MAGNETIC FLUX DENSITY—MAXWELL'S EQUATION

The magnetic flux density **B** is similar to the electric flux density **D**. As $D = \varepsilon_0 E$ in free space, the magnetic flux density **B** is related to the magnetic field intensity **H** according to

$$\mathbf{B} = \mu_o \mathbf{H} \qquad 25$$

Where μ_0 is a constant known as the *permeability of free space.* The constant is in Henrys per meter (H/m) and has the value of

$$\mu_o = 4\pi \times 10^{-7} \text{ H/m}$$

The magnetic flux through a surface S is given by

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} \qquad 26$$

Where the magnetic flux Ψ is in webers (Wb) and the magnetic flux density is in webers per square meter (Wb/m^2) or teslas (T).

A magnetic flux line is a path to which **B** is tangential at every point on the line. It is a line along which the needle of a magnetic compass will orient itself if placed in the presence of a magnetic field.

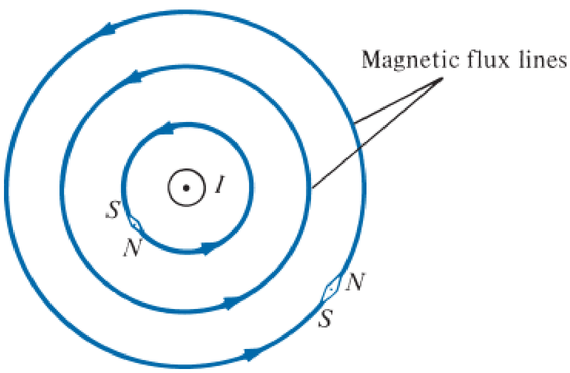


FIGURE 11 Magnetic flux lines due to a straight wire with current coming out of the page.

Magnetic flux lines always close upon themselves as in Figure 12. This is because *it is not possible to have isolated magnetic poles (or magnetic charges).*

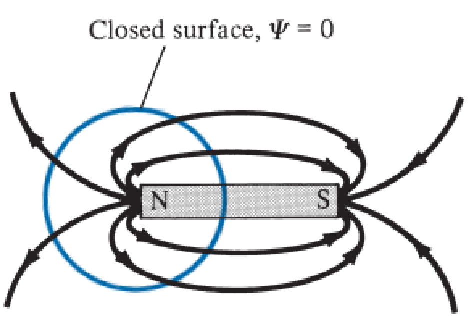


Figure 12.

An isolated magnetic charge does not exist.

Thus the total flux through a closed surface in a magnetic field must be zero; that is,

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0 \qquad 27$$

This equation is referred to as the *law of conservation of magnetic flux* or *Gauss's law for magnetostatic fields*. Although the magnetostatic field is not conservative, magnetic flux is conserved. By applying the divergence theorem to eq. (27), we obtain

$$\oint_S \mathbf{B} \cdot d\mathbf{S} = \int_v \nabla \cdot \mathbf{B}\, dv = 0 \qquad 28$$

Or

$$\nabla \cdot \mathbf{B} = 0 \qquad 29$$

This equation is the fourth Maxwell's equation.

MAXWELL'S EQUATIONS FOR STATIC FIELDS

It should be noted that Maxwell's equations as in Table 2 are only for static electric and magnetic fields.

TABLE 2 Maxwell's Equations for Static Electric and Magnetic Fields

Differential (or Point) Form	Integral Form	Remarks
$\nabla \cdot \mathbf{D} = \rho_v$	$\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_v \rho_v\, dv$	Gauss's law
$\nabla \cdot \mathbf{B} = 0$	$\oint_S \mathbf{B} \cdot d\mathbf{S} = 0$	Nonexistence of magnetic monopole
$\nabla \times \mathbf{E} = \mathbf{0}$	$\oint_L \mathbf{E} \cdot d\mathbf{l} = 0$	Conservative nature of electrostatic field
$\nabla \times \mathbf{H} = \mathbf{J}$	$\oint_L \mathbf{H} \cdot d\mathbf{l} = \int_S \mathbf{J} \cdot d\mathbf{S}$	Ampère's law

MAGNETIC SCALAR AND VECTOR POTENTIALS

$$\nabla \times (\nabla V) = \mathbf{0}$$
$$\nabla \cdot (\nabla \times \mathbf{A}) = 0 \qquad 30\ a,b$$

We define the magnetic scalar potential Vm (in amperes) as related to H according to;

$$\boxed{\mathbf{H} = -\nabla V_m} \qquad \text{if } \mathbf{J} = \mathbf{0} \qquad 31$$

. Combining eq. 31 with

$$\nabla \times \mathbf{H} = \mathbf{J}$$

Leads to

$$\mathbf{J} = \nabla \times \mathbf{H} = \nabla \times (-\nabla V_m) = \mathbf{0} \qquad 32$$

since *Vm* must satisfy the condition in eq. (30,a).thus the magnetic scalar potential *Vm* is only defined in a region where **J** = **0** as in eq. (31). We should also note that *Vm* satisfies Laplace's equation just as *V* does for electrostatic fields; hence

$$\nabla^2 V_m = 0, \quad (\mathbf{J} = \mathbf{0}) \qquad 33$$

We know that for a magnetostatic field,$\nabla . B = 0$ as stated in eq. (29). To satisfy eqs. (79) and (30,b) simultaneously, we can define the *magnetic vector potential* **A** (in Wb/m) such that

$$\mathbf{B} = \nabla \times \mathbf{A} \qquad 34$$

Just as we defined

$$V = \int \frac{dQ}{4\pi\varepsilon_o R} \qquad 35$$

we can define

$$\mathbf{A} = \int_L \frac{\mu_o I \, d\mathbf{l}}{4\pi R} \text{ for line current} \qquad 36$$

$$\mathbf{A} = \int_S \frac{\mu_o \mathbf{K} \, dS}{4\pi R} \text{ For surface current} \qquad 37$$

$$\mathbf{A} = \int_v \frac{\mu_o \mathbf{J} \, dv}{4\pi R} \text{ For Volume current} \qquad 38$$

An alternative approach of obtaining equations 36, 37, and 38. is given as follows.

