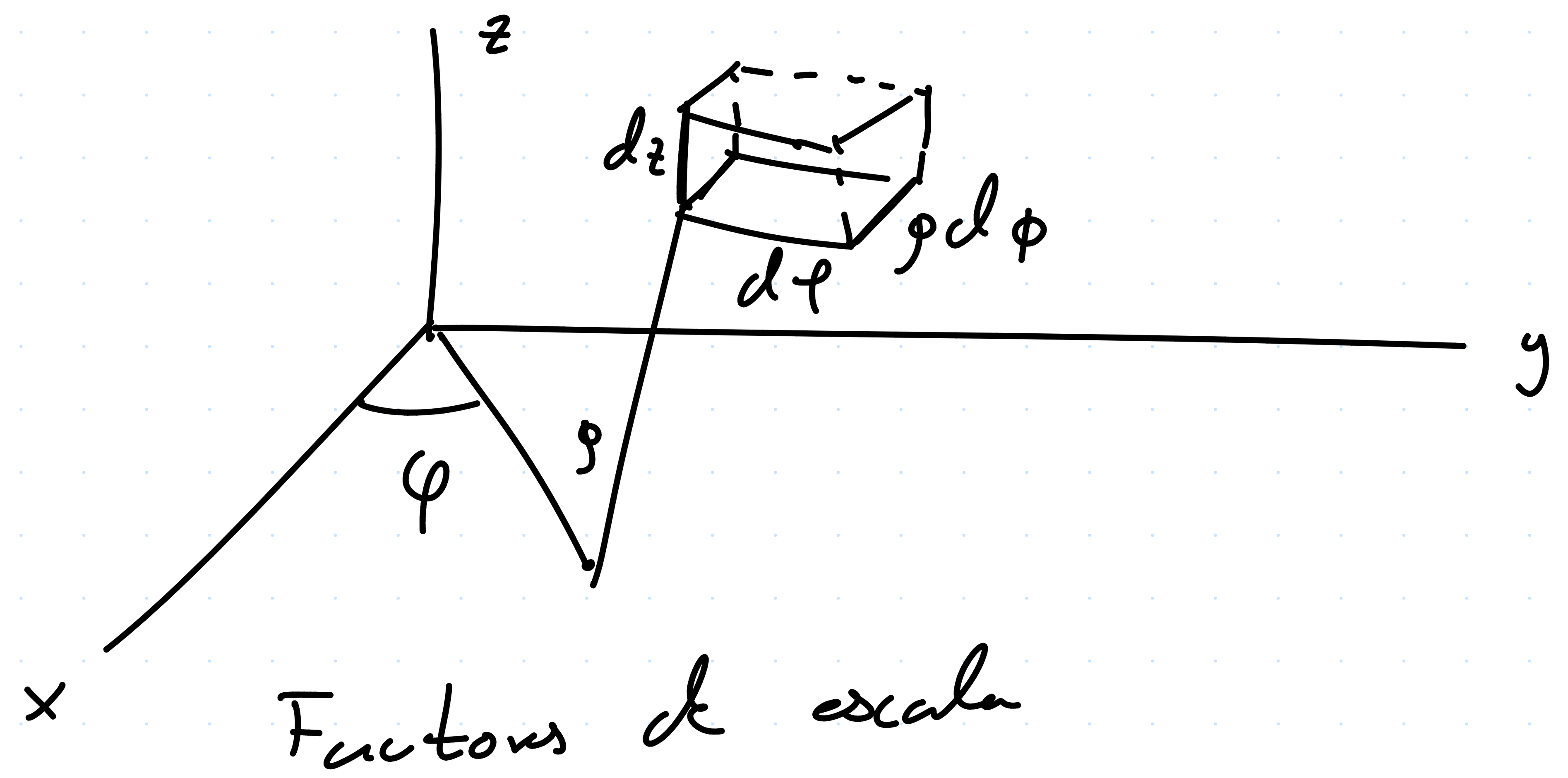
Funciones especiales

Clase # 4

12/08/2025



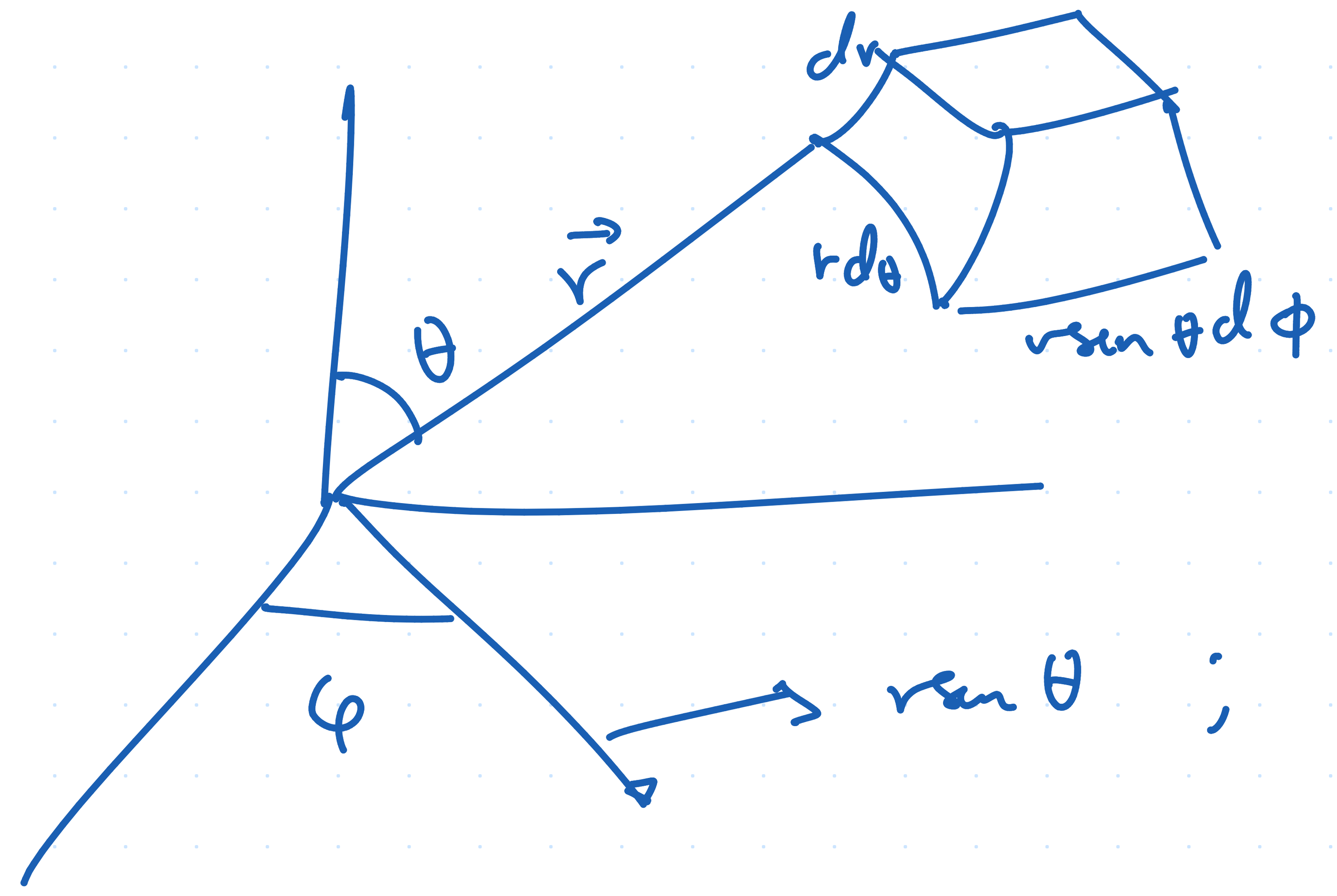
Factores de escala

$h_1 = 1$; $h_2 = \rho$; $h_3 = 1$

$$\boxed{V = \rho \, d\rho \, d\phi \, dz}$$

$$\nabla \cdot \vec{F} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(F_1 h_2 h_3\right) + \frac{\partial}{\partial q_2}\left(F_2 h_1 h_3\right) + \frac{\partial}{\partial q_3}\left(F_3 h_2 h_1\right)\right]$$

$$\nabla^2 \phi = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2 h_3}{h_1}\frac{\partial \phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1 h_3}{h_2}\frac{\partial \phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_2 h_1}{h_3}\frac{\partial \phi}{\partial q_3}\right)\right]$$



