the latter approach is thus to find the potential function between conductors, given values of potential on the boundaries, along with possible volume charge densities in the region of interest. The mathematical tools that enable this to happen are Poisson's and Laplace's equations. Problems involving one to three dimensions can be solved either analytically or numerically. Laplace's and Poisson's equations, when compared to other methods, are probably the most widely useful because many problems in engineering practice involve devices in which applied potential differences are known, and in which constant potentials occur at the boundaries.

Obtaining Poisson's equation is exceedingly simple, for from the point form of Gauss's law,

$$\nabla \cdot \mathbf{D} = \rho_v \tag{21}$$

the definition of **D**,

$$\mathbf{D} = \epsilon \mathbf{E} \tag{22}$$

and the gradient relationship,

$$\mathbf{E} = -\nabla V \tag{23}$$

by substitution we have

$$\nabla \cdot \mathbf{D} = \nabla \cdot (\epsilon \mathbf{E}) = -\nabla \cdot (\epsilon \nabla V) = \rho_v$$

or

$$\nabla \cdot \nabla V = -\frac{\rho_v}{\epsilon} \tag{24}$$

for a homogeneous region in which ϵ is constant.

Equation (24) is *Poisson's equation,* but the "double ∇" operation must be interpreted and expanded, at least in rectangular coordinates, before the equation can be useful. In rectangular coordinates,

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla V = \frac{\partial V}{\partial x}\mathbf{a}_x + \frac{\partial V}{\partial y}\mathbf{a}_y + \frac{\partial V}{\partial z}\mathbf{a}_z$$

and therefore

$$\nabla \cdot \nabla V = \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial y}\right) + \frac{\partial}{\partial z}\left(\frac{\partial V}{\partial z}\right)$$

$$= \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} \tag{25}$$

Usually the operation $\nabla \cdot \nabla$ is abbreviated ∇^2 (and pronounced "del squared"), a good reminder of the second-order partial derivatives appearing in Eq. (5), and we have

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = -\frac{\rho_v}{\epsilon}} \tag{26}$$

in rectangular coordinates.

If $\rho_v = 0$, indicating zero *volume* charge density, but allowing point charges, line charge, and surface charge density to exist at singular locations as sources of the field, then

$$\boxed{\nabla^2 V = 0} \tag{27}$$

which is *Laplace's equation*. The ∇^2 operation is called the *Laplacian of V*.

In rectangular coordinates Laplace's equation is

$$\boxed{\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0 \qquad \text{(rectangular)}} \tag{28}$$

and the form of $\nabla^2 V$ in cylindrical and spherical coordinates may be obtained by using the expressions for the divergence and gradient already obtained in those coordinate systems. For reference, the Laplacian in cylindrical coordinates is

$$\boxed{\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho \frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\left(\frac{\partial^2 V}{\partial \phi^2}\right) + \frac{\partial^2 V}{\partial z^2} \qquad \text{(cylindrical)}} \tag{29}$$

and in spherical coordinates is

$$\boxed{\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta \frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \qquad \text{(spherical)}} \tag{30}$$

D6.5. Calculate numerical values for V and ρ_v at point P in free space if: (a) $V = \dfrac{4yz}{x^2+1}$, at $P(1,2,3)$; (b) $V = 5\rho^2 \cos 2\phi$, at $P(\rho = 3, \phi = \dfrac{\pi}{3}$, $z = 2)$; (c) $V = \dfrac{2\cos\phi}{r^2}$, at $P(r = 0.5, \theta = 45^\circ, \phi = 60^\circ)$.

Ans. 12 V, −106.2 pC/m^3; −22.5 V, 0; 4 V, 0

Solution of the laplace equation:

$$\frac{d^2V}{dx^2} = 0$$

We integrate twice, obtaining

$$\frac{dV}{dx} = A$$

and

$$V = Ax + B \tag{31}$$

EXAMPLE:

Start with the potential function, Eq. (31), and find the capacitance of a parallel-plate capacitor of plate area S, plate separation d, and potential difference V_0 between plates.

Solution. Take $V = 0$ at $x = 0$ and $V = V_0$ at $x = d$. Then from Eq. (31),

$$A = \frac{V_0}{d} \qquad B = 0$$

and

$$\boxed{V = \frac{V_0 x}{d}} \tag{32}$$

We still need the total charge on either plate before the capacitance can be found. We should remember that when we first solved this capacitor problem, the sheet of charge provided our starting point. We did not have to work very hard to find the charge, for all the fields were expressed in terms of it. The work then was spent in finding potential difference. Now the problem is reversed (and simplified).

The necessary steps are these, after the choice of boundary conditions has been made:

1. Given V, use $\mathbf{E} = -\nabla V$ to find $\mathbf{E}$.
2. Use $\mathbf{D} = \epsilon \mathbf{E}$ to find $\mathbf{D}$.
3. Evaluate $\mathbf{D}$ at either capacitor plate, $\mathbf{D} = \mathbf{D}_S = D_N \mathbf{a}_N$.
4. Recognize that $\rho_S = D_N$.
5. Find Q by a surface integration over the capacitor plate, $Q = \int_S \rho_S \, dS$.

Here we have

$$V = V_0 \frac{x}{d}$$

$$\mathbf{E} = -\frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{D}_S = \mathbf{D}\big|_{x=0} = -\epsilon \frac{V_0}{d}\mathbf{a}_x$$

$$\mathbf{a}_N = \mathbf{a}_x$$

$$D_N = -\epsilon \frac{V_0}{d} = \rho_S$$

$$Q = \int_S \frac{-\epsilon V_0}{d} dS = -\epsilon \frac{V_0 S}{d}$$

and the capacitance is

$$\boxed{C = \frac{|Q|}{V_0} = \frac{\epsilon S}{d}} \tag{33}$$

4 The Steady Magnetic Field

At this point, the concept of a field should be a familiar one. Since we first accepted the experimental law of forces existing between two point charges and defined electric field intensity as the force per unit charge on a test charge in the presence of a second charge, we have discussed numerous fields. These fields possess no real physical basis, for physical measurements must always be in terms of the forces on the charges in the detection equipment. Those charges that are the source cause measurable forces to be exerted on other charges, which we may think of as detector charges. The fact that we attribute a field to the source charges and then determine the effect of this field on the detector charges amounts merely to a division of the basic problem into two parts for convenience.

We will begin our study of the magnetic field with a definition of the magnetic field itself and show how it arises from a current distribution.

The relation of the steady magnetic field to its source is more complicated than is the relation of the electrostatic field to its source. We will find it necessary to accept several laws temporarily on faith alone. The proof of the laws does exist and is available in the internet for the disbelievers or the more advanced student.

4.1 BIOT-SAVART LAW

The source of the steady magnetic field may be a permanent magnet, an electric field changing linearly with time, or a direct current. We will largely ignore the permanent magnet and the time-varying electric field. Our present study will concern the magnetic field produced by a differential dc element in free space.

We may think of this differential current element as a vanishingly small section of a current-carrying filamentary conductor, where a filamentary conductor is the limiting case of a cylindrical conductor of circular cross section as the radius approaches zero.

We assume a current I flowing in a differential vector length of the filament $d\mathbf{L}$. The law of Biot-Savart1 then states that at any point P the magnitude of the magnetic field intensity produced by the differential element is proportional to the product of the current, the magnitude of the differential length, and the sine of the angle lying between the filament and a line connecting the filament to the point P at which the field is desired; also, the magnitude of the magnetic field intensity is inversely proportional to the square of the distance from the differential element to the point P.

The direction of the magnetic field intensity is normal to the plane containing the differential filament and the line drawn from the filament to the point *P*. Of the two possible normals, that one to be chosen is the one which is in the direction of progress of a right-handed screw turned from $d\mathbf{L}$ through the smaller angle to the line from the filament to *P*. Using rationalized mks units, the constant of proportionality is $1/4\pi$.

