Chapter 3

Bilinear Transfer Function

3.1 Introduction

A transfer function whose numerator and denominator polynomials are linear functions of s is called a *bilinear transfer function*. It is $bi-$ because of the two linear functions of s, the numerator and the denominator. Bilinear transfer functions are *first order* transfer functions, because the highest order of each polynomial is 1.

Going back to Eq. 2.8, a bilinear transfer function is obtained by setting $m = n = 1$. The bilinear transfer function becomes

$$T(s) = \frac{N(s)}{D(s)} = \frac{b_1 s + b_0}{a_1 s + a_0} = K\frac{s + z_1}{s + p_1} \tag{3.1}$$

where $K = \frac{b_1}{a_1}$, $z_1 = \frac{b_0}{b_1}$, $p_1 = \frac{a_0}{a_1}$, the zero is $s = -z_1$, and the pole is $s = -p_1$. Since a_i and b_j are all real, therefore the pole and the zero both fall on the real axis of the *s*-plane. For one to be able to effectively control the response of transfer function, all the poles of the transfer function have to be on the left-hand side of the *s*-plane. This makes the transfer function stable, which essentially means you can control the output (response) using an carefully selected excitation (input). In this unit, we will only touch on the concept of stability in passing. However, it is important to keep in mind that stability is very important in science and engineering. You have and you will continue to interract with stability in numerous areas of theory and practice.

3.1.1 Poles and zeros in the *s*-plane

In Eq. 3.2c, three examples of bilinear transfer functions are given. Their corresponding poles and zeros in the s-plane are shown in Fig. 3.1. The pole locations are marked using crosses $(\times)$ and the zero locations are marked using circles (○).

$$T_a(s) = K_a \frac{s+3}{s+2} \tag{3.2a}$$
$$T_b(s) = K_b \frac{s+1}{s+4} \tag{3.2b}$$
$$T_c(s) = K_c \frac{s-2}{s+2} \tag{3.2c}$$

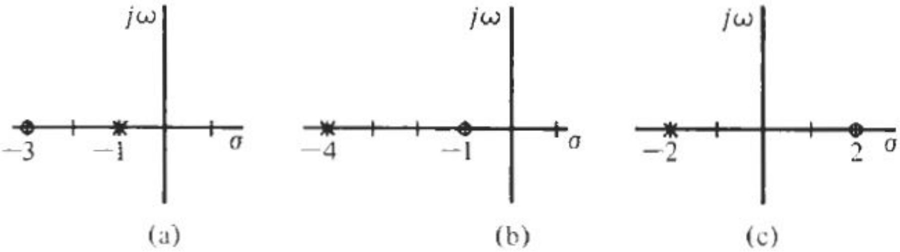

Fig. 3.1: Representations of the poles (×) and zeros (○) of the transfer functions in Eq. 3.2c on the s-plane. The poles and zeros of $T_a(s)$, $T_b(s)$, and $T_c(s)$ are shown in (a), (b), and (c), respectively.

Other examples of bilinear transfer functions are shown in the Table shown in Fig. 3.3. The number of

T(s)	*Pole location*	*Zero location*
$K_1 s$	$s = \infty$	$s = 0$
$K_2(s + z_1)$	$s = \infty$	$s = -z_1$
$\frac{K_3}{s}$	$s = 0$	$s = \infty$
$\frac{K_4}{s + p_1}$	$s = -p_1$	$s = \infty$
$\frac{K_5 s}{s + p_1}$	$s = -p_1$	$s = 0$

Fig. 3.2: Other examples of bilinear transfer functions and their poles and zeros

zeros and poles can always be made equal by writing each of the transfer functions in the format given in Eq. 3.1, and explicitly substituting zeros for the coefficients that are not written in the simplified form of the transfer function. For instance, for the transfer function $K_1 s$ can be written as

$$\begin{aligned} T_a(s) &= K_1 \frac{s}{1} \\ &= K_1 \frac{s+0}{0s+1}, \end{aligned}$$

from which the zero becomes $s = 0$ (from $s+0 = 0$) and the pole becomes $s = \infty$ (from $0 \cdot s + 1 = 0$).

Notes on $K_1 s$ and $\frac{K_3}{s}$

$K_1 s$ represents the transfer function of a *differentiator* with gain K_1 ($K_1 \frac{d}{dt}$ function in the time domain). $\frac{K_3}{s}$ represents the transfer function of an *integrator* with gain K_3 ($K_3 \int dt$ in the time domain.)