$\rightarrow r \sin\theta$; $h_1 = 1$, $h_2 = r$, $h_3 = r\sin\theta$

$V = r^2 \sin\theta \, dr \, d\theta \, d\phi$

Tenemos

$$\vec{r} = r\cos\phi \operatorname{sen}\theta\, \hat{i} + r \operatorname{sen}\phi \operatorname{sen}\theta\, \hat{j} + r\cos\theta\, \hat{k}$$

$$\vec{e}_r = \hat{e}_r = \cos\phi \operatorname{sen}\theta\, \hat{i} + \operatorname{sen}\phi \operatorname{sen}\theta\, \hat{j} + \cos\theta\, \hat{k}$$

$$\vec{e}_\theta = r\cos\phi\cos\theta\, \hat{i} + r\operatorname{sen}\phi\cos\theta\, \hat{j} - r\operatorname{sen}\theta\, \hat{k}$$

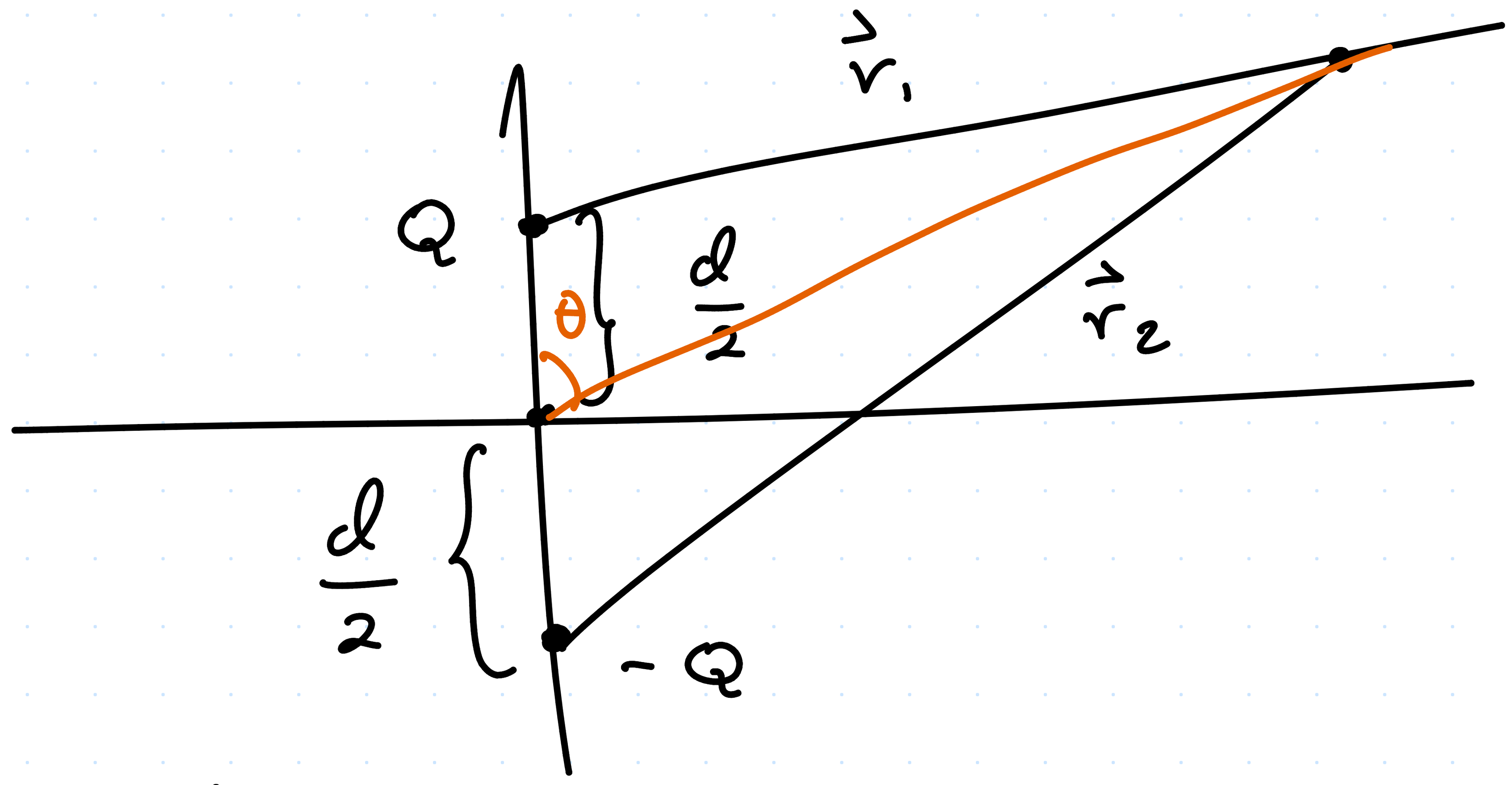
$$\frac{\vec{e}_\theta}{\|\vec{e}_\theta\|} = \hat{e}_\theta = \cos\phi\cos\theta\, \hat{i} + \operatorname{sen}\phi\cos\theta\, \hat{j} - \operatorname{sen}\theta\, \hat{k}$$