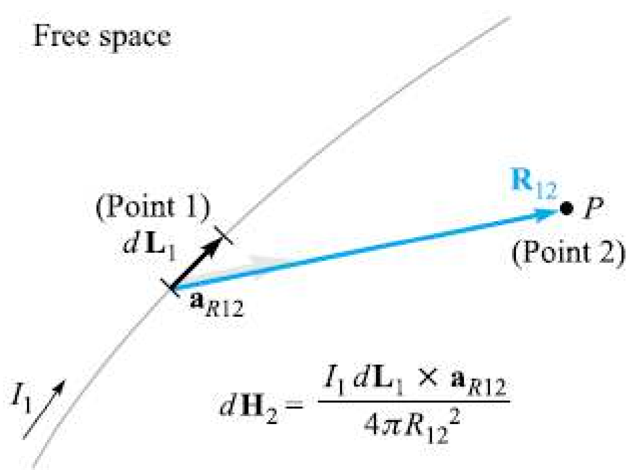
The *Biot-Savart law,* just described in some 150 words, may be written concisely using vector notation as

$$d\mathbf{H} = \frac{I d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2} = \frac{I d\mathbf{L} \times \mathbf{R}}{4\pi R^3}$$

The units of the *magnetic field intensity* **H** are evidently amperes per meter (A/m).

The geometry is illustrated in Figure below. If we locate the current element at point 1 and describe the point *P* at which the field is to be determined as point 2, then

$$\boxed{d\mathbf{H}_2 = \frac{I_1 d\mathbf{L}_1 \times \mathbf{a}_{R12}}{4\pi R_{12}^2}}$$



The law of Biot-Savart is sometimes called Ampere's law for the current element, but we will retain the former name because of possible confusion with Ampere's circuital law, to be discussed later.

In some aspects, the Biot-Savart law is reminiscent of Coulomb's law when that law is written for a differential element of charge,

$$d\mathrm{E}_2 = \frac{dQ_1 \mathbf{a}_{R12}}{4\pi\epsilon_0 R_{12}^2}$$

Both show an inverse-square-law dependence on distance, and both show a linear relationship between source and field. The chief difference appears in the direction of the field.

It is impossible to check experimentally the law of Biot-Savart as expressed by the immediate previous formulations because the differential current element cannot be isolated. We have restricted our attention to direct currents only, so the charge density is not a function of time.

The continuity equation

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

therefore shows that

$$\nabla \cdot \mathbf{J} = 0$$

or upon applying the divergence theorem,

$$\oint_s \mathbf{J} \cdot d\mathbf{S} = 0$$

The total current crossing any closed surface is zero, and this condition may be satisfied only by assuming a current flow around a closed path. It is this current flowing in a closed circuit that must be our experimental source, not the differential element.

It follows that only the integral form of the Biot-Savart law can be verified experimentally,

$$\boxed{\mathbf{H} = \oint \frac{I\,d\mathbf{L} \times \mathbf{a}_R}{4\pi R^2}}$$

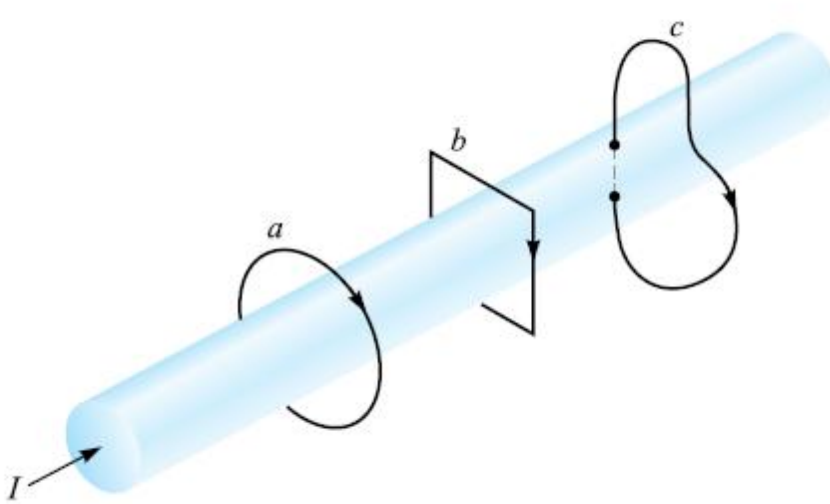
Biot and Savart were colleagues of Ampere, and all three were professors of physics at the College de France at one time or another. The Biot-Savart law was proposed in 1820.

4.2 AMPERE'S CIRCUITAL LAW

Ampere's circuital law, sometimes called Ampere's work law may be derived from the Biot-Savart law.

Ampere's circuital law states that the line integral of **H** about any closed path is exactly equal to the direct current enclosed by that path,

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} = I}$$



A conductor has a total current I. The line integral of **H** about the closed paths a and b is equal to I and the integral around path c is less than I, since the entire current is not enclosed by the path.

We should also consider exactly what is meant by the expression “current enclosed by the path.” Suppose we solder a circuit together after passing the conductor once through a rubber band, which we use to represent the closed path. Some strange and formidable paths can be constructed by twisting and knotting the rubber band, but if neither the rubber band nor the conducting circuit is broken, the current enclosed by the path is that carried by the conductor. Now replace the rubber band by a circular ring of spring steel across which is stretched a rubber sheet. The steel loop forms the closed path, and the current-carrying conductor must pierce the rubber sheet if the current is to be enclosed by the path. Again, we may twist the steel loop, and we may also deform the rubber sheet by pushing our fist into it or folding it in any way we wish. A single current-carrying conductor still pierces the sheet once, and this is the true measure of the current enclosed by the path. If we should thread the conductor once through the sheet from front to back and once from back to front, the total current enclosed by the path is the algebraic sum, which is zero.

In more general language, given a closed path, we recognize this path as the perimeter of an infinite number of surfaces (not closed surfaces). Any current-carrying conductor enclosed by the path must pass through every one of these surfaces once. Certainly some of the surfaces may be chosen in such a way that the conductor pierces them twice in one direction and once in the other direction, but the algebraic total current is still the same.

The simplest surface is, then, that portion of the plane enclosed by the path. We need merely find the total current passing through this region of the plane.

The application of Gauss’s law involves finding the total charge enclosed by a closed surface; the application of Ampere’s circuital law involves finding the total current enclosed by a closed path.

4.3 CURL AS APPLIES TO H, E

We completed our study of Gauss's law by applying it to a differential volume element and were led to the concept of divergence. We now apply Ampere's circuital law to the perimeter of a differential surface element and discuss the third and last of the special derivatives of vector analysis, the curl. Our objective is to obtain the point form of Ampere's circuital law.

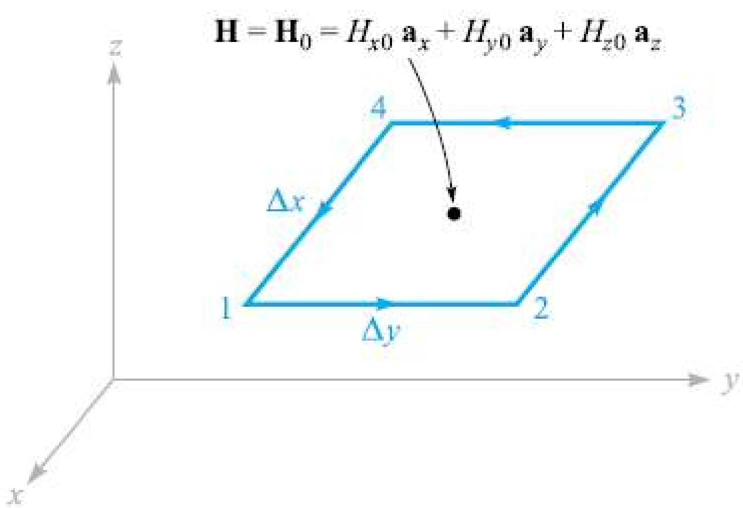
Again we choose rectangular coordinates, and an incremental closed path of sides Δx and Δy is selected (Figure below).We assume that some current, as yet unspecified, produces a reference value for **H** at the *center* of this small rectangle,

$$\mathbf{H}_0 = H_{x0}\mathbf{a}_x + H_{y0}\mathbf{a}_y + H_{z0}\mathbf{a}_z$$

The closed line integral of **H** about this path is then approximately the sum of the four values of $\mathbf{H} \cdot \Delta\mathbf{L}$ on each side. We choose the direction of traverse as 1-2-3-4-1, which corresponds to a current in the $\mathbf{a}_z$ direction, and the first contribution is therefore

$$(\mathbf{H} \cdot \Delta\mathbf{L})_{1-2} = H_{y,1-2}\Delta y$$

The value of H_y *on this section* of the path may be given in terms of the reference value H_{y0} at the center of the rectangle, the rate of change of H_y with x, and the



distance $\Delta x/2$ from the center to the midpoint of side 1–2:

$$H_{y,1-2} \doteq H_{y0} + \frac{\partial H_y}{\partial x}\left(\frac{1}{2}\Delta x\right)$$

