CHAPTER 2

ELECTROSTATICS

2–1 Electric charge. The first observation of the electrification of objects by rubbing is lost in antiquity; however, it is common experience that rubbing a hard rubber comb on a piece of wool endows the rubber with the ability to pick up small pieces of paper. As a result of rubbing the two objects together (strictly speaking, as a result of bringing them into close contact), both the rubber and the wool acquire a new property; they are *charged*. This experiment serves to introduce the concept of *charge*. But charge, itself, is not created during this process; the total charge, or the sum of the charges on the two bodies, is still the same as before electrification. In the light of modern physics we know that microscopic charged particles, specifically electrons, are transferred from the wool to the rubber, leaving the wool positively charged and the rubber comb negatively charged.

Charge is a fundamental and characteristic property of the elementary particles which make up matter. In fact, all matter is composed ultimately of protons, neutrons, and electrons, and two of these particles bear charges. But even though on a microscopic scale matter is composed of a large number of charged particles, the powerful electrical forces associated with these particles are fairly well hidden in a macroscopic observation. The reason is that there are two kinds of charge, positive and negative, and an ordinary piece of matter contains approximately equal amounts of each kind. From the macroscopic viewpoint, then, charge refers to net charge, or excess charge. When we say that an object is charged, we mean that it has an excess charge, either an excess of electrons (negative) or an excess of protons (positive). In this and the following chapters, charge will usually be denoted by the symbol q.

Since charge is a fundamental property of the ultimate particles making up matter, the total charge of a closed system cannot change. From the macroscopic point of view charges may be regrouped and combined in different ways; nevertheless, we may state that *net charge is conserved in a closed system.*

2–2 Coulomb's law. Towards the end of the eighteenth century techniques in experimental science achieved sufficient sophistication to make possible refined observations of the forces between electric charges. The results of these observations, which were extremely controversial at the time, can be summarized in three statements. (a) There are two and only two kinds of electric charge, now known as positive and negative. (b) Two point charges exert on each other forces which act along the line joining them and which are inversely proportional to the square of the distance between them. (c) These forces are also proportional to the product of the charges, are repulsive for like charges, and attractive for unlike charges. The last two statements, with the first as preamble, are known as *Coulomb's law* in honor of Charles Augustin de Coulomb (1736–1806), who was one of the leading eighteenth century students of electricity. Coulomb's law for point charges may be concisely formulated in the vector notation of Chapter 1 as

$$\mathbf{F}_1 = C\,\frac{q_1 q_2}{r_{21}^2}\,\frac{\mathbf{r}_{21}}{r_{21}}, \tag{2-1}$$

where $\mathbf{F}_1$ is the force on charge q_1, $\mathbf{r}_{21}$ is the vector from q_2 to q_1, r_{21} is the magnitude of $\mathbf{r}_{21}$, and C is a constant of proportionality about which more will be said later. In Eq. (2–1) a unit vector in the direction of $\mathbf{r}_{21}$ has been formed by dividing $\mathbf{r}_{21}$ by its magnitude, a device of which frequent use will be made. If the force on q_2 is to be found, it is only necessary to change every subscript 1 to 2 and every 2 to 1. Understanding this notation is important, since in future work it will provide a technique for keeping track of field and source variables.

Coulomb's law applies to point charges. In the macroscopic sense a "point charge" is one whose spatial dimensions are very small compared with any other length pertinent to the problem under consideration, and we shall use the term "point charge" in this sense. To the best of our knowledge, Coulomb's law also applies to the interactions of elementary particles such as protons and electrons. Equation (2–1) is found to hold for the electrostatic repulsion between nuclei at distances greater than about 10^{-14} meter; at smaller distances, the powerful, but short-ranged, nuclear forces dominate the picture.

Equation (2–1) is an experimental law; nevertheless, there is both theoretical and experimental evidence to indicate that the inverse square law is exact, i.e., that the exponent of r_{21} is exactly 2. By an indirect experiment* it has been shown that the exponent of r_{21} can differ from 2 by no more than one part in 10^9.

The constant C in Eq. (2–1) requires some comment, since it determines the system of units. The units of force and distance are presumably those belonging to one of the systems used in mechanics; the most direct procedure here would be to set $C = 1$, and choose the unit of charge such that Eq. (2–1) agrees with experiment. Other procedures are also

* Plimpton and Lawton, *Phys. Rev.* **50**, 1066 (1936). The same experiment was performed earlier by Kelvin and by Maxwell. Maxwell established the exponent of 2 to within one part in 20,000.

possible and may have certain advantages; e.g., the unit of charge may be specified in advance. It was shown by Giorgi in 1901 that all of the common electrical units, such as the ampere, volt, ohm, henry, etc., can be combined with one of the mechanical systems (namely, the mks or meter-kilogram-second system) to form a system of units for all electric and magnetic problems. There is considerable advantage to having the results of calculations come out in the same units as those which are used in the laboratory; hence we shall use the rationalized *mks* or *Giorgi system* of units in the present volume. Since in this system q is measured in coulombs, $\mathbf{r}$ in meters and $\mathbf{F}$ in newtons, it is clear that C must have the dimensions of newton·meters^2/coulomb^2. The size of the unit of charge, the coulomb, is established from magnetic experiments; this requires that $C = 8.9874 \times 10^9$ n·m^2/coul^2. We make the apparently complicated substitution, $C = 1/4\pi\epsilon_0$, in the interest of future simplicity. The constant ϵ_0 will occur repeatedly; it represents a property of free space known as the *permittivity of free space*, and is numerically equal to 8.854×10^{-12} coul^2/n·m^2. In Appendix I the definitions of the coulomb, the ampere, the permeability, and permittivity of free space are related to one another and to the velocity of light in a logical way; since a logical formulation of these definitions requires a knowledge of magnetic phenomena and of electromagnetic wave propagation, it is not appropriate to pursue them now. In Appendix II other systems of electrical units, in particular the gaussian system, are discussed.

If more than two point charges are present, the mutual forces are determined by the repeated application of Eq. (2-1). In particular, if a system of N charges is considered, the force on the ith charge is given by

$$\mathbf{F}_i = q_i \sum_{j \neq i}^{N} \frac{q_j}{4\pi\epsilon_0} \frac{\mathbf{r}_{ji}}{r_{ji}^3}, \tag{2-2}$$

where the summation on the right is extended over all of the charges except the ith. This is, of course, just the superposition principle for forces, which says that the total force acting on a body is the vector sum of the individual forces which act on it.

