Chapter 5

Asymptotic Bode plots

5.1 Bode diagrams

5.2 Introduction

Bode plots represent the frequency response of a system in two separate graphs: a magnitude response graph and a phase response graph. The frequency axes in both graphs are graduated in log scale while the magnitude axis in the magnitude response graph is graduated in dB scale. This makes sketching and analysis of phase and magnitude response plots easier than it would be if the scales were not logarithmic.

5.2.1 Magnitude axis in dB

Any transfer function can be broken down into a product of a constant (K), first order terms ($(\tau s + 1)^{\pm 1}$) and second order terms ($\left(\frac{s^2}{\omega_0^2} + \frac{1}{Q}\frac{s}{\omega_0} + 1\right)^{\pm 1}$). The overall gain becomes a product of the individual terms and the phase becomes a sum of the phases of the individual terms. However, if a $\log$ of the gains is taken, then the total gain becomes a sum of the gains of individual terms. Equation 5.1 shows a transfer function as a product of n smaller transfer functions and Eq. 5.2 shows its corresponding frequency response function. Equation 5.3 shows that the magnitude of a product of transfer functions is a product of the individual magnitudes of the n transfer functions. The phase shown in Eq. 5.4 is already in sum form.

$$T(s) = T_1(s)T_2(s)\cdots T_n(s) \tag{5.1}$$

$$T(j\omega) = T_1(j\omega)T_2(j\omega)\ldots T_n(j\omega) \tag{5.2}$$

$$|T(j\omega)| = |T_1(j\omega)||T_2(j\omega)|\ldots|T_n(j\omega)| \tag{5.3}$$

$$\theta(\omega) = \theta_1(j\omega) + \theta_2(j\omega) + \ldots + \theta_n(j\omega) \tag{5.4}$$

To write the magnitude in sum form, we can express it in dB, or any other logarithmic measure, so that products become sums. This is shown in Eq. 5.6.

$$20\log_{10}|T(j\omega)| = 20\log_{10}\left(|T_1(j\omega)||T_2(j\omega)|\ldots|T_n(j\omega)|\right) \tag{5.5}$$

$$20\log_{10}|T(j\omega)| = 20\log_{10}|T_1(j\omega)| + 20\log_{10}|T_2(j\omega)| + \ldots + 20\log_{10}|T_n(j\omega)| \tag{5.6}$$

Using Eq. 5.6 and Eq. 5.4, the magnitude and phase of a transfer function can be computed by summing the contributions of its terms.

5.2.2 Frequency axis in $\log$ scale

When the frequency axis is also presented in $\log$ scale, slopes in the magnitude and phase plots become linear. Let us illustrate this using frequency dependent first and second order terms.

5.2.2.1 First order terms

Let us consider a generic bilinear transfer function

$$T_k(s) = K\frac{s + z_k}{s + p_k}. \tag{5.7}$$

The idea for Bode diagrams is to separate the $T_k(s)$ into a gain term and frequency dependent terms. The frequency dependent components should have a low frequency gain of 1 (0 dB), so that the method is applicable to all similarly expressed terms. Re-writing Eq. 5.7 into its constituent gain and gain-normalized frequency dependent terms, the expression becomes

$$T(s) = \left(K\frac{z_k}{p_k}\right)\frac{\frac{s}{z_k}+1}{\frac{s}{p_k}+1}. \tag{5.8}$$

The corresponding frequency response function becomes

$$T(j\omega) = \left(K\frac{z_k}{p_k}\right)\frac{\frac{j\omega}{z_k}+1}{\frac{j\omega}{p_k}+1}, \tag{5.9}$$

from which the magnitude and phase responses can be written as

$$|T(j\omega)| = \left|K\frac{z_k}{p_k}\right|\frac{|\frac{j\omega}{z_k}+1|}{|\frac{j\omega}{p_k}+1|}, \tag{5.10}$$

$$\theta(j\omega) = \angle\left(K\frac{z_k}{p_k}\right) + \angle\left(\frac{j\omega}{z_k}+1\right) - \angle\left(\frac{j\omega}{p_k}+1\right). \tag{5.11}$$

Equations 5.10 and 5.11 can be re-written as

$$20\log_{10}|T(j\omega)| = 20\log_{10}\left(K\frac{z_k}{p_k}\right) + 10\log_{10}\left(\left(\frac{\omega}{z_k}\right)^2+1\right) - 10\log_{10}\left(\left(\frac{\omega}{p_k}\right)^2+1\right), \tag{5.12}$$

$$\theta(j\omega) = 0^\circ \text{ or } 180^\circ + \tan^{-1}\left(\frac{j\omega}{z_k}\right) - \tan^{-1}\left(\frac{j\omega}{p_k}\right), \tag{5.13}$$

where the phase of the constant term is 0° if $K > 0$, and it is 180° if $K < 0$. The second and third terms on the right hand side of Eq. 5.12 and Eq. 5.13 are similar apart from their signs and root location. To examine the behavior of the terms with changes in frequency, we choose to work with the second term in both equations and then extend the results to the third term.

$$|T_2(j\omega)| = 10\log_{10}\left(\left(\frac{\omega}{z_k}\right)^2+1\right), \tag{5.14}$$

$$\theta_2(j\omega) = \tan^{-1}\left(\frac{j\omega}{z_k}\right) \tag{5.15}$$

Examining Eq. 5.14 and Eq. 5.15, the magnitude and phase responses can be obtained for $\omega \ll z_k$, $\omega = z_k$, and $\omega \gg z_k$ as shown in Table 5.1. At high frequencies ($\omega \gg z_k$), the magnitude response asymptote is a linear function of $\log_{10}\omega$, crossing the 0 dB axis at $\omega = z_k$. The low frequency asymptote of the magnitude response is at 0 dB. Phase response on the other hand has a low frequency asymptote

Table 5.1: Response for a first order term.