$$\vec{e}_\phi = -r\operatorname{sen}\phi\operatorname{sen}\theta\, \hat{i} + r\cos\phi\operatorname{sen}\theta\, \hat{j}$$

$$\frac{\vec{e}_\phi}{\|\vec{e}_\phi\|} = -\operatorname{sen}\phi\, \hat{i} + \cos\phi\, \hat{j} \quad ; \qquad h_3 = r\operatorname{sen}\theta$$

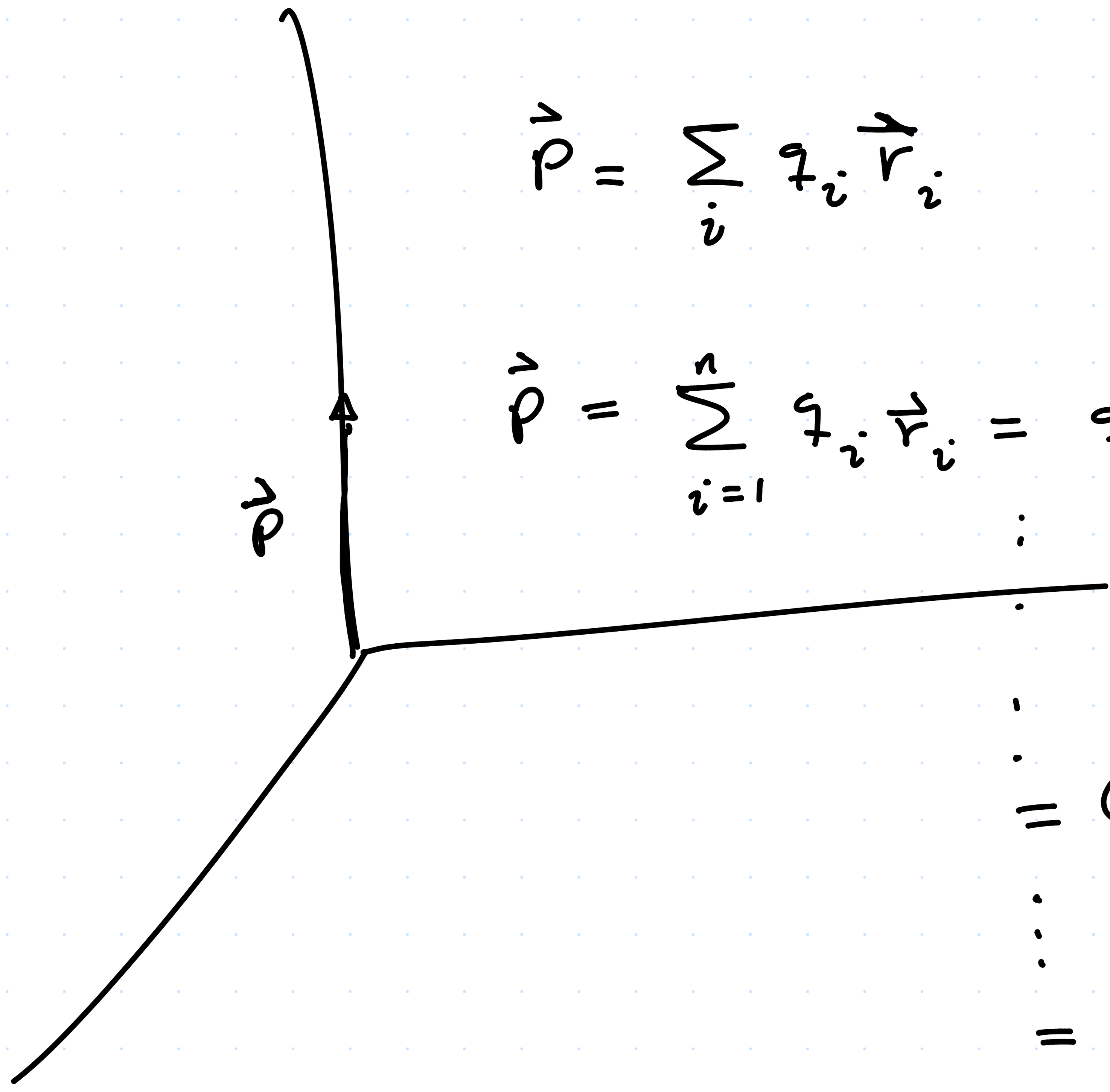
$$\vec{r} = r\hat{e}_r$$

Dipolo



Expansión multipolar de

las cargas.



$$\vec{p} = \sum_i q_i \vec{r}_i$$

$$\vec{p} = \sum_{i=1}^{n} q_i \vec{r}_i = q_1 \vec{r}_1 + q_2 \vec{r}_2$$

$$= Q\left(\frac{d}{2}\right)\hat{k} + (Q)(-\hat{k})$$

$$= \underbrace{Qd}_{p}\hat{k}$$

$$\phi(r, \theta) = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^3}$$

$\underline{\nabla\phi}$

$$\nabla\phi(r,\theta) = \frac{3\hat{r}(\vec{p}\cdot\hat{r}) - \vec{p}}{4\pi\varepsilon_0 r^3}$$

$$-\nabla\phi(r,\theta) = \vec{F}$$ ← en algún punto.

<u>Tenemos</u>

Tomamos el potencial:

$$\phi(r,\theta) = \frac{\vec{p}\cdot\hat{r}}{4\pi\varepsilon_0 r^3} = \frac{p\cos\theta}{4\pi\varepsilon r^3}$$

Proyecciones del campo eléctrico

Gradiente para $\nabla\phi$:

$$\nabla \phi = \frac{1}{h_1}\frac{\partial \phi}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial q_2} e_2 + \frac{1}{h_3}\frac{\partial \phi}{\partial q_3}\hat{e}_3$$

Tenemos

$$E_r = -\frac{\partial}{\partial r}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 r^3}\right) = \frac{3p\cos\theta}{4\pi\varepsilon_0 r^4}$$

$$E_\theta = -\frac{1}{r}\frac{\partial}{\partial \theta}\left(\frac{p\cos\theta}{4\pi\varepsilon_0 r^3}\right) = \frac{p\,\text{sen}\,\theta}{4\pi\varepsilon_0 r^{4}}$$

$$\vec{E} = \frac{3p\cos\theta\,\hat{r} - p\,\text{sen}\,\theta\,\hat{\theta}}{4\pi\varepsilon_0 r^4}$$

$$\vec{p} = p\hat{k}$$

base canónica.

$$\cos\theta\,\hat{r} = \cos\phi\,\text{sen}\,\theta\cos\theta\,\hat{\imath} + \text{sen}\,\phi\,\text{sen}\,\theta\cos\theta\,\hat{\jmath} + \cos^2\theta\,\hat{k}$$

$$-\text{sen}\,\theta\,\hat{\theta} = -\cos\phi\cos\theta\,\text{sen}\,\theta - \text{sen}\,\phi\cos\theta\,\text{sen}\,\theta\,\hat{\jmath} + \text{sen}^2\theta\,\hat{k}$$

$$\hat{k} = \cos\phi\,\hat{r} - \text{sen}\,\theta\,\hat{\theta}$$

$$\vec{p} = p\hat{k} = p\cos\theta\,\hat{r} - p\,\text{sen}\,\theta\,\hat{\theta}$$

$$\vec{p}\cdot\hat{r} = p\cos\theta$$

Electrostática $\longrightarrow$ Ecuación de Laplace.