Thus

$$(\mathbf{H} \cdot \Delta\mathbf{L})_{1-2} \doteq \left(H_{y0} + \frac{1}{2}\frac{\partial H_y}{\partial x}\Delta x\right)\Delta y$$

Along the next section of the path we have

$$(\mathbf{H} \cdot \Delta\mathbf{L})_{2-3} \doteq H_{x,2-3}(-\Delta x) \doteq -\left(H_{x0} + \frac{1}{2}\frac{\partial H_x}{\partial y}\Delta y\right)\Delta x$$

Continuing for the remaining two segments and adding the results,

$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x \Delta y$$

By Ampère's circuital law, this result must be equal to the current enclosed by the path, or the current crossing any surface bounded by the path. If we assume a general current density $\mathbf{J}$, the enclosed current is then $\Delta I \doteq J_z \Delta x \Delta y$, and

$$\oint \mathbf{H} \cdot d\mathbf{L} \doteq \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\Delta x \Delta y \doteq J_z \Delta x \Delta y$$

or

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} \doteq \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} \doteq J_z$$

As we cause the closed path to shrink, the preceding expression becomes more nearly exact, and in the limit we have the equality

$$\lim_{\Delta x, \Delta y \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta x \Delta y} = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y} = J_z \tag{18}$$

After beginning with Ampère's circuital law equating the closed line integral of **H** to the current enclosed, we have now arrived at a relationship involving the closed line integral of **H** *per unit area* enclosed and the current *per unit area* enclosed, or current density. We performed a similar analysis in passing from the integral form of Gauss's law, involving flux through a closed surface and charge enclosed, to the point form, relating flux through a closed surface *per unit volume* enclosed and charge *per unit volume* enclosed, or volume charge density. In each case a limit is necessary to produce an equality.

If we choose closed paths that are oriented perpendicularly to each of the remaining two coordinate axes, analogous processes lead to expressions for the x and y components of the current density,

$$\lim_{\Delta y, \Delta z \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta y \Delta z} = \frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z} = J_x \tag{19}$$

and

$$\lim_{\Delta z, \Delta x \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta z \Delta x} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} = J_y \tag{20}$$

Comparing (18)–(20), we see that a component of the current density is given by the limit of the quotient of the closed line integral of **H** about a small path in a plane normal to that component and of the area enclosed as the path shrinks to zero. This limit has its counterpart in other fields of science and long ago received the name of *curl*. The curl of any vector is a vector, and any component of the curl is given by the limit of the quotient of the closed line integral of the vector about a small path in a plane normal to that component desired and the area enclosed, as the path shrinks to zero. It should be noted that this definition of curl does not refer specifically to a particular coordinate system. The mathematical form of the definition is

$$\boxed{(\text{curl } \mathbf{H})_N = \lim_{\Delta S_N \to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{\Delta S_N}} \tag{21}$$

where ΔS_N is the planar area enclosed by the closed line integral. The N subscript indicates that the component of the curl is that component which is *normal* to the surface enclosed by the closed path. It may represent any component in any coordinate system.

In rectangular coordinates, the definition (21) shows that the x, y, and z components of the curl **H** are given by (18)–(20), and therefore

$$\boxed{\text{curl } \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z} \tag{22}$$

This result may be written in the form of a determinant,

$$\boxed{\operatorname{curl} \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ H_x & H_y & H_z \end{vmatrix}} \tag{23}$$

and may also be written in terms of the vector operator,

$$\boxed{\operatorname{curl} \mathbf{H} = \nabla \times \mathbf{H}} \tag{24}$$

Equation (22) is the result of applying the definition (21) to the rectangular coordinate system. We obtained the z component of this expression by evaluating Ampère's circuital law about an incremental path of sides Δx and Δy, and we could have obtained the other two components just as easily by choosing the appropriate paths. Equation (23) is a neat method of storing the rectangular coordinate expression for curl; the form is symmetrical and easily remembered. Equation (24) is even more concise and leads to (22) upon applying the definitions of the cross product and vector operator.

$$\boxed{\begin{aligned} \nabla \times \mathbf{H} = & \left(\frac{1}{\rho}\frac{\partial H_z}{\partial \phi} - \frac{\partial H_\phi}{\partial z}\right)\mathbf{a}_\rho + \left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right)\mathbf{a}_\phi \\ & + \left(\frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho} - \frac{1}{\rho}\frac{\partial H_\rho}{\partial \phi}\right)\mathbf{a}_z \quad \text{(cylindrical)} \end{aligned}}$$

$$\boxed{\begin{aligned} \nabla \times \mathbf{H} = & \frac{1}{r\sin\theta}\left(\frac{\partial(H_\phi \sin\theta)}{\partial \theta} - \frac{\partial H_\theta}{\partial \phi}\right)\mathbf{a}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial H_r}{\partial \phi} - \frac{\partial(rH_\phi)}{\partial r}\right)\mathbf{a}_\theta \\ & + \frac{1}{r}\left(\frac{\partial(rH_\theta)}{\partial r} - \frac{\partial H_r}{\partial \theta}\right)\mathbf{a}_\phi \quad \text{(spherical)} \end{aligned}}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}}$$

$$\boxed{\nabla \times \mathbf{E} = 0}$$

Although we have described curl as a line integral per unit area, this does not provide everyone with a satisfactory physical picture of the nature of the curl operation, for the closed line integral itself requires physical interpretation. This integral was first met in the electrostatic field, where we saw that $\oint \mathbf{E} \cdot d\mathbf{L} = 0$. Inasmuch as the integral was zero, we did not belabor the physical picture. More recently we have discussed the closed line integral of $\mathbf{H}$, $\oint \mathbf{H} \cdot d\mathbf{L} = I$. Either of these closed line integrals is also known by the name of *circulation,* a term borrowed from the field of fluid dynamics.

(*a*) (*b*)

The circulation of **H**, or $\oint \mathbf{H} \cdot d\mathbf{L}$, is obtained by multiplying the component of **H** parallel to the specified closed path at each point along it by the differential path length and summing the results as the differential lengths approach zero and as their number becomes infinite. We do not require a vanishingly small path. Ampère's circuital law tells us that if **H** does possess circulation about a given path, then current passes through this path. In electrostatics we see that the circulation of **E** is zero about every path, a direct consequence of the fact that zero work is required to carry a charge around a closed path.

We may describe curl as *circulation per unit area*. The closed path is vanishingly small, and curl is defined at a point. The curl of **E** must be zero, for the circulation is zero. The curl of **H** is not zero, however; the circulation of **H** per unit area is the current density by Ampère's circuital law [or (18), (19), and (20)].