Starting with equation (2)

$$\mathbf{H} = \int_L \frac{I \, d\mathbf{l} \times \mathbf{a}_R}{4\pi R^2} \quad \text{(line current)}$$ and equation (25)

$$\mathbf{B} = \mu_o \mathbf{H}$$

We can write;

$$\mathbf{B} = \frac{\mu_o}{4\pi} \int_L \frac{I \, d\mathbf{l}' \times \mathbf{R}}{R^3} \qquad 39$$

where **R** is the distance vector from the line element d**l'** at the source point (x', y', z') to the field point (x, y, z) as shown in Figure 13 and $R = |R|$, that is,

$$R = |\mathbf{r} - \mathbf{r}'| = [(x - x')^2 + (y - y')^2 + (z - z')^2]^{1/2} \qquad 40$$

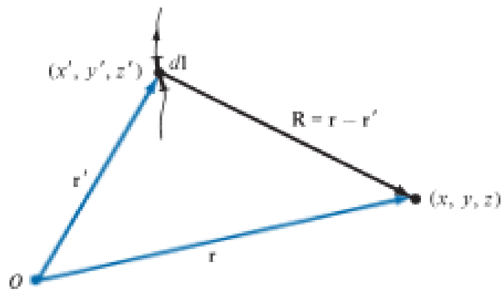


FIGURE 13 Illustration of the source point (x', y', z') and the field point (x, y, z).

Hence,

$$\nabla\left(\frac{1}{R}\right) = -\frac{(x - x')\mathbf{a}_x + (y - y')\mathbf{a}_y + (z - z')\mathbf{a}_z}{[(x - x')^2 + (y - y')^2 + (z - z')^2]^{3/2}} = -\frac{\mathbf{R}}{R^3}$$

Or

$$\frac{\mathbf{R}}{R^3} = -\nabla\left(\frac{1}{R}\right) \quad \left(= \frac{\mathbf{a}_R}{R^2}\right) \qquad 41$$

where the differentiation is with respect to x, y, and z. Substituting this into eq. (39), we obtain

$$\mathbf{B} = -\frac{\mu_o}{4\pi}\int_L I\, d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) \qquad 42$$

We apply the vector identity

$$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + (\nabla f) \times \mathbf{F} \qquad 43$$

where f is a scalar field and F is a vector field. Taking $f = 1/R$ and $F = dl'$, we have

$$d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) = \frac{1}{R}\nabla \times d\mathbf{l}' - \nabla \times \left(\frac{d\mathbf{l}'}{R}\right) \qquad 44$$

Since ∇ operates with respect to (x, y, z) while $d\mathbf{l}'$ is a function of (x', y', z'), $\nabla \times d\mathbf{l}' = \mathbf{0}$. Hence,

$$d\mathbf{l}' \times \nabla\left(\frac{1}{R}\right) = -\nabla \times \frac{d\mathbf{l}'}{R} \qquad 45$$

With this equation, eq. (42) reduces to

$$\mathbf{B} = \nabla \times \int_L \frac{\mu_o I \, d\mathbf{l}'}{4\pi R} \qquad 46$$

Comparing eq. (46) with eq. (34) shows that

$$\mathbf{A} = \int_L \frac{\mu_o I \, d\mathbf{l}'}{4\pi R}$$

Verifying eq. (36).

By substituting eq. (7.39) into eq. (7.32) and applying Stokes's theorem, we obtain

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S} = \oint_L \mathbf{A} \cdot d\mathbf{l}$$

Or

$$\Psi = \oint_L \mathbf{A} \cdot d\mathbf{l} \qquad 47$$

Thus the magnetic flux through a given area can be found by using either eq. (26) or (47). Also, the magnetic field can be determined by using either *Vm* or **A**; the choice is dictated by the nature of the given problem except that *Vm* can be used only in a source-free region. The use of the magnetic vector potential provides a powerful, elegant approach to solving EM problems, particularly those relating to antennas.

Example:

Given the magnetic vector potential $A = \frac{-\rho^2}{4} a_z$ Wb/m, calculate the total magnetic flux crossing the surface $\emptyset = {}^{\pi}/_{2}\,,\ 1 \le \rho \le 2m\,,\ 0 \le z \le 5\ m$

Solution:

We can solve this problem in two different ways: using eq. (26) or eq. (47).

Method 1:

$$\mathbf{B} = \nabla \times \mathbf{A}$$

$$= \begin{vmatrix} a_\rho & a_\varphi & a_z \\ \frac{\delta}{\delta\rho} & \frac{\delta}{\delta\varphi} & \frac{\delta}{dz} \\ 0 & 0 & \frac{-\rho^2}{4} \end{vmatrix} = -\frac{\delta}{\delta\rho}\left(\frac{-\rho^2}{4}\right) a_\varphi = \frac{\rho}{2} a_\varphi$$

Hence:

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S}$$

$$d\mathbf{S} = d\rho \, dz \, \mathbf{a}_\phi$$

$$\Psi = \int_S \mathbf{B} \cdot d\mathbf{S} = \frac{1}{2}\int_{z=0}^{5}\int_{\rho=1}^{2} \rho \, d\rho \, dz = \frac{1}{4}\rho^2 \Big|_2^1 (5) = \frac{15}{4}$$

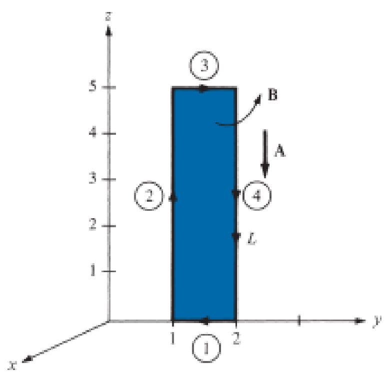
$$\Psi = 3.75 \text{ Wb}$$

Method 2

We use:

$$\Psi = \oint_L \mathbf{A} \cdot d\mathbf{l} = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4$$

where L is the path bounding surface S; Ψ_1, Ψ_2, Ψ_3, and Ψ_4 are, respectively,



Since **A** has only a *z*-component,

$$\Psi_1 = 0 = \Psi_3$$

Therefore,

$$\Psi = \Psi_2 + \Psi_4 = -\frac{1}{4}\left[(1)^2\int_0^5 dz + (2)^2\int_5^0 dz\right]$$

$$= -\frac{1}{4}(1 - 4)(5) = \frac{15}{4}$$

$$= 3.75 \text{ Wb}$$

FORCES DUE TO MAGNETIC FIELDS

There are at least three ways in which force due to magnetic fields can be experienced. The force can be:

(a) due to a moving charged particle in a **B** field.

(b) On a current element in an external **B** field.

(c) Between two current elements.

A. Force on a Charged Particle

The electric force $\mathbf{F}_e$ on a stationary or moving electric charge Q in an electric field is given by Coulomb's experimental law and is related to the electric field intensity **E** as

$$F_e = QE \qquad 48$$

This shows that if Q is positive, F_e and E have the same direction.