3.2 Classification of magnitude and phase responses

Different forms of bilinear transfer functions can be obtained by changing the coefficients of the numerator and denominator polynomials. Such changes can lead to various forms of magnitude and phase responses. Some of the main respnose types for bilinear transfer functions are discussed in the subsections that follow.

3.2.1 Magnitude responses

Some of the main magnitude responses for bilinear transfer functions are

1. lowpass,
2. highpass, and
3. all-pass.

In addition to these three, there are two special forms of lowpass and highpass magnitude responses respectively referred to as

1. integrator, and
2. differentiator.

Each of the magnitude resposes are briefly highlighted in the sections that follow.

3.2.1.1 Lowpass filter

Figure 3.3 shows a theoretical brickwall lowpass magnitude response as well a realizable lowpass magnitude response for bilinear transfer functions. Comparing the brickwall magnitude response in Fig.

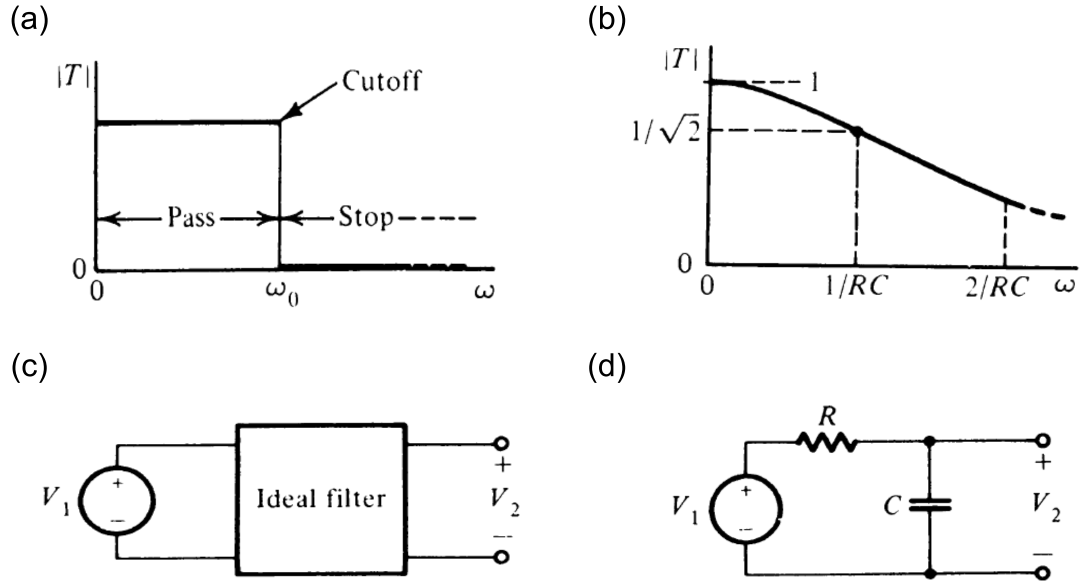

Fig. 3.3: Lowpass magnitude response. The brickwall (a) and realizable (b) lowpass magnitude responses, along with a general two-port block diagram of a filter (c), and realization network for the response using a capacitor and a resistor (d).

3.3(a) with the realizable response Fig. 3.3(b), some of the key points in the magnitude response are at $\omega = 0$ (representing low frequecies), $\omega = \omega_0$ (representing the half-power frequency), and $\omega = \infty$ (representing the high frequencies). For lowpass (and highpass) magnitude responses, the half-power point (where $T = \frac{1}{\sqrt{2}}$). For the resistor-capacitor (RC) realization of the magnitude response in Fig. 3.3(d), the transfer function is given by

$$T(s) = K_1 \frac{\frac{1}{RC}}{s + \frac{1}{CR}}, \tag{3.3}$$

from which

$$|T(\omega)| = \frac{\frac{1}{RC}}{\sqrt{\omega^2 + \frac{1}{(CR)^2}}}, \tag{3.4}$$

and $\omega_0 = \frac{1}{RC}$. This is just one way to realize a practical bilinear lowpass magnitude response.

It is sometimes convenient to *normalize the frequency* by dividing through every term with ω_0 then substituting $\omega_n = \dfrac{\omega}{\omega_0}$ and writing

$$|T(\omega_n)| = \frac{1}{\sqrt{w_n^2 + 1}}, \tag{3.5}$$

where ω_n is the normalized frequency.