A simple extension of the ideas of N interacting point charges is the interaction of a point charge with a continuous charge distribution. We deliberately choose this configuration to avoid certain difficulties which may be encountered when the interaction of two continuous charge distributions is considered. Before proceeding further the meaning of a continuous distribution of charge should be examined. It is now well known that electric charge is found in multiples of a basic charge, that of the electron. In other words, if any charge were examined in great detail, its magnitude would be found to be an integral multiple of the magnitude of the electronic charge. For the purposes of macroscopic physics this discreteness of charge causes no difficulties simply because the electronic charge has a magnitude of 1.6019×10^{-19} coul, which is extremely small. The smallness of the basic unit means that macroscopic charges are invariably composed of a very large number of electronic charges; this in turn means that in a macroscopic charge distribution any small element of volume contains a large number of electrons. One may then describe a charge distribution in terms of a charge density function defined as the limit of the charge per unit volume as the volume becomes infinitesimal. Care must be used, however, in applying this kind of description to atomic problems, since in these cases only a small number of electrons is involved, and the process of taking the limit is meaningless. Leaving aside these atomic cases, we may proceed as if a segment of charge might be subdivided indefinitely, and describe the charge distribution by means of point functions:

a *volume charge density* defined by

$$\rho = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V}, \tag{2–3}$$

and a *surface charge density* defined by

$$\sigma = \lim_{\Delta S \to 0} \frac{\Delta q}{\Delta S}. \tag{2–4}$$

From what has been said about q, it is evident that ρ and σ are net charge, or excess charge, densities. It is worth while mentioning that in typical solid materials even a very large charge density ρ will involve a change in the local electron density of only about one part in 10^9.

If charge is distributed through a volume V with a density ρ, and on the surface S which bounds V with a density σ, then the force exerted by this charge distribution on a point charge q located at $\mathbf{r}$ is obtained from (2–2) by replacing q_j with $\rho_j\, dv_j'$ (or with $\sigma_j\, da_j'$) and proceeding to the limit:

$$\mathbf{F}_q = \frac{q}{4\pi\epsilon_0}\int_V \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\rho(r')\,dv' + \frac{q}{4\pi\epsilon_0}\int_S \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\,\sigma(r')\,da'. \tag{2–5}$$

The variable $\mathbf{r}'$ is used to locate a point within the charge distribution, that is, it plays the role of the source point $\mathbf{r}_j$ in Eq. (2–2). It may appear at first sight that if point $\mathbf{r}$ falls inside the charge distribution, the first integral of (2–5) should diverge. This is not the case; the region of integration in the vicinity of $\mathbf{r}$ contributes a negligible amount, and the integral is well behaved (see Problem 2–5).

It is clear that the force on q as given by Eq. (2–5) is proportional to q; the same is true in Eq. (2–2). This observation leads us to introduce a vector field which is independent of q, namely, the force per unit charge.

This vector field, known as the *electric field,* is considered in detail in the following section.

2-3 The electric field. The electric field at a point is defined as the limit of the following ratio: the force on a test charge placed at the point, to the charge of the test charge, the limit being taken as the magnitude of the test charge goes to zero. The customary symbol for the electric field is **E**. In vector notation the definition of **E** becomes

$$\mathbf{E} = \lim_{q \to 0} \frac{\mathbf{F}_q}{q}. \tag{2-6}$$

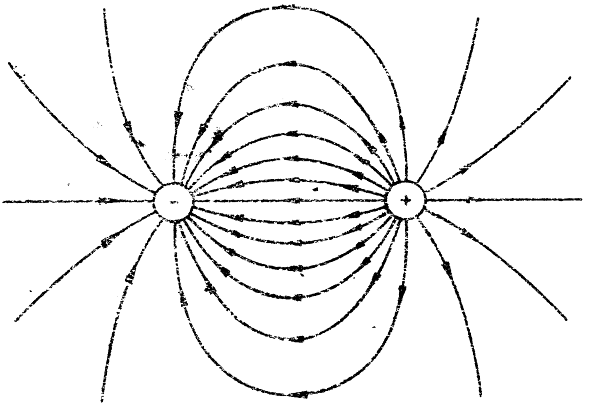
The limiting process is included in the definition of **E** to ensure that the test charge does not affect the charge distribution which produces **E**. If, for example, positive charge is distributed on the surface of a conductor (a conductor is a material in which charge is free to move), then bringing a test charge into the vicinity of the conductor will cause the charge on the conductor to redistribute itself. If the electric field were calculated using the ratio of force to charge for a finite test charge, the field obtained would be that due to the redistributed charge rather than that due to the original charge distribution. In the special case where one of the charges of the charge distribution can be used as a test charge the limiting process is unnecessary. In this case the electric field at the location of the test charge will be that produced by all of the rest of the charge distribution; there will, of course, be no redistribution of charge, since the proper charge distribution obtains under the influence of the entire charge distribution, including the charge being used as test charge. In certain other cases, notably those in which the charge distribution is *specified,* the force will be proportional to the size of the test charge. In these cases, too, the limit is unnecessary; however, if any doubt exists, it is always safe to use the limiting process.

Equations (2-2) and (2-5) provide a ready means for obtaining an expression for the electric field due to a given distribution of charge. Let the charge distribution consist of N point charges $q_1, q_2, \ldots, q_N$ located at the points $\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_N$ respectively, and a volume distribution of charge specified by the charge density $\rho(\mathbf{r}')$ in the volume V and a surface distribution characterized by the surface charge density $\sigma(\mathbf{r}')$ on the surface S. If a test charge q is located at the point $\mathbf{r}$, it experiences a force $\mathbf{F}$ given by

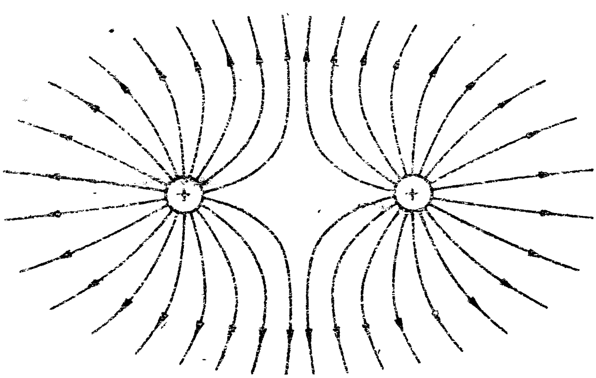
$$\mathbf{F} = \frac{q}{4\pi\epsilon_0} \sum_{i=1}^{N} q_i \frac{\mathbf{r} - \mathbf{r}_i}{|\mathbf{r} - \mathbf{r}_i|^3} + \frac{q}{4\pi\epsilon_0} \int_V \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \rho(\mathbf{r}')\, dv' + \frac{q}{4\pi\epsilon_0} \int_S \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} \sigma(\mathbf{r}')\, da', \tag{2-7}$$

due to the given charge distribution. The electric field at $\mathbf{r}$ is the limit of the ratio of this force to the test charge q. Since the ratio is independent of q, the electric field at $\mathbf{r}$ is just

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N} q_i \frac{\mathbf{r}-\mathbf{r}_i}{|\mathbf{r}-\mathbf{r}_i|^3} + \frac{1}{4\pi\epsilon_0}\int_V \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\rho(\mathbf{r}')\,dv' + \frac{1}{4\pi\epsilon_0}\int_S \frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}\sigma(\mathbf{r}')\,da'. \quad (2\text{–}8)$$



(a)



(b)

FIG. 2–1 The mapping of an electric field with the aid of lines of force.