Skilling[5] suggests the use of a very small paddle wheel as a "curl meter." Our vector quantity, then, must be thought of as capable of applying a force to each blade of the paddle wheel, the force being proportional to the component of the field normal to the surface of that blade. To test a field for curl, we dip our paddle wheel into the field, with the axis of the paddle wheel lined up with the direction of the component of curl desired, and note the action of the field on the paddle. No rotation means no curl; larger angular velocities mean greater values of the curl; a reversal in the direction of spin means a reversal in the sign of the curl. To find the direction of the vector curl and not merely to establish the presence of any particular component, we should place our paddle wheel in the field and hunt around for the orientation which produces the greatest torque. The direction of the curl is then along the axis of the paddle wheel, as given by the right-hand rule.

As an example, consider the flow of water in a river. Figure above *a* shows the longitudinal section of a wide river taken at the middle of the river. The water velocity is zero at the bottom and increases linearly as the surface is approached. A paddle wheel placed in the position shown, with its axis perpendicular to the paper, will turn in a clockwise direction, showing the presence of a component of curl in the direction of an inward normal to the surface of the page. If the velocity of water does not change

as we go up- or downstream and also shows no variation as we go across the river

(or even if it decreases in the same fashion toward either bank), then this component

is the only component present at the center of the stream, and the curl of the water

velocity has a direction into the page.

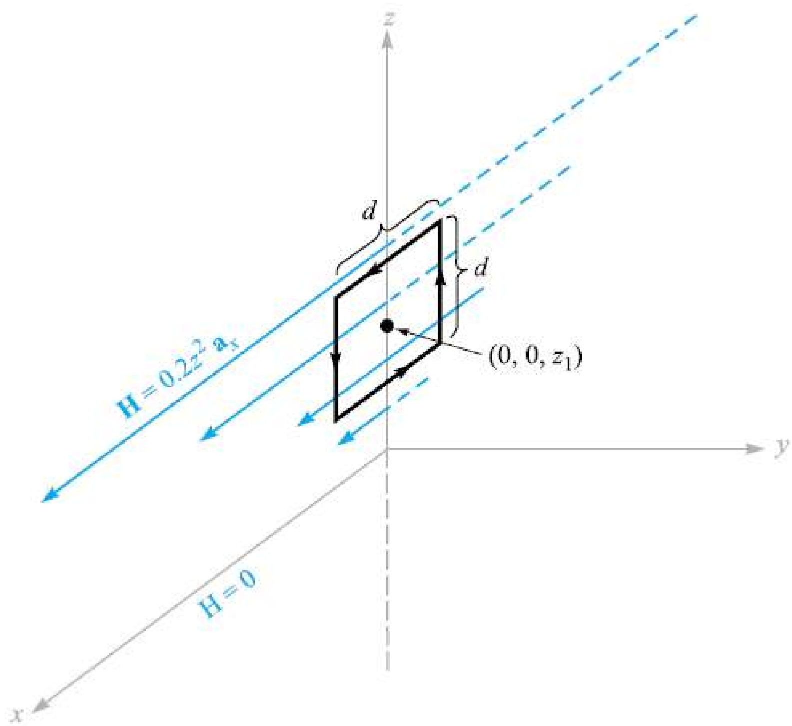
In Figure above *b*, the streamlines of the magnetic field intensity about an infinitely long filamentary conductor are shown. The curl meter placed in this field of curved lines shows that a larger number of blades have a clockwise force exerted on them but that this force is in general smaller than the counterclockwise force exerted on the smaller number of blades closer to the wire. It seems possible that if the curvature of the streamlines is correct and also if the variation of the field strength is just right, the net torque on the paddle wheel may be zero. Actually, the paddle wheel does not rotate in this case, for since

$$\mathbf{H} = (I/2\pi\rho)\mathbf{a}_\phi$$

$$\text{curl } \mathbf{H} = -\frac{\partial H_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho H_\phi)}{\partial \rho}\mathbf{a}_z = 0$$

EXAMPLE

As an example of the evaluation of curl **H** from the definition and of the evaluation of another line integral, suppose that $\mathbf{H} = 0.2z^2\mathbf{a}_x$ for $z > 0$, and $\mathbf{H} = 0$ elsewhere, as shown in Figure 7.15. Calculate $\oint \mathbf{H} \cdot d\mathbf{L}$ about a square path with side d, centered at $(0, 0, z_1)$ in the $y = 0$ plane where $z_1 > d/2$.



Solution. We evaluate the line integral of **H** along the four segments, beginning at the top:

$$\oint \mathbf{H} \cdot d\mathbf{L} = 0.2\left(z_1 + \tfrac{1}{2}d\right)^2 d + 0 - 0.2\left(z_1 - \tfrac{1}{2}d\right)^2 d + 0$$
$$= 0.4z_1 d^2$$

In the limit as the area approaches zero, we find

$$(\nabla \times \mathbf{H})_y = \lim_{d\to 0} \frac{\oint \mathbf{H} \cdot d\mathbf{L}}{d^2} = \lim_{d\to 0} \frac{0.4z_1 d^2}{d^2} = 0.4z_1$$

The other components are zero, so $\nabla \times \mathbf{H} = 0.4z_1\mathbf{a}_y$.

To evaluate the curl without trying to illustrate the definition or the evaluation of a line integral, we simply take the partial derivative indicated by (23):

$$\nabla \times \mathbf{H} = \begin{vmatrix} \mathbf{a}_x & \mathbf{a}_y & \mathbf{a}_z \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 0.2z^2 & 0 & 0 \end{vmatrix} = \frac{\partial}{\partial z}(0.2z^2)\mathbf{a}_y = 0.4z\mathbf{a}_y$$

which checks with the preceding result when $z = z_1$.

Returning now to complete our original examination of the application of Ampère's circuital law to a differential-sized path, we may combine (18)–(20), (22), and (24),

$$\text{curl } \mathbf{H} = \nabla \times \mathbf{H} = \left(\frac{\partial H_z}{\partial y} - \frac{\partial H_y}{\partial z}\right)\mathbf{a}_x + \left(\frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x}\right)\mathbf{a}_y + \left(\frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\right)\mathbf{a}_z = \mathbf{J} \tag{27}$$

and write the *point form of Ampère's circuital law,*

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J}} \tag{28}$$

This is the second of Maxwell's four equations as they apply to non-time-varying conditions. We may also write the third of these equations at this time; it is the point form of $\oint \mathbf{E} \cdot d\mathbf{L} = 0$, or

$$\boxed{\nabla \times \mathbf{E} = 0} \tag{29}$$

D7.4. (*a*) Evaluate the closed line integral of **H** about the rectangular path $P_1(2, 3, 4)$ to $P_2(4, 3, 4)$ to $P_3(4, 3, 1)$ to $P_4(2, 3, 1)$ to P_1, given $\mathbf{H} = 3z\mathbf{a}_x - 2x^3\mathbf{a}_z$ A/m. (*b*) Determine the quotient of the closed line integral and the area enclosed by the path as an approximation to $(\nabla \times \mathbf{H})_y$. (*c*) Determine $(\nabla \times \mathbf{H})_y$ at the center of the area.

Ans. 354 A; 59 A/m^2; 57 A/m^2

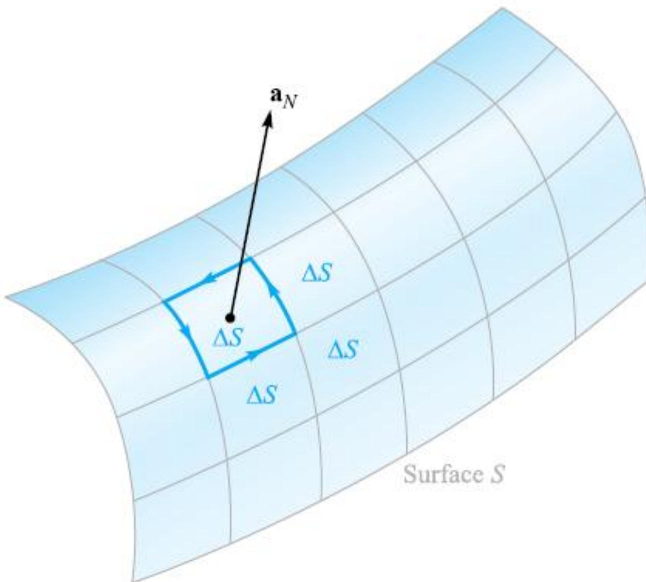
D7.5. Calculate the value of the vector current density: (*a*) in rectangular coordinates at $P_A(2, 3, 4)$ if $\mathbf{H} = x^2z\mathbf{a}_y - y^2x\mathbf{a}_z$; (*b*) in cylindrical coordinates at $P_B(1.5, 90^\circ, 0.5)$ if $\mathbf{H} = \dfrac{2}{\rho}(\cos 0.2\phi)\mathbf{a}_\rho$; (*c*) in spherical coordinates at $P_C(2, 30^\circ, 20^\circ)$ if $\mathbf{H} = \dfrac{1}{\sin\theta}\mathbf{a}_\theta$.

