Гама разпределение

Деф. Казваме, че НСВ X е гама разпределена с параметри $\alpha, \beta > 0$ и бележим $X \in \Gamma(\alpha, \beta)$, ако има плътност $f_X(x) = \begin{cases} \frac{\beta^{\alpha} . x^{\alpha-1} . e^{-\beta . x}}{\Gamma(\alpha)} & , x > 0 \\ 0 & , x \le 0 \end{cases}$, където

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} . e^{-x} dx$$

$\Gamma(n) = (n-1)!, n \in \mathbb{N}$

$\Gamma(\alpha+1) = \alpha \Gamma(\alpha), \alpha > 0$

$\overset{X \sim}{\Rightarrow} \Gamma(1, \beta), f_X(x) = \begin{cases} \beta . e^{-\beta x} & , x > 0 \\ 0 & , x \le 0 \end{cases} \Rightarrow \Gamma(1, \beta) = Exp(\beta)$

$\Gamma(1, 1) = Exp(1)$

Твърдение Ако $X_i \sim \Gamma(\alpha_i, \beta), i = \overline{1, n}$ и $X_1, \dots, X_n$ са незав. в съвкупност, то

$$Y = \sum_{i=1}^{n} X_i \sim \Gamma\left(\sum_{i=1}^{n} \alpha_i, \beta\right)$$

Доказателство за $n=2$ $\quad X_1 \sim \Gamma(\alpha_1, \beta), X_2 \sim \Gamma(\alpha_2, \beta)$

$$\begin{matrix} Y = X_1 + X_2 \\ Z = X_2 \end{matrix} \Rightarrow \begin{matrix} Y = \cancel{X_1} + \cancel{Z} \\ Z = X_2 \end{matrix} \Rightarrow \begin{matrix} X_1 = Y - Z \\ X_2 = Z \end{matrix} \qquad J(y,z) = \left| \det \begin{vmatrix} \frac{\partial}{\partial y} x_1 & \frac{\partial}{\partial z} x_1 \\ \frac{\partial}{\partial y} x_2 & \frac{\partial}{\partial z} x_2 \end{vmatrix} \right| = \begin{vmatrix} 1 & * \\ 0 & 1 \end{vmatrix} = 1$$

$$f_{X_1, X_2}(x_1, x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \frac{\beta^{\alpha_1} . x_1^{\alpha_1 - 1} e^{-\beta x_1}}{\Gamma(\alpha_1)} . \frac{\beta^{\alpha_2} . x_2^{\alpha_2 - 1} e^{-\beta x_2}}{\Gamma(\alpha_2)} = \frac{\beta^{\alpha_1+\alpha_2} . x_1^{\alpha_1 - 1} . x_2^{\alpha_2 - 1} . e^{-\beta . (x_1 + x_2)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}, \begin{matrix} x_1 > 0 \\ x_2 > 0 \end{matrix}$$

$$f_{Y,Z}(y,z) = f_{X_1, X_2}(y-z, z) |J(y,z)| = \frac{\beta^{\alpha_1+\alpha_2} (y-z)^{\alpha_1 - 1} . z^{\alpha_2 - 1} . e^{-\beta(y - z + z)}}{\Gamma(\alpha_1)\Gamma(\alpha_2)}, \quad y > z > 0$$

$$f_Y(y) = \frac{\beta^{\alpha_1+\alpha_2} e^{-\beta y}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \int_0^y (y-z)^{\alpha_1 - 1} z^{\alpha_2 - 1} dz \overset{\substack{z = py \\ \frac{dz}{dp} = y}}{=} \frac{\beta^{\alpha_1+\alpha_2} . e^{-\beta y}}{\Gamma(\alpha_1)\Gamma(\alpha_2)} y . y^{\alpha_1 - 1} . y^{\alpha_2 - 1} \underbrace{\int_0^1 (1-p)^{\alpha_1 - 1} . p^{\alpha_2 - 1} dp}$$

$$= \frac{\beta^{\alpha_1+\alpha_2} . e^{-\beta y}}{\cancel{\Gamma(\alpha_1)\Gamma(\alpha_2)}} . y^{\alpha_1 + \alpha_2 - 1} . \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1 + \alpha_2)} = \frac{\beta^{(\alpha_1+\alpha_2)} . y^{(\alpha_1+\alpha_2) - 1} . e^{-\beta . y}}{\Gamma(\alpha_1 + \alpha_2)}$$