Equation (2-8) is very general; in most cases one or more of the terms will not be needed.

The quantity we have just defined, the electric field, may be calculated at each point in space in the vicinity of a system of charges or of a charge distribution. Thus $\mathbf{E} = \mathbf{E}(\mathbf{r})$ is a vector point function, or a vector field. This field has a number of interesting mathematical properties which we shall proceed to develop in the following sections and in the next chapter. As an aid to visualizing the electric field structure associated with a particular distribution of charge, Michael Faraday (1791-1867) introduced the concept of *lines of force*. A line of force is an imaginary line (or curve) drawn in such a way that its direction at any point is the direction of the electric field at that point.

Consider, for example, the electric field structure associated with a single positive point charge q_1. The lines of force are radial lines radiating outward from q_1. Similarly, the lines of force associated with an isolated negative point charge are also radial lines, but this time the direction is inward (i.e., toward the negative charge). These two examples are extremely simple, but they nevertheless illustrate an important property of the field lines: the lines of force terminate on the sources of the electric field, i.e., upon the charges which produce the electric field.

Figure 2-1 shows several simple electric fields which have been mapped with the aid of lines of force.

2-4 The electrostatic potential. It has been noted in Chapter 1 that if the curl of a vector vanishes, then the vector may be expressed as the gradient of a scalar. The electric field given by Eq. (2-8) satisfies this criterion. To verify this, we note that taking the curl of Eq. (2-8) involves differentiating with respect to r. This variable appears in the equation only in functions of the form $(\mathbf{r} - \mathbf{r}')/|\mathbf{r} - \mathbf{r}'|^3$, and hence it will suffice to show that functions of this form have zero curl. Using the formula from Table 1-1 for the curl of the product (vector times scalar) gives

$$\operatorname{curl} \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3} = \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} \operatorname{curl} (\mathbf{r} - \mathbf{r}') + \left[\operatorname{grad} \frac{1}{|\mathbf{r} - \mathbf{r}'|^3}\right] \times [\mathbf{r} - \mathbf{r}']. \tag{2-9}$$

A direct calculation (see Problem 1-13) shows that

$$\operatorname{curl} (\mathbf{r} - \mathbf{r}') = 0, \tag{2-10}$$

and (see Problem 1-16) that

$$\operatorname{grad} \frac{1}{|\mathbf{r} - \mathbf{r}'|^3} = -3 \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^5}. \tag{2-11}$$

These results, together with the observation that the vector product of a vector with a parallel vector is zero, suffice to prove that

$$\mathbf{curl}\,\frac{\mathbf{r}-\mathbf{r}'}{|\mathbf{r}-\mathbf{r}'|^3}=0. \tag{2-12}$$

Since each contribution of Eq. (2–8) to the electric field is of this form, we have demonstrated that the curl of the electric field is zero. Equation (2–12) indicates that a scalar function exists whose gradient is the electric field; it remains to find this function. That is, we now know that a function exists which satisfies

$$\mathbf{E}(\mathbf{r})=-\mathbf{grad}\,U(\mathbf{r}), \tag{2-13}$$

but we have yet to find the form of the function U. It should be noted that it is conventional to include the minus sign in Eq. (2–13) and to call U the electrostatic potential.

It is easy to find the electrostatic potential due to a point charge q_1; it is just

$$U(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\frac{q_1}{|\mathbf{r}-\mathbf{r}_1|}, \tag{2-14}$$

as is readily verified by direct differentiation. With this as a clue it is easy to guess that the potential which gives the electric field of Eq. (2–8) is

$$U(\mathbf{r})=\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\mathbf{r}-\mathbf{r}_i|}+\frac{1}{4\pi\epsilon_0}\int_V\frac{\rho(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,dv' + \frac{1}{4\pi\epsilon_0}\int_S\frac{\sigma(\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|}\,da', \tag{2-15}$$

which is also easily verified by direct differentiation. It may seem that Eqs. (2–14) and (2–15) were obtained in a rather arbitrary fashion; however, since all that is required of U is that it satisfy (2–13), and since this has been verified directly, the means by which U was obtained is immaterial.

The electrostatic potential U can be obtained directly as soon as its existence is established. Since U is known to exist, we may write

$$\int_{\text{ref}}^{\mathbf{r}}\mathbf{E}(\mathbf{r}')\cdot d\mathbf{r}'=-\int_{\text{ref}}^{\mathbf{r}}\mathbf{grad}\,U\cdot d\mathbf{r}', \tag{2-16}$$

where ref stands for a reference point at which U is zero. From the definition of the gradient,

$$\mathbf{grad}\,U\cdot d\mathbf{r}'=dU. \tag{2-17}$$

Using (2–17) in Eq. (2–16) converts it into the integral of a perfect differential, which is easily done. The result is

$$-\int_{\text{ref}}^{\mathbf{r}} \mathbf{grad}\, U \cdot d\mathbf{r}' = -U(\mathbf{r}) = \int_{\text{ref}}^{\mathbf{r}} \mathbf{E}(\mathbf{r}') \cdot d\mathbf{r}', \tag{2-18}$$

which is really the inverse of Eq. (2–13). If the electric field due to a point charge is used in equation (2–18), and the reference point or lower limit in the integral is taken at infinity, with the potential there zero, the result is

$$U(\mathbf{r}) = \frac{q}{4\pi\epsilon_0 r} \cdot \tag{2-19}$$

This, of course, is just a special case of Eq. (2–14), namely, the case where $\mathbf{r}_1$ is zero. This derivation can be extended to obtain Eq. (2–15); however, the procedure is too cumbersome to include here.

Another interesting and useful aspect of the electrostatic potential is its close relation to the potential energy associated with the conservative electrostatic force. The potential energy associated with an arbitrary conservative force is

$$W(\mathbf{r}) = -\int_{\text{ref}}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}', \tag{2-20}$$

where $W(\mathbf{r})$ is the potential energy at $\mathbf{r}$ relative to the reference point at which the potential energy is arbitrarily taken to be zero. Since in the electrostatic case $\mathbf{F} = q\mathbf{E}$, it follows that if the same reference point is chosen for the electrostatic potential and for the potential energy, then the electrostatic potential is just the potential energy per unit charge. This idea is sometimes used to introduce the electrostatic potential; we feel, however, that the introduction by means of Eq. (2–13) emphasizes the importance of the electrostatic potential in determining the electrostatic field. There is, of course, no question about the ultimate equivalence of the two approaches.

The utility of the electrostatic potential in calculating electric fields can be seen by contrasting Eqs. (2–8) and (2–15). Equation (2–8) is a vector equation; to obtain the electric field from it, it is necessary to evaluate three sums or three integrals for each term. At best this is a tedious procedure; in some cases it is almost impossible to do the integrals. Equation (2–15), on the other hand, is a scalar equation and involves only one sum or integral per term. Furthermore, the denominators appearing in this equation are all of the form $|\mathbf{r} - \mathbf{r}'|$, which simplifies the integrals compared with those of Eq. (2–8). This simplification is sometimes sufficient to make the difference between doing the integrals and not doing them. It may be objected that after doing the integrals of Eq. (2–15) it is still necessary to differentiate the result; this objection is readily answered by observing that differentiation can always be accomplished if the derivatives exist, and is in fact usually much easier than integration. In Chapter 3 it will be seen that the electrostatic potential is even more important in those problems where the charge distribution is not specified, but must rather be determined in the process of solving the problem.