Ans. $-16\mathbf{a}_x + 9\mathbf{a}_y + 16\mathbf{a}_z$ A/m^2; $0.055\mathbf{a}_z$ A/m^2; $\mathbf{a}_\phi$ A/m^2

4.4 STOKE'S THEOREM

Consider the surface S of Figure below, which is broken up into incremental surfaces of area ΔS.

If we apply the definition of the curl to one of these incremental surfaces, then



$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H})_N$$

where the N subscript again indicates the right-hand normal to the surface. The subscript on $d\mathbf{L}_{\Delta S}$ indicates that the closed path is the perimeter of an incremental area ΔS. This result may also be written

$$\frac{\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S}}{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N$$

or

$$\oint \mathbf{H} \cdot d\mathbf{L}_{\Delta S} \doteq (\nabla \times \mathbf{H}) \cdot \mathbf{a}_N \Delta S = (\nabla \times \mathbf{H}) \cdot \Delta \mathbf{S}$$

where $\mathbf{a}_N$ is a unit vector in the direction of the right-hand normal to ΔS.

Now let us determine this circulation for every ΔS comprising S and sum the results. As we evaluate the closed line integral for each ΔS, some cancellation will occur because every *interior* wall is covered once in each direction. The only boundaries on which cancellation cannot occur form the outside boundary, the path enclosing S. Therefore we have

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} \equiv \int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S}} \tag{30}$$

where $d\mathbf{L}$ is taken only on the perimeter of S.

Equation (30) is an identity, holding for any vector field, and is known as *Stokes' theorem*.

D7.6. Evaluate both sides of Stokes' theorem for the field $\mathbf{H} = 6xy\mathbf{a}_x - 3y^2\mathbf{a}_y$ A/m and the rectangular path around the region, $2 \le x \le 5$, $-1 \le y \le 1$, $z = 0$. Let the positive direction of $d\mathbf{S}$ be $\mathbf{a}_z$.

Ans. -126 A; -126 A

4.5 MAGNETIC FLUX AND MAGNETIC FLUX DENSITY

In free space, let us define the *magnetic flux density* $\mathbf{B}$ as

$$\boxed{\mathbf{B} = \mu_0 \mathbf{H}} \quad \text{(free space only)} \tag{32}$$

where $\mathbf{B}$ is measured in webers per square meter (Wb/m^2) or in a newer unit adopted in the International System of Units, tesla (T). An older unit that is often used for magnetic flux density is the gauss (G), where 1 T or 1Wb/m^2 is the same as 10, 000 G. The constant μ_0 is not dimensionless and has the *defined value* for free space, in henrys per meter (H/m), of

$$\boxed{\mu_0 = 4\pi \times 10^{-7} \text{ H/m}} \tag{33}$$

The name given to μ_0 is the *permeability* of free space.

We should note that since $\mathbf{H}$ is measured in amperes per meter, the weber is dimensionally equal to the product of henrys and amperes. Considering the henry as a new unit, the weber is merely a convenient abbreviation for the product of henrys and amperes. When time-varying fields are introduced, it will be shown that a weber is also equivalent to the product of volts and seconds.

The magnetic-flux-density vector **B**, as the name weber per square meter implies, is a member of the flux-density family of vector fields. One of the possible analogies between electric and magnetic fields[7] compares the laws of Biot-Savart and Coulomb, thus establishing an analogy between **H** and **E**. The relations $\mathbf{B} = \mu_0 \mathbf{H}$ and $\mathbf{D} = \epsilon_0 \mathbf{E}$ then lead to an analogy between **B** and **D**. If **B** is measured in teslas or webers per square meter, then magnetic flux should be measured in webers. Let us represent magnetic flux by Φ and define Φ as the flux passing through any designated area,

$$\boxed{\Phi = \int_S \mathbf{B} \cdot d\mathbf{S} \text{ Wb}} \tag{34}$$

Our analogy should now remind us of the electric flux Ψ, measured in coulombs, and of Gauss's law, which states that the total flux passing through any closed surface is equal to the charge enclosed,

$$\Psi = \oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

The charge Q is the source of the lines of electric flux and these lines begin and terminate on positive and negative charges, respectively.

No such source has ever been discovered for the lines of magnetic flux. In the example of the infinitely long straight filament carrying a direct current I, the **H** field formed concentric circles about the filament. Because $\mathbf{B} = \mu_0 \mathbf{H}$, the **B** field is of the same form. The magnetic flux lines are closed and do not terminate on a "magnetic charge." For this reason Gauss's law for the magnetic field is

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0} \tag{35}$$

and application of the divergence theorem shows us that

$$\boxed{\nabla \cdot \mathbf{B} = 0} \tag{36}$$

5 Maxwell's four equations as they apply to static electric fields and steady magnetic fields

$$\boxed{\begin{aligned} \nabla \cdot \mathbf{D} &= \rho_v \\ \nabla \times \mathbf{E} &= 0 \\ \nabla \times \mathbf{H} &= \mathbf{J} \\ \nabla \cdot \mathbf{B} &= 0 \end{aligned}}$$

Constitutive relations:

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E}}$$

$$\boxed{\mathbf{B} = \mu_0 \mathbf{H}}$$

We have also found it helpful to define an electrostatic potential,

$$\boxed{\mathbf{E} = -\nabla V}$$

The corresponding set of four integral equations that apply to static electric fields and steady magnetic fields is:

$$\boxed{\begin{aligned} &\oint_S \mathbf{D} \cdot d\mathbf{S} = Q = \int_{\text{vol}} \rho_v dv \\ &\oint \mathbf{E} \cdot d\mathbf{L} = 0 \\ &\oint \mathbf{H} \cdot d\mathbf{L} = I = \int_S \mathbf{J} \cdot d\mathbf{S} \\ &\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \end{aligned}}$$

With a good knowledge of vector analysis, such as we should now have, either set (integral or differential forms of the Maxwell's equations) may be readily obtained from the other by applying the divergence theorem or Stokes' theorem. The various experimental laws can be obtained easily from these equations.

6 Summary of Maxwell's equations in time-varying fields

6.1 Summary of Maxwell's equations in time-varying fields in point form

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}}$$

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}}$$

The other two equations remain as they were in static fields:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_v}$$

Equation above essentially states that charge density is a source (or sink) of electric flux lines.

$$\boxed{\nabla \cdot \mathbf{B} = 0}$$

Equation above acknowledges the fact that magnetic flux is always found in closed loops and never diverges from a point source.

These four equations form the basis of all electromagnetic theory. They are partial differential equations and relate the electric and magnetic fields to each other and to their sources (namely charge and current density).

