Funciones especiales Clase #6 15/08/2025

Potencial:

$$V = \frac{ze}{4\pi \varepsilon_0 r} \quad ; \quad \vec{F} = -\nabla V$$

$$\nabla r^n = ?$$

$$\nabla f(r) = \left[\hat{r} \frac{\partial}{\partial r} + \frac{\hat{\theta}}{r} \frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r \operatorname{sen}\theta} \frac{\partial}{\partial \phi} \right] f(r)$$

$$\vdots$$

$$= \hat{r} \frac{df}{dr}$$

$$\nabla r^n = \hat{r}\, n r^{n-1}$$

Divergencia

$$\nabla \cdot (\hat{r} f(r))$$

$$\frac{1}{r^2 \operatorname{sen}\theta} \left[\frac{\partial}{\partial r} \left(r^2 \operatorname{sen}\theta f(r) \right) + \cancel{\frac{\partial}{\partial \theta} \left(\quad \right)} + \cancel{\frac{\partial}{\partial \phi}} \right]$$

$$\frac{\operatorname{sen}\theta}{r^2 \operatorname{sen}\theta} \left[\frac{d}{dr} \left(r^2 f(r) \right) \right]$$

$$= \frac{1}{r^2}\left[2r f(r) + r^2 \frac{df}{dr}\right]$$

$$= \frac{2}{r} f(r) + \frac{df}{dr}$$

$$\nabla \cdot (\hat{r}\, r^n) = \frac{2}{r} r^n + n r^{n-1}$$

$$\vdots$$

$$= (2+n)\, r^{n-1}$$

Divergencia del potencial:

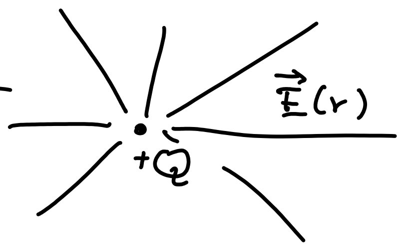
$$\nabla V = \frac{Ze}{4\pi\varepsilon_0} \nabla\left(\frac{1}{r}\right) = \frac{Ze}{4\pi\varepsilon_0} \nabla(r^{-1}) = -\frac{Ze}{4\pi\varepsilon_0 r^2}\hat{r} = \vec{E}$$

$$\nabla \cdot \vec{E} = -\frac{Ze}{4\pi\varepsilon_0} \nabla \cdot (\hat{r}\, r^{-2}) = 0$$

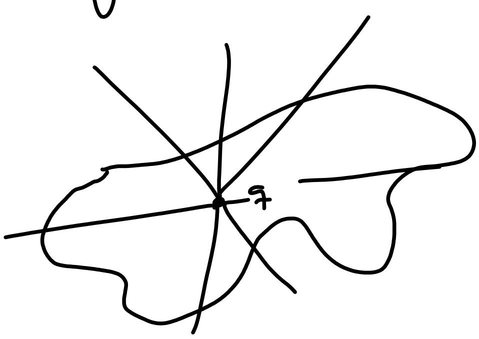
$$\nabla \cdot \vec{E} = 0 \longrightarrow$$ Ley de Gauss

"líneas de campo"

$$\oint \vec{E} \cdot d\vec{s} = Q_{enc}$$



Dirección radial

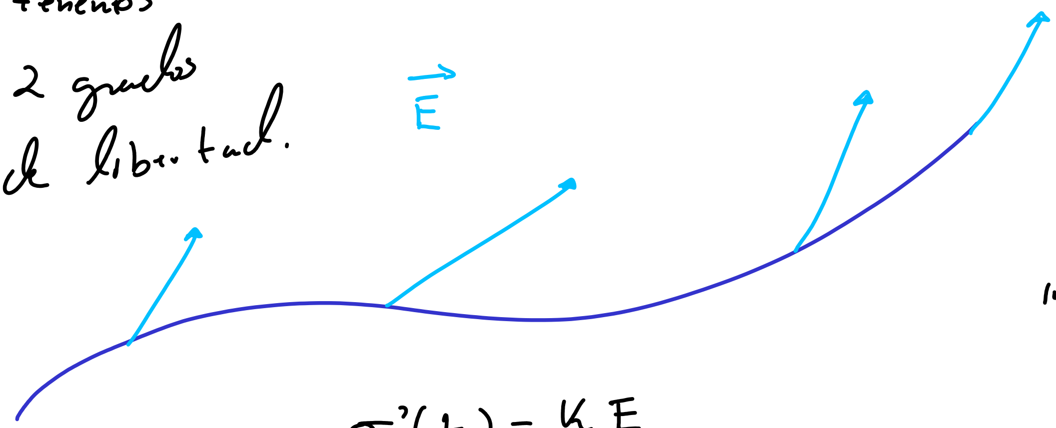


Checar leyes de maxwell

¿Dónde queda la parte vectorial?

