EEE 3102 Circuit and Network Theory III

Department of Electrical and Electronic Engineering
B.Sc. in Electrical and Electronic Engineering
School of Engineering

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Contents

Chapter 1

The Course

Course outline

Prerequisites

- EEE 2204 Circuit and Network Theory II
- EEE 4103 Analog Electronics III (operational amplifiers)

Purpose of the course

The aim of this course is to enable the student to understand coupled circuits, two port networks, network functions and responses.

Expected learning outcomes

By the end of this course, the learner should be able to;

1. Design various types of filters
2. Relate various two port parameters and transform them

Course content

1. *Response*: transfer function, classification of magnitude and phase response.
2. *Bilinear transfer functions*: Classification of magnitude frequency response and phase frequency response, Bode plots.
3. *Biquad transfer functions*: The biquad circuit, classification of magnitude frequency responses and phase frequency responses, bode plots, active biquads.
4. *Scaling*: magnitude and frequency scaling.
5. *Sensitivity*: introduction, Bode sensitivity, magnitude of sensitivity in frequency domain, sensitivity comparison of typical circuits.
6. *Butterworth filter*: response, pole location and design.
7. *Chebyshev filter*: magnitude response, pole location, Guilemin's algorithm.
8. *Transfer function realization*: realization of first order and second order filters using the Sallen Key circuits.

Laboratory/Practical Exercises

1. Lowpass filter, highpass filter, bandpass filter
2. Biquadratic transfer function
3. Bilinear transfer function

Mode of delivery

- two (2) hour lectures per week
- two (2) hour tutorial per week
- at least five (5) 3-hour laboratory sessions per semester organized on a rotational basis

Course assessment

A total of 100%, the components of which are listed below.

1. Cats **10%**
2. Assignments **5%**
3. Labs **15%**
4. Exam **70%**

Core textbook

1. R. Schaumann and M. E. Van Valkenburg, *Design of Analog Filters*, Oxford University Press, 2001.

Reference textbooks

1. M. E. Van Valkenburg, (1982). *Analog Filter Design*, Saunders College Publishing, 1982.
2. K. L. Su, *Analog Filters*, Springer New York, 2002.
3. L. D. Paarmann, *Design and Analysis of Analog Filters: A Signal Processing Approach*, Kluwer Academic Publishers, 2001.
4. C. K. Alexander and M. N. O. Sadiku, *Fundamentals of electric circuits*, McGraw-Hill Higher Education Boston, 2021.

Chapter 2

Response

This unit introduces transfer functions and analog electric wave filters. Simply put, a transfer function is a ratio of some *output* of some *input* in a system. The system can be an electrical network, a mechanical system, an electromechanical system, an electroacoustic system, or any other system in science, engineering and other disciplines.

Transfer functions have a *response* (that can be expressed in time or frequency) that is associated with them. In our context, filters are electric circuits that are deliberately designed to have responses that are *frequency selective*. Therefore, one way of classifying filters is using the frequency selective behavior of their responses. The variables used to characterize these responses are the *magnitude* (ratio of output to input amplitudes) and the *phase* (difference between output and input phase) of the transfer function, evaluated at different frequencies.

The frequency response of a circuit is simply the variation of its output with changes in the input frequency. There are a few ways of visually representing the frequency response of a circuit. In this unit, bode plots are introduced. While there are other approaches that can be used to visualize frequency (such as the Nyquist and Nichols plots), bode plots are the most widely used because they are uncomplicated to sketch and to analyze.

This chapter introduces the transfer function and various types of responses.