In the mks system the unit of energy is the newton-meter or joule. The unit of potential is joule/coulomb, but this unit occurs so frequently that it is given a special name, the volt. The unit of the electric field is the newton/coulomb or the volt/meter.

2-5 Conductors and insulators. So far as their electrical behavior is concerned, materials may be divided into two categories: *conductors* of electricity and *insulators* (*dielectrics*). Conductors are substances, like the metals, which contain large numbers of essentially free charge carriers. These charge carriers (electrons in most cases) are free to wander throughout the conducting material; they respond to almost infinitesimal electric fields, and they continue to move as long as they experience a field. These free carriers carry the electric current when an electric field is maintained in the conductor by an external source of energy.

Dielectrics are substances in which all charged particles are bound rather strongly to constituent molecules. The charged particles may shift their positions slightly in response to an electric field, but they do not leave the vicinity of their molecules. Strictly speaking, this definition applies to an ideal dielectric, one which shows no conductivity in the presence of an externally maintained electric field. Real physical dielectrics may show a feeble conductivity, but in a typical dielectric the conductivity is 10^{20} times smaller than that of a good conductor. Since 10^{20} is a tremendous factor, it is usually sufficient to say that dielectrics are nonconductors.

Certain materials (semiconductors, electrolytes) have electrical properties intermediate between conductors and dielectrics. So far as their behavior in a static electric field is concerned, these materials behave very much like conductors. However, their transient response is somewhat slower; i.e., it takes longer for these materials to reach equilibrium in a static field.

In this and the following four chapters we shall be concerned with materials in *electrostatic* fields. Dielectric polarization, although a basically simple phenomenon, produces some rather complicated effects; hence we shall delay its study until Chapter 4. Conductors, on the other hand, may be treated quite easily in terms of concepts which have already been developed.

Since charge is free to move in a conductor, even under the influence of very small electric fields, the charge carriers (electrons or ions) move until they find positions in which they experience no net force. When they come to rest, the interior of the conductor must be a region devoid of an electric field; this must be so because the charge carrier population in the interior is by no means depleted, and if a field persisted, the carriers would continue to move. *Thus, under static conditions, the electric field in a conductor vanishes.* Furthermore, since $\mathbf{E} = 0$ in a conductor, the potential is the same at all points in the conducting material. In other words, *under static conditions, each conductor forms an equipotential region of space.*

2-6 Gauss' law. An important relationship exists between the integral of the normal component of the electric field over a closed surface and the total charge enclosed by the surface. This relationship, known as Gauss' law, will now be investigated in more detail. The electric field at point $\mathbf{r}$ due to a point charge q located at the origin is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\mathbf{r}}{r^3}. \tag{2-21}$$

Consider the surface integral of the normal component of this electric field over a closed surface (such as that shown in Fig. 2-2) which encloses the origin and, consequently, the charge q; this integral is just

$$\oint_S \mathbf{E}\cdot\mathbf{n}\,da = \frac{q}{4\pi\epsilon_0}\oint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,da. \tag{2-22}$$

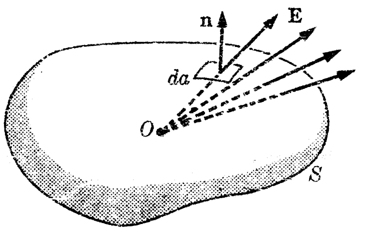


FIG. 2-2. An imaginary closed surface S which encloses a point charge at the origin.

The quantity $(\mathbf{r}/r)\cdot\mathbf{n}\,da$ is the projection of da on a plane perpendicular to $\mathbf{r}$. This projected area divided by r^2 is the solid angle subtended by da, which is written $d\Omega$. It is clear from Fig. 2-3 that the solid angle subtended by da is the same as the solid angle subtended by da', an element of the surface area of the sphere S' whose center is at the origin and whose radius is r'. It is then possible to write

$$\oint_S \frac{\mathbf{r}\cdot\mathbf{n}}{r^3}\,da = \oint_{S'}\frac{\mathbf{r}'\cdot\mathbf{n}}{r'^3}\,da' = 4\pi,$$

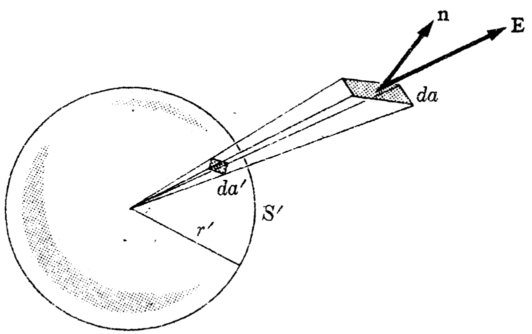


FIG. 2–3. Construction of the spherical surface S' as an aid to evaluation of the solid angle subtended by da.

which shows that

$$\oint_S \mathbf{E} \cdot \mathbf{n}\, da = \frac{q}{4\pi\epsilon_0} 4\pi = \frac{q}{\epsilon_0} \tag{2–23}$$

in the special case described above. If q lies outside of S, it is clear from Fig. 2–4 that S can be divided into two areas S_1 and S_2 each of which subtends the same solid angle at the charge q. For S_2, however, the direction of the normal is towards q, while for S_1 it is away from q. Therefore the contributions of S_1 and S_2 to the surface integral are equal and opposite, and the total integral vanishes. Thus if the surface surrounds a point charge q, the surface integral of the normal component of the electric field is q/ϵ_0, while if q lies outside the surface the surface integral is zero.

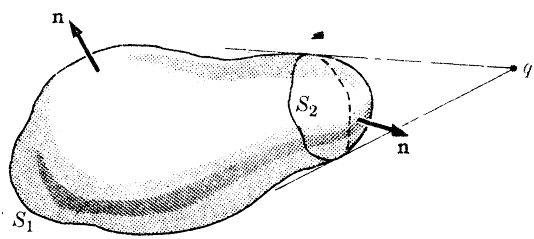


FIG. 2–4. The closed surface S may be divided into two surfaces, S_1 and S_2, each of which subtend the same solid angle at q.

The preceding statement applies to any closed surface, even to so-called re-entrant ones. A study of Fig. 2–5 is sufficient to verify that this is indeed the case.

