Разпределение на Бернули

$X \in Ber(p)$, ако имаме разпределението $\begin{array}{c|c|c} X & 0 & 1 \\ \hline \mathbb{P} & q & p \end{array}$, където $p+q=1$

$$\left.\begin{array}{l} \mathbb{E}[X] = 0.q + 1.p = p \\ \mathbb{E}[X^2] = 0^2.q + 1^2.p = p \end{array}\right\} \Rightarrow \mathbb{D}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = p - p^2 = p(1-p) = p.q$$

$$g_X(s) = \mathbb{E}[s^X] = s^0.q + s^1.p = q + p.s = 1 - p + p.s$$

Биномно разпределение

$X \sim Bin(n,p)$, ако X е броят успехи измежду първите n опита в схема на Бернули.

$$X = \sum_{j=1}^{n} X_j, \quad X_j \sim Ber(p)$$

Твърдение Ако $X \sim Bin(n,p)$:

а) $g_X(s) = (q+ps)^n$

б) $\mathbb{E}[X] = n.p$; $\mathbb{D}[X] = n.p.q$

в) $\mathbb{P}(X=k) = \binom{n}{k}.p^k.q^{n-k}$, $k = \overline{0,n}$

Доказателство

а) $X = \sum_{j=1}^{n} X_j \xRightarrow[]{Тв. (X_1 \overset{d}{=} X_j, j=\overline{1,n})} g_X(s) = (g_{X_1}(s))^n = (1-p+p.s)^n = (q+p.s)^n$

б) $g_X'(1) = \frac{d}{ds}(q+ps)^n = np(q+ps)^{n-1}\Big|_{s=1} = np$

или

$$\mathbb{E}[X] = \mathbb{E}\Big[\sum_{j=1}^{n} X_j\Big] \overset{лин.}{=} \sum_{j=1}^{n} [\mathbb{E}[X_j]] = \sum_{j=1}^{n} p = n.p$$

$$\mathbb{D}[X] = \mathbb{D}\Big[\sum_{j=1}^{n} X_j\Big] \overset{X_i \perp\!\!\!\perp X_j, j=\overline{1,n}}{=} \sum_{j=1}^{n} \mathbb{D}[X_j] = n.p.q$$

$$g_X''(1) = \frac{d}{ds} np(q+ps)^{n-1} = n(n-1)p^2(q+ps)^{n-2}\Big|_{s=1} = n(n-1)p^2$$

$$\mathbb{D}[X] = g_X''(1) + g_X'(1) - [g_X'(1)]^2 = n(n-1)p^2 + np - n^2p^2 = np[(n-1)p + 1 - np] =$$
$$= np[np - p + 1 - np] = np(1-p) = npq$$

в) $k!\,\mathbb{P}(X=k) = g_X^{(k)}(0)$, $k = \overline{0,n}$

$$g_X^{(k)}(0) = \frac{\partial^k}{\partial s^k}(q+ps)^n\Big|_{s=0} = n(n-1)\dots(n-k+1)\,q^{n-k}.p^k$$

$$\Rightarrow \mathbb{P}(X=k) = \frac{n(n-1)\dots(n-k+1)q^{n-k}.p^k}{k!} = \frac{n!\,p^k q^{n-k}}{(n-k)!\,k!} = \binom{n}{k}p^k.q^{n-k}$$

Геометрично разпределение

$X \sim Ge(p)$ $\quad X = \min\{n \geq 1 : \sum_{j=1}^{n} X_j = 1\} - 1$ $\rightarrow$ броят неуспехи до първи успех

Твърдение Ако $X \sim Ge(p)$. Тогава:

а) $\mathbb{P}(X=k) = q^k \cdot p$, $k \geq 0$

б) $g_X(s) = \dfrac{p}{1-s \cdot q}$, $|s| \leq 1$

Доказателство

а) $\mathbb{P}(X=0) = \mathbb{P}(X_1=1) = p$

$\mathbb{P}(X=1) = \mathbb{P}(X_1=0; X_2=1) = p \cdot q$

$\dots\dots$

$\mathbb{P}(X=k) = \mathbb{P}(X_1=0; X_2=0; \dots; X_{k+1}=1) = \mathbb{P}(X_1=0) \cdot \mathbb{P}(X_2=0) \cdot \mathbb{P}(X_3=0) \dots \mathbb{P}(X_{k+1}=1) = q^k \cdot p$

б) $g_X(s) = \sum_{k=0}^{\infty} s^k \cdot p \cdot q^k = p \sum_{k=0}^{\infty} (sq)^k \overset{|sq|<1}{=} \dfrac{p}{1-qs}$

