Funciones especiales — clase #3 — 08/08/2025

$$\nabla \phi(q_1, q_2, q_3) = \frac{1}{h_1}\frac{\partial \phi}{\partial q_1}\hat{e}_1 + \frac{1}{h_2}\frac{\partial \phi}{\partial q_2}\hat{e}_2 + \frac{1}{h_3}\frac{\partial \phi}{\partial q_3}\hat{e}_3$$

$h_i = \|\vec{e}_i\|$; $i = 1, 2, 3$

Cartesianas: $h_1 = 1,\ h_2 = 1,\ h_3 = 1$

Polares: $h_1 = 1,\ h_2 = \rho$

Cilíndricas: $h_1 = 1,\ h_2 = \rho,\ h_3 = 1$

Esféricas ?

$\nabla \cdot \vec{F}$

$\nabla \times \vec{F}$

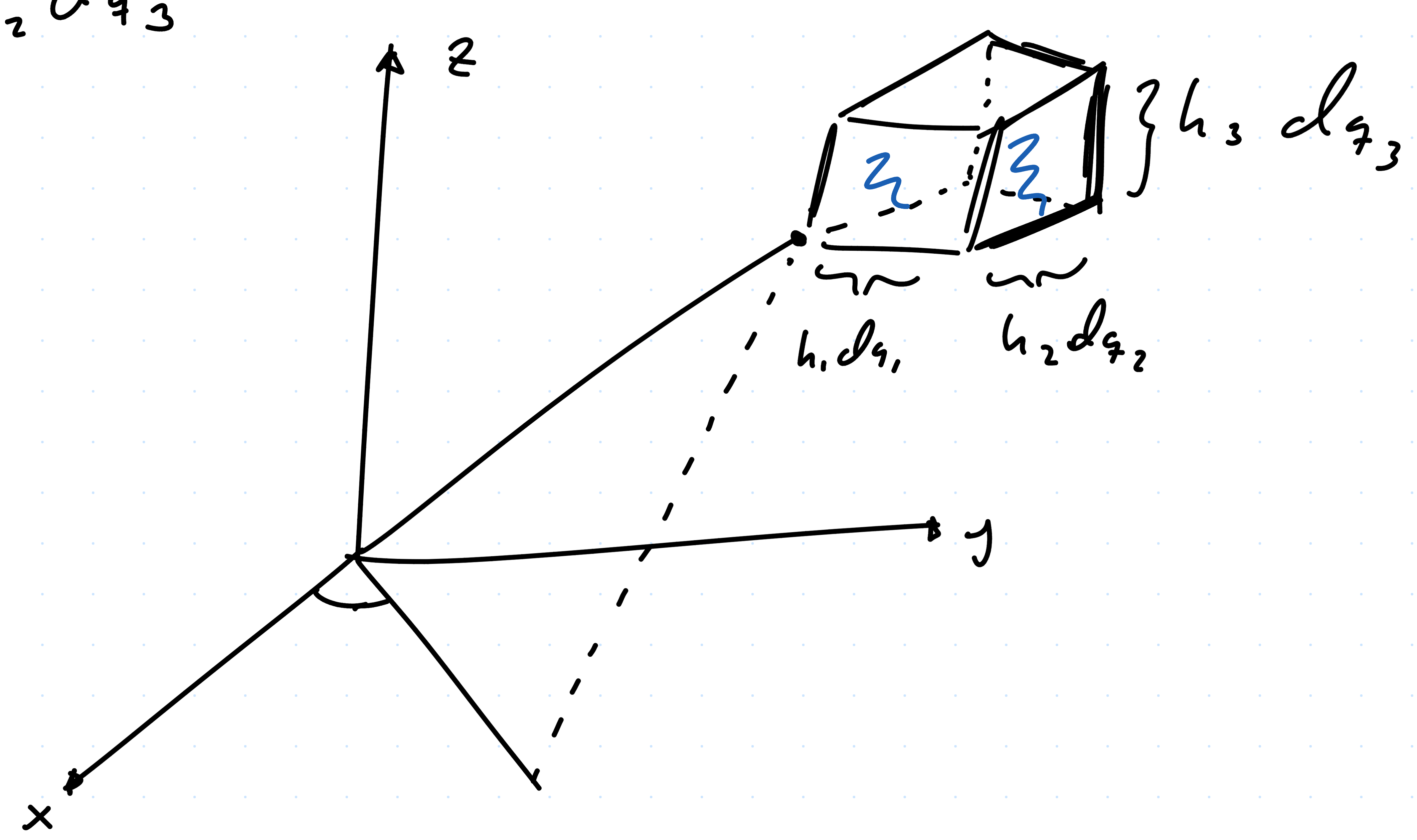
Vol. $dx\,dy\,dz$

