Условно математическо очакване (УМО)

Знаем, че $\min_{a\in\mathbb{R}}(X-a)^2 = \mathbb{E}[(X-\mathbb{E}[X])^2] = \mathbb{D}[X]$, $a=\mathbb{E}[X]$

Ако $Y \sim Ber(p)$, $Y = \begin{cases} 0, & 1-p \\ 1, & p \end{cases}$, означава $A = \{Y=1\}$, $A^c = \{Y=0\} \Rightarrow Y = \mathbb{1}_A$, $1-Y = \mathbb{1}_{A^c}$

За $C \subseteq \Omega$: $\mathbb{E}[\mathbb{1}_C] = \mathbb{P}(C)$

Имаме сл. вел. X и набл. $Y = \{^{1;\,p}_{0;\,1-p}$, $Y = \mathbb{1}_A$.

$G: \{0,1\} \to \mathbb{R}$, $\min_G \mathbb{E}[(X-G(Y))^2] = ?$

$G(Y) = a.\mathbb{1}_A + b.\mathbb{1}_{A^c}$

$$f(a,b) = \mathbb{E}[(X-G(Y))^2] = \mathbb{E}[(X-a.\mathbb{1}_A - b.\mathbb{1}_{A^c})^2] =$$

$$= \mathbb{E}[X^2 + a^2.\mathbb{1}_A + b^2.\mathbb{1}_{A^c} + ab.\underbrace{\mathbb{1}_A.\mathbb{1}_{A^c}}_{0} - 2aX\mathbb{1}_A - 2bX\mathbb{1}_{A^c}] =$$

$$= \mathbb{E}[X^2] + a^2\mathbb{P}(A) + b^2\mathbb{P}(A^c) - 2a\mathbb{E}[\mathbb{1}_A X] - 2b\mathbb{E}[\mathbb{1}_{A^c}X]$$

Интересуваме се от $\min_{a,b} f(a,b) \Rightarrow \begin{cases} 0 = \frac{\partial}{\partial a}f(a,b) = 2a\mathbb{E}[\mathbb{1}_A] - 2\mathbb{E}[\mathbb{1}_A X] \\ 0 = \frac{\partial}{\partial b}f(a,b) = 2b\mathbb{E}[\mathbb{1}_{A^c}] - 2\mathbb{E}[X\mathbb{1}_{A^c}] \end{cases} \Rightarrow \begin{cases} a = \frac{\mathbb{E}[X\mathbb{1}_A]}{\mathbb{E}[\mathbb{1}_A]} \\ b = \frac{\mathbb{E}[X\mathbb{1}_{A^c}]}{\mathbb{E}[\mathbb{1}_{A^c}]} \end{cases}$

$$\Rightarrow G(Y) = \frac{\mathbb{E}[X.\mathbb{1}_A]}{\mathbb{E}[\mathbb{1}_A]}.\mathbb{1}_A + \frac{\mathbb{E}[X\mathbb{1}_{A^c}]}{\mathbb{E}[\mathbb{1}_{A^c}]}.\mathbb{1}_{A^c}$$

Частен случай: $X = \mathbb{1}_B \Rightarrow G(Y) = \frac{\mathbb{E}[\mathbb{1}_B\mathbb{1}_A]}{\mathbb{E}[\mathbb{1}_A]}.\mathbb{1}_A + \frac{\mathbb{E}[\mathbb{1}_B.\mathbb{1}_{A^c}]}{\mathbb{E}[\mathbb{1}_{A^c}]}.\mathbb{1}_{A^c} = \frac{\mathbb{P}(A\cap B)}{\mathbb{P}(A)}\mathbb{1}_A + \frac{\mathbb{P}(A^c\cap B)}{\mathbb{P}(A^c)}.\mathbb{1}_{A^c} =$

$$= \mathbb{P}(B|A).\mathbb{1}_A + \mathbb{P}(B|A^c).\mathbb{1}_{A^c}$$

Деф (УМО)

Нека X и Y са две сл. вел. Тогава $\mathbb{E}[X|Y] = G^*(Y)$, която минимизира

$$\min_G \mathbb{E}[(X-G(Y))^2] = \mathbb{E}[(X-\mathbb{E}[X|Y])^2] = \mathbb{E}[(X-G^*(Y))^2]$$

Твърдение Нека X и Y са сл. вел., където Y е дискретна.

Тогава условно очакване на X при зададена ст-ст на Y, $Y=y_k$ се разбира

$$\mathbb{E}[X|Y=y_k] = \sum_i{}' x_i\mathbb{P}(X=x_i|Y=y_k)$$, когато X също е дискретно

$$= \frac{\mathbb{E}[X.\mathbb{1}_{A_j}]}{\mathbb{E}[\mathbb{1}_{A_j}]}$$

Твърдение X и Y са сл. вел, като Y е дискретна. Тогава $\mathbb{E}[X|Y]$ е дискр. сл. вел

$$\mathbb{E}[X|Y] = \sum_j{}' \mathbb{E}[X|Y=y_j].\mathbb{1}_{A_j}$$

Лема) Нека X и Y са сл. вел., където Y е дискретно. Тогава: $\left(Y = \sum_k' y_k \mathbb{1}_{A_k}\right)$