Следствие $X \sim Ge(p) \Rightarrow \mathbb{E}[X] = \dfrac{q}{p}$ и $\mathbb{D}[X] = \dfrac{q}{p^2}$

Доказателство $g_X(s) = \dfrac{p}{1-q \cdot s}$

$$\mathbb{E}[X] = g_X'(s) = \frac{d}{ds}\frac{p}{1-qs} = p\frac{d}{ds}(1-qs)^{-1} = \frac{pq}{(1-qs)^2}\Big|_{s=1} = \frac{pq}{(1-q)^2} = \frac{q}{p}$$

$$g_X''(s) = \frac{d}{ds}\frac{pq}{(1-qs)^2} = \frac{pq \cdot 2q}{(1-qs)^3}\Big|_{s=1} = \frac{2pq^2}{p^3} = \frac{2q^2}{p^2}$$

$$\mathbb{D}[X] = g_X''(1) + g_X'(1) - (g_X'(1))^2 = \frac{2q^2}{p^2} + \frac{q}{p} - \frac{q^2}{p^2} = \frac{q^2}{p^2} + \frac{q}{p} = \frac{q^2+qp}{p^2} = \frac{q(\overset{1}{\overbrace{p+q}})}{p^2} = \frac{q}{p^2}$$

Твърдение (Безпаметност на геометрично разпределение)

Нека $X \in Ge(p)$. Тогава $\forall m \geq 0$ и $k \geq 0$: $\mathbb{P}(X \geq m+k \mid X \geq m) = \mathbb{P}(X \geq k) = q^k$

Доказателство

$$\mathbb{P}(X \geq l) = \sum_{j=l}^{\infty} p \cdot q^j = p \cdot q^l \sum_{j=0}^{\infty} q^j \overset{|q|<1}{=} \frac{p \cdot q^l}{1-q} = q^l$$

$$\mathbb{P}(X \geq m+k \mid X \geq m) = \frac{\mathbb{P}(X \geq m+k \cap X \geq m)}{\mathbb{P}(X \geq m)} = \frac{\mathbb{P}(X \geq m+k)}{\mathbb{P}(X \geq m)} = \frac{q^{m+k}}{q^m} = q^k = \mathbb{P}(X \geq k)$$

Отрицателно биномно разпределение

$X \in NB(r,p)$ и $X = \min\{n \geq 1 : \sum_{i=1}^{n} X_i = r\} - r \rightarrow$ брой неуспехи до r-ти успех

Твърдение Ако $X \sim NB(r,p)$, то $X = \sum_{j=1}^{r} Y_j$, където $Y_j \in Ge(p)$, $j = \overline{1,r}$ и Y_j са независими в съвкупност

Доказателство Ще проверим, че $Y_1 \perp\!\!\!\perp Y_2$, където $Y_1, Y_2 \in Ge(p)$ и $X = Y_1 + Y_2$

$$\mathbb{P}(Y_1 = l \cap Y_2 = m) \stackrel{?}{=} \mathbb{P}(Y_1 = l)\mathbb{P}(Y_2 = m), \ \forall l, m \geq 0$$

$$\mathbb{P}(Y_1 = l \cap Y_2 = m) = \mathbb{P}(\underbrace{X_1 = 0, \dots, X_l = 0, X_{l+1} = 1, X_{l+2} = 0, \dots, X_{l+m} = 0, X_{l+m+1} = 1}_{\text{независими } X_i \in Ber(p)}) =$$

$$= \mathbb{P}(X_1 = 0)\mathbb{P}(X_2 = 0) \dots \mathbb{P}(X_{l+1} = 1)\mathbb{P}(X_{l+2} = 0) \dots \mathbb{P}(X_{l+m+1} = 1) = \mathbb{P}(Y_1 = l)\mathbb{P}(Y_2 = m) \Rightarrow$$

$$\Rightarrow Y_1 \perp\!\!\!\perp Y_2 \ \square$$

Твърдение Ако $X \sim NB(r,p)$, то $g_X(s) = \left(\frac{p}{1-qs}\right)^r$, $\mathbb{E}[X] = \frac{rq}{p}$ и $\mathbb{D}[X] = r\frac{q}{p^2}$