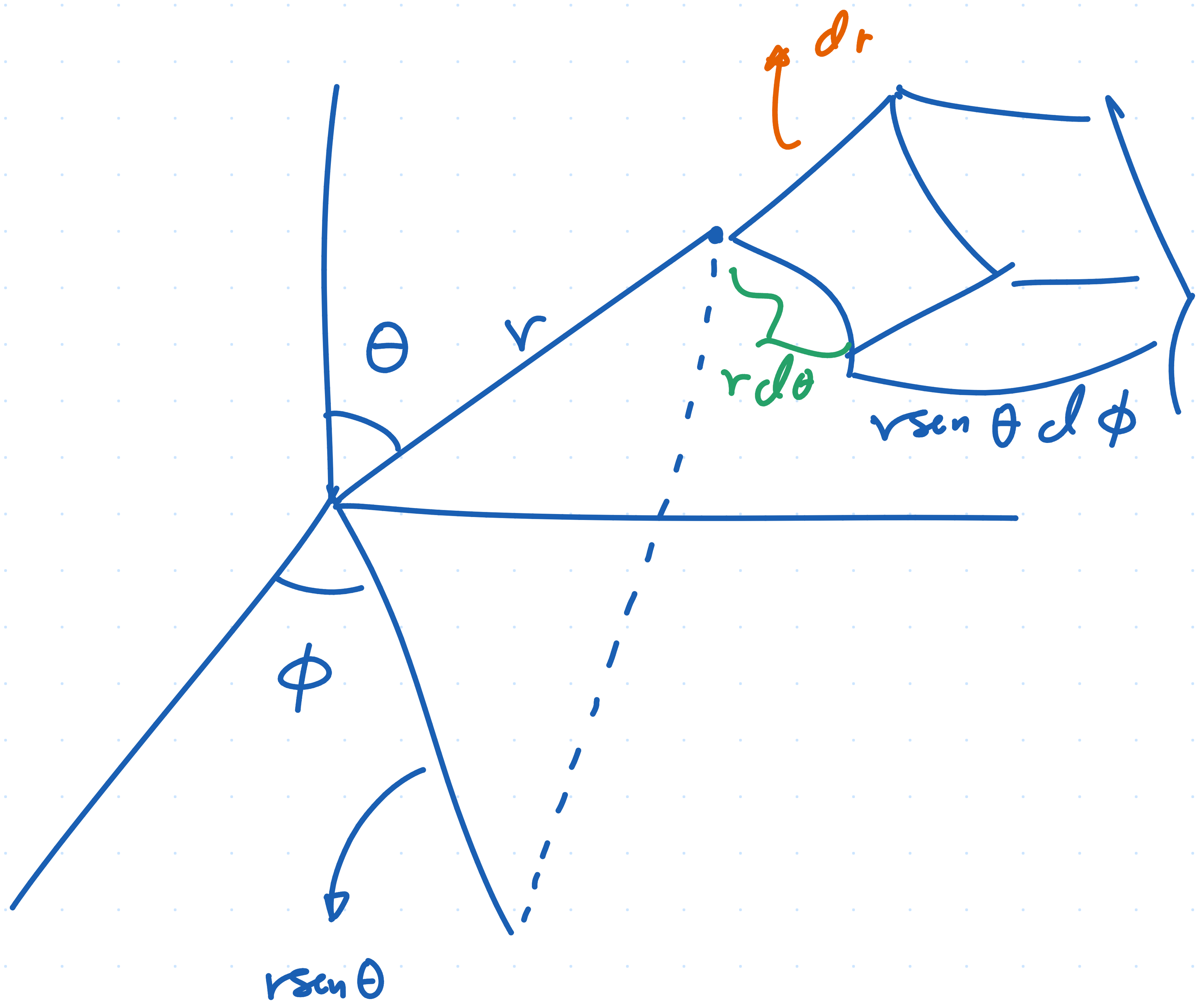
Vol: $\rho\, d\rho\, d\phi\, dz$

Vol: $r^2 \operatorname{sen}\theta\, dr\, d\theta\, d\phi$

Vol. $h_1 h_2 h_3\, dq_1\, dq_2\, dq_3$

Gráfico:





$$\nabla \cdot \vec{F} = \lim_{V \to 0} \frac{\oint_s \vec{F} \cdot d\vec{s}}{V}$$

$$\int \Big(f_1 h_2 h_3 \, dq_2 \, dq_3 + \frac{\partial}{\partial q_1}(f_1 h_2 h_3) dq_1 \, dq_2 dq_3 \Big) - \int f_1 h_2 h_3 dq_2 dq_3$$

$$\frac{\partial}{\partial q_1}(f_1 h_2 h_3) dq_1 dq_2 dq_3 + \frac{\partial}{\partial q_2}(f_2 h_1 h_3)\, dq_2 dq_1 dq_3$$