6.2 Summary of Maxwell's equations in integral form

Faraday's law:

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}}$$

(Obtained by integrating the corresponding point form equation over a surface and applying Stokes' theorem)

Ampere's circuital law:

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}}$$

(Obtained by integrating the corresponding point form equation over a surface and applying Stokes' theorem)

Gauss's laws for the electric and magnetic fields are obtained by integrating the corresponding point form equations throughout a volume and using the divergence theorem; yielding:

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v dv}$$

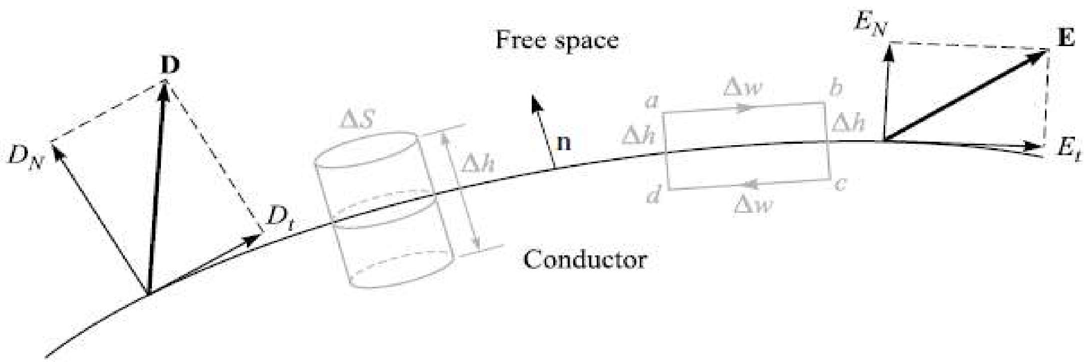
$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0}$$

These four integral equations enable us to find the boundary conditions on **B**, **D**, **H**, and **E**, which are necessary to evaluate the constants obtained in solving Maxwell's equations in partial differential form.

7 Boundary conditions

7.1 ELECTRIC BOUNDARY CONDITIONS

7.1.1 BETWEEN FREE SPACE AND A CONDUCTOR



$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

around the small closed path *abcda*. The integral must be broken up into four parts

$$\int_a^b + \int_b^c + \int_c^d + \int_d^a = 0$$

Remembering that $\mathbf{E} = 0$ within the conductor, we let the length from a to b or c to d be Δw and from b to c or d to a be Δh, and obtain

$$E_t \Delta w - E_{N,\text{at}\, b} \tfrac{1}{2} \Delta h + E_{N,\text{at}\, a} \tfrac{1}{2} \Delta h = 0$$

As we allow Δh to approach zero, keeping Δw small but finite, it makes no difference whether or not the normal fields are equal at a and b, for Δh causes these products to become negligibly small. Hence, $E_t \Delta w = 0$ and, therefore, $E_t = 0$.

The condition on the normal field is found most readily by considering D_N rather than E_N and choosing a small cylinder as the gaussian surface. Let the height be Δh and the area of the top and bottom faces be ΔS. Again, we let Δh approach zero. Using Gauss's law,

$$\oint_S \mathbf{D} \cdot d\mathbf{S} = Q$$

we integrate over the three distinct surfaces

$$\int_{\text{top}} + \int_{\text{bottom}} + \int_{\text{sides}} = Q$$

and find that the last two are zero (for different reasons). Then

$$D_N \Delta S = Q = \rho_S \Delta S$$

or

$$D_N = \rho_S$$

These are the desired *boundary conditions* for the conductor-to-free-space boundary in electrostatics,

$$\boxed{D_t = E_t = 0}$$

$$\boxed{D_N = \epsilon_0 E_N = \rho_S}$$

The tangential electric field is continuous across the boundary.

The normal electric field is discontinuous.

An immediate and important consequence of a zero tangential electric field intensity is the fact that a conductor surface is an equipotential surface. The evaluation of the potential difference between any two points on the surface by the line integral leads to a zero result, because the path may be chosen on the surface itself where $\mathbf{E} \cdot d\mathbf{L} = 0$.

To summarize the principles which apply to conductors in electrostatic fields, we may state that

1. The static electric field intensity inside a conductor is zero.
2. The static electric field intensity at the surface of a conductor is everywhere directed normal to that surface.
3. The conductor surface is an equipotential surface.

EXAMPLE:

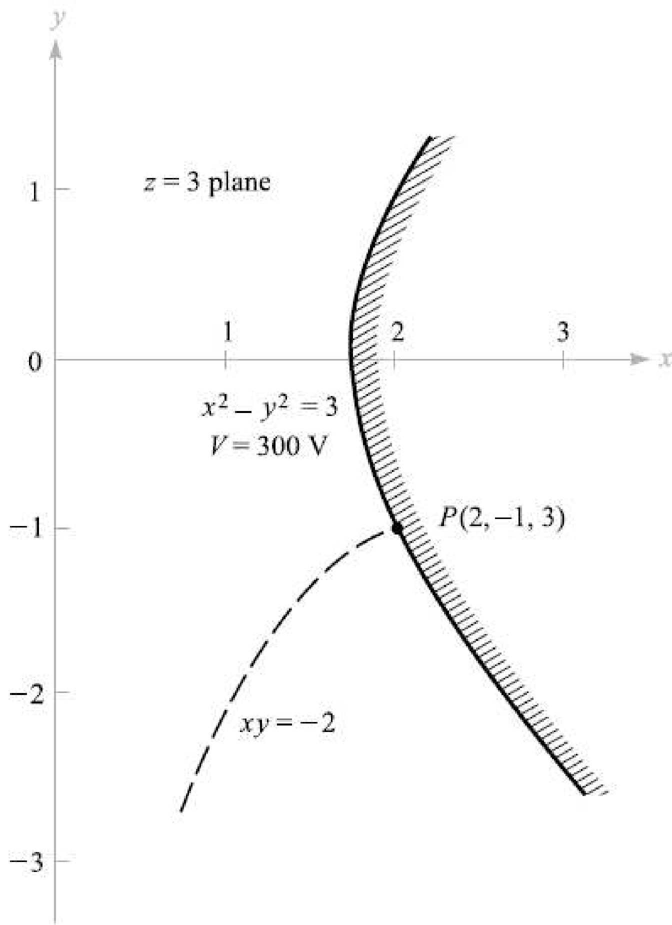
Given the potential,

$$V = 100(x^2 - y^2)$$

and a point $P(2, -1, 3)$ that is stipulated to lie on a conductor-to-free-space boundary, find V, $\mathbf{E}$, $\mathbf{D}$, and ρ_S at P, and also the equation of the conductor surface.

Solution. The potential at point P is

$$V_P = 100[2^2 - (-1)^2] = 300 \text{ V}$$



Because the conductor is an equipotential surface, the potential at the entire surface must be 300 V. Moreover, if the conductor is a solid object, then the potential everywhere in and on the conductor is 300 V, for $\mathbf{E} = 0$ within the conductor.

The equation representing the locus of all points having a potential of 300 V is

$$300 = 100(x^2 - y^2)$$

or

$$x^2 - y^2 = 3$$

This is therefore the equation of the conductor surface; it happens to be a hyperbolic cylinder, as shown in Figure above. Let us assume arbitrarily that the solid conductor lies above and to the right of the equipotential surface at point *P*, whereas free space is down and to the left.

Next, we find $\mathbf{E}$ by the gradient operation,

$$\mathbf{E} = -100\nabla(x^2 - y^2) = -200x\mathbf{a}_x + 200y\mathbf{a}_y$$

At point P,

$$\mathbf{E}_p = -400\mathbf{a}_x - 200\mathbf{a}_y \text{ V/m}$$

Because $\mathbf{D} = \epsilon_0\mathbf{E}$, we have

$$\mathbf{D}_P = 8.854 \times 10^{-12}\mathbf{E}_P = -3.54\mathbf{a}_x - 1.771\mathbf{a}_y \text{ nC/m}^2$$

The field is directed downward and to the left at P; it is normal to the equipotential surface. Therefore,

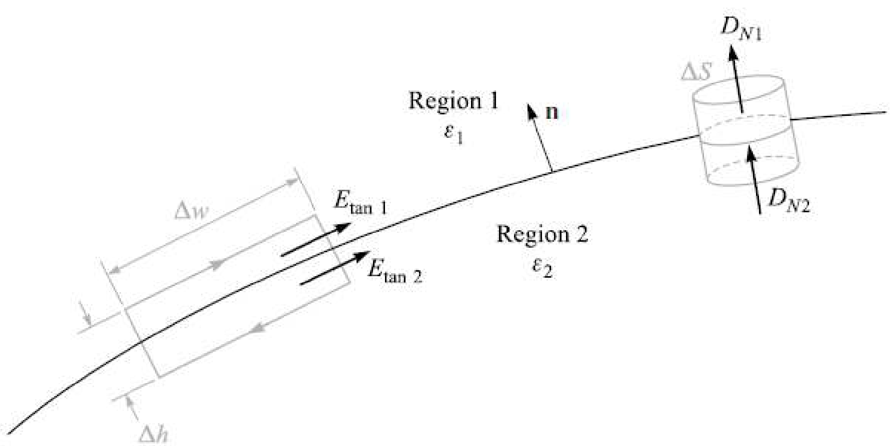
$$D_N = |\mathbf{D}_P| = 3.96 \text{ nC/m}^2$$

Thus, the surface charge density at P is

$$\rho_{S,P} = D_N = 3.96 \text{ nC/m}^2$$

Note that if we had taken the region to the left of the equipotential surface as the conductor, the $\mathbf{E}$ field would *terminate* on the surface charge and we would let $\rho_S = -3.96 \text{ nC/m}^2$.