Доказателство

$$\mathbb{E}[X] = \mathbb{E}[\sum_{j=1}^{r} \underbrace{Y_j}_{\substack{\text{незав} \\ Ge(p)}}] = \sum_{j=1}^{r} \mathbb{E}[Y_j] = r.\frac{q}{p}$$

$$\mathbb{D}[X] = \mathbb{D}[\sum_{j=1}^{r} Y_j] \stackrel{\text{незав.}}{=} \sum_{j=1}^{r} \mathbb{D}[Y_j] = r.\frac{q}{p^2}$$

$$g_X(s) \xlongequal[\substack{\text{еднакво} \\ \text{разпр.}}]{\text{незав.}} \prod_{j=1}^{r} g_{Y_j}(s) = \prod_{j=1}^{r} \frac{p}{1-qs} = \left(\frac{p}{1-qs}\right)^r$$

Твърдение $X \sim NB(r,p)$, то $\mathbb{P}(X = k) = \binom{r+k-1}{r} p^r q^k$

Доказателство $g_X(s) = \left(\frac{p}{1-qs}\right)^r$

$$k!\,\mathbb{P}(X = k) = g_X^{(k)}(s)\Big|_{s=0}$$

$$g_X(s) = \frac{p^r}{(1-qs)^r} \Rightarrow \frac{d^k}{ds^k} \frac{p^r}{(1-qs)^r} = p^r \frac{d^k}{ds^k} \frac{1}{(1-qs)^r}\Big|_{s=0} = p^r.q^k.\frac{r(r+1)\dots(r+k-1)}{(1-q.0)^{r+k}} = \binom{r+k-1}{r} p^r.q^k$$

Поасоново разпределение

Нека $\lambda > 0$. Казваме, че $X \sim Pois(\lambda)$, ако $P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}, k \geq 0$

$$1 \stackrel{?}{=} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \cdot \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{\text{развитие в ред на Тейлър за exp.}} = e^{-\lambda} \cdot e^{\lambda} = 1$$

Твърдение $X \sim Pois(\lambda)$. Тогава:

а) $g_X(s) = e^{-\lambda + \lambda s}$, за $|s| \leq 1$

б) $\mathbb{E}[X] = \mathbb{D}[X] = \lambda$

Доказателство

а) $$g_X(s) = \sum_{k=0}^{\infty} s^k \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!}}_{\text{Тейлър}} = e^{-\lambda + \lambda s}$$

б) $$\mathbb{E}[X] = g_X'(1) = \frac{d}{ds} e^{-\lambda + \lambda s} = \lambda . e^{-\lambda + \lambda s} \Big|_{s=1} = \lambda$$

$$g_X''(1) = \frac{d}{ds} \lambda e^{-\lambda + \lambda s} = \lambda^2 . e^{-\lambda + \lambda s} \Big|_{s=1} = \lambda^2$$

$$\mathbb{D}[X] = g_X''(1) + g_X'(1) - (g_X'(1))^2 = \cancel{\lambda^2} + \lambda - \cancel{\lambda^2} = \lambda$$

Теорема (Поасон)

Нека $\forall n \geq 1$: $X_n \sim Bin(n, p_n)$, където $p_n = \frac{\lambda}{n} + \frac{V_n}{n}$, $\lambda > 0$ и $\lim_{n \to \infty} |V_n| = 0$. Тогава $\forall k \geq 0$ е изпълнено, че $\lim_{n \to \infty} \mathbb{P}(X_n = k) = \mathbb{P}(Y = k) = \frac{\lambda^k}{k!} . e^{-\lambda}$, където $Y \sim Pois(\lambda)$ или $X_n \xrightarrow[n \to \infty]{d} Y$

⊕ $Y \sim Bin(n, p)$, $\lambda = np$ и Тогава $P(Y=k) = \binom{n}{k} p^k . q^{n-k} \approx \frac{\lambda^k}{k!} . e^{-\lambda}$

При $\lambda = n.p \leq 20$ и $n \geq 100$ приближението е добро

Доказателство

$\lim_{n \to \infty} \mathbb{P}(X_n = k) = \mathbb{P}(Y = k)$ е изпълнено, ако $\lim_{n \to \infty} g_{X_n}(s) = g_Y(s)$

$$g_{X_n}(s) = (q_n + p_n . s)^n = (1 - p_n + p_n . s)^n = \left(1 - \frac{\lambda}{n} - \frac{V_n}{n} + \frac{\lambda}{n} s + \frac{V_n}{n} s\right)^n$$

$$\lim_{n \to \infty} \left(1 - \frac{\lambda}{n} \overset{-\frac{V_n}{n}}{} + \frac{\lambda}{n} s \overset{+\frac{V_n}{n} s}{}\right)^n = \lim_{n \to \infty} \left(1 + \frac{1}{n} \lambda (s-1)\right)^n = e^{\lambda(s-1)} = g_Y(s)$$