Writing transfer functions in the normalized form

1. makes design independent of ω_0,
2. simplifies the expressions for the transfer function,
3. simplifies component values during design,
4. makes is possible to use pre-computed standard transfer function coefficients for various magnitude responses, and
5. makes approximation of transfer functions independent of ω_0.

Frequency normalization and scaling

Going back to the transfer function in s in Ea. 3.3, with ω_0 known, it can also be normalized by dividing through by ω_0 and substituting $s_n = \dfrac{s}{\omega_0}$ to obtain

$$T(s_n) = \frac{1}{s_n + 1}, \tag{3.6}$$

where s_n represents the normalized frequency in s-domain. Carrying out design with normalized frequency is usually the norm. After the design, frequency *scaling* is carried out using

$$s = \omega_0 s_n, \tag{3.7}$$

in order to meet the actual design specifications. We will revisit this approach during design and scaling of magnitude and frequency.

During design, it is common to drop the subscript n and it is implicitly understood when normalization and scaling is required, as you are likely to find in a number of books.

In practical filter specifications, there is a transition band between the passband and the stopband frequencies.

This is just one realization of bilinear lowpass magnitude response.

3.2.1.2 Highpass filter

Figure 3.4 shows a theoretical brickwall highpass magnitude response as well a realizable highpass magnitude response for bilinear transfer functions. Carrying out an analysis similar to what we did with

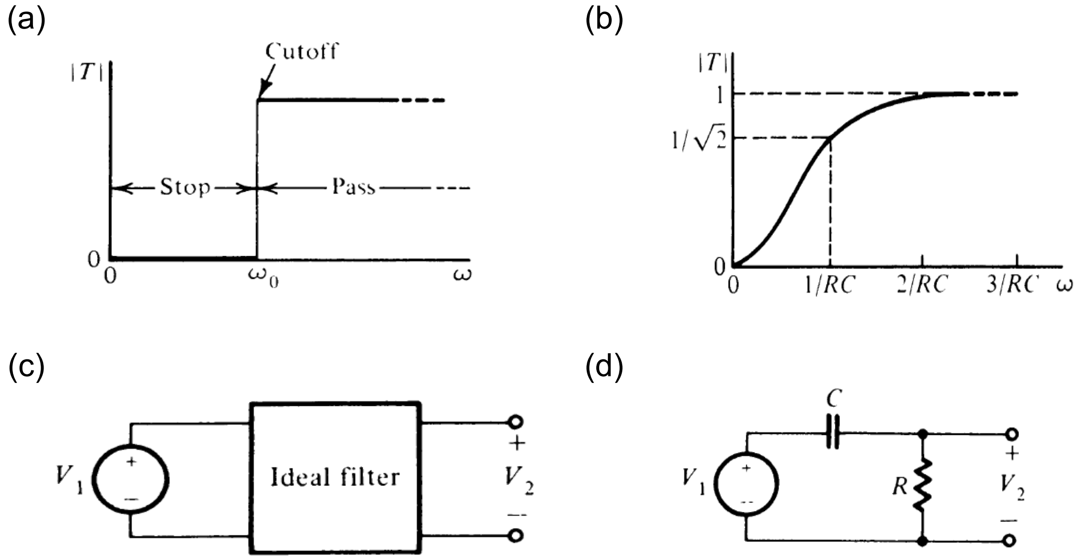

Fig. 3.4: Highpass magnitude response. The brickwall (a) and realizable (b) lowpass magnitude responses, along with a general two-port block diagram of a filter (c), and a realization network for the response using a capacitor and a resistor (d).

the low-pass response, the transfer function is expressed using

$$T(s) = \frac{s}{s + \frac{1}{CR}}, \tag{3.8}$$

and the magnitude response is

$$|T(\omega)| = \frac{\omega}{\sqrt{\omega^2 + \frac{1}{(CR)^2}}}. \tag{3.9}$$

Once again, $\omega_0 = \frac{1}{RC}$. This is just one realization of a bilinear highpass magnitude response.