Response for $T_2(j\omega) = \frac{j\omega}{z_k} + 1$			
Response	$\omega \ll z_k$	$\omega = z_k$	$\omega \gg z_k$
$\mathbf{20}\log_{\mathbf{10}}\lvert\mathbf{T_2(j}\omega)\rvert$	0	3.04 dB	$20\log_{10}\omega - 20\log_{10} z_k$
$\theta_{\mathbf{2}}(\mathbf{j}\omega)$	$0°$	$45°$	$90°$

Table 5.2: Estimating low- and high-frequency asymptotes for a first order term.

Response for $T_2(j\omega) = \frac{j\omega}{z_k} + 1$

Response	$\omega = 0.1z_k$	$\omega = z_k$	$\omega = 10z_k$
$\mathbf{20}\log_{\mathbf{10}}\lvert\mathbf{T_2(j}\omega)\rvert$	0.04	3.01 dB	20.04
$\theta_{\mathbf{2}}(\mathbf{j}\omega)$	$5.72°$	$45°$ dB	$84.29°$

at $0°$ and a high frequency asymptote at $90°$.

For $\omega \gg z_k$, the magnitude plot changes by 20 dB for a change $\times 10$ change in ω (from the expression $20\log_{10}\omega - 20\log_{10} z_k$). This $\times 10$ change in frequency is called a ***decade***. Therefore, the slope of the magnitude plot is ***20 dB/decade***. Another way of interpreting the change in magnitude after every doubling in frequency. This $\times 2$ change in frequency is called an ***octave***. For an octave, the change in gain is $20\log_{10} 2 \approx 6$ dB. Therefore, ***20 dB/decade*** is equivalent to ***6 db/octave***.

For the purposes of sketching, onset of $\omega \ll z_k$ is estimated with $\omega = 0.1z_k$ and the onset of $\omega \gg z_k$ is estimated with $\omega = 10z_1$. Where the high-frequency and low-frequency asymptotes meet for the magnitude response is called the corner frequency, $\omega = z_k$. Table 5.2 shows the computed values at $\omega = 0.1z_k$, $\omega = z_k$, and $\omega = 10z_k$. From Table 5.2, the low frequency and high frequency asymptotes of $\theta_2(j\omega)$ are joined by a $45°$ per decade line from $\omega = 0.1z_k$ to $\omega = 10z_k$. The asymptotes do not give the exact response, but they provide good enough approximations for sketches.

The analysis for a first order zero term also applies for a first order pole term. The difference is that the magnitudes and the phase are negative i.e. magnitude and phase response are mirror images of the first order zero term's responses at 0 dB line for the magnitude and $0°$ for the phase.

Table 5.3: Response for a second order term.

Response for $T_2(j\omega) = 1 - \left(\frac{\omega}{\omega_0}\right)^2 + j\frac{\omega}{\omega_0 Q}$

Response	$\omega \ll \omega_0$	$\omega = \omega_0$	$\omega \gg \omega_0$
$\mathbf{20}\log_{\mathbf{10}}\lvert\mathbf{T_2(j}\omega\mathbf{)}\rvert$	0 dB	$-20\log_{10} Q$ dB	$40\log_{10}\omega - 40\log_{10}\omega_0$
$\theta_{\mathbf{2}}(\mathbf{j}\omega)$	$0°$	$90°$	$180°$

5.2.2.2 Second order terms

For a second order term given by $T(s) = s^2 + \frac{\omega_0}{Q}s + \omega_0^2$ (where $Q > 0.5$ leading to complex conjugate poles), the response is written in terms of a constant gain and a frequency dependent part with a gain of 1 at $s = 0$ by factoring out ω_0^2 to obtain

$$T(s) = \omega_0^2\left(\left(\frac{s}{\omega_0}\right)^2 + \frac{1}{Q}\frac{s}{\omega_0} + 1\right) = \omega_0^2 T_2(s) \tag{5.16}$$

and the magnitude and phase responses for the non-constant part

$$20\log_{10}\lvert T_2(j\omega)\rvert = 10\log_{10}\left(\left(1 - \left(\frac{\omega}{\omega_0}\right)^2\right)^2 + \frac{1}{Q^2}\left(\frac{\omega}{\omega_0}\right)^2\right) \tag{5.17}$$

$$\theta_2(j\omega) = \tan^{-1}\left[\frac{\frac{\omega}{\omega_0 Q}}{1 - \left(\frac{\omega}{\omega_0}\right)^2}\right] \tag{5.18}$$

From which the high and low frequencies can be computed as shown in Table 5.3.

Examining the magnitude response, the high frequency asymptote is ***40 dB/decade*** or ***12 dB/octave***, crossing 0 dB axis at ω_0. For the phase plot, the low frequency and high frequency asymptotes should be joined by a line from $\omega = 10^{-\frac{1}{2Q}}\omega_0$ to $\omega = 10^{\frac{1}{2Q}}\omega_0$. However, an approximation using a line from $\omega = 0.1\omega_0$ to $\omega = 10\omega_0$ can be used.