Líneas de flujo

- tenemos 2 grados de libertad.



cuántica

interacción

fotón – fotón

$$\sigma'(t) = kE$$

curva parametrizada

- Las líneas de campo no se cruzan
  - Fuentes y sumideros.

Laplaciano

$$\nabla^2 f(r)$$

$$= \frac{1}{r^2 \operatorname{sen}\theta}\left[\frac{\partial}{\partial r}\left(\frac{r \operatorname{sen}\theta}{1}\frac{\partial f}{\partial r}\right) + \frac{\partial}{\partial \theta}\left[\frac{(1)\, r \operatorname{sen}\theta}{r}\cancel{\frac{\partial}{\partial \theta}\left(f(r)\right)}\right]\right]$$

0

$\vdots$

$$= \frac{1}{r^2}\left[\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right)\right]$$

$\vdots$

$$= \frac{1}{r^2}\left[\frac{df}{dr} + r\frac{d^2f}{dr^2}\right] \rightarrow \frac{1}{r^2}\frac{df}{dr} + \frac{1}{r}\frac{d^2f}{dr^2}$$

$$\nabla^2 r^n = \frac{1}{r^2} n r^{n-1} + \frac{1}{r} n(n-1) r^{n-2}$$

$\vdots$

$$= n r^{n-3} + n(n-1) r^{n-3}$$

$\vdots$

$$= [n + n(n-1)] r^{n-3}$$

correspondencia cuántica clásica.

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operadores adjuntos.

Torca: derivada temporal de $\vec{L}$

$$\vec{L} = \vec{r} \times \vec{p}$$

$\longrightarrow$ corchetes de Poisson $\rightarrow$ Estructura simpléctica

$$\left.\begin{array}{l} \{x, x\} = 0, \quad \{x, p_x\} = 1 \\ \{y, p_y\} = 1 \\ \{z, p_z\} = 1 \end{array}\right\} \quad \begin{array}{l} p_x \rightarrow \hat{p}_x = i\frac{\partial}{\partial x} \\ \hat{p}_y = -i\frac{\partial}{\partial y} \\ \hat{p}_z = -i\frac{\partial}{\partial z} \end{array}$$

Tenemos

$$\vec{p} = -i\nabla$$

$$\longrightarrow \quad \vec{p} = -i\left[\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial\theta} + \frac{\hat{\phi}}{r\,\text{sen}\theta}\frac{\partial}{\partial\phi}\right]$$

Momento angular en esféricas...

Cartesianas

$$\vec{r} = r\hat{r} = \frac{r}{r}(x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

$$\vdots$$

$$= (x\hat{\imath} + y\hat{\jmath} + z\hat{k})$$

$$\vec{L}_j = -i\,\varepsilon_{jkl}\, r_k \partial_l$$

$$\vec{L}_x = -i\,\varepsilon_{123}\, y\partial_z - i\,\varepsilon_{132}\, z\partial_y$$

$$\Rightarrow \quad \vec{L}_x = -i\left[y\frac{\partial}{\partial x} - z\frac{\partial}{\partial y}\right]$$ ← operador de momento angular x.

Probemos el conmutador:

$$\boxed{[\hat{L}_i, \hat{L}_j] = -i\,\varepsilon_{ijk}\hat{L}_k}$$

$$\hat{L}_y = -i\,\varepsilon_{2jk}\, r_k\, \partial_k$$

$$\hat{L}_y = -i\left[\varepsilon_{213}\, x\,\partial_z + \varepsilon_{23}\, z\,\partial_x\right]$$

$$\vdots$$

$$= -i\left[-x\,\partial_z + z\,\partial_x\right]$$

$$\hat{L}_y = -i\left[z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right]$$

Tenemos ...

$$\hat{L}_z = -i\,\varepsilon_{3jk}\, r_j\, \partial_k$$

$$\hat{L}_z = -i\left[\varepsilon_{312}\, x\,\partial_y + \varepsilon_{321}\, y\,\partial_x\right)$$

$$\boxed{\hat{L}_z = -i\left[x\,\partial_y - y\,\partial_x\right]}$$

Conmutador → conmutatividad o no conmutatividad (Lie algebra)