$$\nabla^2\phi = 0$$

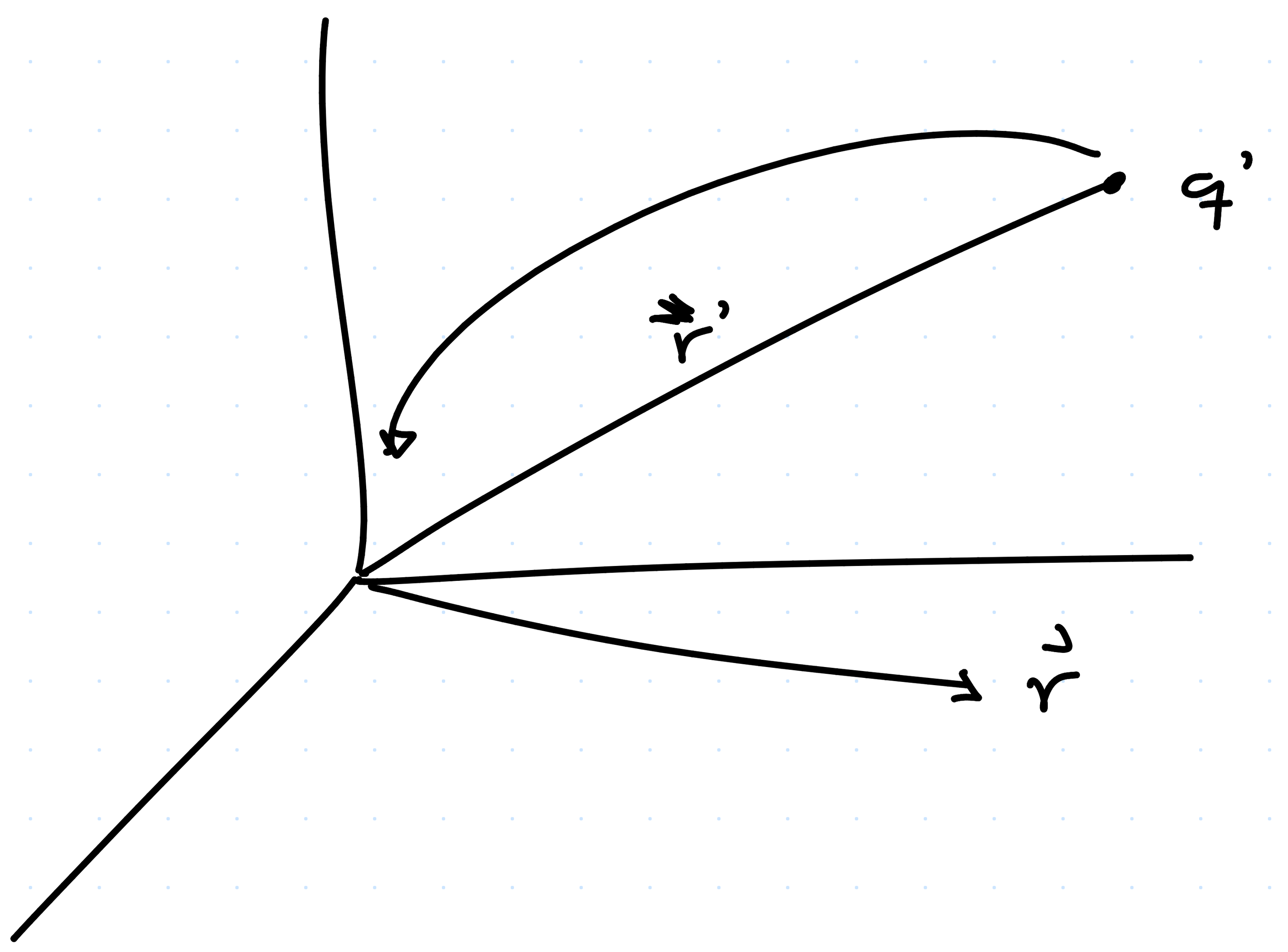
Ecuación de Poisson.

$$\nabla^2\phi = -4\pi\rho$$

$$\phi = \frac{1}{|r-\vec{r}'|}$$

Potencial electrostático

$$\theta = \frac{q'}{4\pi\varepsilon_0|\vec{r}-r'|}$$



Laplaciano de ϕ en esféricas:

$\phi(r) = \frac{1}{r}$

Tenemos

$$\nabla^2\phi = \frac{1}{r^2 \operatorname{sen}\theta}\left[\frac{\partial}{\partial r}\left(\frac{r^2 \operatorname{sen}\theta}{1}\frac{\partial\phi}{\partial r}\right)\right]$$

$\vdots$

$$= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\right)\right]$$

$\vdots$

- Hom → Espacios vectoriales
- Discontinuidad del potencial.

$$= -\frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right)\right]$$

si $r \neq 0$

↳ integrales de superficie alrededor del origen

Integramos en un volumen el laplaciano:

$$\int_V \nabla^2 \left(\frac{1}{r}\right) dv = \int_V \nabla \cdot \nabla \left(\frac{1}{r}\right) dv$$