Once again, for complex poles, the analysis is similar, but the slopes and the angles are all negative.

5.2.2.3 Differentiator and integrator

For differentiator ($T(s) = s$), the phase is always $90°$ and the gain is always $20\log_{10}\omega$ for all frequencies. Therefore, its magnitude response has a slope of ***20 dB/decade*** for all frequencies, and it crosses the 0 dB line at $\omega = 1$. The response is the exact opposite: phase of $-90°$ and magnitude response of ***-20 dB/dec***, crossing the 0 dB line at $\omega = 1$.

5.3 Summary of rules for sketching bode diagrams

Table **??** summarizes what you are required to do to create an asymptotic bode plot for different kinds of terms. You can find these rules along with examples and other tools and insight here.

Table 5.4: Summary of rules for creating asymptotic bode.

Summary of rules

	Term	Magnitude	Phase
1	Constant: K	$20\log_{10}\|K\|$	1. $K > 0: \ 0°$ 2. $K < 0: \ \pm180°$
2	Integrator: $\frac{1}{s}$	-20 dB/dec passing through 0 dB at $\omega = 1$	$-90°$
3	Differentiator: s	$+20$ dB/dec passing through 0 dB at $\omega = 1$	$90°$

4	Real pole: $\frac{1}{\frac{s}{\omega_0}+1}$	1. low frequency asymptote: at 0 db, 0 dB/decade. 2. high frequency asymptote: -20 dB/decade. 3. corner frequency: ω_0	1. low requency asymptote: at $0°$. 2. high frequency asymptote: at $-90°$. 3. straight line between $0.1\omega_0$ to $10\omega_0$.
5	Real zero: $\frac{s}{\omega_0}+1$	1. low frequency asymptote: at 0 db, 0 dB/decade. 2. high frequency asymptote: $+20$ dB/decade. 3. corner frequency: ω_0	1. low requency asymptote: at $0°$. 2. high frequency asymptote: at $+90°$. 3. straight line between $0.1\omega_0$ to $10\omega_0$.
6	complex poles: $\frac{1}{\left(\frac{s}{\omega_0}\right)^2+\frac{1}{Q}\left(\frac{s}{\omega_0}\right)+1}$, $Q > 0.5$	1. low frequency asymptote: at 0 db, 0 dB/decade. 2. high frequency asymptote: -40 dB/decade. 3. corner frequency: ω_0 4. if $Q \geq 1$, $T(\omega_0) = 20\log_{10}(Q)$	1. low requency asymptote: at $0°$. 2. high frequency asymptote: at $-180°$. 3. straight line from $0°$ to $-180°$ between $\omega = 0.1\omega_0$ and $\omega = 10\omega_0$ (or use the more accurate $\omega = \frac{\omega_0}{10^{\frac{1}{2Q}}}$ to $\omega = \omega_0 \times 10^{\frac{1}{2Q}}$).

7	complex zeros: $\left(\frac{s}{\omega_0}\right)^2 + \frac{1}{Q}\left(\frac{s}{\omega_0}\right) + 1$, $Q > 0.5$	1. low frequency asymptote: at 0 db, 0 dB/decade. 2. high frequency asymptote: $+40$ dB/decade. 3. corner frequency: ω_0 4. if $Q \geq 1$, $T(\omega_0) = -20\log_{10}(Q)$	1. low frequency asymptote: at $0°$. 2. high frequency asymptote: at $+180°$. 3. straight line from $0°$ to $180°$ between $\omega = 0.1\omega_0$ and $\omega = 10\omega_0$ (or use the more accurate $\omega = \frac{\omega_0}{10^{\frac{1}{2Q}}}$ to $\omega = \omega_0 \times 10^{\frac{1}{2Q}}$).

If there are repeated poles or zeros, the slope is multiplied by the order of the root, and the maximum phase is also multiplied by the order of the root. For instance, if you have s^m, the slope becomes $20m$ dB/dec and the phase varies from $0°$ to $90° \times m$. Another example is that of $1/\left(\frac{s}{\omega_0} + 1\right)$. The corner frequency stays at ω_0, but the gain slope changes to $-20m$ dB/dec and the phase varies from $0°$ to $-90° \times m$.

5.4 Procedure for sketching the Bode diagram of a transfer function

The following procedure can be followed when sketching bode diagrams for a transfer function.

1. Write the transfer function in standard form by
   (a) Factoring out the numerator and denominator of a transfer function into first order terms with real zeros and poles, and second order terms with complex poles and zeros. The factored components of your transfer function should appear in three forms (in the numerator or the denominator):
      - s (for root(s) at 0),

- $s + c$ (where c is the root), and
- $s^2 + \frac{\omega_0}{Q}s + \omega_0^2$, when $Q > 0.5$.

(b) Factoring out the corner frequencies for the first and second order terms as follows

- $c\left(\frac{s}{c} + 1\right)$
- $\omega_0^2\left(\left(\frac{s}{\omega_0}\right)^2 + \frac{s}{\omega_0 Q} + 1\right)$

(c) Multiplying the factored out terms by the gain K.

2. drawing the magnitude and phase response asymptotes for each term written in standard form.