After normalizing the frequency, the magnitude response becomes

$$|T(\omega_n)| = \frac{\omega_n}{\sqrt{w_n^2 + 1}}. \tag{3.10}$$

Similarly, the transfer function can also be normalized and written as

$$T(s_n) = \frac{s_n}{s_n + 1}. \tag{3.11}$$

3.2.1.3 Allpass filter

Figure 3.5 shows an allpass magnitude response along with its realization using a bilinear transfer function. For the realization in Fig. 3.5(b) and (c), the transfer function is

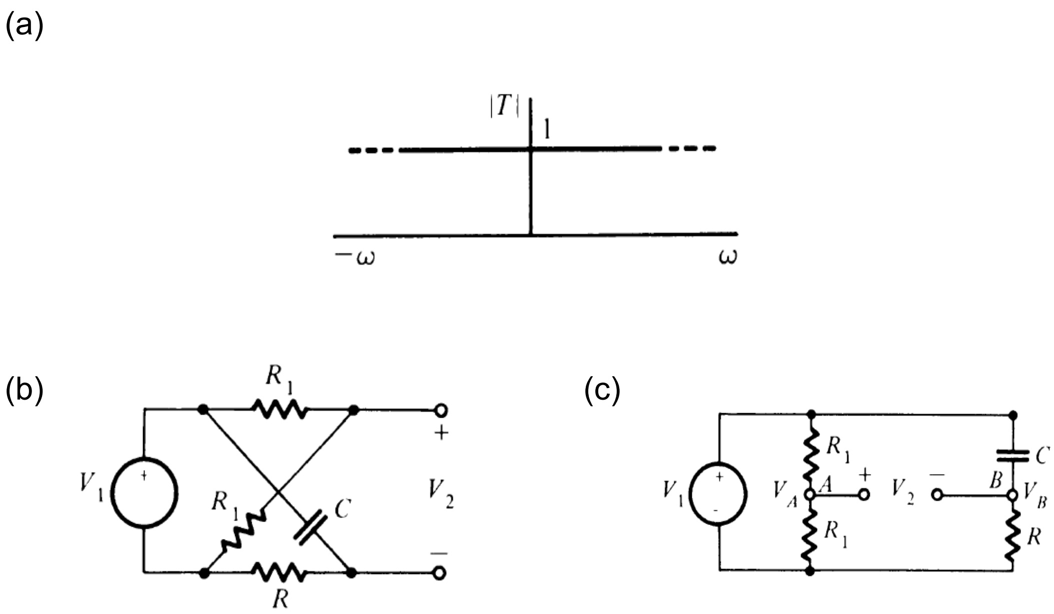

Fig. 3.5: Allpass magnitude response (a), realization using a *lattice* structure, and the circuit in (b) redrawn to make analysis easier (c).

$$T(s) = \frac{V_2(s)}{V_1(s)} = -\frac{s - \frac{1}{CR}}{s + \frac{1}{CR}}, \tag{3.12}$$

leading to a magnitude response given by

$$|T(\omega)| = 1. \tag{3.13}$$

This is an allpass magnitude response and cutoff does not hold any significance.

Calculating dB

You may wonder why one would need an allpass filter at all, since ideally id does not change the signal's amplitude. The answer lies in what it does to the phase of the signal. For instance, in communication systems, phase shifted versions of a signal may be required during mixing. All pass filters can be useful in such scenarios.

3.2.1.4 Integrator

The output of an ideal integrator is the time integral of the input. The transfer function is given by

$$T(s) = \frac{K}{s}, \tag{3.14}$$

and the magnitude response is given by

$$|T(\omega)| = 1/\omega. \tag{3.15}$$

At $\omega = 0$, $|T(\omega)| = \infty$ and at $\omega = \infty$, $|T(\omega)| = 0$, which is a lowpass magnitude response. When sketching the response, it is important to note that $|T(1)| = K$. In practice, the integrator is just a low pass filter, with the integral action happening within the transition range of frequencies (beyond ω_0) between the passband and the stopband.

An illustration of the ideal response of an integrator is shown in Figure 3.6(a).

Some applications of integrators include integration in analog computers, integral control in control engineering, analog-to-digital converter circuits, waveform generators, as well as wave shaping circuits.