2.1 Transfer Function

2.1.1 Definition

Consider the two-port circuit shown in Fig. 2.1. The input (*excitation*), is applied across terminals 1-1′ (port 1) and the output (*response*) is measured across terminals 2-2′ (port 2). The circuits may be implemented in the differential format or ground referenced format as shown in Fig. 2.1(a) and (b), respectively. Assuming the excitation $v_1(t)$ and the response $v_2(t)$ are *sinusoidal* and operating in *steady state*, the two voltages may be written in time domain as:

$$v_1(t) = V_1 \cos(\omega t + \theta_1) \tag{2.1a}$$

$$v_2(t) = V_2 \cos(\omega t + \theta_2). \tag{2.1b}$$

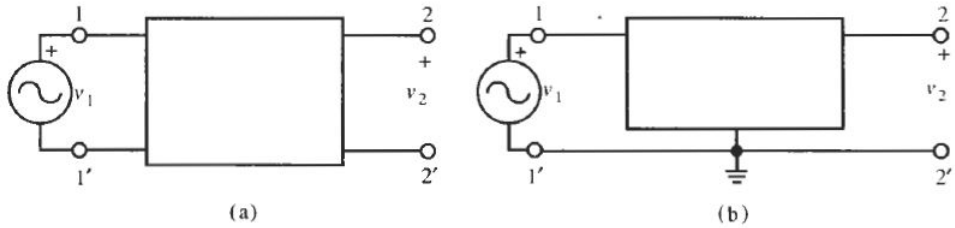

Fig. 2.1: Block diagram representations of differential floating (a) and ground referenced (b) circuits.

The Laplace transforms of the voltages ($V_1(s)$ and $V_2(s)$ for $v_1(t)$ and $v_2(t)$, respectively) can also be used to represent the voltages. Further, the voltages can also be written in terms of frequency dependent phasors by substituting $j\omega$ for s. Generally, $s = \sigma + j\omega$. However, when one is interested a steady state response study (such as frequency response), the σ component of s is ignored because it is responsible for transient response. In the frequency domain, the voltages may be represented using

$$V_1(j\omega) = V_1(s)|_{s=j\omega} = |V_1(j\omega)|e^{j\theta_1(\omega)} \tag{2.2a}$$

$$V_2(j\omega) = V_2(s)|_{s=j\omega} = |V_2(j\omega)|e^{j\theta_2(\omega)}. \tag{2.2b}$$

$V_1(j\omega)$ and $V_2(j\omega)$ can be simply represented using $V_1(jomega)$ and $V_1(\omega)$, respectively.

Transfer function

The ratio of the response of a circuit to an excitation is the transfer function of the circuit. This ratio is usually of two Laplace transforms, and it is typically converted to the frequency-dependent phasors format to study the transfer functions frequency response.

Using an expression, a transfer function, $T(s)$, is defined as

$$T(s) = \frac{\text{outputquantity}}{\text{inputquantity}} = \frac{V_2(s)}{V_1(s)}, \tag{2.3}$$

and the frequency response form becomes

$$\begin{aligned} T(\omega) &= \frac{V_2(\omega)}{V_1(\omega)} \\ &= \frac{|V_2(\omega)|e^{j\theta_2(\omega)}}{|V_1(\omega)|e^{j\theta_1(\omega)}} \\ &= \frac{|V_2(\omega)|}{|V_1(\omega)|}e^{j[\theta_2(\omega)-\theta_1(\omega)]} \\ &= |T(\omega)|e^{j\theta(\omega)}. \end{aligned} \tag{2.4}$$

The *magnitude*

$$|T(\omega)| = \frac{|V_2(\omega)|}{|V_1(\omega)|}, \tag{2.5}$$

and the *phase*

$$\theta(\omega) = \theta_2(\omega) - \theta_1(\omega), \tag{2.6}$$

are used to characterize the transfer function's frequency response.

Must the transfer function be a ratio of voltages?

Other circuit variables such as currents can also be used to represent transfer functions. For example, a ratio of output current to input voltage can be used to represent transfer admittance. The choice of the input and output variables to use depends on the type of analysis that is to be carried out.