If several point charges $q_1, q_2, \ldots, q_N$ are enclosed by the surface S, then the total electric field is given by the first term of Eq. (2–8). Each

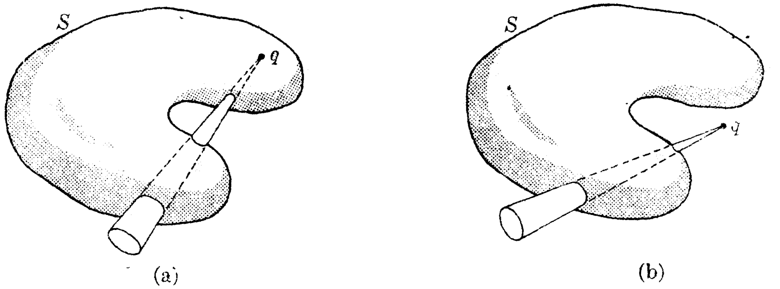


FIG. 2-5. An element of solid angle cutting the surface S more than once.

charge subtends a full solid angle (4π); hence Eq. (2-23) becomes

$$\oint_S \mathbf{E} \cdot \mathbf{n}\, da = \frac{1}{\epsilon_0} \sum_{i=1}^{N} q_i. \tag{2-24}$$

This result can be readily generalized to the case of a continuous distribution of charge characterized by a charge density. If each element of charge $\rho\, dv$ is considered as a point charge, it contributes $\rho\, dv/\epsilon_0$ to the surface integral of the normal component of the electric field provided it is inside the surface over which we integrate. The total surface integral is then the sum of all contributions of this form due to the charge inside the surface. Thus if S is a closed surface which bounds the volume V,

$$\oint_S \mathbf{E} \cdot \mathbf{n}\, da = \frac{1}{\epsilon_0} \int_V \rho\, dv. \tag{2-25}$$

Equations (2-24) and (2-25) are known as Gauss' law. The term on the left, the integral of the normal component of the electric field over the surface S, is sometimes called the *flux* of the electric field through S.

Gauss' law may be expressed in yet another form by using the divergence theorem. The divergence theorem (1-37) states that

$$\oint_S \mathbf{F} \cdot \mathbf{n}\, da = \int_V \operatorname{div} \mathbf{F}\, dv.$$

If this theorem is applied to the surface integral of the normal component of $\mathbf{E}$, it yields

$$\oint_S \mathbf{E} \cdot \mathbf{n}\, da = \int_V \operatorname{div} \mathbf{E}\, dv, \tag{2-26}$$

which, when substituted into Eq. (2-25), gives

$$\int_V \operatorname{div} \mathbf{E}\, dv = \frac{1}{\epsilon_0} \int_V \rho\, dv. \tag{2-27}$$

Equation (2–27) must be valid for all volumes, that is, for any choice of the volume V. The only way in which this can be true is if the integrands appearing on the left and on the right in the equation are equal. Thus the validity of Eq. (2–27) for any choice of V implies that

$$\operatorname{div} \mathbf{E} = \frac{1}{\epsilon_0}\rho. \tag{2–28}$$

This result may be thought of as a differential form of Gauss' law.

2–7 Application of Gauss' law. Equation (2–28) or, more properly, a modified form of this equation which will be derived in Chapter 4, is one of the basic differential equations of electricity and magnetism. In this role it is important, of course; but Gauss' law also has practical utility. This practicality of the law lies largely in providing a very easy way to calculate electric fields in sufficiently symmetric situations. In other words, in certain highly symmetric situations of considerable physical interest, the electric field may be calculated by using Gauss' law instead of by the integrals given above or by the procedures of Chapter 3. When this can be done, it accomplishes a major saving in effort.

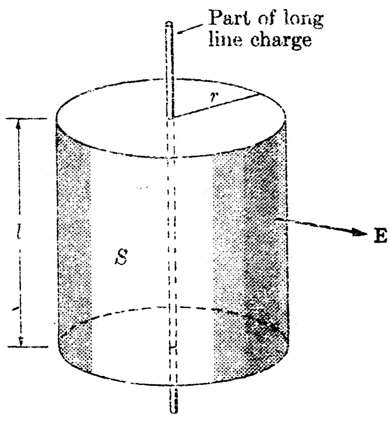


FIG. 2–6. A cylindrical surface to be used with Gauss' law to find the electric field produced by a long line charge.

In order that Gauss' law be useful in calculating the electric field, it must be possible to choose a closed surface such that the electric field has a normal component which is either zero or a single fixed value at every point on the surface. As an example, consider a very long line charge of charge density λ per unit length, as shown in Fig. 2–6. The symmetry of the situation clearly indicates that the electric field is radial and independent of both position along the wire and angular position around the wire. These observations lead us to choose the surface shown in Fig. 2-6. For this surface it is easy to evaluate the integral of the normal component of the electric field. The circular ends contribute nothing, since the electric field is parallel to them. The cylindrical surface contributes $2\pi r l E_r$ since $\mathbf{E}$ is radial and independent of the position of the cylindrical surface. Gauss' law then takes the form

$$2\pi r l E_r = \frac{\lambda l}{\epsilon_0}. \tag{2-29}$$

Equation (2-29) can be solved for E_r to give

$$E_r = \frac{\lambda}{2\pi\epsilon_0 r}. \tag{2-30}$$

The saving of effort accomplished by the use of Gauss' law will be more fully appreciated by solving Problem 2-4, which involves direct application of Eq. (2-8).

Another important result of Gauss' law is that the charge (net charge) of a charged conductor resides on its surface. We saw in Section 2-5 that the electric field inside a conductor vanishes. We may construct a gaussian surface anywhere inside the conductor; by Gauss' law, the net charge enclosed by each of these surfaces is zero. Finally, we construct the gaussian surface S of Fig. 2-7; again the net charge enclosed is zero. The only place left for the charge which is not in contradiction with Gauss' law is for it to reside on the surface of the conductor.

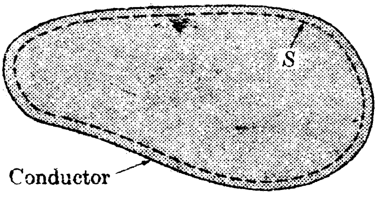


FIG. 2-7. A gaussian surface S constructed inside a charged conductor.

The electric field just outside a charged conductor must be normal to the surface of the conductor. This follows because the surface is an equipotential, and $\mathbf{E} = -\mathbf{grad}\, U$. Let us assume that the charge on a conductor is given by the surface density function σ. If Gauss' law is applied to the small pillbox-shaped surface S of Fig. 2-8, then

$$E\,\Delta S = \left(\frac{\sigma}{\epsilon_0}\right)\Delta S,$$

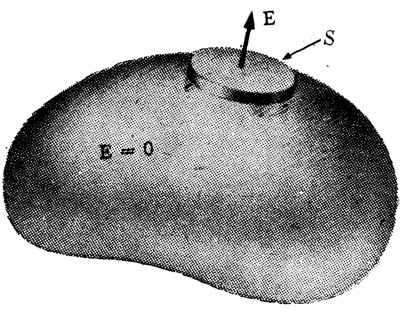


FIG. 2-8. Application of Gauss' law to the closed, pillbox-shaped surface S which intersects the surface of a charged conductor.

