Експоненциално разпределена НСВ

Деф. Казваме, че $X \sim Exp(\lambda)$, $\lambda > 0$, ако X има плътност от вида

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} &, x \geq 0 \\ 0 &, x < 0 \end{cases}$$

Проверка:

$$\underline{1} = \int_{-\infty}^{\infty} f_X(x)\,dx = \int_0^{\infty} \lambda e^{-\lambda x}\,dx \overset{u = \lambda x}{\underset{\frac{du}{dx} = \lambda}{=}} \int_0^{\infty} e^{-u}\,du = \left[-e^{-u}\right]_0^{\infty} = \underline{1}$$

$$F_X(x) = \int_0^{x} \lambda e^{-\lambda y}\,dy \overset{u = \lambda y}{\underset{\frac{du}{dy} = \lambda}{=}} \int_0^{\lambda x} e^{-u}\,du = \left[-e^{-u}\right]_0^{\lambda x} = \underline{1} - e^{-\lambda x} \cdot \underline{1}_{\{x>0\}}$$

$$\overline{F}_X(x) = \mathbb{P}(X > x) = e^{-\lambda x} \cdot 1_{\{x>0\}}$$

$$\mathbb{E}[X] = \int_0^{\infty} x \cdot \lambda \cdot e^{-\lambda x}\,dx \overset{x = \frac{u}{\lambda},\ u = \lambda x}{\underset{\frac{du}{dx} = \lambda}{=}} \int_0^{\infty} \frac{u}{\lambda} e^{-u}\,du = \frac{1}{\lambda} \int_0^{\infty} -u\,de^{-u} = \frac{1}{\lambda}\Big[\underbrace{\left[-ue^{-u}\right]_0^{\infty}}_{0} + \int_0^{\infty} e^{-u}\,du\Big] =$$

$$= \frac{1}{\lambda}\left[-e^{-u}\right]_0^{\infty} = \frac{1}{\lambda}$$

$$\mathbb{E}[X^2] = \int_0^{\infty} x^2 \cdot \lambda \cdot e^{-\lambda x}\,dx \overset{x = \frac{u}{\lambda},\ u = \lambda x}{\underset{\frac{du}{dx} = \lambda}{=}} \int_0^{\infty} \frac{u^2}{\lambda^2} e^{-u}\,du = \frac{1}{\lambda^2} \int_0^{\infty} -u^2\,de^{-u} = \frac{1}{\lambda^2}\Big[\underbrace{\left[-u^2 e^{-u}\right]_0^{\infty}}_{0} + \int_0^{\infty} e^{-u}\,du^2\Big] =$$

$$= \frac{1}{\lambda^2} \int_0^{\infty} 2ue^{-u}\,du = \frac{2}{\lambda^2} \underbrace{\int_0^{\infty} ue^{-u}\,du}_{1} = \frac{2}{\lambda^2}$$

$$\mathbb{D}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$$

Безпаметност $\forall y, x > 0$, $\mathbb{P}(X > x + y \mid X > y) = \mathbb{P}(X > x)$

Доказателство $\mathbb{P}(X > x+y \mid X > y) = \dfrac{\mathbb{P}(X > x+y \cap X > y)}{\mathbb{P}(X > y)} = \dfrac{\mathbb{P}(X > x+y)}{\mathbb{P}(X > y)} = \dfrac{e^{-\lambda(x+y)}}{e^{-\lambda y}} = e^{-\lambda x} \cdot 1_{\{x>0\}} = \mathbb{P}(X > x)$

Съвместно непрекъснато разпределение

Деф. Казваме, че $X=(X_1,\dots,X_n)$ е вектор от Н.С.В., ако $\exists f_X=f_{X_1,\dots,X_n}:\mathbb{R}^n\to[0,\infty)$ такава, че:

а) $f_X(x_1,\dots,x_n)\geq 0,\ \forall(x_1,\dots,x_n)\in\mathbb{R}^n$

б) $\int_{-\infty}^{\infty}\dots\int_{-\infty}^{\infty} f_X(x_1,\dots,x_n)\,dx_1\dots dx_n=1$

в) $D\subseteq\mathbb{R}^n$ $\quad P(X\in D)=\int_D f_X(x_1,\dots,x_n)\,dx_1\dots dx_n$

Деф. (Носител на сл. вел.)