Следствие Нека $X_1, \dots, X_n$ са независими в съвкупност $Exp(\lambda)$. Тогава

$$Y = \sum_{i=1}^{n} X_i \sim \Gamma(n, \lambda)$$

Теорема | Ако $X \sim \Gamma(\alpha, \beta)$, то $\mathbb{E}[X] = \frac{\alpha}{\beta}$, $\mathbb{D}[X] = \frac{\alpha}{\beta^2}$

Доказателство |

$$\mathbb{E}[X] = \int_0^\infty x \cdot \frac{\beta^\alpha \cdot x^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\beta x} dx = \frac{1}{\Gamma(\alpha)} \cdot \beta^\alpha \int_0^\infty x^\alpha \cdot e^{-\beta x} dx =$$

$$= \frac{\Gamma(\alpha+1)\, \beta^{\alpha+1}}{\Gamma(\alpha)\, \beta} \int_0^\infty \frac{x^\alpha \cdot e^{-\beta x}}{\Gamma(\alpha+1)} dx = \frac{\alpha \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}\beta} \int_0^\infty \underbrace{\frac{\beta^{\alpha+1} \cdot x^\alpha \cdot e^{-\beta x}}{\Gamma(\alpha+1)}}_{\text{плътност на } \Gamma(\alpha+1, \beta)} dx = \frac{\alpha}{\beta}$$

$$\mathbb{E}[X^2] = \int_0^\infty \frac{x^2 \cdot \beta^\alpha \cdot x^{\alpha-1}}{\Gamma(\alpha)} \cdot e^{-\beta x} dx = \frac{\beta^{\alpha}}{\Gamma(\alpha)} \int_0^\infty x^{\alpha+1} \cdot e^{-\beta x} dx =$$

$$= \frac{\Gamma(\alpha+2)\, \beta^{\alpha+2}}{\Gamma(\alpha+2)\Gamma(\alpha)\beta^2} \int_0^\infty x^{\alpha+1} \cdot e^{-\beta x} dx = \frac{\Gamma(\alpha+2)}{\Gamma(\alpha)} \cdot \frac{1}{\beta^2} \int_0^\infty \underbrace{\frac{\beta^{\alpha+2} x^{\alpha+1} \cdot e^{-\beta x}}{\Gamma(\alpha+2)}}_{f_{\Gamma(\alpha+2, \beta)}} dx =$$

$$= \frac{(\alpha+1)\alpha \cancel{\Gamma(\alpha)}}{\cancel{\Gamma(\alpha)}} \cdot \frac{1}{\beta^2} = \frac{\alpha^2+\alpha}{\beta^2}$$

$$\mathbb{D}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{\alpha^2+\alpha}{\beta^2} - \frac{\alpha^2}{\beta^2} = \frac{\alpha}{\beta^2}$$

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Хи-квадрат разпределение

Деф. Казваме, че $X \sim \chi^2(n)$, $n \geq 1$, ако $X \sim \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$

$$f_X(x) = \begin{cases} \dfrac{\left(\frac{1}{2}\right)^{\frac{n}{2}} \cdot x^{\frac{n}{2}-1} \cdot e^{-\frac{x}{2}}}{\Gamma\left(\frac{n}{2}\right)}, & x > 0 \\ 0, & x \leq 0 \end{cases}$$

Следствие $X \sim \chi^2(n)$, то $\mathbb{E}[X] = \dfrac{\frac{n}{2}}{\frac{1}{2}} = n$, $\mathbb{D}[X] = \dfrac{\frac{n}{2}}{\left(\frac{1}{2}\right)^2} = 2n$

Твърдение

Нека $X_1, \dots, X_n$ са незав. в съвкупност стандартно нормално разпределени сл. вел. $X_i \sim N(0,1)$, $i = \overline{1,n}$. Тогава $Y = \sum_{i=1}^{n} X_i^2 \sim \chi^2(n)$

Доказателство Ще док., че $X_i^2 \sim \Gamma\left(\frac{1}{2}, \frac{1}{2}\right) = \chi^2(1)$