3. summing the asymptotes to get the combined estimate of the response.

You can find examples here, along with more elaborate explanations.

Why learn sketching bode diagrams by hand?

As with many design and analysis tools for engineering, science, and other disciplines, there are computer programs that can be used to plot and analyze Bode diagrams. Nonetheless, to accurately interpret bode diagrams during design and analysis, engineers and scientists require the background theory and the ability to sketch the plot themselves. Upon developing Bode diagram interpretation skills, it should be possible to estimate a system's transfer function from it's bode diagram.

5.5 Examples

Exercise 19:

Plot the bode diagram for the following function.

$$T(s) = 2.5\frac{s^2 + s + 100}{s^3 + 25s^2}$$

Solution 19:

In the standard form:

$$T(s) = \left(2.5 \times \frac{100}{25}\right)\left(\frac{s^2}{100} + \frac{s}{100} + 1\right) \times \frac{1}{s^2} \times \frac{1}{s+25}$$

Individual terms of the transfer function are

$$\begin{aligned}
K &= 10 \\
T_1(s) &= \frac{s^2}{100} + \frac{s}{100} + 1 \\
T_2(s) &= \frac{1}{s^2} \\
T_3(s) &= \frac{1}{\frac{s}{25} + 1}
\end{aligned}$$

The bode diagram for each term (as well as the sum) is approximated as shown in Fig. 5.1. $T_1(s)$ is a second order term with corner frequency at $\omega_0 = 10$, $T_2(s)$ is a repeated pole (order 2) at 0, and $T_3(s)$ is a first order pole with its corner frequency at $\omega = 25$. In the magnitude plot,

1. K gain is constant at $20 \log_{10} |K| = 20$ dB,
2. $T_1(s)$ (complex conjugate zeros) the low and high frequency asymptotes of magnitude response meet at $\omega = 10$, and the low and high frequency asymptotes are at 0 dB and a slope of 40 dB/decade, respectively,
3. $T_2(s) = 1/s^2$ (repeated pole at 0), the slope of the magnitude response is $2 \times (-20) = -40$ db/decade, and it crosses 0 dB line at $\omega = 1$, and
4. $T_3(s)$ (simple non-zero pole), low and high frequency asymptotes are 0 dB and -20 db/decade, and they intersect at $\omega = 25$.

In the combined magnitude plot, we start at low frequency. The important operation here is shifting of slopes (by the gain) and summation of slopes at each frequency.

1. Before the first corner frequency, the gain shifts the $-40 + 0 + 0 = -40$ dB/decade graph up by 20 dB.

2. At $\omega = 10$, where a 40 dB/decade slope starts, total slope becomes $-40+40+0 = 0$ db/decade.

3. The 0 dB/decade slope meets another corner frequency at $\omega = 25$, where the total slope changes to $-40 + 40 - 20 = -20$ dB/decade.

For the phase plot,

1. K gain is positive, so it has a phase of $0°$,

2. $T_1(s)$ (complex conjugate zeros) the low and high frequency asymptotes are $0°$ and $180°$, and they are joined by a line from $\omega = 1$ to $\omega = 100$,

3. $T_2(s) = 1/s^2$ (repeated pole at 0), the phase response is constant at $-90° \times 2 = -180°$, and

4. $T_3(s)$ (simple non-zero pole), low and high frequency asymptotes are $0°$ and $-90°$, and they are jointed by a line from $\omega = 2.5$ to $\omega = 250$.

In the phase plot, we start at low frequency.

1. At low frequencies, the total phase is $-180°$ with a $0°$ per decade slope.

2. At $\omega = 0.1$, where a $90°$ per decade slope starts, total slope becomes $90 + 0 + 0 = 90°$ per decade.

3. At $\omega = 2.5$, where another slope starts ($-45°$ per decade) the total slope changes to $90-45 = 45°$ per decade.

4. At $\omega = 100$, $90°$ per decade slope ends, the total slope becomes $0 - 45 = -45°$ per decade.

5. At $\omega = 250$, $-45°$ per decade slope ends, the total slope becomes $0 + 0 = 0°$ per decade.

Exercise 20:

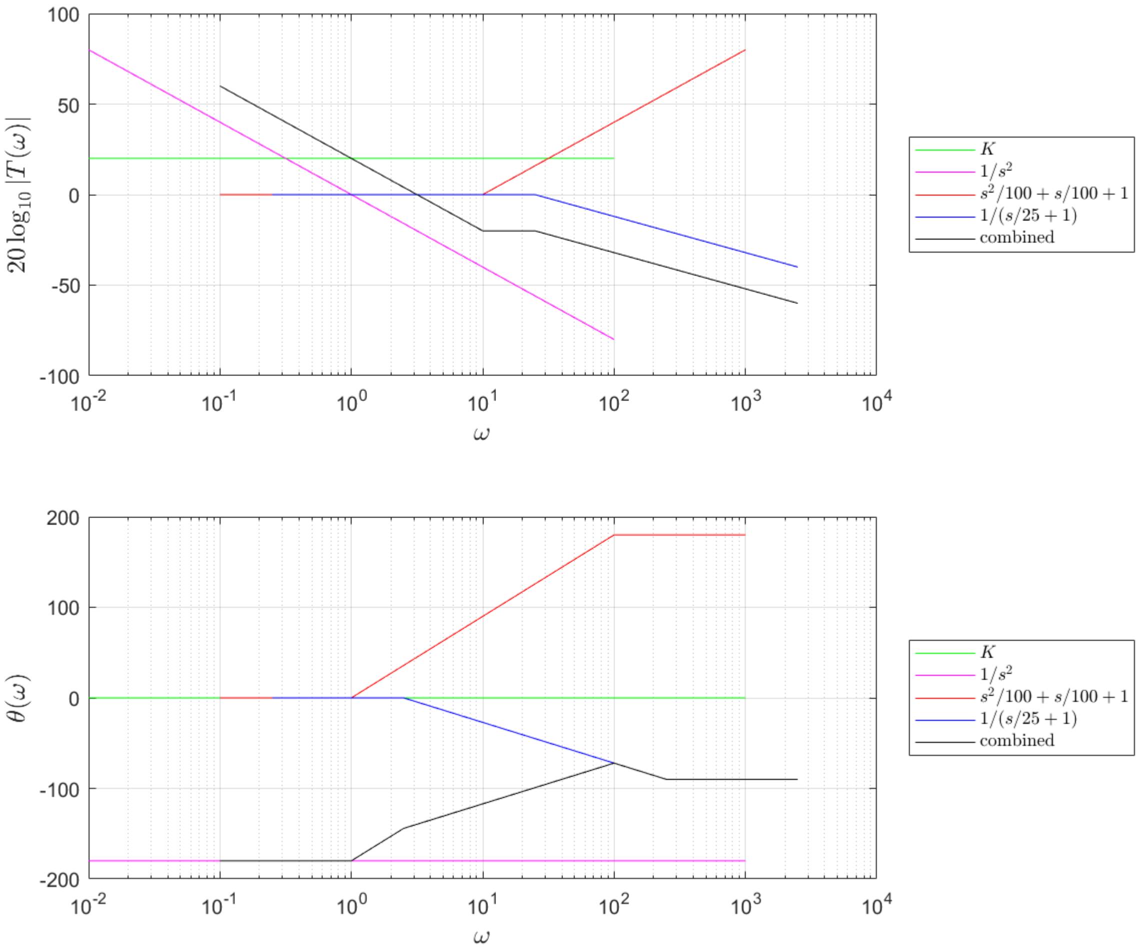

Fig. 5.1: Bode diagram for $T(s) = 5\frac{s^2+s+100}{s^3+25s^2}$. The low frequency and high frequency phase asymptotes for $T_1(s) = \frac{s^2}{100} + \frac{s}{100} + 1$ are connected by a slope of $90°$ per decade from $\omega = 0.1\omega_0$ to $\omega = 10\omega_0$. This approximation is still acceptable.

Sketch the Bode diagrams for the following transfer functions.

$$T(s) = \frac{10s}{s^3 + 12s^2 + 21s + 10}$$
$$T(s) = \frac{1000s}{s + 10}$$
$$T(s) = \frac{100s + 100}{s^2 + 11s + 100}$$
$$T(s) = \frac{100s^2 + 100s + 5000}{s^3 + 200s^2 + 1000s}$$

Solution 20:

Let us start with $T(s) = \frac{10s}{s^3+12s^2+21s+10}$. In standard form:

$$\begin{aligned} T(s) &= 10\,(s)\left(\frac{1}{s+1}\right)^2\left(\frac{1}{s+10}\right) \\ &= \left(10\frac{1}{10}\right)\left(\frac{1}{s+1}\right)^2\left(\frac{1}{s/10+1}\right) \end{aligned}$$

The terms are:

$$\begin{aligned} K &= 1 \\ T_1(s) &= s \\ T_2(s) &= \left(\frac{1}{s+1}\right)^2 \\ T_3(s) &= \frac{1}{s/10+1} \end{aligned}$$

The corresponding asymptotic bode diagram is shown in Fig. 5.2

The asymptotic plots for the other functions are Figs. 5.3 to 5.5 for $T(s) = \frac{1000s}{s+10}$, $T(s) = \frac{100s+100}{s^2+11s+100}$, and $T(s) = \frac{100s^2+100s+5000}{s^3+200s^2+1000s}$, respectively.

Visit here and here to view some more plots and to get more insight. You can check if your hand drawn attempts for other functions are correct using the tool provided here.

Exercise 21:

Can you estimate the transfer function resulting in the Bode diagram shown in Fig. 5.6.

Solution 21:

Try this for fun. You can get material elsewhere explaining how you can estimate transfer functions from Bode plots.