7.1.2 BOUNDARY CONDITIONS FOR PERFECT DIELECTRIC MATERIALS



We first examine the tangential components by using

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0$$

around the small closed path on the left, obtaining

$$E_{\tan 1}\, \Delta w - E_{\tan 2}\, \Delta w = 0$$

The small contribution to the line integral by the normal component of **E** along the sections of length Δh becomes negligible as Δh decreases and the closed path crowds the surface. Immediately, then,

$$\boxed{E_{\tan 1} = E_{\tan 2}} \tag{32}$$

Evidently, Kirchhoff's voltage law is still applicable to this case. Certainly we have shown that the potential difference between any two points on the boundary that are separated by a distance Δw is the same immediately above or below the boundary.

If the tangential electric field intensity is continuous across the boundary, then tangential **D** is discontinuous, for

$$\frac{D_{\tan 1}}{\epsilon_1} = E_{\tan 1} = E_{\tan 2} = \frac{D_{\tan 2}}{\epsilon_2}$$

or

$$\frac{D_{\tan 1}}{D_{\tan 2}} = \frac{\epsilon_1}{\epsilon_2} \tag{33}$$

The boundary conditions on the normal components are found by applying Gauss's law to the small "pillbox" shown at the right in Figure 5.10. The sides are again very short, and the flux leaving the top and bottom surfaces is the difference

$$D_{N1}\Delta S - D_{N2}\Delta S = \Delta Q = \rho_S \Delta S$$

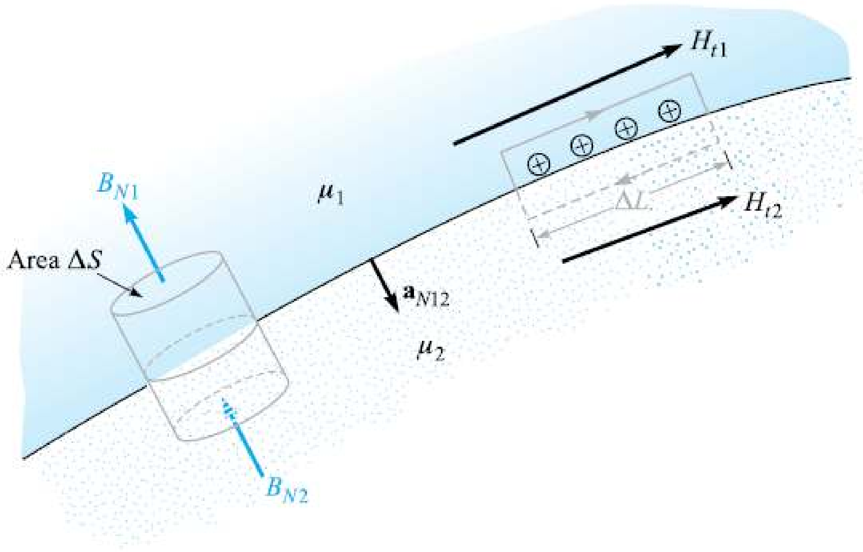
from which

$$\boxed{D_{N1} - D_{N2} = \rho_S} \tag{34}$$

What is this surface charge density? It cannot be a *bound* surface charge density, because we are taking the polarization of the dielectric into effect by using a dielectric constant different from unity; that is, instead of considering bound charges in free space, we are using an increased permittivity. Also, it is extremely unlikely that any *free* charge is on the interface, for no free charge is available in the perfect dielectrics we are considering. This charge must then have been placed there deliberately, thus unbalancing the total charge in and on this dielectric body. Except for this special case, then, we may assume ρ_S is zero on the interface and

$$\boxed{D_{N1} = D_{N2}} \tag{35}$$

7.2 MAGNETIC BOUNDARY CONDITIONS



The boundary condition on the normal components is determined by allowing the surface to cut a small cylindrical gaussian surface.

Applying Gauss's law for the magnetic field:

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$$

we find that

$$B_{N1}\Delta S - B_{N2}\Delta S = 0$$

or

$$\boxed{B_{N2} = B_{N1}} \tag{32}$$

Thus

$$H_{N2} = \frac{\mu_1}{\mu_2} H_{N1} \tag{33}$$

The normal component of **B** is continuous, but the normal component of **H** is discontinuous by the ratio μ_1/μ_2.

The relationship between the normal components of **M**, of course, is fixed once the relationship between the normal components of **H** is known. For linear magnetic materials, the result is written simply as

$$M_{N2} = \chi_{m2}\frac{\mu_1}{\mu_2} H_{N1} = \frac{\chi_{m2}\mu_1}{\chi_{m1}\mu_2} M_{N1} \tag{34}$$

Next, Ampère's circuital law

$$\oint \mathbf{H} \cdot d\mathbf{L} = I$$

is applied about a small closed path in a plane normal to the boundary surface, as shown to the right in Figure 8.10. Taking a clockwise trip around the path, we find that

$$H_{t1}\Delta L - H_{t2}\Delta L = K\Delta L$$

where we assume that the boundary may carry a surface current **K** whose component normal to the plane of the closed path is K. Thus

$$\boxed{H_{t1} - H_{t2} = K} \tag{35}$$

The directions are specified more exactly by using the cross product to identify the tangential components,

$$(\mathbf{H}_1 - \mathbf{H}_2) \times \mathbf{a}_{N12} = \mathbf{K}$$

where $\mathbf{a}_{N12}$ is the unit normal at the boundary directed from region 1 to region 2. An equivalent formulation in terms of the vector tangential components may be more convenient for **H**:

$$\mathbf{H}_{t1} - \mathbf{H}_{t2} = \mathbf{a}_{N12} \times \mathbf{K}$$

For tangential **B**, we have

$$\frac{B_{t1}}{\mu_1} - \frac{B_{t2}}{\mu_2} = K \tag{36}$$

The boundary condition on the tangential component of the magnetization for linear materials is therefore

$$M_{t2} = \frac{\chi_{m2}}{\chi_{m1}} M_{t1} - \chi_{m2} K \tag{37}$$

The last three boundary conditions on the tangential components are much simpler, of course, if the surface current density is zero. This is a free current density, and it must be zero if neither material is a conductor.

EXAMPLE:

To illustrate these relationships with an example, let us assume that $\mu = \mu_1 = 4\,\mu$H/m in region 1 where $z > 0$, whereas $\mu_2 = 7\,\mu$H/m in region 2 wherever $z < 0$. Moreover, let $\mathbf{K} = 80\mathbf{a}_x$ A/m on the surface $z = 0$. We establish a field, $\mathbf{B}_1 = 2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z$ mT, in region 1 and seek the value of $\mathbf{B}_2$.