$n = 1$, $Y = X_1^2$ $g(x_1) = x_1^2$ (не е монотонна в $(-\infty, \infty)$)

$$\mathbb{P}(Y \leq y) = \mathbb{P}(X_1^2 \leq y) = \mathbb{P}(-\sqrt{y} \leq X_1 \leq \sqrt{y}) = \frac{1}{\sqrt{2\pi}} \int_{-\sqrt{y}}^{\sqrt{y}} e^{-\frac{x^2}{2}} dx = \frac{2}{\sqrt{2\pi}} \int_0^{x=\sqrt{y}} e^{-\frac{x^2}{2}} dx \;\underset{\frac{dy}{dx} = 2x = 2\sqrt{y}}{\overset{y = x^2}{=}}\; \frac{2}{\sqrt{2\pi}} \int_0^{y} \frac{e^{-\frac{y}{2}}}{2\sqrt{y}} dy$$

$$\frac{d}{dy}\mathbb{P}(Y \leq y) = \frac{2}{\sqrt{2\pi}} \cdot \frac{1}{2\sqrt{y}} \cdot e^{-\frac{y}{2}} = \frac{1}{\sqrt{2\pi}} \cdot y^{-\frac{1}{2}} \cdot e^{-\frac{y}{2}} =$$

$$= \frac{1}{\sqrt{2\pi}} y^{\frac{1}{2}-1} \cdot e^{-\frac{y}{2}} = \frac{\left(\frac{1}{2}\right)^{\frac{1}{2}}}{\Gamma\left(\frac{1}{2}\right)} \cdot y^{\frac{1}{2}-1} \cdot e^{-\frac{y}{2}}$$

$\square$

Деф. (t-разпределение)

Нека $Z \sim N(0,1)$, $X \sim \chi^2(n)$, $n \geq 1$, $Z \perp\!\!\!\perp X$

$$Y = \frac{Z}{\sqrt{\frac{X}{n}}} \sim t(n), n \geq 1$$

Деф. (Сходимост почти сигурно (п.с.))

Казваме, че $X_n \xrightarrow[n\to\infty]{п.с.} X \Leftrightarrow \mathbb{P}(L) = 1$, където $L \in \mathcal{A} = \{\lim\limits_{n\to\infty} X_n = X\} = \{w \in \Omega : \lim\limits_{n\to\infty} X_n(w) = X(w)\}$

където $X_n = (X_1, \dots, X_n)$ е редица от сл. вел. деф. в едно вер. пр-во $U = (\Omega, \mathcal{A}, \mathbb{P})$ и X е сл. вел. деф. в същото вер. пр-во.

Деф. (Сходимост по вероятност)

Казваме, че редицата $X_n = (X_1, \dots, X_n)$ от сл. вел. деф. във $U = (\Omega, \mathcal{A}, \mathbb{P})$ се схожда по вероятност към X деф. в същото вер. пр-во, ако $X_n \xrightarrow[n\to\infty]{P} X \Leftrightarrow$

$\forall \varepsilon > 0, \lim\limits_{n\to\infty} \mathbb{P}(A_{n,\varepsilon}) = 0$, където

$A_{n,\varepsilon} = \{|X_n - X| > \varepsilon\} = \{w \in \Omega : |X_n - X| > \varepsilon\}$

Деф. Ако F_X е ф-я на разпр., то с C_{F_X} означаваме всички точки x, за които F е непрекъсната в x. $C_{F_X} = \{x \in \mathbb{R} : F_X \text{ е непр. в } x\}$ и $x \in C_{F_X} \Leftrightarrow \mathbb{P}(X = x) = 0$

Деф. (Сходимост по разпределение)

Нека $X_n = (X_1, \dots, X_n)$ е редица от сл. вел. и X е сл. вел. Тогава $X_n \xrightarrow[n\to\infty]{d} X$, ако $\forall x \in C_{F_X}$ е в сила $\lim\limits_{n\to\infty} \mathbb{P}(X_n \leq x) = \lim\limits_{n\to\infty} F_{X_n}(x) = F_X(x) = \mathbb{P}(X \leq x)$

Ако $X_n \xrightarrow[n\to\infty]{d} X$, то всяка непр. и огр. ф-я $f : \lim\limits_{n\to\infty} \mathbb{E}[f_{X_n}(X_n)] = \mathbb{E}[f_X]$

Теорема Нека $X_n = (X_1, \dots, X_n)$ е редица от сл. вел. деф. в едно вер. пр-во $(\Omega, \mathcal{A}, \mathbb{P})$ и нека X е сл. вел. деф. в същото вер. пр-во.