Propiedad de antisimetría

$$[\hat{L}_x, \hat{L}_y] = i\,\varepsilon_{123}\,\hat{L}_3 = -\hat{L}_z$$

$$\vdots$$

$$= \hat{L}_x\hat{L}_y - \hat{L}_y\hat{L}_x$$

$$\vdots$$

$$= \hat{L}_x\left[-i\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right] - \hat{L}_y\left[-i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right]$$

$$\vdots$$

$$= -i\left\{-i\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right) - (-i)\right.$$

$$\left.\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right\}$$

$$\vdots$$

$$= -i\left\{-i\, y\left(\frac{\partial}{\partial x} + z\frac{\partial^2}{\partial z\partial x}\right) - y\left(x\frac{\partial^2}{\partial z^2}\right) - z\left(z\frac{\partial^2}{\partial y\partial x} - x\frac{\partial^2}{\partial y\partial x}\right)\right.$$

$$- (-i)\left[z\left(y\frac{\partial^2}{\partial x\partial z}\right) - z\left(z\frac{\partial^2}{\partial x\partial y}\right) - x\left(y\frac{\partial^2}{\partial z^2}\right)\right.$$

$$+ x\left(\frac{\partial}{\partial y} + z\frac{\partial^2}{\partial z\partial y}\right)$$

$$= -i\left\{-i\left(y\partial_x - x\partial_y\right)\right\} \quad ; \quad \left[\hat{L}_x, \hat{L}_z\right] = \hat{L}_y$$

$$\vdots$$

$$= -i\hat{L}_z \qquad\qquad \left[\hat{L}_y, \hat{L}_z\right] = \hat{L}_x$$

Problema <u>2.5 Arfken.</u>

$$\frac{\partial}{\partial x} = \operatorname{sen}\theta\cos\phi\frac{\partial}{\partial r} + \cos\theta\cos\varphi\,\frac{1}{r}\frac{\partial}{\partial\theta} - \frac{\operatorname{sen}\varphi}{r\operatorname{sen}\theta}\frac{\partial}{\partial\phi}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\operatorname{sen}\theta}{r}\frac{\partial}{\partial\theta}$$

Tenemos

$$\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial x}$$

Átomo de hidrógeno

- cuantizado
- $L = n\hbar$

Resolver:

$$\nabla_{xx} = \nabla_{r\theta\phi} \rightsquigarrow \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z} \right)$$

$$= \left(\hat{r}\frac{\partial}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial}{\partial \theta} + \frac{\hat{\phi}}{r\,\mathrm{sen}\,\theta}\frac{\partial}{\partial \phi} \right]$$

Igualar componente a componente.

$$\hat{r} = \mathrm{sen}\,\theta\cos\phi\,\hat{\imath} + \mathrm{sen}\,\theta\,\mathrm{sen}\,\phi\,\hat{\jmath} + \cos\theta\,\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\,\hat{\imath} + \cos\theta\,\mathrm{sen}\,\phi\,\hat{\jmath} - \mathrm{sen}\,\theta\,\hat{k}$$

$$\hat{\phi} = -\mathrm{sen}\,\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath} + 0\,\hat{k}$$

$$\frac{\partial}{\partial x} = \mathrm{sen}\,\theta\cos\phi\frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\phi\frac{\partial}{\partial \theta} - \frac{\mathrm{sen}\,\phi}{r\,\mathrm{sen}\,\theta}\frac{\partial}{\partial \phi}$$

$$-i\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right] = -i\frac{\partial}{\partial \phi}$$

$$L_z = -i\frac{\partial}{\partial \phi}$$

$$-i\left(r\,\mathrm{sen}\,\theta\cos\phi\left[\mathrm{sen}\,\theta\,\mathrm{sen}\,\phi\frac{\partial}{\partial r} + \cos\theta\frac{\mathrm{sen}\,\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\varphi}{r\,\mathrm{sen}\,\theta}\frac{\partial}{\partial \varphi} \right] \right.$$

$$-\left(r \operatorname{sen}\theta \operatorname{sen}\phi \left[\operatorname{sen}\theta \cos\varphi \frac{\partial}{\partial r} + \frac{1}{r}\cos\theta\cos\varphi\frac{\partial}{\partial\theta} - \frac{\operatorname{sen}\phi}{r\operatorname{sen}\theta}\frac{\partial}{\partial\phi}\right]\right)$$

$$-i\left[\frac{\partial}{\partial\varphi}\right]$$

$$\frac{\partial}{\partial y} = \operatorname{sen}\theta \operatorname{sen}\phi \frac{\partial}{\partial r} + \cos\theta \frac{\operatorname{sen}\phi}{r}\frac{\partial}{\partial\theta} + \frac{\cos\varphi}{r\operatorname{sen}\theta}\frac{\partial}{\partial\varphi}$$

Nuevo tema

Ecuaciones diferenciales:

$$f(x, y, y', y'', \ldots, y^{n}) = 0$$

$$f(x, y, y') = 0$$

$$\frac{dy}{dx} = f(x,y) = f(x)\cdot g(y)$$

Función homogénea de orden 0

$\Rightarrow$ separable...