where ΔS is the area of one of the pillbox bases. Hence, for the electric field just outside a conductor,

$$E = \frac{\sigma}{\epsilon_0}. \tag{2-31}$$

2-8 The electric dipole. Two equal and opposite charges separated by a small distance form an electric dipole. The electric field and potential distribution produced by such a charge configuration can be investigated with the aid of the formulas of Sections 2-3 and 2-4. Suppose that a charge $-q$ is located at the point $\mathbf{r}'$ and a charge q is located at $\mathbf{r}' + \mathbf{l}$, as shown in Fig. 2-9; then the electric field at an arbitrary point $\mathbf{r}$ may be found by direct application of Eq. (2-8). The electric field at $\mathbf{r}$ is found to be

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left\{\frac{\mathbf{r} - \mathbf{r}' - \mathbf{l}}{|\mathbf{r} - \mathbf{r}' - \mathbf{l}|^3} - \frac{\mathbf{r} - \mathbf{r}'}{|\mathbf{r} - \mathbf{r}'|^3}\right\}. \tag{2-32}$$

This is the correct electric field for any value of q and any value of the separation $\mathbf{l}$; however, it is not easy to interpret. What we want is the dipole field, and in the dipole the separation $\mathbf{l}$ is small compared with $\mathbf{r} - \mathbf{r}'$; hence we may expand Eq. (2-32), keeping only the first nonvanishing term. Since this procedure is of general utility it will be considered in detail. The primary difficulty in making this expansion is caused by the denominator of the first term of Eq. (2-32). The reciprocal of this denominator can be rewritten as

$$\begin{aligned}|\mathbf{r} - \mathbf{r}' - \mathbf{l}|^{-3} &= [(\mathbf{r} - \mathbf{r}')^2 - 2(\mathbf{r} - \mathbf{r}') \cdot \mathbf{l} + l^2]^{-3/2} \\ &= |\mathbf{r} - \mathbf{r}'|^{-3}\left[1 - \frac{2(\mathbf{r} - \mathbf{r}') \cdot \mathbf{l}}{|\mathbf{r} - \mathbf{r}'|^2} + \frac{l^2}{|\mathbf{r} - \mathbf{r}'|^2}\right]^{-3/2}\end{aligned}$$

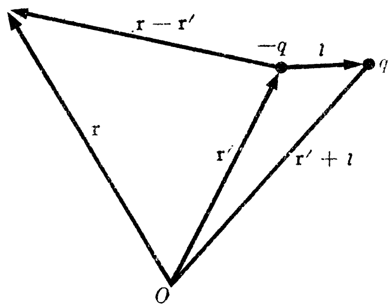


FIG. 2-9. Geometry involved in calculating the electric field $\mathbf{E}(\mathbf{r})$ due to two point charges.

In the last form it is easy to expand by the binomial theorem, keeping only terms linear in $\boldsymbol{l}$. The result of this expansion is

$$|\mathbf{r} - \mathbf{r}' - \boldsymbol{l}|^{-3} = |\mathbf{r} - \mathbf{r}'|^{-3}\left\{1 + \frac{3(\mathbf{r} - \mathbf{r}') \cdot \boldsymbol{l}}{|\mathbf{r} - \mathbf{r}'|^2} + \cdots\right\}, \tag{2-33}$$

where terms involving l^2 have been dropped. Using Eq. (2-33) in Eq. (2-32) and again keeping only terms linear in $\boldsymbol{l}$ gives

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left\{\frac{3(\mathbf{r} - \mathbf{r}') \cdot \boldsymbol{l}}{|\mathbf{r} - \mathbf{r}'|^5}(\mathbf{r} - \mathbf{r}') - \frac{\boldsymbol{l}}{|\mathbf{r} - \mathbf{r}'|^3} + \cdots\right\}. \tag{2-34}$$

Equation (2-34) gives that part of the electric field, due to a finite electric dipole, which is proportional to the separation of the charges. There are other contributions proportional to the square, the cube, and higher powers of the separation. If, however, the separation is small, these higher powers contribute very little. In the limit as $\boldsymbol{l}$ goes to zero, all of the terms vanish unless the charge becomes infinite. In the limit as $\boldsymbol{l}$ goes to zero while q becomes infinite, in such a way that $q\boldsymbol{l}$ remains constant, all terms except the term linear in $\boldsymbol{l}$ vanish. In this limit a point dipole is formed. A point dipole has no net charge, no extent in space, and is completely characterized by its dipole moment, which is the limit of $q\boldsymbol{l}$ as $\boldsymbol{l}$ goes to zero. We use the symbol $\mathbf{p}$ to represent the electric dipole moment, and write

$$\mathbf{p} = q\boldsymbol{l}. \tag{2-35}$$

In terms of the dipole moment, Eq. (2-34) may be written

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left\{\frac{3(\mathbf{r} - \mathbf{r}') \cdot \mathbf{p}}{|\mathbf{r} - \mathbf{r}'|^5}(\mathbf{r} - \mathbf{r}') - \frac{\mathbf{p}}{|\mathbf{r} - \mathbf{r}'|^3}\right\}. \tag{2-36}$$

The potential distribution produced by a point dipole is also important. This could be found by looking for a function with gradient equal to the right side of Eq. (2–36). It is, however, easier to apply Eq. (2–15) to the charge distribution consisting of two point charges separated by a small distance. Using the notation of Eq. (2–32), the potential distribution is given by

$$U(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{|\mathbf{r} - \mathbf{r}' - \boldsymbol{l}|} - \frac{1}{|\mathbf{r} - \mathbf{r}'|}\right]. \tag{2–37}$$

By expanding the first term in exactly the same way as was done for the first term of (2–32) and retaining only the linear term in $\boldsymbol{l}$, Eq. (2–37) can be put in the form

$$U(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\,\frac{(\mathbf{r} - \mathbf{r}')\cdot\boldsymbol{l}}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{2–38}$$

This equation is valid to the same approximation as Eq. (2–34); namely, terms proportional to l^2 and to higher powers of l are neglected. For a point dipole, $\mathbf{p}$, Eq. (2–38) is exact; however, it is better written as

$$U(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\,\frac{\mathbf{p}\cdot(\mathbf{r} - \mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|^3}. \tag{2–39}$$

Equation (2–39) gives the potential $U(\mathbf{r})$ produced by an electric dipole; from this potential the electric field (Eq. 2–36) may be determined. It is also interesting to inquire about the potential energy of an electric dipole which is placed in an *external electric field*. In the case of two charges, $-q$ at $\mathbf{r}$ and q at $\mathbf{r} + \boldsymbol{l}$, in an electric field described by the potential function $U_{\text{ext}}(\mathbf{r})$, the potential energy is just

$$W = -qU_{\text{ext}}(\mathbf{r}) + qU_{\text{ext}}(\mathbf{r} + \boldsymbol{l}). \tag{2–40}$$