A magnetic field can exert force only on a moving charge. From experiments, it is found that the magnetic force $\mathbf{F}_m$ experienced by a charge Q moving with a velocity **u** in a magnetic field **B** is

$$\mathbf{F}_m = Q\mathbf{u} \times \mathbf{B} \qquad 49$$

This clearly shows that F_m is perpendicular to both u and B.

From eqs. (48) and (49), a comparison between the electric force $\mathbf{F}_e$ and the magnetic force $\mathbf{F}_m$ can be made. We see that $\mathbf{F}_e$ is independent of the velocity of the charge and can perform work on the charge and change its kinetic energy. Unlike $\mathbf{F}_e$, $\mathbf{F}_m$ depends on the charge velocity and is normal to it. However, $\mathbf{F}_m$ cannot perform work because it is at right angles to the direction of motion of the charge ($F_m \cdot dl = 0$); it does not cause an increase in kinetic energy of the charge. The magnitude of $\mathbf{F}_m$ is generally small in comparison to $\mathbf{F}_e$ except at high velocities.

For a moving charge Q in the presence of both electric and magnetic fields, the total force on the charge is given by,

$$\mathbf{F} = \mathbf{F}_e + \mathbf{F}_m$$

Or

$$\mathbf{F} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad 50$$

It relates mechanical force to electrical force and is known as *Lorentz force equation.*

If the mass of the charged particle moving in **E** and **B** fields is *m*, by Newton's second law of motion

$$\mathbf{F} = m\frac{d\mathbf{u}}{dt} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B}) \qquad 51$$

TABLE 3 Force on a Charged Particle

State of Particle	E Field	B Field	Combined E and B Fields
Stationary	$Q\mathbf{E}$	—	$Q\mathbf{E}$
Moving	$Q\mathbf{E}$	$Q\mathbf{u} \times \mathbf{B}$	$Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$

B. Force on a Current Element

To determine the force on a current element $I d\mathbf{l}$ of a current-carrying conductor due to the magnetic field **B**, we modify eq. (49) using the fact that for convection current (does not involve conductors)

$$\mathbf{J} = \rho_v \mathbf{u} \qquad 52$$

We recall the relationship between current elements:

$$I\, d\mathbf{l} = \mathbf{K}\, dS = \mathbf{J}\, dv \qquad 53$$

Combining eqs. (52) and (53) yields

$$I\, d\mathbf{l} = \rho_v \mathbf{u}\, dv = dQ\, \mathbf{u}$$

Alternatively,

$$I\, d\mathbf{l} = \frac{dQ}{dt} d\mathbf{l} = dQ \frac{d\mathbf{l}}{dt} = dQ\, \mathbf{u}$$

Hence;

$$I\, d\mathbf{l} = dQ\, \mathbf{u} \qquad 54$$

This shows that an elemental charge dQ moving with velocity **u** (thereby producing convection current element $dQ\,\mathbf{u}$) is equivalent to a conduction current element $I d\mathbf{l}$. Thus the force on a current element $I\, d\mathbf{l}$ in a magnetic field **B** is found from eq. (49) by merely replacing $Q\mathbf{u}$ by $I d\mathbf{l}$; that is

$$d\mathbf{F} = I\, d\mathbf{l} \times \mathbf{B} \qquad 55$$

If the current I is through a closed path L or circuit, the force on the circuit is given by;

$$\mathbf{F} = \oint_L I\, d\mathbf{l} \times \mathbf{B} \qquad 56$$

C. Force between Two Current Elements

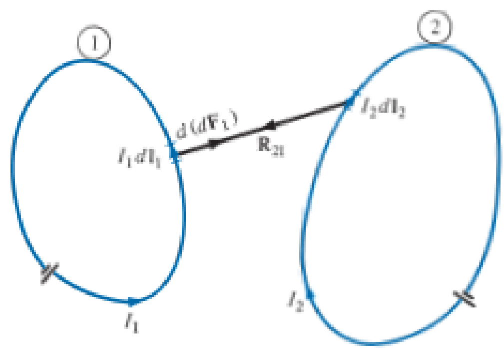


FIGURE 14 Force between two current loops.

Consider the force between two elements $I_1 d\mathbf{l}_1$ and $I_2 d\mathbf{l}_2$

According to Biot– Savart's law, both current elements produce magnetic fields. So we may find the force $d(d\mathbf{F}_1)$ on element $I_1 d\mathbf{l}_1$ due to the field $d\mathbf{B}_2$ produced by element $I_2 d\mathbf{l}_2$ as shown in Figure 14.

From eq. (56),

$$d(d\mathbf{F}_1) = I_1\, d\mathbf{l}_1 \times d\mathbf{B}_2 \qquad 57$$

But from Biot–Savart's law,

$$d\mathbf{B}_2 = \frac{\mu_o I_2\, d\mathbf{l}_2 \times \mathbf{a}_{R_{21}}}{4\pi R_{21}^2} \qquad 58$$

Hence:

$$d(d\mathbf{F}_1) = \frac{\mu_o I_1\, d\mathbf{l}_1 \times (I_2\, d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{4\pi R_{21}^2} \qquad 59$$

This equation is essentially the law of force between two current elements and is analogous to Coulomb's law, which expresses the force between two stationary charges. From eq. (59), we obtain the total force $\mathbf{F}_1$ on current loop 1 due to current loop 2 shown in Figure 14 as

$$\mathbf{F}_1 = \frac{\mu_o I_1 I_2}{4\pi} \oint_{L_1} \oint_{L_2} \frac{d\mathbf{l}_1 \times (d\mathbf{l}_2 \times \mathbf{a}_{R_{21}})}{R_{21}^2} \qquad 60$$

The force $\mathbf{F}_2$ on loop 2 due to the magnetic field $\mathbf{B}_1$ from loop 1 is obtained from eq. (60) by interchanging subscripts 1 and 2. Obeys Newton's third law that action and reaction are equal and opposite.

Example:

A charged particle moves with a uniform velocity $4a_x$ m/s in a region where $E = 20a_y$ V/m and $B = B_0 a_z$Wb/m^2. Determine B_o such that the velocity of the particle remains constant.