Considering input and output currents and voltages, the possible transfer functions are

$$\text{voltage gain, } T(\omega) = \frac{V_2(\omega)}{V_1(\omega)} \tag{2.7a}$$

$$\text{voltage gain, } T(\omega) = \frac{I_2(\omega)}{I_1(\omega)} \tag{2.7b}$$

$$\text{transfer impedance, } Z(\omega) = \frac{V_2(\omega)}{I_1(\omega)} \tag{2.7c}$$

$$\text{transfer admittance, } Y(\omega) = \frac{I_2(\omega)}{V_1(\omega)}. \tag{2.7d}$$

Throughout this unit our transfer functions will be voltage gains, unless otherwise stated.

2.1.2 Poles and zeros of a transfer function

A transfer function (written in s) can be expressed in terms of a numerator, $N(s)$, and a denominator, $D(s)$. $N(s)$ and $D(s)$ are polynomials of s of various degrees. A transfer function in terms of $N(s)$ and $D(s)$ is shown in Eq. 2.8, where the numerator polynomial is of degree m and the denominator polynomial is of degree n. The coefficients $a_i,\ i = 0, \ldots, n$ and $b_j,\ j = 0 \ldots, n$ are all real numbers and they determine the behaviour of the transfer function's frequency response. The coefficients can take positive, negative, and zero values.

$$T(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \ldots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \ldots + a_1 s + a_0}. \tag{2.8}$$

Poles and zeros

The *poles* of a transfer function are the roots of the equation $D(s) = 0$ and the *zeros* of a transfer function are the roots of the equation $N(s) = 0$. In other words, poles are values of s that make a $T(s) = \infty$ while zeros are values of s that make $T(s) = 0$. Later, we will look at the influence of poles and zeros on responses.

To determine the frequency response of a circuit with a given transfer function, the magnitude and phase responses can be obtained after substituting $j\omega$ for s in Eq. 2.8. The results for $|T(j\omega)|$ and $\theta(j\omega)$ after the substitution are shown in Eqns 2.9d(a) and 2.9d(d), respectively. $\mathfrak{R}(\cdot)$ and $\mathfrak{I}(\cdot)$ are the real and imaginary parts of the complex numerator, $N(j\omega)$, and complex denominator, $D(j\omega)$, polynomials of ω.

$$|T(\omega)| = \frac{|N(\omega)|}{|D(\omega)|} \tag{2.9a}$$

$$\theta_N(\omega) = \tan^{-1}\left[\frac{\mathfrak{I}(N(\omega))}{\mathfrak{R}(N(\omega))}\right] \tag{2.9b}$$

$$\theta_D(\omega) = \tan^{-1}\left[\frac{\mathfrak{I}(D(\omega))}{\mathfrak{R}(D(\omega))}\right] \tag{2.9c}$$

$$\theta(\omega) = \theta_N(\omega) - \theta_D(\omega). \tag{2.9d}$$

Analysis and Design/Synthesis

The process of determining $T(s)$ for a given circuit is called *circuit analysis*. The process of determining the values of coefficients a_i and b_j in Eq. 2.8 and developing a circuit to achieve a desired circuit behavior is called *circuit design* or *circuit synthesis*. Even during synthesis, circuit analysis is always required to check the working of a synthesized circuit.

The following exercises are modified from examples and practice problems in [**?**, Chapter 14].

Exercise 1:

Obtain the transfer function, $T(s)$, of Fig. 2.2. The input is $v_s(t)$ and the output is $v_o(t)$. From the $T(s)$, obtain the poles and the zeros of the transfer function. Additionally, obtain $T(\omega)$ and the expressions for the magnitude, $|T(\omega)|$, and phase, $\theta(\omega)$.