If $\boldsymbol{l}$ is small compared with $\mathbf{r}$, $U_{\text{ext}}(\mathbf{r} + \boldsymbol{l})$ may be expanded in a power series in $\boldsymbol{l}$ and only the first two terms kept. The expansion gives

$$U_{\text{ext}}(\mathbf{r} + \boldsymbol{l}) = U_{\text{ext}}(\mathbf{r}) + \boldsymbol{l}\cdot\operatorname{grad} U_{\text{ext}}, \tag{2–41}$$

where the value of the gradient at point $\mathbf{r}$ is to be used. If this expansion is used in Eq. (2–40), the result is

$$W = q\boldsymbol{l}\cdot\operatorname{grad} U_{\text{ext}}. \tag{2–42}$$

Going to the limit of a point dipole gives simply

$$W(\mathbf{r}) = \mathbf{p}\cdot\operatorname{grad} U_{\text{ext}}, \tag{2–43}$$

which is, of course, exact. Since the electric field is the negative gradient of the electrostatic potential, an alternative form of Eq. (2–43) is

$$W(\mathbf{r}) = -\mathbf{p} \cdot \mathbf{E}_{\text{ext}}(\mathbf{r}). \tag{2–44}$$

This, then, is the potential energy of a dipole $\mathbf{p}$ in an external electric field $\mathbf{E}_{\text{ext}}$, where $\mathbf{E}_{\text{ext}}(\mathbf{r})$ is evaluated at the location of the dipole.

It is important to note that two potentials have been discussed in this section. In Eqs. (2–37), (2–38), and (2–39), the electrostatic potential *produced by* an electric dipole is considered. In Eqs. (2–40) through (2–43), the dipole is considered to be in an *existing electric field* described by a potential function $U_{\text{ext}}(\mathbf{r})$. This electric field is due to charges other than those comprising the dipole; in fact, the dipole field must be excluded to avoid an infinite result. This statement could lead us to rather complicated questions concerning self-forces and self-energies which we cannot discuss here; however, it may be noted that the potential energy resulting from the interaction of an electric dipole with its own field arises from forces exerted on the dipole by itself. Such forces, known in dynamics as internal forces, do not affect the motion of the dipole as a whole. For our purposes further consideration of this question will be unnecessary.

2–9 Multipole expansion of electric fields. It is apparent from the definition of dipole moments given above that certain aspects of the potential distribution produced by a specified distribution of charge might well be expressed in terms of its electric dipole moment. In order to do this it is necessary, of course, to define the electric dipole moment of an arbitrary charge distribution. Rather than make an unmotivated definition, we shall consider a certain expansion of the electrostatic potential due to an arbitrary charge distribution. To reduce the number of position coordinates, a charge distribution in the neighborhood of the origin of coordinates will be considered. The further restriction will be made that the charge distribution can be entirely enclosed by a sphere of radius a which is small compared with the distance to the point of observation. An arbitrary point within the charge distribution will be designated by $\mathbf{r}'$, the charge density at that point by $\rho(\mathbf{r}')$, and the observation point by $\mathbf{r}$ (see Fig. 2–10). The potential at $\mathbf{r}$ is given by

$$U(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_{V'} \frac{\rho(\mathbf{r}')}{|\mathbf{r} - \mathbf{r}'|}\, dv', \tag{2–45}$$

where dv' is used to designate an element of volume in the charge distribution and V' denotes the entire volume occupied by the charge distribution. In view of the restriction made above to points of observation which are remote from the origin, the quantity $|\mathbf{r} - \mathbf{r}'|^{-1}$ can be expanded in a series of ascending powers of r'/r. The result of such an expansion is

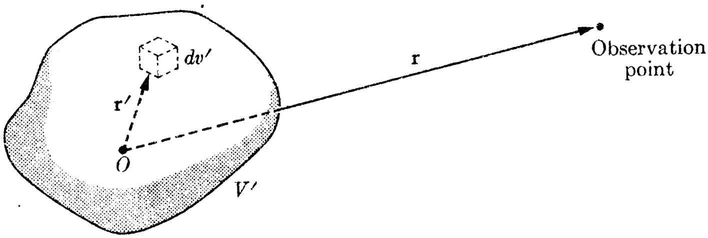


FIG. 2-10. The charge is localized in the volume V with charge density $\rho(\mathbf{r}')$. The electric field is to be calculated at point $\mathbf{r}$.

$$\begin{aligned}|\mathbf{r} - \mathbf{r}'|^{-1} &= (r^2 - 2\mathbf{r}\cdot\mathbf{r}' + r'^2)^{-1/2} \\ &= \frac{1}{r}\left\{1 - \frac{1}{2}\left[-\frac{2\mathbf{r}\cdot\mathbf{r}'}{r^2} + \frac{r'^2}{r^2}\right] + \frac{1}{2}\,\frac{1}{2}\,\frac{3}{2}[\quad]^2 + \cdots\right\},\end{aligned} \tag{2-46}$$

where only the first three terms are explicitly indicated. It should be noted that while $(r'/r)^2$ is negligible compared with $2\mathbf{r}'\cdot\mathbf{r}/r^2$, it may not be dropped in the first set of brackets because it is of the same order as the dominant term in the second set of brackets. Using Eq. (2-46) in Eq. (2-45) and omitting terms involving the cube and higher powers of $\mathbf{r}'$ yields

$$U(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_{V'}\left\{\frac{1}{r} + \frac{\mathbf{r}\cdot\mathbf{r}'}{r^3} + \frac{1}{2}\left[\frac{3(\mathbf{r}\cdot\mathbf{r}')^2}{r^5} - \frac{r'^2}{r^3}\right] + \cdots\right\}\rho(r')\,dv'. \tag{2-47}$$

Since $\mathbf{r}$ does not involve the variable of integration $\mathbf{r}'$, all of the $\mathbf{r}$ dependence may be taken from under the integral sign, to obtain

$$\begin{aligned}U(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\Bigg\{&\frac{1}{r}\int_{V'}\rho(r')\,dv' + \frac{\mathbf{r}}{r^3}\cdot\int_{V'}\mathbf{r}'\rho(\mathbf{r}')\,dv' \\ &+ \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\,\frac{x_ix_j}{r^5}\int_{V'}(3x'_ix'_j - \delta_{ij}r'^2)\rho(\mathbf{r}')\,dv',\end{aligned} \tag{2-48}$$

where x_i, x_j are cartesian components of $\mathbf{r}$, x'_i, x'_j are the cartesian components of $\mathbf{r}'$, and δ_{ij} is defined as follows:

$$\delta_{ij} = \begin{Bmatrix}0, & i \neq j \\ 1, & i = j\end{Bmatrix}.$$

It is easy to interpret Eq. (2-48). The first integral in the equation is clearly the total charge, and the first term is the potential which would result if this total charge were concentrated at the origin. The second integral is very similar to the dipole moment defined in Section 2-7, and so it is called the dipole moment of the charge distribution. As a definition, this represents a generalization of the definition given for two equal and opposite point charges; it is easy to show, however, that both definitions give the same result for two equal and opposite point charges. The second term in Eq. (2-48) is the potential which would result if a point dipole equal to the dipole moment of the charge distribution were located at the origin of coordinates. It is interesting to note that the dipole moment of a charge distribution is independent of the origin of coordinates if the total charge is zero. To verify this, consider a new coordinate system with *origin* at **R** in the old system. Denoting a point with respect to the old system by **r**′ and the same point with respect to the new system by **r**″, we have

$$\mathbf{r}' = \mathbf{r}'' + \mathbf{R}. \tag{2-49}$$

The dipole moment with respect to the old system is

$$\mathbf{p} = \int_{V'} \mathbf{r}'\rho(\mathbf{r}')\,dv' = \int_{V'} (\mathbf{r}'' + \mathbf{R})\rho(r')\,dv' = \int_{V'} \mathbf{r}''\rho\,dv' + \mathbf{R}Q, \tag{2-50}$$

which proves the statement above.