Solution:

If the particle moves with a constant velocity, it is implied that its acceleration is zero. In other words, the particle experiences no net force. Hence,

$$\mathbf{0} = \mathbf{F} = m\mathbf{a} = Q(\mathbf{E} + \mathbf{u} \times \mathbf{B})$$

$$\mathbf{0} = Q(20\mathbf{a}_y + 4\mathbf{a}_x \times B_o\mathbf{a}_z)$$

Or

$$-20\mathbf{a}_y = -4B_o\mathbf{a}_y$$

Thus $B_0 = 5$

Example 2

A charged particle of mass 2 kg and charge 3 C starts at point (1, -2, 0) with velocity $4a_x + 3a_z$ m/s in an electric field $12a_x + 10a_y$ V/m. At time t = 1 s, determine;

(a) The acceleration of the particle

(b) Its velocity

(c) Its kinetic energy

(d) Its position

Solution:

(a) This is an initial-value problem because initial values are given. According to Newton's second law of motion,

$$\mathbf{F} = m\mathbf{a} = Q\mathbf{E}$$

Where **a** is the acceleration of the particle. Hence,

$$\mathbf{a} = \frac{Q\mathbf{E}}{m} = \frac{3}{2}(12\mathbf{a}_x + 10\mathbf{a}_y) = 18\mathbf{a}_x + 15\mathbf{a}_y \text{ m/s}^2$$

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = \frac{d}{dt}(u_x, u_y, u_z) = 18\mathbf{a}_x + 15\mathbf{a}_y$$

(b) Equating components and then integrating, we obtain;

$$\frac{du_x}{dt} = 18 \rightarrow u_x = 18t + A \qquad 1$$

$$\frac{du_y}{dt} = 15 \rightarrow u_y = 15t + B \qquad 2$$

$$\frac{du_z}{dt} = 0 \rightarrow u_z = C \qquad 3$$

where A, B, and C are integration constants. But at t = 0, $u = 4a_x + 3a_z$. Hence,

$$u_x(t = 0) = 4 \rightarrow 4 = 0 + A \quad \text{or} \quad A = 4$$

$$u_y(t = 0) = 0 \rightarrow 0 = 0 + B \quad \text{or} \quad B = 0$$

$$u_z(t = 0) = 3 \rightarrow 3 = C$$

Substituting the values of A, B, and C into eqs. (1) to (3) gives

$$\mathbf{u}(t) = (u_x, u_y, u_z) = (18t + 4, 15t, 3)$$

Hence

$$\mathbf{u}(t = 1 \text{ s}) = 22\mathbf{a}_x + 15\mathbf{a}_y + 3\mathbf{a}_z \text{ m/s}$$

(c) Kinetic energy (K.E.)

$$(\text{K.E.}) = \frac{1}{2}m\,|\mathbf{u}|^2 = \frac{1}{2}(2)(22^2 + 15^2 + 3^2)$$
$$= 718\ \text{J}$$

(d) $\mathbf{u} = \dfrac{d\mathbf{l}}{dt} = \dfrac{d}{dt}(x, y, z) = (18t + 4, 15t, 3)$

Equating components yields;

$$\frac{dx}{dt} = u_x = 18t + 4 \rightarrow x = 9t^2 + 4t + A_1 \qquad 4$$

$$\frac{dy}{dt} = u_y = 15t \rightarrow y = 7.5t^2 + B_1 \qquad 5$$

$$\frac{dz}{dt} = u_z = 3 \rightarrow z = 3t + C_1 \qquad 6$$

At $t = 0$, $(x, y, z) = (1, -2, 0)$; hence,

$$x(t = 0) = 1 \rightarrow 1 = 0 + A_1 \quad \text{or} \quad A_1 = 1$$
$$y(t = 0) = -2 \rightarrow -2 = 0 + B_1 \quad \text{or} \quad B_1 = -2$$
$$z(t = 0) = 0 \rightarrow 0 = 0 + C_1 \quad \text{or} \quad C_1 = 0$$

Substituting the values of A_1, B_1, and C_1 into eqs. (4) to (6), we obtain

$$(x, y, z) = (9t^2 + 4t + 1, 7.5t^2 - 2, 3t)$$

Hence, at $t = 1$, $(x, y, z) = (14, 5.5, 3)$

MAGNETIC TORQUE AND MOMENT

The concept of a current loop experiencing a torque in a magnetic field is of paramount importance in understanding the behavior of orbiting charged particles, dc motors, and generators. If the loop is placed parallel to a magnetic field, it experiences a force that tends to rotate it.

The **torque T** (or mechanical moment of force) on the loop is the vector product of the moment arm **r** and the force **F**

$$\mathbf{T} = \mathbf{r} \times \mathbf{F} \qquad 1$$

and its units are newton-meters (N.m).

Applying this to a rectangular loop of length l and width w placed in a uniform magnetic field B as shown in Figure 15(a)

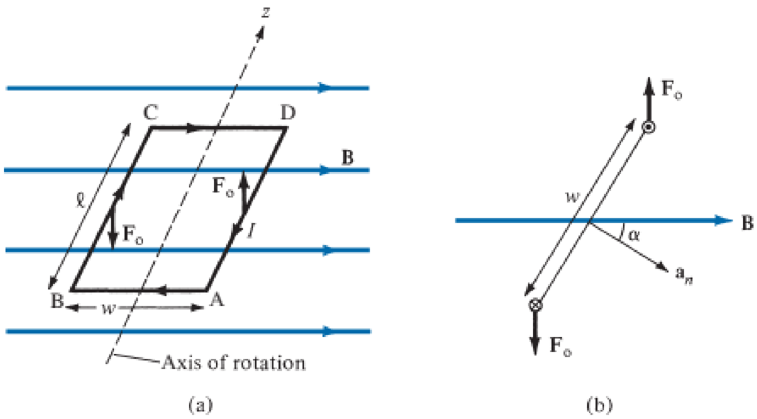


FIGURE 15 (**a**) Rectangular planar loop in a uniform magnetic field. (**b**) Cross-sectional view of part (a).

From Figure 15(a), we notice that $d\mathbf{l}$ is parallel to **B** along sides AB and CD of the loop and no force is exerted on those sides. Thus

$$\mathbf{F} = I\int_{B}^{C} d\mathbf{l} \times \mathbf{B} + I\int_{D}^{A} d\mathbf{l} \times \mathbf{B}$$

$$= I\int_{0}^{\ell} dz\, \mathbf{a}_z \times \mathbf{B} + I\int_{\ell}^{0} dz\, \mathbf{a}_z \times \mathbf{B}$$

Or

$$\mathbf{F} = \mathbf{F}_o - \mathbf{F}_o = 0 \qquad 2$$

Where $|F_0| = IBl$, because **B** is uniform. Thus, no force is exerted on the loop as a whole. However, $\mathbf{F}_o$ and $-\mathbf{F}_o$ act at different points on the loop, thereby creating a couple. If the normal to the plane of the loop makes an angle α with **B**, as shown in the cross-sectional view of Figure 15(b), the torque on the loop is;

$$|\mathbf{T}| = |\mathbf{F}_o|\, w \sin\alpha$$

Or

$$T = BI\ell w \sin\alpha \qquad 3$$

But ,$lw = S$, the area of the loop. Hence,

$$T = BIS \sin\alpha \qquad 4$$

We define the quantity

$$\mathbf{m} = IS\mathbf{a}_n \qquad 5$$

As the *magnetic dipole moment* $(in\ A \cdot m^2)$ of the loop. In eq. (5), a_n is a unit normal vector to the plane of the loop and its direction is determined by the right-hand rule: fingers in the direction of current and thumb along $\boldsymbol{a_n}$.

