Теорема (Централна гранична теорема)

Нека $\bar{X} = (X_1, \dots, X_n)$ е редица от незав. и еднакво разпр. сл. вел.
Нека $\mathbb{E}[X_1] = \mu$, $\mathbb{D}[X_1] = \sigma^2$. Тогава е в сила, ако

$S_n = \sum_{j=1}^{n} X_j$, то $\dfrac{S_n - n.\mu}{\sigma\sqrt{n}} \xrightarrow[n \to \infty]{d} Z \sim N(0,1)$

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{S_n - n.\mu}{\sigma\sqrt{n}} \leq x\right) = \mathbb{P}(Z \leq x) = \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{y^2}{2}} dy$$

$$Z_n = \frac{S_n - n.\mu}{\sigma\sqrt{n}} = \frac{\frac{S_n}{n} - \mu}{\frac{\sigma}{\sqrt{n}}} \Rightarrow \frac{S_n}{n} = \mu + \frac{\sigma}{\sqrt{n}} . Z_n \approx \mu + \frac{\sigma}{\sqrt{n}} . Z$$

$$\left|\frac{S_n}{n} - \mu\right| = \left|\frac{\sigma}{\sqrt{n}} . Z_n\right| \approx \frac{\sigma}{\sqrt{n}} |Z|$$

$$\mathbb{P}\left(a < \frac{S_n - n.\mu}{\sigma\sqrt{n}} \leq b\right) \xrightarrow[n\to\infty]{d} \Phi(b) - \Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{a}^{b} e^{-\frac{y^2}{2}} dy$$

Деф. (функция на моментите)
Нека X е сл. вел и $\mathbb{E}[e^{tX}]<\infty$, за $|t|<\varepsilon$, за някое $\varepsilon>0$. Тогава $M_X(t)=\mathbb{E}[e^{tX}]$, $|t|<\varepsilon$ се нарича ф-я на моментите

Деф. (k-ти момент)
Нека X е сл. вел. Ако $\mathbb{E}[X^k]$ съществува, то $\mathbb{E}[X^k]$ се нарича k-ти момент на X.

Деф. Нека X е сл. вел. и k-тия момент съществува. Тогава $\mathbb{E}[(X-\mathbb{E}[X])^k]$ е централен k-ти момент, а $\mathbb{E}[|X-\mathbb{E}[X]|^k]$ е централен абсолютен k-ти момент.

Свойства

а) $M_X(0)=1$

б) $M_X(t)=\sum_{k=0}^{\infty}\frac{t^k}{k!}\mathbb{E}[X^k]$

в) $M_X^{(k)}(0)=\mathbb{E}[X^k]$

г) $X \perp\!\!\!\perp Y$ и M_X, M_Y са добре деф. за $|t|<\varepsilon$, то $M_{X+Y}(t)=M_X(t)M_Y(t)$

д) Ако $\lim_{n\to\infty} M_{X_n}(t)=M_X(t)$, $\forall |t|\le\varepsilon$, то $X_n \xrightarrow[n\to\infty]{d} X$

е) Ако $M_X=M_Y$, $\forall |t|<\varepsilon \Rightarrow X \stackrel{d}{=} Y$

ж) Ако $Y=aX+b$, то $M_Y(t)=e^{bt}\cdot M_X(at)$

Доказателство

а) $M_X(0)=\mathbb{E}[e^{0X}]=\mathbb{E}[1]=1$

б) $M_X(t)=\mathbb{E}[e^{tX}]=\mathbb{E}\left[\sum_{k=0}^{\infty}\frac{t^k.X^k}{k!}\right]=\sum_{k=0}^{\infty}\frac{t^k}{k!}\mathbb{E}[X^k]$

в) $M_X^{(k)}(t)=\frac{d^k}{dt^k}\cancel{\mathbb{E}}\sum_{k=0}^{\infty}\frac{t^k.X^k}{k!}=\frac{d^k}{dt^k}\mathbb{E}\left[\sum_{k=0}^{\infty}\frac{t^k.X^k}{k!}\right]=\cancel{\sum_{k=0}^{\infty}\frac{t^k}{k!}\frac{d^k}{dt^k}\mathbb{E}[X^k]}=$

$$=\sum_{k=0}^{\infty}\frac{d^k}{dt^k}\frac{t^k}{k!}\cdot\mathbb{E}[X^k]=\mathbb{E}[X^k]+\sum_{k=1}^{\infty}0=\mathbb{E}[X^k]$$

г) $M_{X+Y}(t)=\mathbb{E}[e^{t(X+Y)}]=\mathbb{E}[e^{tX}e^{tY}]\stackrel{X\perp\!\!\!\perp Y}{=}\mathbb{E}[e^{tX}]\mathbb{E}[e^{tY}]=M_X(t)M_Y(t)$

ж) $M_Y(t)=\mathbb{E}[e^{t(aX+b)}]=\mathbb{E}[e^{taX}e^{tb}]=e^{bt}\mathbb{E}[e^{atX}]=M_X(at)e^{bt}$

$\oplus$ $X\sim N(\mu,\sigma^2)$, то $M_X(t)=e^{\mu t}e^{\frac{\sigma^2t^2}{2}}$, $\forall t\in\mathbb{R}$