Нека f_X е съвм. плътност на X. Тогава $D_{f_X}=\{(x_1,\dots,x_n)\in\mathbb{R}^n: f_X(x_1,\dots,x_n)>0\}$ се нарича носител на X

Деф. (Маргинална плътност)

Нека f_X е съвм. плътност на $X=(X_1,\dots,X_n)$. Тогава:

$$f_{X_j}(x_j)=\int_{x_1=-\infty}^{\infty}\dots\int_{x_{j-1}=-\infty}^{\infty}\int_{x_{j+1}=-\infty}^{\infty}\dots\int_{x_n=-\infty}^{\infty} f_X(x_1,\dots,x_{j-1},x_j,x_{j+1},\dots,x_n)\,dx_1\dots dx_{j-1}dx_{j+1}\dots dx_n$$

е маргиналната плътност на X_j за някое $j=\overline{1,n}$

Деф. (Условна плътност, $n=2$)

Нека f_X е съвм. плътност на $X=(X_1,X_2)$. Тогава, ако $f_{X_1}(x_1)>0$,

$$f_{X_2|X_1}(x_2|x_1)=\frac{f_X(x_1,x_2)}{f_{X_1}(x_1)}$$ е условна плътност на X_n при X_1

Деф. Нека $X=(X_1,\dots,X_n)$ е в-р от сл. вел. Тогава $F_X(x_1,\dots,x_n)=\mathbb{P}(X_1\leq x_1;X_2\leq x_2;\dots;X_n\leq x_n)$ е съвм. ф-я на разпределение.

Ако f_X съществува, $F_X(x_1,\dots,x_n)=\int_{y_1=0}^{x_1}\int_{y_2=0}^{x_2}\dots\int_{y_n=0}^{x_n} f_X(y_1,\dots,y_n)\,dy_1\dots dy_n$

$\frac{\partial^n}{\partial x_1\dots\partial x_n}F_X=f_X$

Деф. Казваме, че $X_1 \perp\!\!\!\perp X_2$, ако $F_X(x_1,x_2)=F_{X_1}(x_1)F_{X_2}(x_2)$ $\quad (X=(X_1,X_2))$

$\mathbb{P}(X_1\leq x_1;X_2\leq x_2)=\mathbb{P}(X_1\leq x_1)\mathbb{P}(X_2\leq x_2),\ \forall(x_1,x_2)\in\mathbb{R}^2$

Също така, ако $\forall(x_1,x_2)\in\mathbb{R}^2$: $f_X(x_1,x_2)=f_X(x_1)f_X(x_2)$, то $X_1 \perp\!\!\!\perp X_2$

Деф. (Независимост в съвкупност)

Нека $X = (X_1, \dots, X_n)$ е в-р от непр. сл. вел. Тогава казваме, че $X_1, \dots, X_n$ са независими в съвкупност, ако $f_X(x_1, \dots, x_n) = \prod_{j=1}^{n} f_{X_j}(x_j), \forall (x_1, \dots, x_n)$

Нека X има f_X и $g: \mathbb{R}^n \to \mathbb{R}$, то

$$\mathbb{E}[g(X)] = \int_{x_1=-\infty}^{\infty} \dots \int_{x_n=-\infty}^{\infty} g(x_1, \dots, x_n) f_X(x_1, \dots, x_n)\, dx_1 \dots dx_n$$

Следствие Нека $X = (X_1, X_2)$ е вектор от НСВ, т.ч. $\mathbb{E}[X_1] < \infty$ и $\mathbb{E}[X_2] < \infty$

Тогава $\mathbb{E}[X_1 + X_2] = \mathbb{E}[X_1 + X_2]$

Доказателство $g(x_1, x_2) = x_1 + x_2$

$$\mathbb{E}[X_1 + X_2] = \int_{x_1=-\infty}^{\infty} \int_{x_2=-\infty}^{\infty} (x_1 + x_2) f_X(x_1, x_2)\, dx_2\, dx_1 = \int_{x_1=-\infty} x_1 \underbrace{\int_{x_2=-\infty} f_X(x_1, x_2)\, dx_2}_{f_{X_1}(x_1)} dx_1 + \int_{x_2=-\infty} x_2 \underbrace{\int_{x_1=-\infty} f_X(x_1, x_2)\, dx_1}_{f_{X_2}(x_2)} dx_2 =$$

$$= \int_{x_1=-\infty}^{\infty} f_{X_1}(x_1)\, dx_1 + \int_{x_2=-\infty}^{\infty} f_{X_2}(x_2)\, dx_2 = \mathbb{E}[X_1] + \mathbb{E}[X_2]$$

Следствие Нека $X = (X_1, X_2)$ е в-р от НСВ, т.ч. $X_1 \perp\!\!\!\perp X_2 \Rightarrow \mathbb{D}[X_1 + X_2] = \mathbb{D}[X_1] + \mathbb{D}[X_2]$

Незав. в съвкупност $(X_1, \dots, X_n) \Rightarrow \mathbb{D}[X_1 + \dots + X_n] = \sum_{j=1}^{n} \mathbb{D}[X_j]$

Теорема (Смяна на променливите)

Нека $X=(X_1,X_2)$ е в-р от НСВ и има $g: \mathbb{R}^2 \to \mathbb{R}^2$

Нека f_X е съвм. плътност.