Solution. The normal component of $\mathbf{B}_1$ is

$$\mathbf{B}_{N1} = (\mathbf{B}_1 \cdot \mathbf{a}_{N12})\mathbf{a}_{N12} = [(2\mathbf{a}_x - 3\mathbf{a}_y + \mathbf{a}_z)\cdot(-\mathbf{a}_z)](-\mathbf{a}_z) = \mathbf{a}_z \text{ mT}$$

Thus,

$$\mathbf{B}_{N2} = \mathbf{B}_{N1} = \mathbf{a}_z \text{ mT}$$

We next determine the tangential components:

$$\mathbf{B}_{t1} = \mathbf{B}_1 - \mathbf{B}_{N1} = 2\mathbf{a}_x - 3\mathbf{a}y \text{ mT}$$

and

$$\mathbf{H}_{t1} = \frac{\mathbf{B}_{t1}}{\mu_1} = \frac{(2\mathbf{a}_x - 3\mathbf{a}_y)10^{-3}}{4 \times 10^{-6}} = 500\mathbf{a}_x - 750\mathbf{a}_y \text{ A/m}$$

Thus,

$$\begin{aligned}\mathbf{H}_{t2} &= \mathbf{H}_{t1} - \mathbf{a}_{N12} \times \mathbf{K} = 500\mathbf{a}_x - 750\mathbf{a}_y - (-\mathbf{a}_z) \times 80\mathbf{a}_x \\ &= 500\mathbf{a}_x - 750\mathbf{a}_y + 80\mathbf{a}_y = 500\mathbf{a}_x - 670\mathbf{a}_y \text{ A/m}\end{aligned}$$

and

$$\mathbf{B}_{t2} = \mu_2\mathbf{H}_{t2} = 7 \times 10^{-6}(500\mathbf{a}_x - 670\mathbf{a}_y) = 3.5\mathbf{a}_x - 4.69\mathbf{a}_y \text{ mT}$$

Therefore,

$$\mathbf{B}_2 = \mathbf{B}_{N2} + \mathbf{B}_{t2} = 3.5\mathbf{a}_x - 4.69\mathbf{a}_y + \mathbf{a}_z \text{ mT}$$

D8.8. Let the permittivity be $5\,\mu$H/m in region A where $x < 0$, and $20\,\mu$H/m in region B where $x > 0$. If there is a surface current density $\mathbf{K} = 150\mathbf{a}_y - 200\mathbf{a}_z$ A/m at $x = 0$, and if $H_A = 300\mathbf{a}_x - 400\mathbf{a}_y + 500\mathbf{a}_z$ A/m, find: (*a*) $|\mathbf{H}_{tA}|$; (*b*) $|\mathbf{H}_{NA}|$; (*c*) $|\mathbf{H}_{tB}|$; (*d*) $|\mathbf{H}_{NB}|$.

Ans. 640 A/m; 300 A/m; 695 A/m; 75 A/m

8 APPENDIX

Table B.1 Names and units of the electric and magnetic quantities in the International System (in the order in which they appear in the text)

Symbol	Name	Unit	Abbreviation
v	Velocity	meter/second	m/s
F	Force	newton	N
Q	Charge	coulomb	C
r, R	Distance	meter	m
ϵ_0, ϵ	Permittivity	farad/meter	F/m
E	Electric field intensity	volt/meter	V/m
ρ_v	Volume charge density	coulomb/meter^3	C/m^3
v	Volume	meter^3	m^3
ρ_L	Linear charge density	coulomb/meter	C/m
ρ_S	Surface charge density	coulomb/meter^2	C/m^2
Ψ	Electric flux	coulomb	C
D	Electric flux density	coulomb/meter^2	C/m^2
S	Area	meter^2	m^2
W	Work, energy	joule	J
L	Length	meter	m
V	Potential	volt	V
p	Dipole moment	coulomb-meter	C·m
I	Current	ampere	A
J	Current density	ampere/meter^2	A/m^2
μ_e, μ_h	Mobility	meter^2/volt-second	m^2/V·s
e	Electronic charge	coulomb	C
σ	Conductivity	siemens/meter	S/m
R	Resistance	ohm	Ω
P	Polarization	coulomb/meter^2	C/m^2
$\chi_{e,m}$	Susceptibility		
C	Capacitance	farad	F
R_s	Sheet resistance	ohm per square	Ω
H	Magnetic field intensity	ampere/meter	A/m
K	Surface current density	ampere/meter	A/m
B	Magnetic flux density	tesla (or weber/meter^2)	T (or Wb/m^2)
μ_0, μ	Permeability	henry/meter	H/m
Φ	Magnetic flux	weber	Wb
V_m	Magnetic scalar potential	ampere	A
A	Vector magnetic potential	weber/meter	Wb/m
T	Torque	newton-meter	N·m
m	Magnetic moment	ampere-meter^2	A·m^2
M	Magnetization	ampere/meter	A/m
$\mathcal{R}$	Reluctance	ampere-turn/weber	A·t/Wb
L	Inductance	henry	H
M	Mutual inductance	henry	H

(Continued)

Table B.1 *(Continued)*

Symbol	Name	Unit	Abbreviation
ω	Radian frequency	radian/second	rad/s
c	Velocity of light	meter/second	m/s
λ	Wavelength	meter	m
η	Intrinsic impedance	ohm	Ω
k	Wave number	meter^{-1}	m^{-1}
α	Attenuation constant	neper/meter	Np/m
β	Phase constant	radian/meter	rad/m
f	Frequency	hertz	Hz
S	Poynting vector	watt/meter^2	W/m^2
P	Power	watt	W
δ	Skin depth	meter	m
Γ	Reflection coefficient		
s	Standing wave ratio		
γ	Propagation constant	meter^{-1}	m^{-1}
G	Conductance	siemen	S
Z	Impedance	ohm	Ω
Y	Admittance	siemen	S
Q	Quality factor		

A.3 VECTOR IDENTITIES

The vector identities that follow may be proved by expansion in rectangular (or general curvilinear) coordinates. The first two identities involve the scalar and vector triple products, the next three are concerned with operations on sums, the following three apply to operations when the argument is multiplied by a scalar function, the next three apply to operations on scalar or vector products, and the last four concern the second-order operations.

$$(\mathbf{A} \times \mathbf{B}) \cdot \mathbf{C} \equiv (\mathbf{B} \times \mathbf{C}) \cdot \mathbf{A} \equiv (\mathbf{C} \times \mathbf{A}) \cdot \mathbf{B} \tag{A.6}$$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \equiv (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \tag{A.7}$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) \equiv \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \tag{A.8}$$

$$\nabla(V + W) \equiv \nabla V + \nabla W \tag{A.9}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) \equiv \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \tag{A.10}$$

$$\nabla \cdot (V\mathbf{A}) \equiv \mathbf{A} \cdot \nabla V + V\nabla \cdot \mathbf{A} \tag{A.11}$$

$$\nabla(VW) \equiv V\nabla W + W\nabla V \tag{A.12}$$

$$\nabla \times (V\mathbf{A}) \equiv \nabla V \times \mathbf{A} + V\nabla \times \mathbf{A} \tag{A.13}$$

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) \equiv \mathbf{B} \cdot \nabla \times \mathbf{A} - \mathbf{A} \cdot \nabla \times \mathbf{B} \tag{A.14}$$

$$\begin{aligned}\nabla(\mathbf{A} \cdot \mathbf{B}) \equiv\ & (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) \\ & + \mathbf{B} \times (\nabla \times \mathbf{A})\end{aligned} \tag{A.15}$$

$$\nabla \times (\mathbf{A} \times \mathbf{B}) \equiv \mathbf{A}\nabla \cdot \mathbf{B} - \mathbf{B}\nabla \cdot \mathbf{A} + (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} \tag{A.16}$$

$$\nabla \cdot \nabla V \equiv \nabla^2 V \tag{A.17}$$

$$\nabla \cdot \nabla \times \mathbf{A} \equiv 0 \tag{A.18}$$

$$\nabla \times \nabla V \equiv 0 \tag{A.19}$$

$$\nabla \times \nabla \times \mathbf{A} \equiv \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} \tag{A.20}$$