а) $X_n \xrightarrow[n\to\infty]{п.с.} X \Rightarrow X_n \xrightarrow[n\to\infty]{P} X$

б) $X_n \xrightarrow[n\to\infty]{P} X \Rightarrow X_n \xrightarrow[n\to\infty]{d} X$

Доказателство) а) $L_X = \{\lim_{n\to\infty} X_n = X\}$, знаем, че $\mathbb{P}(L_X) = 1$

$X_n \xrightarrow[n\to\infty]{п.с.} X$, искаме да покажем, че $\lim_{n\to\infty} \mathbb{P}(A_{n,\varepsilon}) = 0$

$A_{n,\varepsilon} = \{|X_n - X| > \varepsilon\}$, $\forall \varepsilon = \frac{1}{r}, r \geq 1$

$L_X = \bigcap_{r=1}^{\infty} \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A^c_{k,\frac{1}{r}}$, $A^c_{k,\frac{1}{r}} = \{|X_k - X| \leq \frac{1}{r}\}$

За всяко r, съществува n, т.ч. ако $k \geq n$ е изп. $|X_k - X| \leq \frac{1}{r}$

$L_X^c = \bigcup_{r=1}^{\infty} \underbrace{\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_{k,\frac{1}{r}}}_{B_r}$; $\mathbb{P}(L_X^c) = 0 \Rightarrow \mathbb{P}(\bigcup_{r=1}^{\infty} B_r) = 0 \Rightarrow \mathbb{P}(B_r) = 0, \forall r \geq 1$

$\mathbb{P}(\bigcap_{n=1}^{\infty} \underbrace{\bigcup_{k=n}^{\infty} A_{k,\frac{1}{r}}}_{C_{n,r}}) = 0 \Rightarrow \mathbb{P}(\bigcap_{n=1}^{\infty} C_{n,r}) = 0 = \lim_{n\to\infty} \mathbb{P}(C_{n,r})$ - това е вярно, защото

$C_{n,r}$ е намаляващ за по-голяма n и $C_{n,r} \supseteq C_{n+1,r} \supseteq \dots$ (свойство от втора лекция)

$C_{n,r} = \bigcup_{k=n}^{\infty} A_{k,\frac{1}{r}} \supseteq A_{n,\frac{1}{r}}$

$\mathbb{P}(C_{n,r}) = \mathbb{P}(\bigcup_{k=n}^{\infty} A_{k,\frac{1}{r}}) \geq \mathbb{P}(A_{n,\frac{1}{r}})$

$\mathbb{P}(C_{n,r}) \xrightarrow{n\to\infty} 0$, $\mathbb{P}(A_{n,\frac{1}{r}}) \xrightarrow{n\to\infty} 0$

б) Имаме, че $\lim_{n\to\infty} \mathbb{P}(A_{n,\varepsilon}) = 0, \forall \varepsilon > 0$. Целим да док., че $\forall x \in C_{F_X}$, $\lim_{n\to\infty} F_{X_n}(x) = F_X(x)$

$\mathbb{P}(X_n \leq x) \xrightarrow[n\to\infty]{} \mathbb{P}(X \leq x)$. Нека фиксираме $\varepsilon > 0$

$\mathbb{P}(X_n \leq x) = \underbrace{\mathbb{P}(X_n \leq x; A_{n,\varepsilon})}_{\leq \mathbb{P}(A_{n,\varepsilon}) \xrightarrow{n\to\infty} 0} + \mathbb{P}(X_n \leq x; A^c_{n,\varepsilon}) = \mathbb{P}(X_n \leq x; A^c_{n,\varepsilon}) = \mathbb{P}(X_n \leq x \cap |X_n - X| \leq \varepsilon)$

$\lim_{n\to\infty} \mathbb{P}(X_n \leq x) = \lim_{n\to\infty} \mathbb{P}(X_n \leq x; A^c_{n,\varepsilon}) \leq \mathbb{P}(X \leq x + \varepsilon)$ $(\{X_n \leq x \cap A^c_{n,\varepsilon}\} \subseteq \{X \leq x + \varepsilon\})$

