EEE 2204

CIRCUITS & NETWORK THEORY 11

Lecture 8

Laplace Transform

- When using phasors for the analysis of circuits, we transform the circuit from the time domain to the frequency or phasor domain.
- Once we obtain the phasor result, we transform it back to the time domain.
- The Laplace transform method follows the same process.
- We use Laplace transformation to transform the circuit from the time domain to the frequency domain, obtain the solution and apply inverse Laplace transform to the result to transform it back to the time domain.
- The Laplace transform is an integral transformation of a function f(t) from the time domain into the complex frequency domain, giving F(s).

Significance of LT

- It can be applied to a wider variety of inputs than phasor analysis.
- It provides an easy way to solve circuit problems involving initial conditions, because it allows us to work with algebraic equations instead of differential equations.
- The Laplace transform is capable of providing us, in one single operation, the total response of the circuit comprising both the natural and forced responses.

Definition of the Laplace Transform

Given a function $f(t)$, its Laplace transform, denoted by $F(s)$ or $\mathcal{L}[f(t)]$ is defined by:

$$\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}dt \qquad (1\text{–}1)$$

where s is a complex variable given by:

$$s = \sigma + j\omega$$

The Laplace transform in Eq. 1–1 is known as the *one-sided (or unilateral)* Laplace transform.

A companion to the direct Laplace transform in Eq. 1–1 is the *inverse* Laplace transform given by:

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}ds \qquad (1\text{–}2)$$

We will however not use Eq. 1–2 to find the inverse Laplace transform but a look-up table shown in Fig. 1.1.The functions $f(t)$ and $F(s)$ are regarded as a Laplace transform pair where:

$$f(t) \Leftrightarrow F(s)$$

meaning that there is one-to-one correspondence between $f(t)$ and $F(s)$.

Laplace pairs

Laplace transform pairs.*

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
$t^n e^{-at}$	$\frac{n!}{(s+a)^{n+1}}$

$f(t)$	$F(s)$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t+\theta)$	$\frac{s\sin\theta+\omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t+\theta)$	$\frac{s\cos\theta-\omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t)=0$, for $t<0$.

Determine the Laplace transform of each of the following functions:

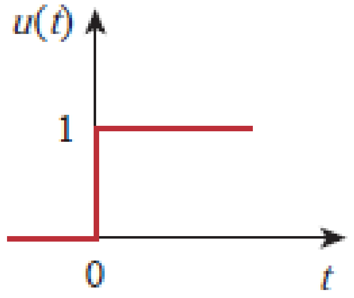
(a) $u(t)$, (b) $e^{-at}u(t)$, $a \geq 0$,

Solution:

(a) For the unit step function $u(t)$, shown in Fig.(a), the Laplace transform is

$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} 1e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty}$$

$$= -\frac{1}{s}(0) + \frac{1}{s}(1) = \frac{1}{s}$$

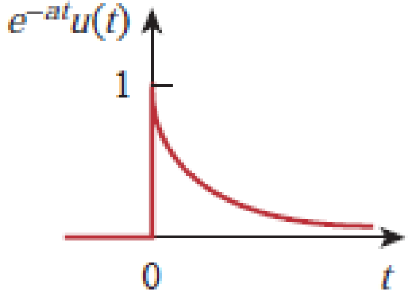


(a)

(b) For the exponential function, shown in Fig. (b), the Laplace transform is

$$\mathcal{L}[e^{-at}u(t)] = \int_{0^-}^{\infty} e^{-at}e^{-st}\,dt$$

$$= -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a}$$



(b)

Determine the Laplace transform of $f(t) = \sin \omega t u(t)$.

Solution:

$$F(s) = \mathcal{L}[\sin \omega t] = \int_0^\infty (\sin \omega t) e^{-st}\, dt = \int_0^\infty \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j}\right) e^{-st}\, dt$$

$$= \frac{1}{2j} \int_0^\infty \left(e^{-(s-j\omega)t} - e^{-(s+j\omega)t}\right) dt$$

$$= \frac{1}{2j}\left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega}\right) = \frac{\omega}{s^2 + \omega^2}$$

Find the Laplace transform of $f(t) = 10 \cos \omega t u(t)$.

Answer: $10s/(s^2 + \omega^2)$.