The **magnetic dipole moment** is the product of current and area of the loop; its direction is normal to the loop.

Introducing eq. (5) in eq. (4), we obtain

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} \qquad 6$$

Although this expression was obtained by using a rectangular loop, it is generally applicable in determining the torque on a planar loop of any arbitrary shape. The only limitation is that the magnetic field must be uniform. It should be noted that the torque is in the direction of the axis of rotation (the *z*-axis in the case of Figure 8.5(a)). It is directed with the aim of reducing a so that **m** and **B** are in the same direction. In an equilibrium position (when **m** and **B** are in the same direction), the loop is perpendicular to the magnetic field and the torque will be zero as well as the sum of the forces on the loop.

Example:

Determine the magnetic moment of an electric circuit formed by the triangular loop of Figure 16.

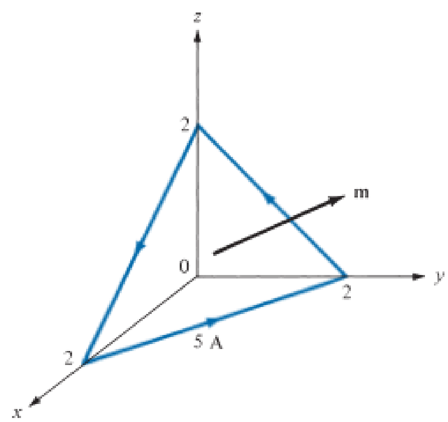


FIGURE 16; Triangular loop

Solution:

If a plane intercepts the coordinate axes at (*a*, 0, 0), (0, *b*, 0), and (0, 0, *c*), its equation is given by

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1 \longrightarrow bcx + cay + abz = abc$$

For the present problem, *a* = *b* = *c* = 2. Hence $x + y + z = 2$

Thus, we can use

$$\mathbf{m} = IS\mathbf{a}_n$$

Where;

$$S = \text{loop area} = \frac{1}{2} \times \text{base} \times \text{height} = \frac{1}{2}(2\sqrt{2})(2\sqrt{2})\sin 60°$$

$$= 4 \sin 60°$$

If we define the plane surface by a function

$$f(x, y, z) = x + y + z - 2 = 0$$

$$\mathbf{a}_n = \pm \frac{\nabla f}{|\nabla f|} = \pm \frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}}$$

We choose the plus sign in view of the direction of the current in the loop (using the right-hand rule, **m** is directed as in Figure 16). Hence

$$\mathbf{m} = 5(4 \sin 60°)\frac{(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)}{\sqrt{3}}$$

$$= 10(\mathbf{a}_x + \mathbf{a}_y + \mathbf{a}_z)\ \text{A} \cdot \text{m}^2$$

A MAGNETIC DIPOLE

A magnetic dipole consists of a bar magnet or small current-carrying loop.
Let us determine the magnetic field **B** at an observation point $P(r, \theta, \emptyset)$ due to a circular loop carrying current *I* as in Figure 17.

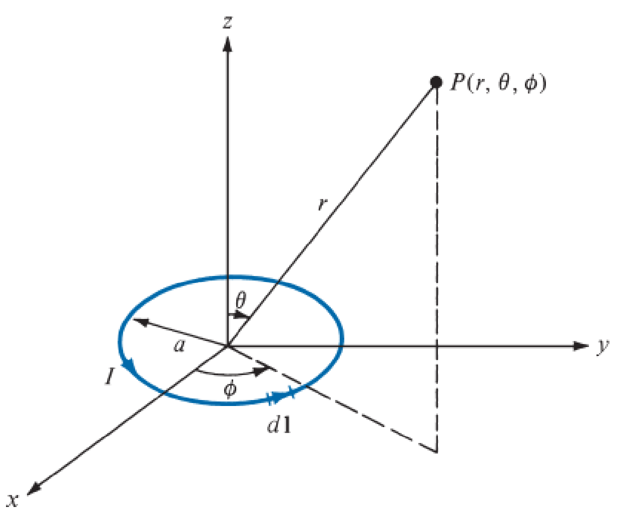


FIGURE 17 Magnetic field at *P* due to a current loop.

The magnetic vector potential at P is

$$\mathbf{A} = \frac{\mu_o I}{4\pi} \oint \frac{d\mathbf{l}}{r} \tag{7}$$

It can be shown that in the far field $r \gg a$, so that the loop appears small at the observation point, **A** has only ∅ component and it is given by

$$\mathbf{A} = \frac{\mu_o I \pi a^2 \sin\theta \, \mathbf{a}_\phi}{4\pi r^2} \tag{8}$$