$$+ \frac{\partial}{\partial q_3}(f_3 h_1 h_2)\, dq_3 dq_1 dq_2$$

$$\lim_{V \to 0} \frac{\int \left\{ \frac{\partial}{\partial q_1}(f_1 h_2 h_3) + \frac{\partial}{\partial q_2}(f_2 h_1 h_3) + \frac{\partial}{\partial q_3}(f_3 h_1 h_2) \right\} dq_1 dq_2 dq_3}{\int h_1 h_2 h_3 \, dq_1 dq_2 dq_3}$$

$$\Rightarrow \quad \nabla \cdot \vec{F} = \frac{1}{h_1 h_2 h_3} \left[\frac{\partial}{\partial q_1}(f_1 h_2 h_3) + \frac{\partial}{\partial q_2}(f_2 h_1 h_3) + \frac{\partial}{\partial q_3}(f_3 h_1 h_2) \right]$$

En cilindricas: $h_1 = 1$, $h_2 = q_1$; $h_3 = 1$

$$\nabla\cdot\vec{F} = \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(f_1\,\rho) + \frac{\partial}{\partial\phi}(f_2) + \frac{\partial}{\partial z}(f_3\,\rho)\right]$$

$$\nabla^2\phi = \nabla\cdot\nabla\phi = \nabla\cdot\Big[\underbrace{\frac{1}{h_1}\frac{\partial\phi}{\partial q_1}}_{f_1}\hat{e}_1 + \underbrace{\frac{1}{h_2}\frac{\partial\phi}{\partial q_2}}_{f_2}\hat{e}_2 + \underbrace{\frac{1}{h_3}\frac{\partial\phi}{\partial q_3}}_{f_3}\hat{e}_3\Big]$$

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{1}{h_1}\frac{\partial\phi}{\partial q_1}h_2h_3\right] + \frac{\partial}{\partial q_2}\left(\frac{\partial\phi}{\partial q_2}\frac{h_1h_3}{h_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{\partial\phi}{\partial q_3}\frac{h_1h_2}{h_3}\right)\right]$$

$$\nabla^2\phi = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial q_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\phi}{\partial q_1}\right) + \frac{\partial}{\partial q_2}\left(\frac{h_1h_3}{h_2}\frac{\partial\phi}{\partial q_2}\right) + \frac{\partial}{\partial q_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\phi}{\partial q_3}\right)\right]$$

Rotacional

$$\lim_{V\to 0}\frac{\int \vec{F}\times d\vec{s}}{V}$$

$$\nabla\times\vec{F} = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{e}_1h_1 & \hat{e}_2h_2 & \hat{e}_3h_3 \\ \frac{\partial}{\partial q_1} & \frac{\partial}{\partial q_2} & \frac{\partial}{\partial q_3} \\ h_1f_1 & h_2f_2 & h_3f_3 \end{vmatrix}$$

Ecuación de Navier-Stokes

$\nabla\times[\vec{V}\times(\nabla\times\vec{V})]$; $\vec{V} = V(\rho)\,\hat{k}$. Coordenadas cilíndricas.

Tenemos...

Levi-Civita

$$(\nabla \times \vec{V})_i = \frac{1}{\rho} \varepsilon_{ijk} \partial_j \vec{V}_k \;;\; \frac{1}{\rho}\left[\varepsilon_{123}\, \partial_\phi V(\rho) + \varepsilon_{132}\, \partial_z (0)\right] = 0$$

1er

Determinantes

$$\nabla \times \vec{V} = \frac{1}{\rho} \begin{vmatrix} \hat{\rho} & \rho\hat{\phi} & \hat{k} \\ \partial_\rho & \partial\phi & \partial z \\ 0 & 0 & V(\rho) \end{vmatrix}$$

2nd $\rightarrow$ $\frac{\rho}{\rho}\left[\varepsilon_{213}\, \partial_\rho V(\rho) + \varepsilon_{231}\, \partial_z (0)\right] = -\partial_\rho V(\rho)$

3er término : $\frac{1}{\rho}\left[\varepsilon_{312}\, \partial_\rho (0) + \varepsilon_{321}\, \partial_\rho (0)\right] = 0$

Tenemos

$$\nabla \times \vec{V} = -\partial_\rho V(\rho)\, \hat{\phi}$$

Tenemos

$$\vec{V} \times (\nabla \times \vec{V})$$

$$- V(\rho)\, \hat{k} \times \partial_\rho V(\rho)\, \hat{\phi}$$

$$= -V(\rho)\, \partial_\rho V(\rho)\, \hat{k} \times \hat{\phi} = \partial_\rho V(\rho)\, \hat{\rho}$$

$$\left.\begin{aligned} \hat{\rho} \times \hat{\phi} &= \hat{k} \\ \hat{k} \times \hat{\rho} &= \hat{\phi} \\ \hat{\phi} \times \hat{k} &= \hat{\rho} \end{aligned}\right\}$$