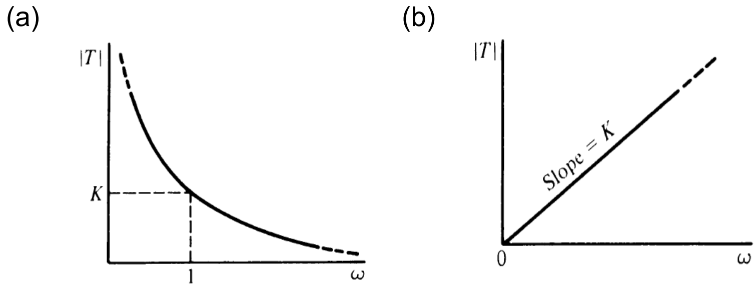

Fig. 3.6: Magnitude responses for an integrator (a) and a differentiator (b).

3.2.1.5 Differentiator

The output of an ideal differentiator is the time derivative of the input. The transfer function is given by

$$T(s) = Ks, \tag{3.16}$$

and the magnitude response is given by

$$|T(\omega)| = \omega. \tag{3.17}$$

At $\omega = 0$, $|T(\omega)| = 0$ and at $\omega = \infty$, $|T(\omega)| = \infty$, which is a highpass magnitude response. When sketching the response, it is important to note that $|T(1)| = K$. In practice, the differentiator can be a high pass filter, with the derivative action happening below ω_0.

Some uses of differentiators are used include demodulators, waveshaping circuits, and as controllers in control engineering.

Exercise 7:

For the circuits shown in Fig. 3.7, obtain $T(s) = \frac{V_2(s)}{V_1(s)}$ and then

1. obtain the poles and the zeros,
2. sketch the magnitude response, and
3. describe the obtained magnitude responses.

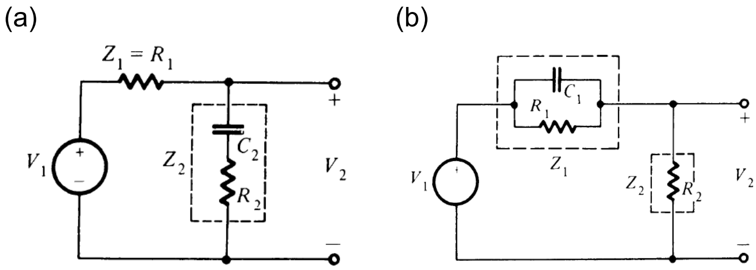

Fig. 3.7: Circuits for analysis

Solution 7:

Exercise 8:

The integrator and the differentiator can be converted into allpass filters. How can this be achieved? (**Hint**: You need to scale the magnitude response appropriately).

Solution 8:

3.2.2 Phase responses

The phase response is classified into either leading or lagging. A lagging phase is negative while a leading phase is positive. Going back to Eq. 3.1, we write $T(\omega)$

$$T(\omega) = K\frac{j\omega + z_1}{j\omega + p_1} \tag{3.18}$$

and the phase is computed using

$$\theta(\omega) = \angle K + \tan^{-1}\frac{\omega}{z_1} - \tan^{-1}\frac{\omega}{p_1}. \tag{3.19}$$

From Eq. 3.19, we see that the poles contribute negative phase while the zeros contribute positive phase. If K is positive, $\angle K = 0°$, and if K is negative, $\angle K = 180°$. The phase contributions of the numerator and the denominator for a bilinear transfer function are shown in Table 2.1 and the corresponding sketches are shown in Fig. 2.8.

When the pole and zero phases are combine, the resultant phase $\theta = \theta_z - \theta_p$ is shown in Fig. 3.9. The resultant phase θ can be leading or lagging depending the relative values of the p_1 and z_1. Two

Table 3.1: Zero and pole phase contributions at selected frequencies.

Phase response

ω	$\theta_z(\omega) = \tan^{-1}\frac{\omega}{z_1}$	$\theta_p(\omega) = -\tan^{-1}\frac{\omega}{p_1}$
0	$0°$	$0°$
z_1	$45°$	
p_1		$-45°$
∞	$90°$	$-90°$

(a) (b)