Нека $Y=(Y_1,Y_2)=g(X)=(g_1(x_1,x_2), g_2(x_1,x_2))$

Нека $g: \mathcal{D}_{f_X} \to \mathbb{R}^2$ е взаимно еднозначна

Нека $g(\mathcal{D}_{f_X}) = \{ y \in \mathbb{R}^2 : y = g(x) \text{ за } x \in \mathcal{D}_{f_X} \}$

Нека $X = h(y) = (h_1(y_1,y_2), h_2(y_1,y_2))$, където $h = g^{-1}$

Нека h и g са непр. и диф. съответно в $g(\mathcal{D}_{f_X})$ и $\mathcal{D}_{f_X}$.

Нека $J(y) = \left| \det \begin{pmatrix} \frac{\partial}{\partial y_1} h_1(y) & \frac{\partial}{\partial y_1} h_2(y) \\ \frac{\partial}{\partial y_2} h_1(y) & \frac{\partial}{\partial y_2} h_2(y) \end{pmatrix} \right| \neq 0, \forall y \in g(\mathcal{D}_{f_X})$

Тогава съвм. пл. на Y е $f_Y(y) = f_X(h(y))|J(y)|$, $\forall y \in g(\mathcal{D}_{f_X})$

Доказателство $A \subseteq g(\mathcal{D}_{f_X})$ и искаме да потвърдим, че $P(Y \in A) = \iint_A f_Y(y_1,y_2)dy_1dy_2$

$$\mathbb{P}(Y\in A) = \mathbb{P}(g(X)\in A) = \mathbb{P}(X \in h(A)) = \int_{X\in h(A)} f_X(x)dx \overset{x=h(y)}{=\!=\!=} \int_{y\in A} f_X(h(y))|J(y)|dy \Rightarrow$$

$\Rightarrow f_X(h(y))|J(y)|$ е плътността на Y.

⊕ Нека $V_1, V_2, \dots, V_n$ са независими в съвкупност НСВ, $V_i \sim N(\mu_i, \sigma_i^2)$. Тогава $\sum_{i=1}^n V_i \sim N(\sum_{i=1}^n \mu_i, \sum_{i=1}^n \sigma_i^2)$. Ще покажем, че е изпълнено за $n=2$. От мат. индукция следва $\forall n \in \mathbb{N}$.

$V_1 + V_2 = \mu_1 + \sigma_1 . Z + \mu_2 + \sigma_2 . Z$, където $Z_1, Z_2 \sim N(0,1)$ и $Z_1 \perp\!\!\!\perp Z_2 \Rightarrow V_1+V_2 = (\mu_1+\mu_2) + \sigma_1 Z_1 + \sigma_2 Z_2$

Поставяме $X_1 = \sigma_1 Z_1 \sim N(0, \sigma_1^2)$ и $X_2 = \sigma_2 Z_2 \sim N(0,\sigma_2^2) \Rightarrow V_1+V_2 = \mu_1+\mu_2+X_1+X_2$

Ако $X_1+X_2 \sim N(0, \sigma_1^2+\sigma_2^2)$, то $V_1+V_2 \sim N(\mu_1+\mu_2, \sigma_1^2+\sigma_2^2)$

$$f_{X_1,X_2}(x_1,x_2) = f_{X_1}(x_1) f_{X_2}(x_2) = \frac{1}{\sqrt{2\pi}\sigma_1} . e^{-\frac{x_1^2}{2\sigma_1^2}} . \frac{1}{\sqrt{2\pi}\sigma_2} . e^{\frac{-x_2^2}{2\sigma_2^2}} > 0$$

$$\left. \begin{array}{l} y_1 = x_1 + x_2 \\ y_2 = x_2 \end{array} \right\} \Rightarrow y = g(x) \Leftrightarrow X = h(y) = (y_1 - y_2, y_2) \qquad J(y) = \left| \det \begin{vmatrix} 1 & -1 \\ 0 & 1 \end{vmatrix} \right| = 1$$