$X=\mu+\sigma.Z \stackrel{(ж)}{\Rightarrow} M_X(t)=e^{\mu t}.M_Z(\sigma t)$

$$M_Z(t)=\mathbb{E}[e^{tZ}]=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{tx}.e^{-\frac{x^2}{2}}dx=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{tx-\frac{x^2}{2}}dx=\frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{t^2}{2}+tx-\frac{x^2}{2}}dx=\frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-\frac{1}{2}(x-t)^2}dx$$

$$\stackrel{\substack{y=(x-t)\\ \frac{dy}{dx}=1}}{=}\frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}}\underbrace{\int_{-\infty}^{\infty}e^{-\frac{y^2}{2}}dy}_{\text{Гаусов}}=\frac{e^{\frac{t^2}{2}}}{\sqrt{2\pi}}.\sqrt{2\pi}=e^{\frac{t^2}{2}}\Rightarrow M_Z(\sigma t)=e^{\frac{\sigma^2t^2}{2}}\Rightarrow M_X(t)=e^{\mu t}.e^{\frac{\sigma^2t^2}{2}}$$

④ $X \sim \Gamma(\alpha, \beta)$

$$M_X(t) = \mathbb{E}[e^{tX}] = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} \cdot e^{-\beta x} \cdot e^{tx} dx = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty x^{\alpha-1} e^{-(\beta-t)x} dx \quad \frac{(\beta-t)x = y}{\frac{dy}{dx} = \beta - t}$$

$$= \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^\infty \frac{y^{\alpha-1}}{(\beta-t)^{\alpha-1}} \cdot e^{-y} dy = \frac{\beta^\alpha}{\Gamma(\alpha)(\beta-t)^\alpha} \underbrace{\int_0^\infty y^{\alpha-1} \cdot e^{-y} dy}_{= \Gamma(\alpha)} = \frac{\beta^\alpha}{(\beta-t)^\alpha}$$

Теорема (ЦГТ)

Нека $(X_n)_{n \geq 1}$ е редица от незав. едн. разпр. сл. вел. в едно вер. пр-во и $\mathbb{E}[X_1] = \mu$, $\mathbb{D}[X_1] = \sigma^2$ свщ. Тогава, ако $S_n = \sum_{j=1}^n X_j$ е изп. ЦГТ и

$$\frac{S_n - n.\mu}{\sigma\sqrt{n}} \xrightarrow[n \to \infty]{d} Z \sim N(0, 1)$$

Тогава $\forall x \in \mathbb{R} = C_\Phi = C_{F_X}$: $\lim_{n \to \infty} \mathbb{P}\left(\frac{S_n - n.\mu}{\sigma\sqrt{n}} \leq x\right) = \mathbb{P}(Z \leq x) = \Phi(x)$

Доказателство

Ще доп., че $M_{X_1}(t)$ е добре деф. $\forall t \in \mathbb{R}$

Центрираме и нормираме X_i: $Y_i = \frac{X_i - \mu}{\sigma}$. Тогава $(Y_n)_{n \geq 1}$ е редица от незав. и еднакво разпр. сл. вел. с $\mathbb{E}[Y_1] = 0$, $\mathbb{D}[Y_1] = 1$

Нека $V_n = \frac{S_n - n.\mu}{\sigma\sqrt{n}} = \frac{\sum_{i=1}^n Y_i}{\sqrt{n}}$ и искаме да док., че $M_{V_n}(t) = M_{\frac{\sum_{i=1}^n Y_i}{\sqrt{n}}}(t) = M_{\sum_{i=1}^n Y_i}\left(\frac{t}{\sqrt{n}}\right) \overset{iid}{=} \left[M_{Y_1}\left(\frac{t}{\sqrt{n}}\right)\right]^n$

Нека разгледаме $M_{Y_1}\left(\frac{t}{\sqrt{n}}\right) = \mathbb{E}\left[e^{\frac{tY_1}{\sqrt{n}}}\right] \overset{\text{Тейлър}}{=}$

$$= \mathbb{E}\left[1 + \frac{t}{\sqrt{n}} Y_1 + \frac{t^2}{2!\,n} Y_1^2 + \frac{t^3}{3!\,n^{\frac{3}{2}}} \mathbb{E}[Y_1^3 \cdot \Theta(Y_1)]\right] = 1 + 0 + \frac{t^2}{2n} + \frac{t^3}{6n^{\frac{3}{2}}} \xrightarrow[n \to \infty]{} e^{\frac{t^2}{2n}}$$

Тогава $M_{V_n}(t) = \left[M_{Y_1}\left(\frac{t}{\sqrt{n}}\right)\right]^n \xrightarrow{n \to \infty} e^{\frac{t^2}{2}} = M_Z(t)$

Следствие Нека $X \sim Bin(n, p)$. Тогава $\forall a < b$

$$\mathbb{P}\left(a < \frac{X - np}{\sqrt{npq}} < b\right) \xrightarrow[n \to \infty]{} \Phi(b) - \Phi(a)$$

Теорема (Бери-Есеен)

Нека $(X_n)_{n\geq 1}$ е редица от незав. едн. разпр. сл. вел. с $\mathbb{E}[X_1]=\mu$, $\mathbb{D}[X_1]=\sigma^2$

Тогава
$$\sup_{x\in\mathbb{R}}\left|\mathbb{P}\left(\frac{S_n-n.\mu}{\sigma\sqrt{n}}\leq x\right)-\mathbb{P}(Z\leq x)\right|\leq 0.4748\cdot\frac{\mathbb{E}\left[\,|X_1-\mathbb{E}[X_1]|^3\right]}{\sigma^{\frac{3}{2}}\sqrt{n}}$$