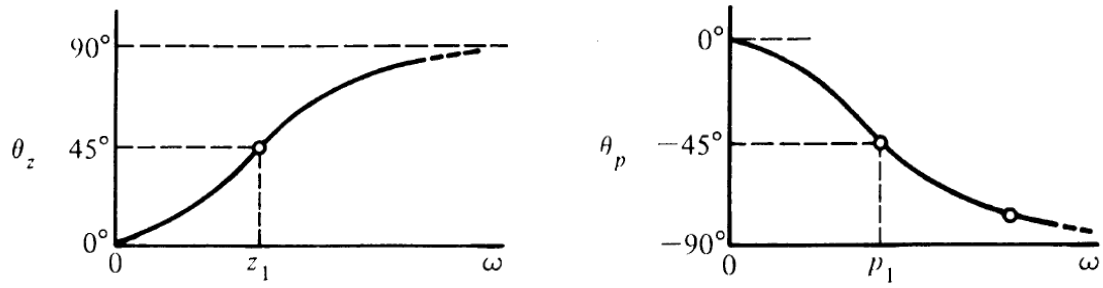

Fig. 3.8: Phase responses for a simple pole and a simple zero.

cases are considered. The first case is shown in Fig. 3.9(a), where $z_1 > p_1$ and the resultant $\theta \geq 0$ (giving phase lead). The second case is shown in Fig. 3.9(b), where $z_1 < p_1$ and $\theta \leq 0$ (giving phase lag). Therefore, when designing first order (bilinear) phase lead circuits, then $z_1 > p_1$. For first order phase lag circuits, $z_1 > p_1$. Phase lead and phase lag circuits are widely applied in control engineering to compensate poor performance.

(a) (b)

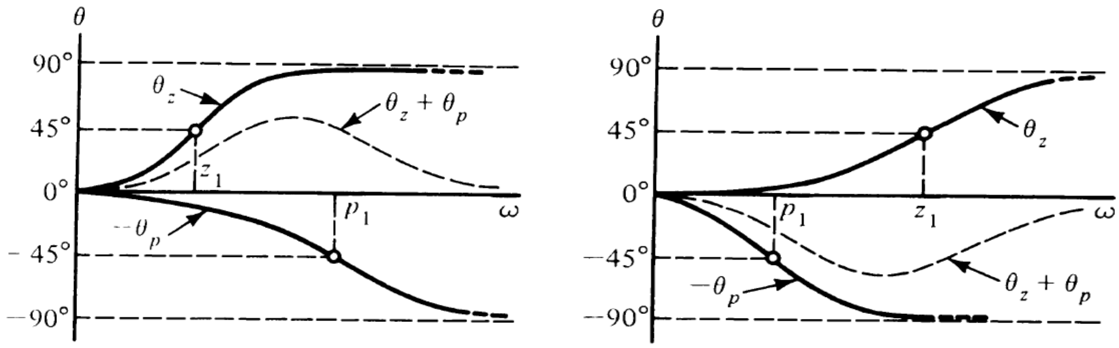

Fig. 3.9: Phase lead and phase lag responses for a bilinear response with a pole and a zero.

Special cases are the integrator and the differentiator, whose phases are $-90°$ and $90°$, respectively. Figure 3.10(a) and (b), illustrates the phase responses the ideal integrator and differentiator, respectively.

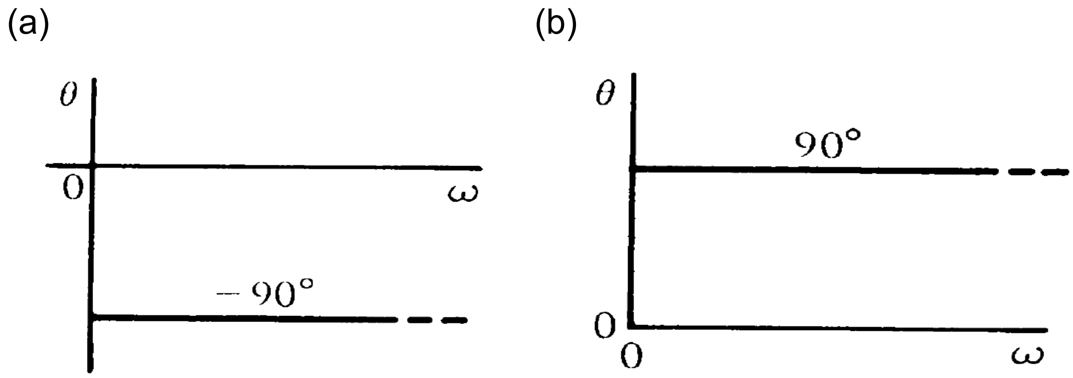

Fig. 3.10: -90° phase for the ideal integrator (a) and 90° phase for the ideal differentiator (b).

A typical response typically combines elements of the numerator and the denominator response.