$$f_Y(y) = f_X(h(y)) . 1 = \frac{1}{2\pi\sigma_1\sigma_2} e^{\frac{-(y_1-y_2)^2}{2\sigma_1^2} - \frac{y_2^2}{2\sigma_2^2}}, \forall y \in \mathbb{R}^2$$

$$f_{Y_1}(y_1) = \frac{1}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} e^{\frac{-(y_1-y_2)^2}{2\sigma_1^2} - \frac{y_2^2}{2\sigma_2^2}} dy_2 \overset{\frac{(y_1-y_2)^2}{\sigma_1^2} + \frac{y_2^2}{\sigma_2^2} = cy_1^2 + (by_2 - ay_1)^2}{=\!=\!=\!=\!=} \frac{e^{-\frac{y_1^2 c}{2}}}{2\pi\sigma_1\sigma_2} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(by_2 - ay_1)^2} dy_2 \overset{by_2 = w}{=\!=\!=}$$

$$= \frac{e^{-\frac{y_1^2 c}{2}}}{2\pi\sigma_1\sigma_2 b} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(w-ay_1)^2} dw \overset{v=w-ay_1}{=\!=\!=} \frac{e^{-\frac{y_1^2 c}{2}}}{2\pi\sigma_1\sigma_2 b} \underbrace{\int_{-\infty}^{\infty} e^{\frac{1}{2}v^2} dv}_{\text{Гаусов} \Rightarrow \sqrt{2\pi}} = \frac{e^{-\frac{y_1^2 c}{2}}}{\sqrt{2\pi}\sigma_1\sigma_2 b}$$

Сега да намерим c и b

$$\frac{(y_1-y_2)^2}{\sigma_1^2}+\frac{y_2^2}{\sigma_2^2}=c\,y_1^2+(b y_2-a y_1)^2 \iff \frac{y_1^2}{\sigma_1^2}-\frac{2y_1y_2}{\sigma_1^2}+\frac{y_2^2}{\sigma_1^2}+\frac{y_2^2}{\sigma_2^2}=y_2^2\underbrace{\left(\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)}_{=b^2}-\frac{2y_1y_2}{\sigma_1^2}+\frac{y_1^2}{\sigma_1^2}$$

$$=\left(y_2^2b^2-\frac{2y_1y_2}{\sigma_1^2}\cdot\frac{b}{b}+\frac{y_1^2}{\sigma_1^4b^2}\right)-\frac{y_1^2}{\sigma_1^4b^2}+\frac{y_1^2}{\sigma_1^2}=\left(by_2-\frac{y_1}{\sigma_1^2b}\right)^2-\frac{y_1^2}{\sigma_1^4b^2}+\frac{y_1^2}{\sigma_1^2}$$

$$=\left(by_2-\underbrace{\frac{y_1}{\sigma_1^2b}}_{=a\cdot y_1}\right)^2+y_1^2\underbrace{\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_1^4b^2}\right)}_{=c}$$

$$b^2=\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}=\frac{\sigma_1^2+\sigma_2^2}{\sigma_1^2\sigma_2^2}$$

$$a^2=\frac{1}{\sigma_1^4b^2}=\frac{\sigma_2^2}{\sigma_1^2(\sigma_1^2+\sigma_2^2)}$$

$$c=\frac{1}{\sigma_1^2}-\frac{1}{\sigma_1^4b^2}=\frac{1}{\sigma_1^2}-\frac{\sigma_2^2}{\sigma_1^2(\sigma_1^2+\sigma_2^2)}=\frac{\cancel{\sigma_1^2}}{\cancel{\sigma_1^2}(\sigma_1^2+\sigma_2^2)}=\frac{1}{\sigma_1^2+\sigma_2^2}$$

$$\Rightarrow f_{Y_1}(y_1)=\frac{e^{-\frac{y_1^2}{2}\cdot c}}{\sqrt{2\pi}\,\sigma_1\sigma_2\,b}=\frac{e^{-\frac{y_1^2}{2}\cdot\frac{1}{\sigma_1^2+\sigma_2^2}}}{\sqrt{2\pi}\,\cancel{\sigma_1\sigma_2}\,\frac{\sqrt{\sigma_1^2+\sigma_2^2}}{\cancel{\sigma_1\sigma_2}}}=\frac{1}{\sqrt{2\pi(\sigma_1+\sigma_2)}}\cdot e^{-\frac{y_1^2}{2(\sigma_1^2+\sigma_2^2)}}$$ – плътност на $N(0,\sigma_1^2+\sigma_2^2)$

$$\Rightarrow Y_1=X_1+X_2\sim N(0,\sigma_1^2+\sigma_2^2)$$

$\square$