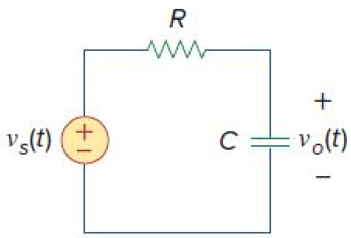

Fig. 2.2: Circuit for transfer function computation

Solution 1:

First, the circuit parameters and variables are converted to their frequency domain equivalent. The capacitive reactance or C is $X_C = \frac{1}{sC}$. Using voltage division to express $V_o(s)$ in terms of $V_s(s)$, the transfer function becomes:

$$T(s) = \frac{V_o(s)}{V_s(s)} = \frac{1}{sCR+1}.$$

The poles are obtained using the numerator by setting $1+sCR = 0$, leading to one finite pole at $-\frac{1}{CR}$. The zeros are obtained using the denominator by setting $0s + 1 = 0$, leading to no finite zero. There is, however, one zero at ∞. In this class, only the finite zeros and poles are of interest to us.

Substituting s for $j\omega$, we get

$$T(s)|_{s=j\omega} = T(\omega) = \frac{1}{j\omega CR+1},$$

and the magnitude and phase become

$$|T(\omega)| = \frac{1}{\sqrt{1+(\omega CR)^2}},$$

and

$$\theta(\omega) = -\tan^{-1}(\omega CR),$$

respectively.

An important feature of the magnitude response, $|T(\omega)|$, is the frequency ω_0 at which $|T(\omega_0)| = \frac{1}{2}$. This is the half power frequency and it is used to mark bandwidth. Rewriting the magnitude and phase responses by substituting $\omega_0 = \frac{1}{RC}$, we get

$$|T(\omega)| = \frac{1}{\sqrt{1+(\frac{\omega}{\omega_0})^2}},$$

and

$$\theta(\omega) = -\tan^{-1}\left(\frac{\omega}{\omega_0}\right).$$

Exercise 2:

Plot the magnitude and phase responses obtained for 2.1.2. To crate the plot without specifying the values for R and C, let the frequency axis be $\frac{\omega}{\omega_0}$. Assign values to $\frac{\omega}{\omega_0}$ and compute the values of $|T(\frac{\omega}{\omega_0})|$ and $\theta\left(\frac{\omega}{\omega_0}\right)$, then create the plot. Plot $|T|$ and θ plots using *linear* frequency axes. Comment on the shapes of the $|T|$ and θ plots.

Solution 2:

Assigning selected values (within the range 0 to ∞) to $\frac{\omega}{\omega_0}$ and computing $|T|$ and θ, leads to the responses in 2.3.

As can be seen in Fig. 2.3, at near zero values of $\frac{\omega}{\omega_0}$, the $|T|$ is close to 1 and θ is close to 0. At $\omega = \omega_0$, $|T| = \frac{1}{\sqrt{2}}$ and $\theta = -45°$. As $\frac{\omega}{\omega_0}$ becomes large, approaching ∞, $|T|$ approaches 0 and θ approaches $-90°$.