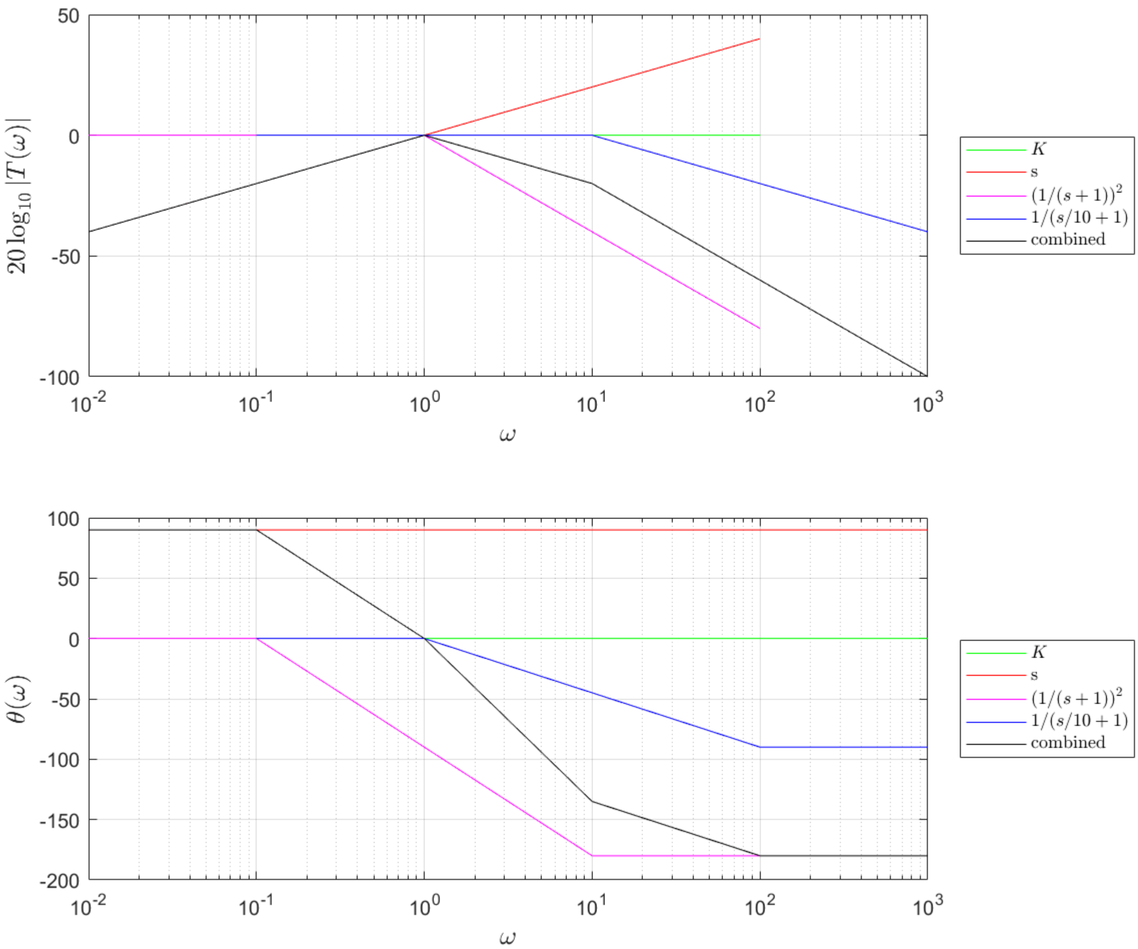

Fig. 5.2: Bode diagram for $T(s) = 10\frac{s}{s^3+12s^2+21s+10}$.