Teorema de la Divergencia

$$\oint \nabla\left(\frac{1}{r}\right) \cdot \hat{n} \, dA$$

$$\oint \left|\nabla\left(\frac{1}{r}\right)\right| \cos\theta \, dA$$

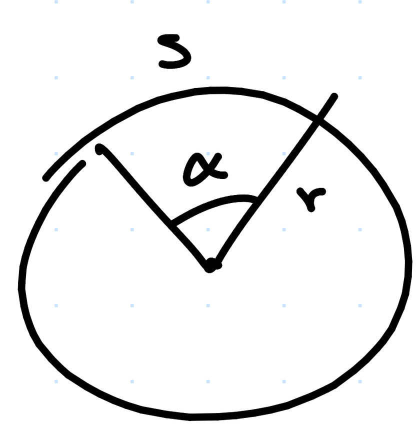
$$\nabla\left(\frac{1}{r}\right) = \frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{r}$$

$$-\oint \frac{\cos\theta \, dA}{r^2} = -\oint \frac{r^2 \operatorname{sen}\theta \, d\theta \, d\phi}{r^2}$$

$$\vdots$$

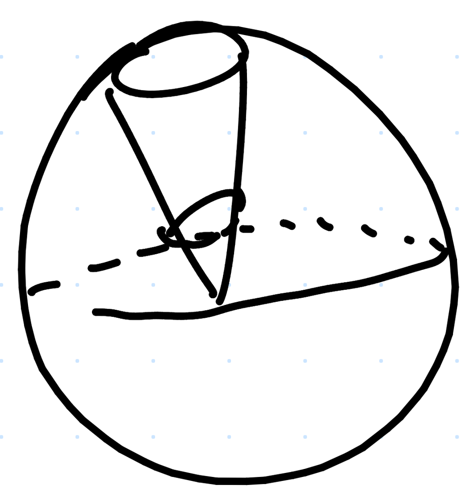
$$= -4\pi$$

Radián:



$$\alpha = \frac{s}{r} = 1$$

Estereoradián



$$\Omega = \frac{s}{R^2} = 1$$

$$d\Omega = \frac{ds}{R^2}$$

Visión aproximada de 2π (ojo humano).

$$\nabla^2 \phi = 0 \; ; \quad \text{si} \quad r \neq 0$$

$$\nabla^2 \phi = -4\pi \; ; \quad \text{si} \quad r = 0$$

Delta de Dirac

Función definida como:

$$\delta(x) = \begin{cases} \infty & \text{si} \quad x = 0 \\ 0 & \text{si} \quad x \neq 0 \end{cases}$$

$$\delta(x-a) = \begin{cases} \infty & \text{si} \quad x = a \\ 0 & \text{si} \quad x \neq a \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(x-a)\, dx = 1$$

Propiedades

- simetría

$$\delta(x-a) = \delta(a-x)$$

$$\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$$

$$\delta(x-a) = \frac{1}{2\pi} + \frac{1}{h} \sum_{n} \cos(n(x-a))$$

Podemos escribir:

$$\nabla^2 \left(\frac{1}{r} \right) = -4\pi\, \delta(r)$$

Fun del origen

$$\nabla^2 \left(\frac{1}{|\vec{r}-\vec{r}'|} \right) = -4\pi\,\delta(\vec{r}-\vec{r}')$$

Potencial

$$\phi = \int \frac{\rho'(r')\,dv'}{4\pi\varepsilon_0 |\vec{r}-\vec{r}'|}$$

$$\nabla^2\phi = \int \frac{\rho(r')\nabla^2}{4\pi\varepsilon_0}\left(\frac{1}{|r-r'|}\right) dv'$$

$$\vdots$$

$$= -\int \frac{\rho(r')\,\nabla^2}{4\pi\varepsilon_0} \frac{1}{|r-\vec{r}'|}\, dv'$$

$$\vdots$$

$$= -\int \frac{\rho(r')}{4\pi\varepsilon_0}\,\nabla^2 \frac{1}{|r-\vec{r}'|}\, dv'$$

$$\vdots$$

$$= -\int \frac{\rho(r')}{4\pi\varepsilon_0}\,\delta(\vec{r}-\vec{r}')\,dv$$

$$\boxed{\nabla^2\phi = -4\pi\rho(r)}$$