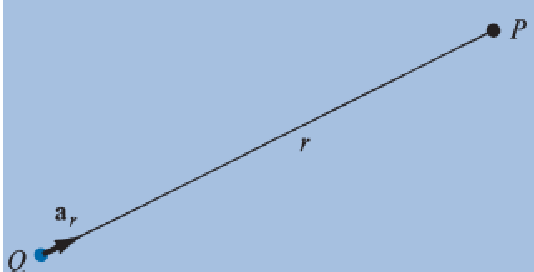
Or

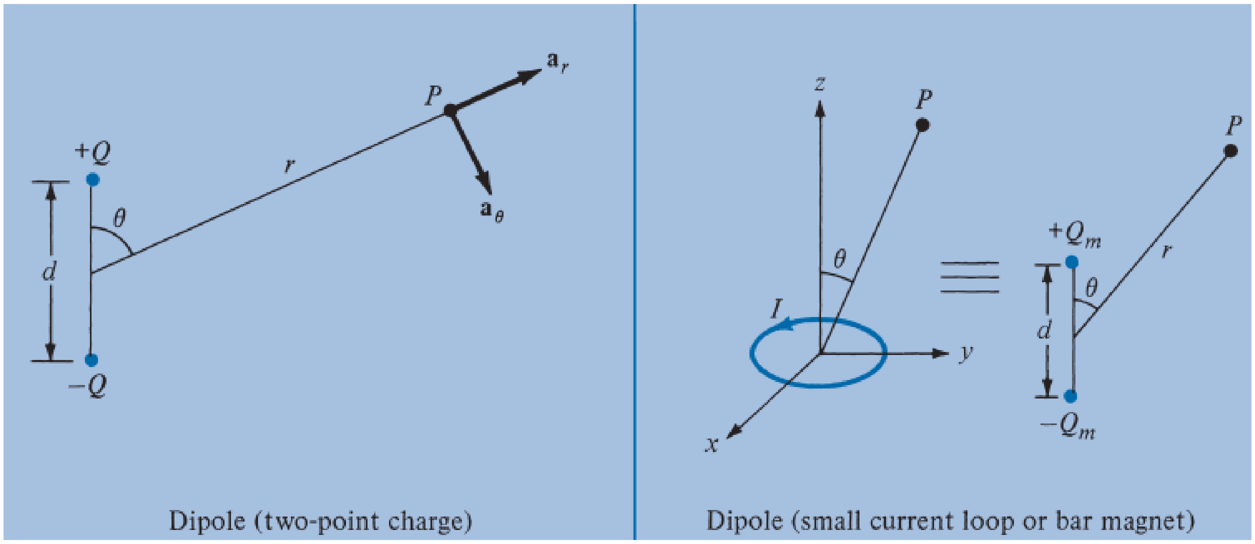
$$\mathbf{A} = \frac{\mu_o \, \mathbf{m} \times \mathbf{a}_r}{4\pi r^2} \tag{9}$$

Where $m = I\pi a^2 a_z$, the magnetic moment of the loop, and $a_z \times a_r = \sin\theta a_\emptyset$. We determine the magnetic flux density B from $B = \nabla \times A$ as;

$$\mathbf{B} = \frac{\mu_o m}{4\pi r^3}(2\cos\theta \, \mathbf{a}_r + \sin\theta \, \mathbf{a}_\theta) \tag{10}$$

TABLE 4 Comparison between Electric and Magnetic Monopoles and Dipoles

Electric	Magnetic
$V = \frac{Q}{4\pi\varepsilon_o r}$ $\mathbf{E} = \frac{Q\mathbf{a}_r}{4\pi\varepsilon_o r^2}$  Monopole (point charge)	Does not exist  Monopole (point charge)
$V = \frac{Q\cos\theta}{4\pi\varepsilon_o r^2}$ $\mathbf{E} = \frac{Qd}{4\pi\varepsilon_o r^3}(2\cos\theta \, \mathbf{a}_r + \sin\theta \, \mathbf{a}_\theta)$	$\mathbf{A} = \frac{\mu_o m \sin\theta \mathbf{a}_\phi}{4\pi r^2}$ $\mathbf{B} = \frac{\mu_o m}{4\pi r^3}(2\cos\theta \mathbf{a}_r + \sin\theta \mathbf{a}_\theta)$



We notice the striking similarities between **B** in the far field due to a small current loop and **E** in the far field due to an electric dipole. It is therefore reasonable to regard a small current loop as a magnetic dipole. The **B** lines due to a magnetic dipole are similar to the **E** lines due to an electric dipole.

Figure 18(a) illustrates the **B** lines around the magnetic dipole $\boldsymbol{m} = \boldsymbol{IS}$.

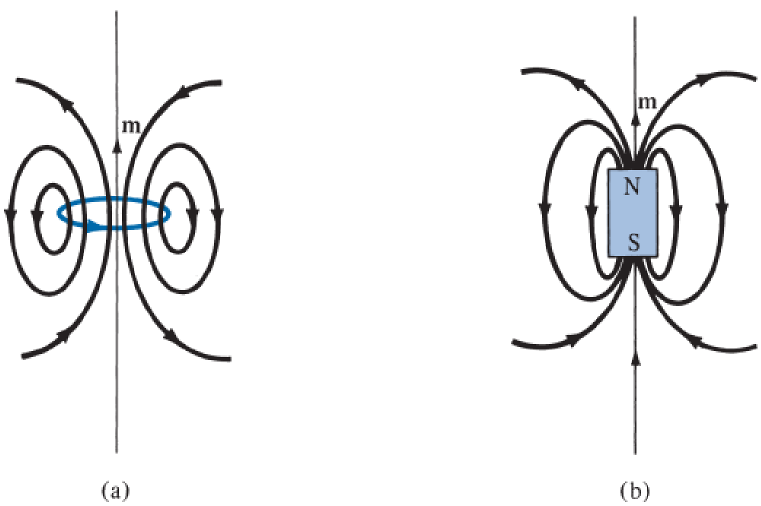


FIGURE 18; The B lines due to magnetic dipoles: (a) a small current loop with $m = IS$.

, (b) a bar magnet with $m = Q_m l$

A short permanent magnetic bar, shown in Figure 18(b), may also be regarded as a magnetic dipole. Observe that the **B** lines due to the bar are similar to those due to a small current loop in Figure 18(a).

Consider the bar magnet of Figure 19.

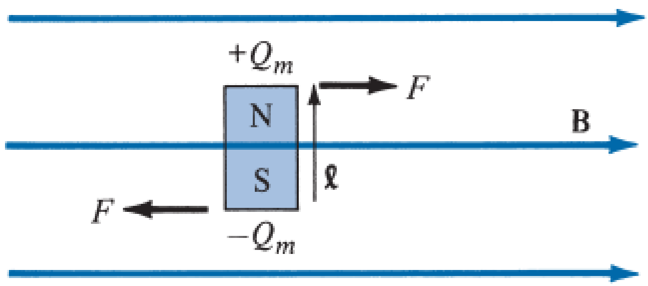


FIGURE 19 A bar magnet in an external magnetic field.

If Q_m is an isolated magnetic charge (pole strength) and l is the length of the bar, the bar has a dipole moment $Q_m l$. (Notice that Q_m does exist; however, it does not exist without an associated$-Q_m$. See Table 4.) When the bar is in a uniform magnetic field **B**, it experiences a torque;

$$\mathbf{T} = \mathbf{m} \times \mathbf{B} = Q_m \boldsymbol{\ell} \times \mathbf{B} \qquad 11$$

Where l points south to north. The torque tends to align the bar with the external magnetic field. The force acting on the magnetic charge is given by;

$$\mathbf{F} = Q_m \mathbf{B} \qquad 12$$

Since both a small current loop and a bar magnet produce magnetic dipoles, they are equivalent if they produce the same torque in a given **B** field, that is, when

$$T = Q_m \ell B = ISB \qquad 13$$

Hence

$$Q_m \ell = IS \qquad 14$$

Showing that they must have the same dipole moment.