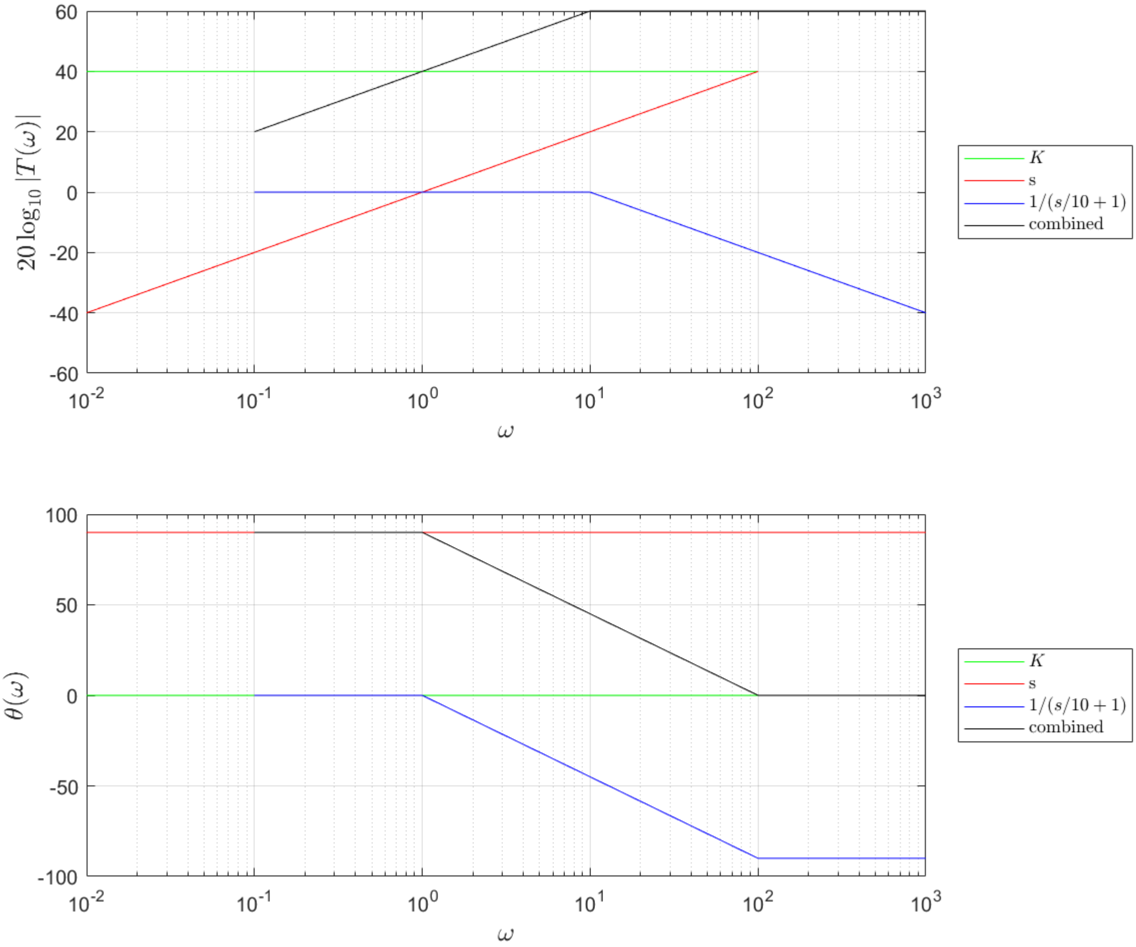

Fig. 5.3: Bode diagram for $T(s) = \frac{1000s}{s+10}$.

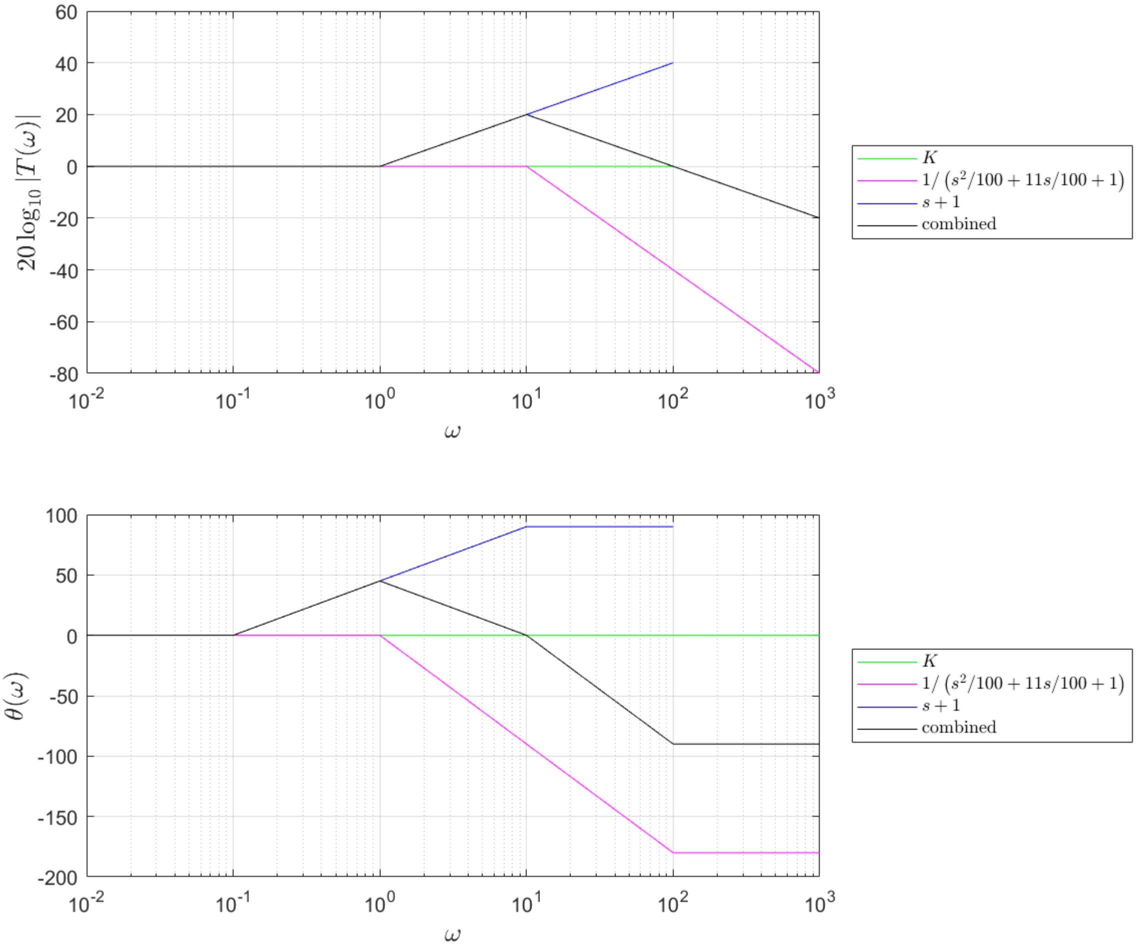

Fig. 5.4: Bode diagram for $T(s) = \frac{100s+100}{s^2+11s+100}$.