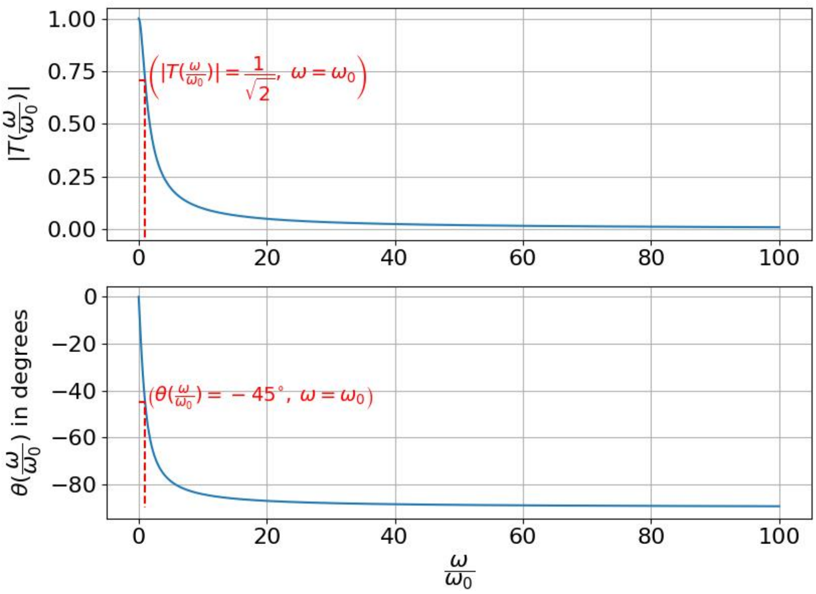

Fig. 2.3: Magnitude and phase responses for Exercise 1. The frequency axis is linear.

Table 2.1: Approximations of response at $\omega = \frac{\omega_0}{10}$, $\omega = \omega_0$, and $\omega = 10\omega_0$

Response

ω	$\lvert T\rvert$	θ	approximation
$\frac{\omega_0}{10}$	0.9950	-5.7106	low-frequency
ω_0	0.7071	-45.0000	mid-frequency
$10\omega_0$	0.0995	-84.2894	high-frequency

Generally, response at low frequency can be approximated using the value at $\omega = \frac{\omega_0}{10}$ and the response at high frequency can be approximated using $\omega = 10\omega_0$, as shown in the table. The approximations will later be used with Bode plots.

Exercise 3:

The frequency response shown in Fig.2.3 does not resolve the plot well in the lower frequency ranges. To improve that resolution, plot $|T|$ and θ against $\log_{10}\left(\frac{\omega}{\omega_0}\right)$.

Solution 3:

It is common to obtain frequency response plots using the *semilog* charts. The frequency axis is divided in $\log_{10}$ scale. Figure 2.4 shows the response similar to that in 2.3, but with the frequency axis divided in the $\log_{10}$ scale. In the $\log_{10}$, it is ratios between two points that is important, not the points themselves. For instance, the distance between $\log_{10} 0.01$ and $\log_{10} 1$ is equal to the distance between $\log_{10} 1$ and $\log_{10} 100$ in the $\log_1 0$ scale. We will use this approach, where it is convenient to better resolve smaller values, throughout this unit.

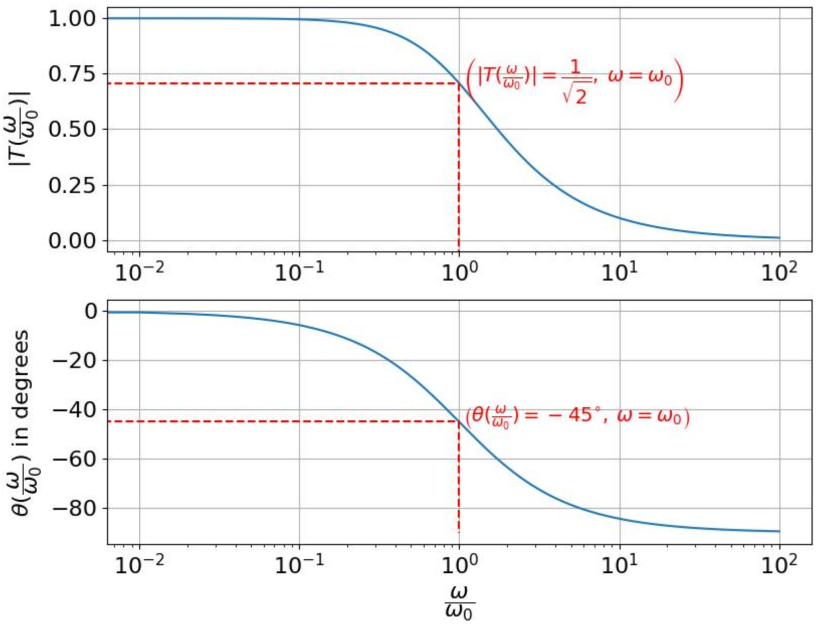

Fig. 2.4: Magnitude and phase responses for Exercise 1. The frequency axis is scaled using $\log_{10}$.