$\lim_{n\to\infty} \mathbb{P}(X_n \leq x) = F_X(x+\varepsilon)$, вярно $\forall \varepsilon > 0$

$\lim_{n\to\infty} \mathbb{P}(X_n \leq x) = \lim_{\varepsilon\to 0} F_X(x+\varepsilon) = F_X(x)$

Цел: $\lim_{n\to\infty} \mathbb{P}(X_n \leq x) \geq F_X(x)$

$\varepsilon > 0$: $\mathbb{P}(X_n \leq x) \geq \mathbb{P}(X_n \leq x, A^c_{n,\varepsilon}) \geq \mathbb{P}(X \leq x - \varepsilon; A^c_{n,\varepsilon})$

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Твърдение) Нека $X_n \xrightarrow[n\to\infty]{d} c$ и $\{X_n = (Y_1, \dots, Y_n)$ са деф. в едно вер. пр-во.

Тогава $X_n \xrightarrow[n\to\infty]{\mathbb{P}} c$

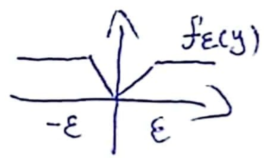
Доказателство) $X_n \xrightarrow[n\to\infty]{d} c \Rightarrow \lim\limits_{n\to\infty} \mathbb{E}[f(X_n)] = \mathbb{E}[f(X)]$, $\forall$ опр. и неогр. ф-я f

$X_n \xrightarrow[n\to\infty]{\mathbb{P}} c$ или $X_n - c \xrightarrow[n\to\infty]{\mathbb{P}} 0 \Rightarrow \forall \varepsilon > 0 \ \lim\limits_{n\to\infty} \mathbb{P}(|X_n - c| > \varepsilon) = 0$

Ако $X = c$, тогава е валидно, че $X_n - c \xrightarrow[n\to\infty]{d} 0$.

$Y_n = X_n - c \xrightarrow[n\to\infty]{d} 0$, то $Y_n \xrightarrow[n\to\infty]{\mathbb{P}} 0$

$\lim\limits_{n\to\infty} \mathbb{E}[f(Y_n)] = \mathbb{E}[f(Y)] = f(0)$



$f_\varepsilon(y) = \min(|y|, \varepsilon)$

$\mathbb{E}[\min(|Y_n|, \varepsilon)] \xrightarrow[n\to\infty]{} 0 = f_\varepsilon(0)$

$\mathbb{E}[\min(|Y_n|, \varepsilon) \cdot \mathbb{1}_{\{|Y_n| > \varepsilon\}}] + \mathbb{E}[\min(|Y_n|, \varepsilon) \cdot \mathbb{1}_{\{|Y_n| \le \varepsilon\}}]$

$\varepsilon \cdot \mathbb{P}(|Y_n| > \varepsilon) + \mathbb{E}[|Y_n|] \cdot \mathbb{1}_{\{|Y_n| \le \varepsilon\}} \ge \varepsilon \cdot \mathbb{P}(|Y_n| > \varepsilon) \xrightarrow[n\to\infty]{} 0$

$\Rightarrow Y_n \xrightarrow{\mathbb{P}} 0$

$\quad \ \|$

$X_n - c \xrightarrow{\mathbb{P}} 0$

Неравенство на Чебишов

Теорема (Чебишев)

Нека X е сл. вел. с очакване $\mathbb{E}[X]<\infty$ и дисперсия $\mathbb{D}[X]<\infty$. Тогава

$$\mathbb{P}(|X-\mathbb{E}[X]|>a) \le \frac{\mathbb{D}[X]}{a^2}$$

Доказателство

$$\mathbb{D}[X] = \mathbb{E}[(X-\mathbb{E}[X])^2].1 = \mathbb{E}[(X-\mathbb{E}[X])^2].1_{\{|X-\mathbb{E}[X]|>a\}} + \mathbb{E}[(X-\mathbb{E}[X])^2].1_{\{X-\mathbb{E}[X]\le a\}}$$

$$\ge \mathbb{E}[(X-\mathbb{E}[X])^2].1_{\{|X-\mathbb{E}[X]|>a\}} \ge a^2\,\mathbb{E}[1.\{|X-\mathbb{E}[X]|\ge a\}] = a^2\,\mathbb{P}(|X-\mathbb{E}[X]|>a)$$