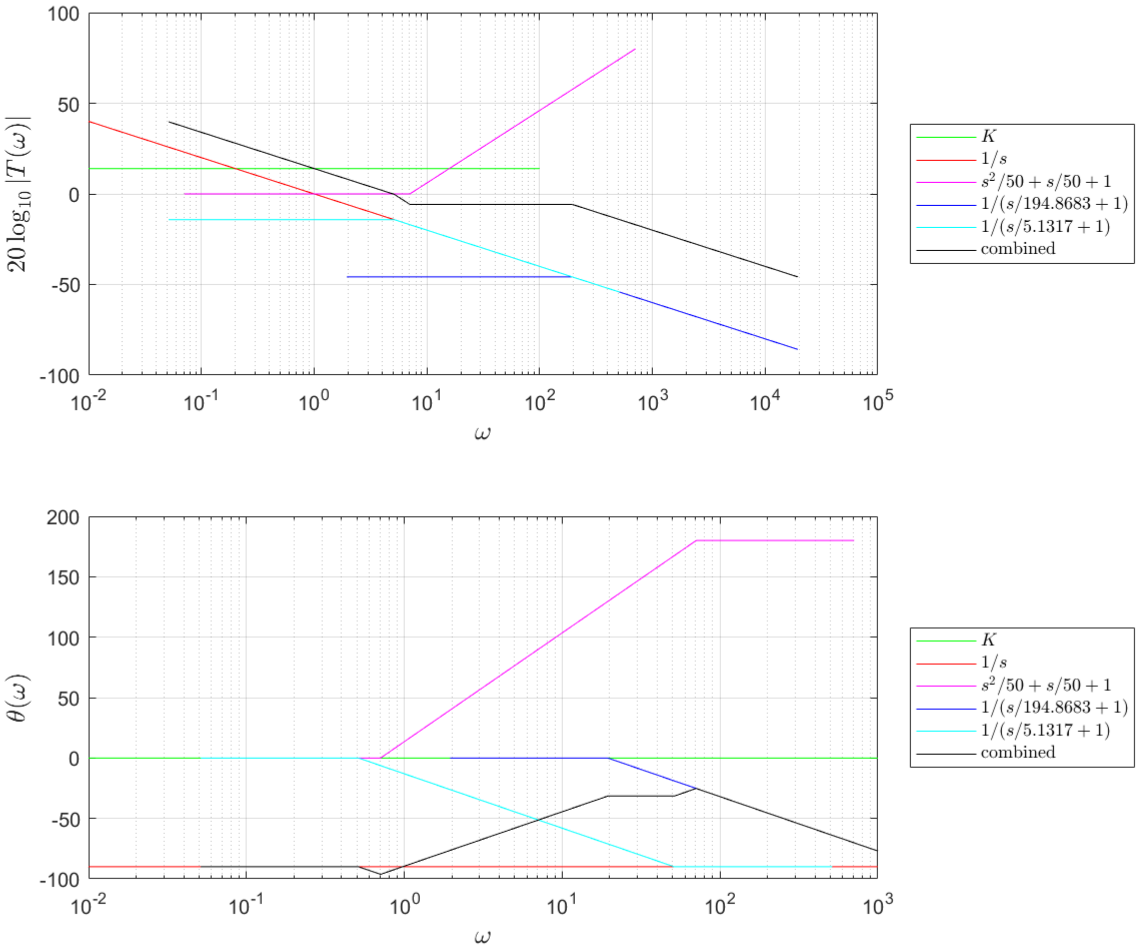

Fig. 5.5: Bode diagram for $T(s) = \frac{100s^2+100s+5000}{s^3+200s^2+1000s}$.

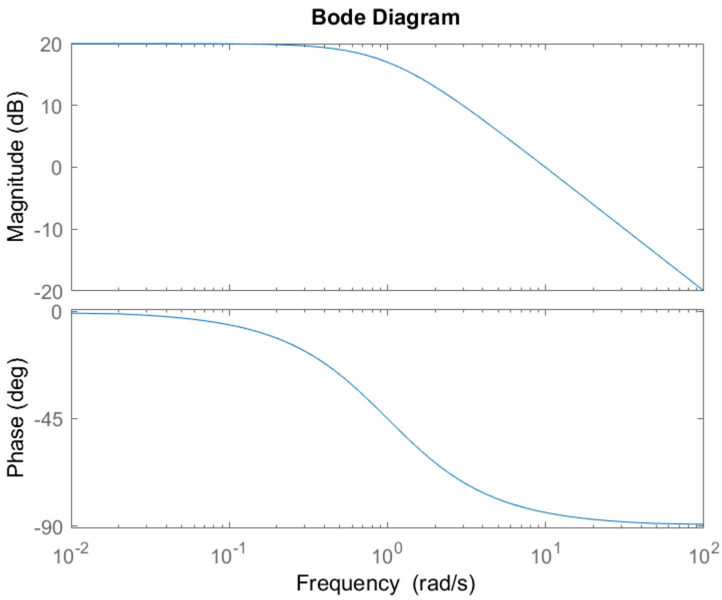

Fig. 5.6: Bode plot.

Chapter 6

Scaling and sensitivity

6.1 Sensitivity

There are many ways of realizing transfer functions. Some considerations that can make a designer select one circuit over another during the design of a transfer function are

- cost of realization, and
- sensitivity.

Design is carried out using nominal values, but in practice, component values change in tme due to environmental factors, aging, defects, mechanical, and chemical changes. These changes in component values away from their nominal values lead to changes in the realized transfer function. Sensitivity is a measure of the extent to which changes in circuit components or parameters change a selected circuit behavior.

To compensate for changes that variations from nominal design lead to, a designer may choose to

1. use a design that compensates for changes from nominal values, or
2. select a design whose desired behavior is less sensitive to variations in component values.

Since it is generally very difficult and costly to develop designs that compensate for variations from nominal design, selecting circuits with low sensitivity to changes is the more practical option for many applications.