Exercise 4:

Consider the circuit in Fig. 2.5. The input is $i_i(t)$ and the output is $i_o(t)$. Use it to do the following questions.

1. Write the transfer function $T(s)$.
2. Obtain the poles and the zeros of $T(s)$.
3. Write $T(\omega)$, $|T(\omega)|$, and $\theta(\omega)$.
4. Draw the circuit's magnitude and phase response using MATLAB or Python. Use a linear ω axis.
5. Draw the circuit's magnitude and phase response using MATLAB or Python. Use the log scale for the ω axis.
6. Which frequencies (low, mid, or high) are suppressed by this circuit?

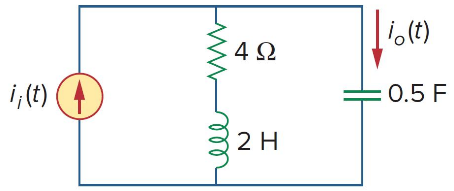

Fig. 2.5: Circuit diagram for Exercise 4.

Solution 4:

Exercise 5:

Repeat what you did in Example 4 but with the circuit shown in Fig. 2.6. The transfer function is $\frac{V_o(\omega)}{I_i(\omega)}$.

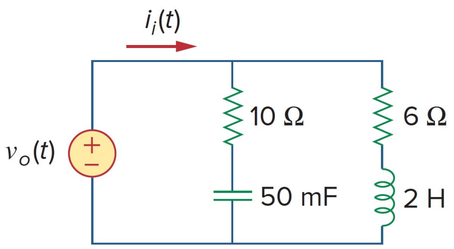

Fig. 2.6: Circuit diagram for Exercise 5.

Solution 5:

2.1.3 The decibel scale

Just as it is convenient to use the log scale for frequency, it is also convenient to measure gain (ratios of similar variables) using the log scale. The log scale for magnitude ratios is called the *bel* (named in honor of Alexander Graham Bell). It is used to measure two power ratios. For two power values, P_1 and P_2, the number of bels (B) is calculated using

$$G = \log_{10} \frac{P_2}{P_1} \text{ B}. \tag{2.10}$$

A more common unit is the decibel (dB) (which is B/10), computed using

$$G_{\text{dB}} = 10 \log_{10} \frac{P_2}{P_1} \text{ dB}. \tag{2.11}$$

Table 2.2: Approximations of gain in dB at $\omega = \frac{\omega_0}{10}$, $\omega = \omega_0$, and $\omega = 10\omega_0$

Response

ω	$\lvert T \rvert$	$20\log_{10}\lvert T \rvert \mathrm{dB}$
$\frac{\omega_0}{10}$	0.9950	-0.0432
ω_0	0.7071	-3.0103
$10\omega_0$	0.0995	-20.0432

Where only voltage and current magnitude ratios are available, the dB is computed using

$$G_{\mathrm{dB}} = 20\log_{10}\frac{V_2}{V_1}\ \mathrm{dB}, \tag{2.12a}$$

$$G_{\mathrm{dB}} = 20\log_{10}\frac{I_2}{I_1}\ \mathrm{dB}, \tag{2.12b}$$

because $P \propto V^2$ and $P \propto I^2$.

Calculating dB

1. The dB is only computed for a ratio of similar variable. For current/current, voltage/voltage, or power/power.
2. Only the magnitudes are used when computing dB.
3. When voltage or current ratios are used, $20\log_{10}(ratio)$ is applied.
4. Where power ratio is used, $10\log_{10}(ratio)$ is applied.

Exercise 6:

For the table in Exercise 2, compute the gain values in dB.

Solution 6:

The solution is shown in the table that follows.