Example

A small current loop L_1 with magnetic moment $5a_z A.m^2$ is located at the origin while another small loop current L_2 with magnetic moment $3a_y A.m^2$ is located at (4, -3, 10). Determine the torque on L_2.

Solution:
The torque $\mathbf{T}_2$ on the loop L_2 is due to the field $\mathbf{B}_1$ produced by loop L_1. Hence,

$$\mathbf{T}_2 = \mathbf{m}_2 \times \mathbf{B}_1$$

Since $\mathbf{m}_1$ for loop L_1 is along $\mathbf{a}_z$, we find $\mathbf{B}_1$ using eq. (10):

$$\mathbf{B} = \frac{\mu_o m}{4\pi r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

We have;

$$\mathbf{B}_1 = \frac{\mu_o m_1}{4\pi r^3}(2\cos\theta\,\mathbf{a}_r + \sin\theta\,\mathbf{a}_\theta)$$

Using equation below, we transform $\mathbf{m}_2$ from Cartesian to spherical coordinates:

$$\mathbf{a}_x = \sin\theta\cos\phi\,\mathbf{a}_r + \cos\theta\cos\phi\,\mathbf{a}_\theta - \sin\phi\,\mathbf{a}_\phi$$
$$\mathbf{a}_y = \sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi$$
$$\mathbf{a}_z = \cos\theta\,\mathbf{a}_r - \sin\theta\,\mathbf{a}_\theta$$

That is;

$$\mathbf{m}_2 = 3\mathbf{a}_y = 3(\sin\theta\sin\phi\,\mathbf{a}_r + \cos\theta\sin\phi\,\mathbf{a}_\theta + \cos\phi\,\mathbf{a}_\phi)$$

At (4.-3,10),

$$r = \sqrt{4^2 + (-3)^2 + 10^2} = 5\sqrt{5}$$

$$\tan\theta = \frac{\rho}{z} = \frac{5}{10} = \frac{1}{2} \rightarrow \sin\theta = \frac{1}{\sqrt{5}}, \quad \cos\theta = \frac{2}{\sqrt{5}}$$

$$\tan\phi = \frac{y}{x} = \frac{-3}{4} \rightarrow \sin\phi = \frac{-3}{5}, \quad \cos\phi = \frac{4}{5}$$

Hence,

$$\mathbf{B}_1 = \frac{4\pi \times 10^{-7} \times 5}{4\pi\, 625\sqrt{5}}\left(\frac{4}{\sqrt{5}}\mathbf{a}_r + \frac{1}{\sqrt{5}}\mathbf{a}_\theta\right)$$

$$= \frac{10^{-7}}{625}(4\mathbf{a}_r + \mathbf{a}_\theta)$$

$$\mathbf{m}_2 = 3\left[-\frac{3\mathbf{a}_r}{5\sqrt{5}} - \frac{6\mathbf{a}_\theta}{5\sqrt{5}} + \frac{4\mathbf{a}_\phi}{5}\right]$$

And

$$\mathbf{T} = \frac{10^{-7}(3)}{625(5\sqrt{5})}(-3\mathbf{a}_r - 6\mathbf{a}_\theta + 4\sqrt{5}\mathbf{a}_\phi) \times (4\mathbf{a}_r + \mathbf{a}_\phi)$$

$$= 4.293 \times 10^{-11}(-8.944\mathbf{a}_r + 35.777\mathbf{a}_\theta + 21\mathbf{a}_\phi)$$

$$= -0.384\mathbf{a}_r + 1.536\mathbf{a}_\theta + 0.9015\mathbf{a}_\phi\ \text{nN}\cdot\text{m}$$

MAGNETIZATION IN MATERIALS

An internal magnetic field is produced by electrons orbiting around the nucleus as in Figure 20(a) or electrons spinning as in Figure 20(b).

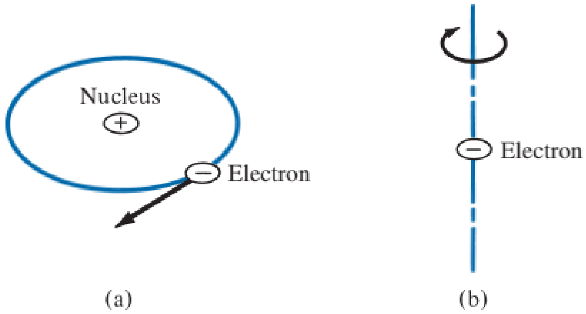


FIGURE 20 (**a**) Electron orbiting around the nucleus. (**b**) Electron spin.

Both these electronic motions produce internal magnetic fields **B**i that are similar to the magnetic field produced by a current loop of Figure 21.

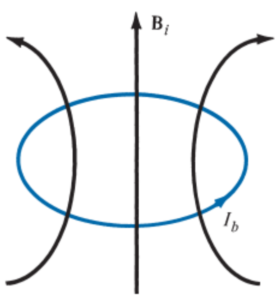


FIGURE 21 Circular current loop equivalent to electronic motion of Figure 20.

The equivalent current loop has a magnetic moment of $m = I_b S a_n$, where S is the area of the loop and I_b is the bound current (bound to the atom).

Without an external B field applied to the material, the sum of m's is zero due to random orientation as in Figure 22(a). When an external B field is applied, the magnetic moments of the electrons more or less align themselves with B so that the net magnetic moment is not zero, as illustrated in Figure 22(b).

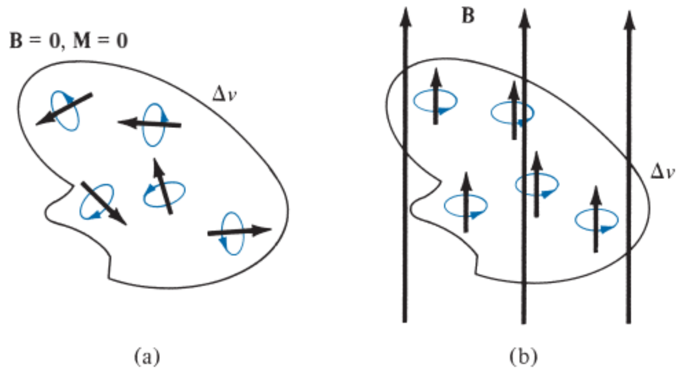


FIGURE 22 Magnetic dipole moment in a volume Δv: (**a**) before **B** is applied, (**b**) after **B** is applied.