The third term of Eq. (2-48) can be written

$$\sum_{i=1}^{3}\sum_{j=1}^{3}\frac{1}{2}\frac{x_i x_j}{r^3}Q_{ij}, \tag{2-51}$$

where Q_{ij} is given by

$$Q_{ij} = \int_{V'} (3x_i'x_j' - \delta_{ij}r'^2)\rho(r')\,dv'. \tag{2-52}$$

There are nine components of Q_{ij} corresponding to i, j equal to 1, 2, 3. Of these nine components six are equal in pairs, leaving six distinct components. This set of quantities form the quadrupole moment tensor and represent an extension of the dipole moment concept. There are, of course, higher-order moments which are generated by keeping higher-order terms in the expansion of Eq. (2-48). These higher-order multipoles are important in nuclear physics, but will not be considered further in this book.

The electric multipoles are used, as Eq. (2-48) indicates, to approximate the electric field of a charge distribution. There are, however, many other uses, all in the framework of approximating a real extended charge distribution by point charges, point dipoles, etc. These approximations often make it possible to solve problems which would otherwise be prohibitively difficult.

Problems

2–1. Two particles, each of mass m and having charge q, are suspended by strings of length l from a common point. Find the angle θ which each string makes with the vertical.

2–2. Two small identical conducting spheres have charges of 2.0×10^{-9} coul and -0.5×10^{-9} coul, respectively. When they are placed 4 cm apart, what is the force between them? If they are brought into contact and then separated by 4 cm, what is the force between them?

2–3. Point charges of 3×10^{-9} coul are situated at each of three corners of a square whose side is 15 cm. Find the magnitude and direction of the electric field at the vacant corner point of the square.

2–4. Given an infinitely long line charge with uniform charge density λ per unit length. Using direct integration, find the electric field at a distance r from the line.

2–5. (a) A circular disk of radius R has a uniform surface charge density σ. Find the electric field at a point on the axis of the disk at a distance z from the plane of the disk. (b) A right circular cylinder of radius R and height L is oriented along the z-axis. It has a nonuniform volume density of charge given by $\rho(z) = \rho_0 + \beta z$ with reference to an origin at the center of the cylinder. Find the force on a point charge q placed at the center of the cylinder.

2–6. A thin, conducting, spherical shell of radius R is charged uniformly with total charge Q. By direct integration, find the potential at an arbitrary point (a) inside the shell, (b) outside the shell.

2–7. Two point charges, $-q$ and $+\frac{1}{2}q$, are situated at the origin and at the point $(a, 0, 0)$ respectively. At what point along the x-axis does the electric field vanish? In the x, y-plane, make a plot of the equipotential surface which goes through the point just referred to. Is this point a true minimum in the potential?

2–8. Show that the $U = 0$ equipotential surface of the preceding problem is spherical in shape. What are the coordinates of the center of this sphere?

2–9. Given a right circular cylinder of radius R and length L containing a uniform charge density ρ. Calculate the electrostatic potential at a point on the cylinder axis but external to the distribution.

2–10. Given a region of space in which the electric field is everywhere directed parallel to the x-axis. Prove that the electric field is independent of the y- and z-coordinates in this region. If there is no charge in this region, prove that the field is also independent of x.

2–11. Given that the dielectric strength of air (i.e., the electric field which produces corona) is 3×10^6 v/m, what is the highest possible potential of an isolated spherical conductor of radius 10 cm?

2–12. A conducting object has a hollow cavity in its interior. If a point charge q is introduced into the cavity, prove that the charge $-q$ is induced on the surface of the cavity. (Use Gauss' law.)

2–13. The electric field in the atmosphere at the earth's surface is approximately 200 v/m, directed downward. At 1400 m above the earth's surface, the electric field in the atmosphere is only 20 v/m, again directed downward. What is the average charge density in the atmosphere below 1400 m? Does this consist predominantly of *positive* or *negative* ions?

2–14. Two infinite parallel conducting plates are separated . y the distance d. If the plates have uniform charge densities σ and $-\sigma$, respectively, on their inside surfaces, obtain an expression for the electric field between the plates. Prove that the electric field in the regions external to the plates is zero. [Two charged parallel conducting plates of finite area produce essentially the same electric field in the region between them as was found above provided the dimensions of the plates are large compared with the separation d; such an arrangement is called a *capacitor* (see Chapter 6).]

2–15. A spherical charge distribution has a volume charge density which is a function only of r, the distance from the center of the distribution. In other words, $\rho = \rho(r)$. If $\rho(r)$ is as given below, determine the electric field as a function of r. Integrate the result to obtain an expression for the electrostatic potential $U(r)$, subject to the restriction that $U(\infty) = 0$.

(a) $\rho = A/r$ with A a constant for $0 \geq r \geq R$;
$\rho = 0$ for $r > R$.

(b) $\rho = \rho_0$ (i.e., constant) for $0 \geq r \geq R$;
$\rho = 0$ for $r \geq R$.

2–16. Using Eq. (2–39) for the potential produced by a dipole $\mathbf{p}$, make a plot of the traces of equipotential surfaces in a plane containing the dipole. For convenience, the dipole may be located at the origin. Use the results obtained to sketch in some of the lines of force. Compare the result with Fig. 2–1.

2–17. (a) Show that the force acting on a dipole $\mathbf{p}$ placed in an external electric field $\mathbf{E}_{\text{ext}}$ is $\mathbf{p} \cdot \nabla \mathbf{E}_{\text{ext}}$. (b) Show that the torque acting on the dipole in this field is

$$\boldsymbol{\tau} = \mathbf{r} \times [\mathbf{p} \cdot \nabla \mathbf{E}_{\text{ext}}] + \mathbf{p} \times \mathbf{E}_{\text{ext}},$$

where $\mathbf{r}$ is the vector distance to the dipole from the point about which the torque is to be measured. The quantity $\mathbf{p} \times \mathbf{E}_{\text{ext}}$, which is independent of the point about which the torque is computed, is called the turning couple acting on the dipole.

2–18. Three charges are arranged in a linear array. The charge $-2q$ is placed at the origin, and two charges, each of $+q$, are placed at $(0, 0, l)$ and $(0, 0, -l)$ respectively. Find a relatively simple expression for the potential $U(\mathbf{r})$ which is valid for distances $|\mathbf{r}| \gg l$. Make a plot of the equipotential surfaces in the x, z-plane.