2.1.4 Attenuation

Attenuation (often denoted by α) is normally used to refer to a gain that is less than 1 in magnitude (that is, $|T| < 1$). Attenuation yields negative gain on the dB scale (because the $\log_{10}$ of a fraction is a negative value). Circuits often attempt to block certain frequency components of a signal by attenuating them, while maintaining or enhancing the gain of other frequency components of the signal. This *frequency selection* behavior is called *filtering*.

In Exercise 6, all the computed gains are attenuation. The circuit attenuates high frequency components, while allowing the low frequency components through with zero to no attenuation.

The point at which the $\alpha \approxeq -3\,\mathrm{dB}$ (i.e. $|T| = \frac{1}{\sqrt{2}} \approxeq 0.7071$) marks the half-power point of the transfer function. The frequency corresponding to this point is used to mark the *bandwidth* of the transfer function, $T(\omega)$.

Specifying attenuation

When specifying α, it is sometimes typical to drop the minus sign because attenuation already implies that the gain is less than 1. For instance, if some attenuation value $\alpha = \alpha_n = 100\,\mathrm{dB}$ is specified, this implies that $|T| = -100\,\mathrm{dB}$.

2.2 Classification of responses

2.2.1 Magnitude responses

Responses can be classified according to the behavior of the ideal magnitude over a frequency range of interest. Responses can *stop* or *pass* selected frequency bands. Frequency bands that are stopped (attenuated) by a response are called a ***stopbands*** while the frequency bands that are passed (allowed through without attenuation) are called ***passbands***.

In the ideal magnitude responses, in the passbands, $|T| = 1$ and $\alpha = 0$ while in the stopbands $|T| = 0$ and $\alpha = \infty$. The patterns of passbands and stopbands in a response give rise to the four main types of responses:

1. *lowpass* - where the passband extends from $\omega = 0$ to a *cutoff frequency* $\omega = \omega_c$,
2. *highpass* - where a stopband extends from $\omega = 0$ to a *cutoff frequency* $\omega = \omega_c$,
3. *bandpass*- where a passband extends from $\omega = \omega_1$ to $\omega = \omega_2$, and
4. *bandstop* or *notch* - where a stopband extends from $\omega = \omega_1$ to $\omega = \omega_2$.

An illustration of ideal and typical practical response types are shown in Fig. 2.7. The responses are used to describe filters.