Exercise 9:

For the circuits previously discussed in the chapter, obtain their frequency response and determine whether they are lead or lag circuits. **Solution 9:**

3.2.3 Summary of bilinear magnitude and phase responses

The Table in Fig. 3.10 provides a summary of classification of the magnitude and phase responses for various bilinear transfer functions.

Exercise 10:

For the circuits previously discussed in the chapter, obtain their frequency response and determine whether they are lead or lag circuits.

Solution 10:

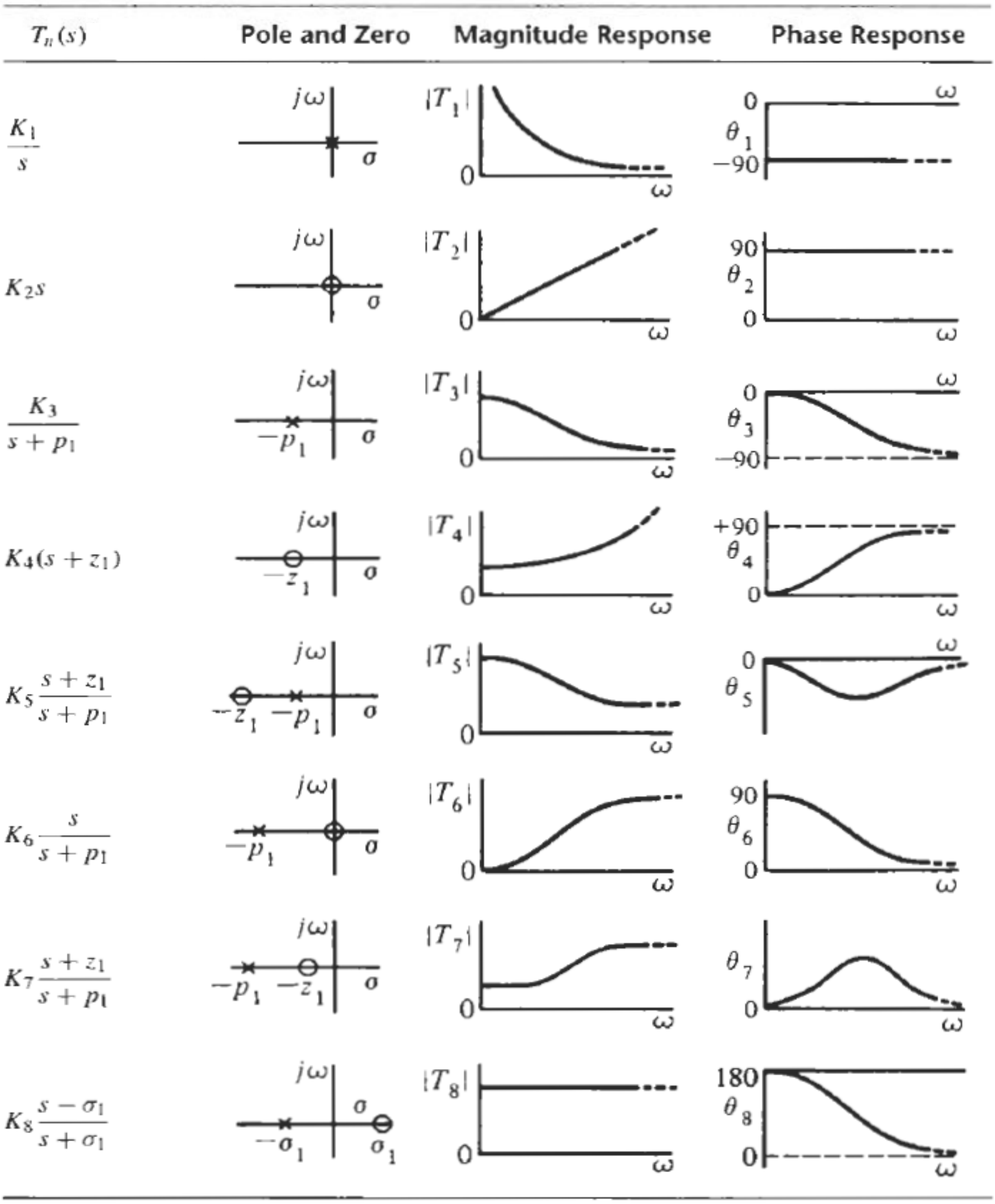

Fig. 3.11: A summary of bilinear magnitude and phase responses.