а) Ако Z е сл. вел., то $\mathbb{E}[aX+bZ|Y] = a.\mathbb{E}[X|Y] + b.\mathbb{E}[Z|Y]$

б) $X \perp\!\!\!\perp Y$, то $\mathbb{E}[X|Y] = \mathbb{E}[X]$

в) $X = f(Y)$, то $\mathbb{E}[X|Y] = f(Y) = X$

г) $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]$

д) $\mathbb{E}[f(X,Y)|Y=y_k] = \mathbb{E}[f(X,y_k)]$, ако $X \perp\!\!\!\perp Y$

Доказателство)

а) $\mathbb{E}[aX+bZ|Y] = \sum_k' \frac{\mathbb{E}[(aX+bZ).\mathbb{1}_{A_k}]}{\mathbb{E}[\mathbb{1}_{A_k}]}.\mathbb{1}_{A_k} = \sum_k' \frac{a\mathbb{E}[X.\mathbb{1}_{A_k}] + b\mathbb{E}[Z\mathbb{1}_{A_k}]}{\mathbb{P}(A_k)}.\mathbb{1}_{A_k} =$

$= a.\sum_k' \frac{\mathbb{E}[X.\mathbb{1}_{A_k}]}{\mathbb{P}(A_k)}.\mathbb{1}_{A_k} + b.\sum_k' \frac{\mathbb{E}[Z.\mathbb{1}_{A_k}]}{\mathbb{P}(A_k)}.\mathbb{1}_{A_k} = a.\mathbb{E}[X|Y] + b.\mathbb{E}[Z|Y]$

б) $\mathbb{E}[X|Y] = \sum_k' \mathbb{E}[X|Y=y_k].\mathbb{1}_{A_k} \overset{\text{Тв.}}{=} \sum_k' \frac{\mathbb{E}[X.\mathbb{1}_{A_k}]}{\mathbb{E}[\mathbb{1}_{A_k}]}.\mathbb{1}_{A_k} \overset{X \perp\!\!\!\perp Y}{=} \sum_k' \frac{\mathbb{E}[X].\cancel{\mathbb{E}[\mathbb{1}_{A_k}]}}{\cancel{\mathbb{E}[\mathbb{1}_{A_k}]}}.\mathbb{1}_{A_k} =$

$= \sum_k' \mathbb{E}[X].\mathbb{1}_{A_k} = \mathbb{E}[X]\underbrace{\sum_k' \mathbb{1}_{A_k}}_{\text{пълна група събития} \Rightarrow =1} = \mathbb{E}[X]$

в) $\mathbb{E}[X|Y] = \sum_k' \frac{\mathbb{E}[f(Y)\mathbb{1}_{A_k}]}{\mathbb{P}(A_k)}.\mathbb{1}_{A_k} = \sum_k' \frac{f(y)\cancel{\mathbb{E}[\mathbb{1}_{A_k}]}}{\cancel{\mathbb{E}[\mathbb{1}_{A_k}]}}.\mathbb{1}_{A_k} = f(Y)\sum_k' \mathbb{1}_{A_k} = f(Y) = X$

г) $\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}\left[\sum_k' \frac{\mathbb{E}[X\mathbb{1}_{A_k}]}{\mathbb{E}[\mathbb{1}_{A_k}]}.\mathbb{1}_{A_k}\right] = \sum_k' \frac{\mathbb{E}[X\mathbb{1}_{A_k}]}{\cancel{\mathbb{E}[\mathbb{1}_{A_k}]}}.\cancel{\mathbb{E}[\mathbb{1}_{A_k}]} = \sum_k' \mathbb{E}[X\mathbb{1}_{A_k}] =$

$= \mathbb{E}\left[\sum_k' X\mathbb{1}_{A_k}\right] = \mathbb{E}[X]$

д) $\mathbb{E}[f(X,Y)|Y=y_k] = \frac{\mathbb{E}[f(X,y_k)\mathbb{1}_{A_k}]}{\mathbb{P}(A_k)} \overset{X \perp\!\!\!\perp Y}{=} \frac{\mathbb{E}[f(X,y_k)].\cancel{\mathbb{E}[\mathbb{1}_{A_k}]}}{\cancel{\mathbb{P}(A_k)}} = \mathbb{E}[f(X,y_k)]$

Твърдение $\sum_i \mathbb{P}(X=x_i \mid Y=y_k) = \underline{1}$

Доказателство Нека $\{X=x_i\} = \mathbb{1}_{A_i}$

$\{Y=y_k\} = \mathbb{1}_{B_k}$

$$\sum_i \mathbb{P}(X=x_i \mid Y=y_k) = \sum_i \frac{\mathbb{P}(X=x_i \cap Y=y_k)}{\mathbb{P}(Y=y_k)} = \sum_i \frac{\mathbb{E}[\mathbb{1}_{A_i} \mathbb{1}_{B_k}]}{\mathbb{E}[B_k]} = \frac{\mathbb{E}[\overbrace{\sum_i \mathbb{1}_{A_i}}^{\stackrel{1}{\shortparallel}} \mathbb{1}_{B_k}]}{\mathbb{E}[B_k]} =$$

$$= \frac{\mathbb{E}[\mathbb{1}_{B_k}]}{\mathbb{E}[\mathbb{1}_{B_k}]} = \underline{1}$$