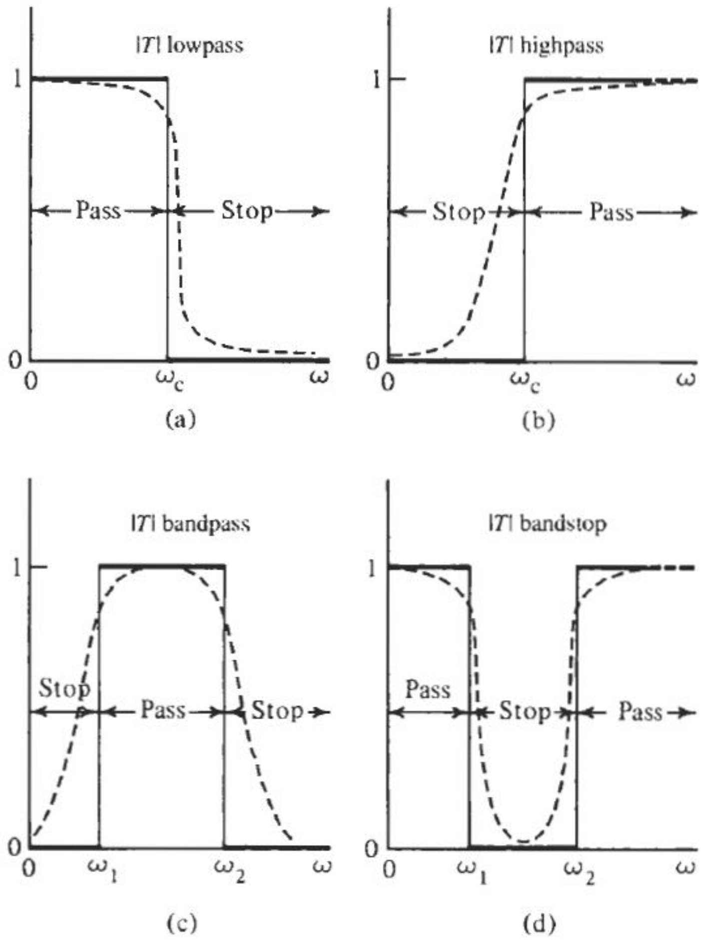

Fig. 2.7: Ideal (continuous) and practical (dashed) response types. The responses are lowpass (a), highpass (b), bandpass (c), and bandstop (d).

Because it is impossible to achieve ideal responses in practice, the following criteria are used to designate stopbands and passbands.

1. The stopbands are the regions where the attenuation exceed a minimum attenuation, $\alpha_{\min}$.
2. The passbands are the regions where the attenuation is below a maximum attenuation, $\alpha_{\max}$.
3. The regions between stopbands and passbands are called *transition* bands.

An illustration of how $\alpha_{\min}$, $\alpha_{\max}$, and transition bands are used to classify responses is provided in Fig. 2.8. As is typical, the minus sign has been dropped for the attenuations in Fig. 2.8.

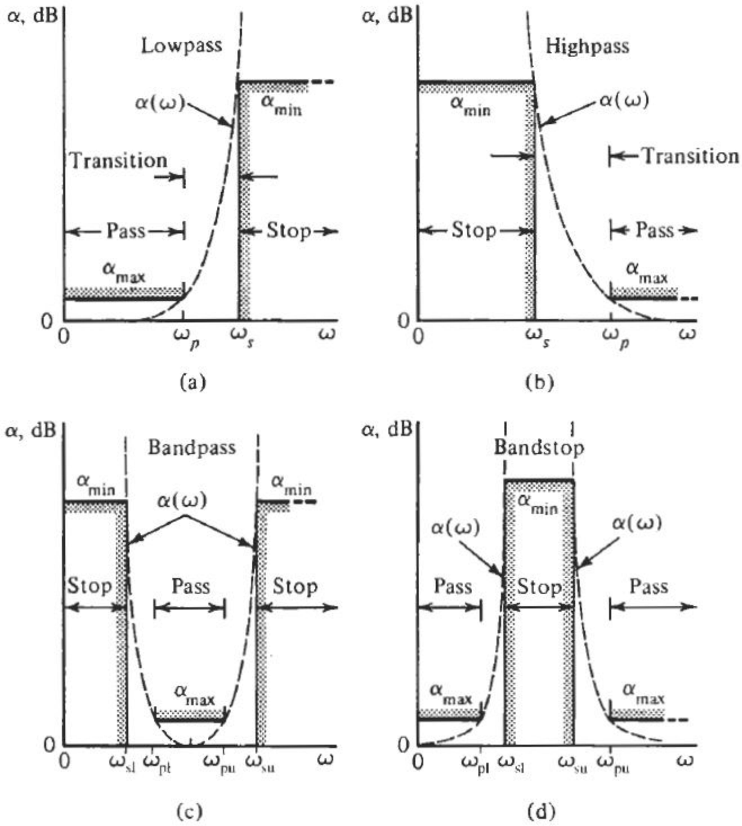

Fig. 2.8: Showing $\alpha_{\min}$, $\alpha_{\max}$, and transition bands in lowpass, highpass, bandpass, and bandstop filters.

We will use these notations later in this unit.

2.2.2 Phase responses

The phase responses can be classified into

1. lead, or
2. lag

responses depending on whether $\theta(\omega) > 0$ (leading) or $\theta(\omega) < 0$ (lagging). For instance, in Fig. 2.4 the phase response is lagging.