Tenemos... $M\,dx + N\,dy = 0$

$$\frac{dy}{dx} = \frac{M}{N}$$

Función homogenea:

Todas las entradas las multificamos por un parámetro:

$$f(x,y) = f(\lambda x, \lambda y) = \lambda^n f(x,y)$$

$$\sqrt{x^2+y^2} = \sqrt{\lambda^2(x^2+y^2)} = \lambda\sqrt{x^2+y^2} = \lambda f(x,y)$$

- Vemos si es homogenea de orden 0.

Tenemos $\dfrac{dy}{dx} = f(x,y)$

$$\Rightarrow f(\lambda x, \lambda y) = f(x,y)$$

$\lambda = \dfrac{1}{x}$; $z = \dfrac{y}{x}$ ← cambio de variable

$$f(1,z) = f(x,y)$$

$$\frac{dy}{dx} = \frac{d}{dx}[xz] = z + x\frac{dz}{dx}$$

$$\Rightarrow z + x\frac{dz}{dx} = f(1,z)$$

$$\frac{x\,dz}{dx} = f(1,z) - z$$

$$\int \frac{dz}{f(1,z)-z} = \int \frac{dx}{x}$$

$$\Rightarrow \int \frac{dz}{f(1,z)-z} = \ln(x)$$. ya la separamos

otro ejemplo:

$$\frac{dy}{dx} = \frac{(x+y)}{x-y} \; ; \quad z = \frac{y}{x}$$

$$z + x\frac{dz}{dx} = \frac{x(1+z)}{x(1-z)}$$

$$x\frac{dz}{dx} = \frac{1+z}{1-z} - z$$

$$x\frac{dz}{dx} = \frac{1+z-z+z^2}{1-z}$$

$$x\frac{dz}{dx} = \frac{1+z^2}{1-z}$$

$$\int \frac{(1-z)\,dz}{1+z^2} = \ln(x) \; ; \quad \int \frac{dz}{1+z^2} - \int \frac{z\,dz}{1-z^2} = \ln(x)$$

Fracciones parciales si existiese un -1

cambio de var.

$u = 1 - z^2$

$du = -2z\,dz$

$$-\frac{du}{2} = z\,dz$$

$$-\frac{1}{2}\int \frac{du}{u} = -\frac{1}{2}\ln\left[1 - z^2\right]$$

Tenemos

$z = \tan\theta$

$dz = \sec^2\theta$

$$\int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \theta \; ; \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

$$\tan^{-1}\left(\frac{y}{x}\right) - \frac{1}{2}\ln\left(1 - z^2\right) = \ln(x)$$

Todo esto para homogenea de orden 0.

Wando no es de orden 0.

Tenemos...

$$Mdx + Ndy = 0$$

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$\exists f$ ...

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

$$\frac{\partial f}{\partial x} = M \; ; \; \frac{\partial f}{\partial y} = N.$$

si no es exacta: tenemos...

$$\alpha \left[Mdx + Ndy \right] = 0$$

$$\frac{\partial}{\partial y}(\alpha M) = \frac{\partial (\alpha N)}{\partial x}$$

$$\alpha(x) = \int e^{\frac{1}{M}\left(\frac{\partial N}{\partial y} - \frac{\partial M}{\partial x}\right)} dx$$

Si no es ninguna de las anteriores:

$$\underbrace{\frac{dy}{dx} + p(x)\, y} + q(x) = 0$$

$$\frac{d}{dx}\left(e^{\int p(x)dx} \cdot y \right) + q(x)\, e^{\int p dx} = 0$$

$$e^{\int p dx} p y + e^{\int p dx} \frac{dy}{dx}$$

$$\int d\left(e^{\int p(x) dx} y \right) = - \int q(x)\, e^{\int p dx} dx$$

$$e^{\int p dx} y = - \int q(x)\, e^{\int p dx} dx$$