The **magnetization M**, in amperes per meter, is the magnetic dipole moment per unit volume.

If there are N atoms in a given volume Δv and the k_{th} atom has a magnetic moment m_k,

$$\mathbf{M} = \lim_{\Delta v \to 0} \frac{\sum_{k=1}^{N} \mathbf{m}_k}{\Delta v} \tag{15}$$

A medium for which **M** is not zero everywhere is said to be magnetized. For a differential volume dv', the magnetic moment is $dm = M dv'$.

From

$$\mathbf{A} = \frac{\mu_o \, \mathbf{m} \times \mathbf{a}_r}{4\pi r^2} \tag{16}$$

The vector magnetic potential due to $d\mathbf{m}$ is;

$$d\mathbf{A} = \frac{\mu_o \mathbf{M} \times \mathbf{a}_R}{4\pi R^2} dv' = \frac{\mu_o \mathbf{M} \times \mathbf{R}}{4\pi R^3} dv' \tag{17}$$

From

$$\frac{\mathbf{R}}{R^3} = \nabla' \frac{1}{R} \tag{18}$$

We can write;

$$\mathbf{A} = \frac{\mu_o}{4\pi} \int_{v'} \mathbf{M} \times \nabla' \frac{1}{R} dv' \tag{19}$$

Using the vector identity;

$$\nabla \times (f\mathbf{F}) = f\nabla \times \mathbf{F} + (\nabla f) \times \mathbf{F}$$

We can write;

$$\mathbf{M} \times \nabla' \frac{1}{R} = \frac{1}{R} \nabla' \times \mathbf{M} - \nabla' \times \frac{\mathbf{M}}{R} \qquad 20$$

Substituting this into eq. (19) yields

$$\mathbf{A} = \frac{\mu_o}{4\pi} \int_{v'} \frac{\nabla' \times \mathbf{M}}{R} dv' - \frac{\mu_o}{4\pi} \int_{v'} \nabla' \times \frac{\mathbf{M}}{R} dv' \qquad 21$$

Applying the vector identity

$$\int_{v'} \nabla' \times \mathbf{F} \, dv' = -\oint_{S'} \mathbf{F} \times d\mathbf{S}$$

To the second integral, we obtain;

$$\mathbf{A} = \frac{\mu_o}{4\pi} \int_{v'} \frac{\nabla' \times \mathbf{M}}{R} dv' + \frac{\mu_o}{4\pi} \oint_{S'} \frac{\mathbf{M} \times \mathbf{a}_n}{R} dS'$$

$$= \frac{\mu_o}{4\pi} \int_{v'} \frac{\mathbf{J}_b \, dv'}{R} + \frac{\mu_o}{4\pi} \oint_{S'} \frac{\mathbf{K}_b \, dS'}{R} \qquad 22$$

Comparing eq. (22) with eqs.

$$\mathbf{A} = \int_S \frac{\mu_o \mathbf{K} \, dS}{4\pi R}$$

And

$$\mathbf{A} = \int_v \frac{\mu_o \mathbf{J} \, dv}{4\pi R}$$

(upon dropping the primes) gives

$$\mathbf{J}_b = \nabla \times \mathbf{M} \qquad 23$$

And

$$\mathbf{K}_b = \mathbf{M} \times \mathbf{a}_n \qquad 24$$

where J_b is the bound volume current density or magnetization volume current density, in amperes per meter squared, K_b is the bound surface current density, in amperes per meter, and a_n is a unit vector normal to the surface.

Equation (22) shows that the potential of a magnetic body is due to a volume current density $\boldsymbol{J_b}$ throughout the body and a surface current $\boldsymbol{K_b}$ on the surface of the body.

The vector **M** is analogous to the polarization **P** in dielectrics and is sometimes called the *magnetic polarization density* of the medium.

In another sense, **M** is analogous to **H** and they both have the same units. In this respect, as $J = \nabla \times H$, so $J_b = \nabla \times M$. Also, $\boldsymbol{J_b}$ and $\boldsymbol{K_b}$ for a magnetized body are similar to ρ_{pv} and ρ_{ps} for a polarized body.

As is evident in eqs. (22) to (24), $\boldsymbol{J_b}$ and $\boldsymbol{K_b}$ can be derived from **M**; therefore, $\boldsymbol{J_b}$ and $\boldsymbol{K_b}$ are not commonly used.

In free space, **M**=**0** and we have;

$$\nabla \times \mathbf{H} = \mathbf{J}_f \quad \text{or} \quad \nabla \times \left(\frac{\mathbf{B}}{\mu_o}\right) = \mathbf{J}_f \qquad 25$$

Where J_f is the free current volume density. In a material medium $\boldsymbol{M} \neq \boldsymbol{0}$, and as a result, **B** changes so that

$$\nabla \times \left(\frac{\mathbf{B}}{\mu_o}\right) = \mathbf{J}_f + \mathbf{J}_b = \mathbf{J} \qquad 26$$

$$= \nabla \times \mathbf{H} + \nabla \times \mathbf{M}$$

Or

$$\mathbf{B} = \mu_o(\mathbf{H} + \mathbf{M}) \qquad 27$$

The relationship in eq. (27) holds for all materials whether they are linear or not. The concepts of linearity, isotropy, and homogeneity apply here for magnetic media. For linear materials, **M** (in A/m) depends linearly on **H** such that

$$\mathbf{M} = \chi_m \mathbf{H} \qquad 28$$

Where X_m is a dimensionless quantity (ratio of *M* to *H*) called *magnetic susceptibility* of the medium. It is more or less a measure of how susceptible (or sensitive) the material is to a magnetic field. Substituting eq. (28) into eq. (27) yields;

$$\mathbf{B} = \mu_o(1 + \chi_m)\mathbf{H} = \mu\mathbf{H}$$

Or

$$\mathbf{B} = \mu_o \mu_r \mathbf{H} \qquad 29$$

Where;

$$\mu_r = 1 + \chi_m = \frac{\mu}{\mu_o} \qquad 30$$

The quantity $\mu = \mu_0 \mu_r$ is called the *permeability* of the material and is measured in henrys per meter. The dimensionless quantity μ_r is the ratio of the permeability of a given material to that of free space and is known as the *relative permeability* of the material.

It should be borne in mind that the relationships in eqs. (24) to (30) hold only for linear and isotropic materials.