Properties of the Laplace Transform

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as} F(s)$
Frequency shift	$e^{-at} f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2 f}{dt^2}$	$s^2 F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3 f}{dt^3}$	$s^3 F(s) - s^2 f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^n f}{dt^n}$	$s^n F(s) - s^{n-1} f(0^-) - s^{n-2} f'(0^-) - \cdots - f^{(n-1)}(0^-)$

Properties of the Laplace Transform

Time integration	$\int_0^t f(t)\,dt$	$\frac{1}{s}F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^{\infty} F(s)\,ds$
Time periodicity	$f(t) = f(t + nT)$	$\frac{F_1(s)}{1 - e^{-sT}}$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

(i) Obtain the Laplace transform of:

$$f(t) = \delta(t) + 2u(t) - 3e^{-2t}u(t)$$

Solution:

$$F(s) = \mathcal{L}[\delta(t)] + 2\mathcal{L}[u(t)] - 3\mathcal{L}[e^{-2t}u(t)]$$

$$= 1 + 2\frac{1}{s} - 3\frac{1}{s+2} = \frac{s^2+s+4}{s(s+2)}$$

(ii). Determine the Laplace transform of:

$$f(t) = t^2 \sin 2t u(t)$$

Solution:

Using the Laplace transform tables:

$$\mathcal{L}[\sin \omega t] = \frac{\omega}{s^2+\omega^2} \Rightarrow \mathcal{L}[\sin 2t] = \frac{2}{s^2+4}$$

Using frequency differentiation given by:

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s)$$

$$F(s) = \mathcal{L}[t^2 \sin 2t] = (-1)^2 \frac{d^2}{ds^2}\left(\frac{2}{s^2+4}\right)$$

$$= \frac{d}{ds}\left(\frac{-4s}{(s^2+4)^2}\right) = \frac{12s^2-16}{(s^2+4)^3}$$

The Inverse Laplace Transform

- Given F(s), we transform it back to the time domain and obtain the corresponding f(t) by matching entries in the table of the Laplace transform pairs.
- Suppose F(s) has the general form of:

$$F(s) = \frac{N(s)}{D(s)}$$

- where N(s) is the numerator polynomial and D(s) is the denominator polynomial.
- The roots of N(s) are called the zeros of F(s), while the roots of D(s) are the poles of F(s).
- We use partial fraction expansion to break F(s) down into simple terms whose inverse transform we obtain from the table of Laplace transform pairs.
- The poles may be simple, repeated or complex.

Partial Fraction Expansion

Simple poles

$$F(s) = \frac{N(s)}{(s+p_1)(s+p_2)\dots(s+p_n)}$$

where

$$s = -p_1, -p_2, \dots, -p_n$$

$$p_i \neq p_j \quad \forall \quad i \neq j$$

for the degree of $N(s)$ smaller than that of $D(s)$ the partial fraction expansion decomposes F(s) as

$$F(s) = \frac{k_1}{s+p_1} + \frac{k_1}{s+p_2} + \dots + \frac{k_1}{s+p_n}$$

The expansion coefficients k_1, $k_2, \dots, k_n$ are known as the residues of F(s). Employing the residue method and multiplying both sides by $(s+p_1)$, we get

$$F(s)(s+p_1) = \frac{k_1(s+p_1)}{s+p_1} + \frac{k_1(s+p_1)}{s+p_2} + \dots + \frac{k_n(s+p_1)}{s+p_n}$$

since $p_i \neq p_j$, setting $s = p_1$ leaves k_1 on the RHS. Hence

$$(s+p_1)F(s)\Big|_{s=p_1} = k_1$$

Generally $k_i = (s+p_i)F(s)\Big|_{s=p_i}$

Example

Find the inverse Laplace transform of

$$F(s) = \frac{3}{s} - \frac{5}{s+1} + \frac{6}{s^2+4}$$

Solution:

The inverse transform is given by:

$$f(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left(\frac{3}{s}\right) - \mathcal{L}^{-1}\left(\frac{5}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{6}{s^2+4}\right)$$

$$= (3 - 5e^{-t} + 3\sin 2t)u(t), \qquad t \geq 0$$

Example

Find $f(t)$ given that:

$$F(s) = \frac{s^2 + 12}{s(s+2)(s+3)}$$

Solution:

Unlike in the previous example where the partial fractions have been provided, we first need to determine the partial fractions. Since there are three poles, we let:

$$\frac{s^2 + 12}{s(s+2)(s+3)} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{s+3} \qquad \textit{(i)}$$

where A, B, and C are the constants to be determined. We can find the constants using two approaches:

Residue Method

- Method 1: Residue Method

$$A = sF(s)|_{s=0} = \frac{s^2+12}{(s+2)(s+3)}|_{s=0} = \frac{12}{(2)(3)} = 2$$

$$B = (s+2)F(s)|_{s=-2} = \frac{s^2+12}{s(s+3)}|_{s=-2} = \frac{4+12}{(-2)(1)} = -8$$

$$C = (s+3)F(s)|_{s=-3} = \frac{s^2+12}{s(s+2)}|_{s=-3} = \frac{9+12}{(-3)(-1)} = 7$$

$$F(s) = \frac{2}{s} - \frac{8}{s+2} + \frac{7}{s+3}$$

By finding the inverse transform of each term, we obtain:

$$f(t) = (2 - 8e^{-2t} + 7e^{-3t})u(t)$$

- **Method 2: Algebraic Method**

  Multiplying both sides of Eq. *(i)* by $s(s+2)(s+3)$ gives:

$$s^2 + 12 = A(s+2)(s+3) + Bs(s+3) + Cs(s+2)$$

$$s^2 + 12 = A(s^2 + 5s + 6) + B(s^2 + 3s) + C(s^2 + 2s)$$

Equating the coefficients of like powers of s gives:

$$\text{Constant} : 12 = 6A \Rightarrow A = 2$$

$$s : 0 = 5A + 3B + 2C \Rightarrow 3B + 2C = -10$$

$$s^2 : 1 = A + B + C \Rightarrow B + C = -1$$

Thus $A = 2, B = -8, C = 7$ and Eq. *(i)* becomes:

$$F(s) = \frac{2}{s} - \frac{8}{s+2} + \frac{7}{s+3}$$

By finding the inverse transform of each term, we obtain:

$$f(t) = (2 - 8e^{-2t} + 7e^{-3t})u(t)$$

Practice

(i) Find $f(t)$ if

$$F(s) = \frac{6(s+2)}{(s+1)(s+3)(s+4)}$$

Answer: $f(t) = (e^{-t} + 3e^{-3t} - 4e^{-4t})u(t).$

(ii) Calculate $v(t)$ given that

$$V(s) = \frac{10s^2 + 4}{s(s+1)(s+2)^2}$$

Answer:

$$v(t) = (1 - 14e^{-t} + 13e^{-2t} + 22te^{-2t})u(t)$$

Practice

Find the inverse transform of the frequency-domain function

$$H(s) = \frac{20}{(s+3)(s^2+8s+25)}$$

Solution:

$H(s)$ has a pair of complex poles at $s = -4 \pm j3$. We let:

$$H(s) = \frac{20}{(s+3)(s^2+8s+25)} = \frac{A}{s+2} + \frac{Bs+C}{s^2+8s+25}$$

Multiplying both sides by $(s+3)(s^2+8s+25)$ gives:

$$\begin{aligned} 20 &= A(s^2+8s+25) + (Bs+C)(s+3) \\ &= A(s^2+8s+25) + B(s^2+3s) + C(s+3) \end{aligned}$$

Equating coefficients gives:

$$\begin{aligned} s^2: \quad & 0 = A + B \Rightarrow A = -B \\ s: \quad & 0 = 8A + 3B + C = 5A + C \Rightarrow C = -5A \\ \text{Constant}: \quad & 20 = 25A + 3C = 25A - 15A \Rightarrow A = 2 \end{aligned}$$

Then, $B = -2, C = -10$.

Thus:

$$H(s) = \frac{2}{s+3} - \frac{2s+10}{(s^2+8s+25)} = \frac{2}{s+3} - \frac{2(s+4)+2}{(s+4)^2+9}$$

$$= \frac{2}{s+3} - \frac{2(s+4)}{(s+4^2)+9} - \frac{2}{3}\frac{3}{(s+4)^2+9}$$

Taking the inverse of each term, we obtain:

$$h(t) = (2e^{-3t} - 2e^{-4t}\cos 3t - \frac{2}{3}e^{-4t}\sin 3t)u(t)$$

Practice

Solve for the response $y(t)$ in the following integro-differential equation.

$$\frac{dy}{dt} + 5y(t) + 6\int_0^t y(\tau)d\tau = u(t), \qquad y(0) = 2 \qquad \textit{(i)}$$

Solution:

We use the two properties of the Laplace transform (time differentiation and time integration) to solve integro-differential equations.

$$\begin{aligned} \frac{df}{dt} &= sF(s) - f(0^-) \\ \frac{d^2f}{dt^2} &= s^2F(s) - sf(0^-) - f'(0^-) \qquad \textit{(ii)} \\ \int_0^t f(t)dt &= \frac{1}{s}F(s) \end{aligned}$$

Taking the Laplace transform of each term in Eq. *(i)* we get:

$$[sY(s) - y(0)] + 5Y(s) + \frac{6}{s}Y(s) = \frac{1}{s}$$

Substituting $y(0) = 2$ and multiplying through by s,

$$Y(s)[s^2 + 5s + 6] = 1 + 2s$$

$$Y(s) = \frac{2s+1}{s^2+5s+6} = \frac{A}{s+2} + \frac{B}{s+3}$$

$$A = (s+2)Y(s)|_{s=-2} = \frac{2s+1}{s+3}|_{s=-2} = \frac{-3}{1} = -3$$

$$B = (s+3)Y(s)|_{s=-3} = \frac{2s+1}{s+2}|_{s=-3} = \frac{-5}{-1} = 5$$

Thus

$$Y(s) = \frac{-3}{s+2} + \frac{5}{s+3}$$

Taking the inverse laplace transform:

$$y(t) = (-3e^{-2t} + 5e^{-3t})u(t)$$