Твърдение Нека X е сл. вел., т.ч. $\mathbb{E}[|X|^n]<\infty$ и цяло число. Тогава $\forall a>0$

$$\mathbb{P}(|X|>a) \le \frac{\mathbb{E}[|X|^n]}{a^n} \quad \text{и} \quad \mathbb{P}(|X-\mathbb{E}[X]|>a) \le \frac{\mathbb{E}[(X-\mathbb{E}[X])^n]}{a^n}$$

Деф. (Закон за големите числа - слаб)

Нека $\bar{X}=(X_1,\dots,X_n)$ е редица от сл. вел. в едно вер. пр-во и $\mathbb{E}[X_i]<\infty, \forall i=\overline{1,n}$

Тогава за $\bar{X}$ е в сила слаб закон за големите числа, ако

$$\frac{\sum_{i=1}^{n}(X_i-\mathbb{E}[X_i])}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

Теорема Нека $\bar{X}=(X_1,\dots,X_n), n\ge 1$ е редица от нез. и едн. разпр-сл. вел. в едно вер. пр-во

Ако $\mathbb{E}[X_i]<\infty$, то $\frac{\sum_{i=1}^{n}X_i}{n} \xrightarrow[n\to\infty]{\mathbb{P}} \mathbb{E}[X_1]$

Доказателство $\mathbb{E}[X_i]=\mathbb{E}[X_1], \forall i=\overline{1,n}$

$$\frac{\sum_{i=1}^{n}(X_i-\mathbb{E}[X_i])}{n} = \frac{\sum_{i=1}^{n}(X_i)-n\,\mathbb{E}[X_1]}{n} = \frac{\sum_{i=1}^{n}X_i}{n} - \mathbb{E}[X_1]$$

$$\mathbb{D}[X_i]=\mathbb{D}[X_1]<\infty$$

$$\frac{\sum_{i=1}^{n}(X_i-\mathbb{E}[X_i])}{n} = \frac{\sum_{i=1}^{n}X_i}{n} - \mathbb{E}[X_1] \xrightarrow[n\to\infty]{\mathbb{P}} 0$$

$X_i-\mathbb{E}[X_i]=Y_i \Rightarrow \frac{\sum_{i=1}^{n}Y_i}{n} \xrightarrow[n\to\infty]{\mathbb{P}} 0$ Цел: $\forall\varepsilon>0, \mathbb{P}\left(\left|\frac{\sum_{i=1}^{n}Y_i}{n}\right|>\varepsilon\right) \overset{?}{\to} 0$

$$\lim_{n\to\infty}\mathbb{P}\left(\left|\sum_{i=1}^{n}Y_i\right|>n\varepsilon\right) = \lim_{n\to\infty}\mathbb{P}\left(\left|\sum_{i=1}^{n}Y_i-\mathbb{E}\sum_{i=1}^{n}Y_i\right|>\varepsilon\right) \le \lim_{n\to\infty}\frac{\mathbb{D}[\sum_{i=1}^{n}Y_i]}{n^2\varepsilon^2} =$$

$$= \lim_{n\to\infty}\frac{\sum_{i=1}^{n}\mathbb{D}[Y_i]}{\varepsilon^2 n^2} = \lim_{n\to\infty}\frac{n\,\mathbb{D}[Y_1]}{\varepsilon^2 n^2} = 0$$

Деф.) (ЗГЧ - силен)

Нека $\bar{X}=(X_1,\dots,X_n)$ е редица от сл. вел. деф. в едно вер. пр-во. Тогава, ако $\mathbb{E}[X_i]<\infty$, $\forall i=\overline{1,n}$, казваме, че редицата удовлетворява УЗГЧ, ако

$$\frac{\sum_{i=1}^{n}(X_i-\mathbb{E}[Y_i])}{n}\xrightarrow[n\to\infty]{п.с.}0$$

Ако $\bar{X}$ е редица от едн. разпр., то тя има УЗГЧ, ако

$$\frac{\sum_{i=1}^{n}X_i}{n}\xrightarrow[n\to\infty]{п.с.}\mathbb{E}[X_1]$$

Теорема) (УЗГЧ)

Нека $\bar{X}$ е редица от нез. и едн. разпр. сл. вел., т.ч. $\mathbb{E}[X_1]<\infty$. Тогава за редицата е валиден УЗГЧ, т.е. $\lim_{n\to\infty}\frac{\sum_{i=1}^{n}X_i}{n}\overset{п.с.}{=}\mathbb{E}[X_1]$

Доказателство) $\mathbb{E}[X_1]<\infty$ и $\mathbb{E}[X_i]\neq 0$

$Y_i=X_i-\mathbb{E}[X_i]$

$$L=\{w\in\Omega:\lim_{n\to\infty}\frac{\sum_{i=1}^{n}X_i(w)}{n}=0\}=\bigcap_{r=1}^{\infty}\bigcup_{n=1}^{\infty}\bigcap_{k=n}^{\infty}\{w\in\Omega:\left|\frac{\sum_{i=1}^{k}X_i(w)}{k}\right|\le\frac{1}{r}\}$$

Цел.: $\mathbb{P}(L^c)=0\Rightarrow\mathbb{P}(L=1)$

$$L^c=\bigcup_{r=1}^{\infty}\underbrace{\bigcap_{n=1}^{\infty}\bigcup_{k=n}^{\infty}\{w\in\Omega:\left|\frac{\sum_{i=1}^{n}X_i}{k}\right|\ge\frac{1}{r}\}}_{B_{n,r}}=\bigcup_{r=1}^{\infty}\bigcap_{n=1}^{\infty}B_{n,r}$$

$$\mathbb{P}(L^c)=\mathbb{P}(\bigcup_{r=1}^{\infty}\bigcap_{n=1}^{\infty}B_{n,r})\le\sum_{r=1}^{\infty}\mathbb{P}(\bigcap_{n=1}^{\infty}B_{n,r})=\sum_{r=1}^{\infty}\lim_{n\to\infty}\mathbb{P}(B_{n,r})=\sum_{r=1}^{\infty}0=0$$

$$\mathbb{P}(\bigcap_{n=1}^{\infty}B_{n,r})=\lim_{n\to\infty}\mathbb{P}(B_{n,r}),\quad B_{n+1,r}\subseteq B_{n,r}$$

$$\mathbb{P}(B_{n,r})=\mathbb{P}(\bigcup_{k=n}^{\infty}\{w\in\Omega:\left|\frac{\sum_{i=1}^{k}X_i(w)}{k}\right|>\frac{1}{r}\})\le\sum_{k=n}^{\infty}\mathbb{P}(|\sum_{i=1}^{k}X_i|\ge\frac{k}{r})\overset{(1)}{\le}\sum_{k=n}^{\infty}\frac{r^4\,\mathbb{E}[|\sum_{i=1}^{k}X_i|^4]}{k^4}$$

Уточнение (1): От следствието на нер. на Чебишов, приложено за $n=4$, $a=\frac{k}{r}$

$$\mathbb{E}[(\sum_{i=1}^{n}X_i)^4]\overset{iid}{=}k.\mathbb{E}[X_1]^4+c\binom{n}{k}\mathbb{E}[X_1^2]$$

$$\Rightarrow\mathbb{P}(B_{n,r})=r^4\sum_{k=n}^{\infty}\frac{k\mathbb{E}[X_1^4]+c\binom{n}{k}\mathbb{E}[X_1^2]}{k^4}\le r^4\mathbb{E}[X_1^4]\sum_{k=n}^{\infty}\frac{1}{k^3}+\frac{cr^4}{2}\sum_{k=n}^{\infty}\frac{k(k-1)}{k^4}$$

Понеже $\sum_{k=1}^{\infty}\frac{1}{k^3}$ е сходящ ред, то $\sum_{k=n}^{\infty}\frac{1}{k^3}\xrightarrow[n\to\infty]{}0$ и $\sum_{k=n}^{\infty}\frac{k(k-1)}{k^4}\le\sum_{k=n}^{\infty}\frac{1}{k^2}\xrightarrow[n\to\infty]{}0$

Получаваме: $0\le\lim_{n\to\infty}\mathbb{P}(B_{n,r})\le r^4.\mathbb{E}[X_1^4].0+\frac{cr^4}{2}.0=0$

Тогава $\mathbb{P}(L^c)=0$ и $\mathbb{P}(L)=1$

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