Доверителни интервали

Постановка: X, $F_X(x,\theta)$, $\theta \in \mathbb{R}$ и $\vec{X} \Rightarrow \hat{\theta} = \hat{\theta}(\vec{X})$

$I_1 = I_1(\vec{X})$; $I_2 = I_2(\vec{X})$

$\mathbb{P}(I_1 \le \theta \le I_2) = \gamma$ -> ниво на доверие ($\gamma = 0.95, 0.99, 0.9$)

(I_1, I_2) се нарича дов. инт. с ниво на дов. γ.

Деф. (**Централна статистика**)

$T = T(\vec{X}, \theta)$ е ф-я, тогава T е централна статистика, ако:

а) T е монотонна по θ

б) $\mathbb{P}(T < x) = H(x)$ не зависи от θ, $\forall x \in \mathbb{R}$

(+) Нека T е централна статистика и $T \uparrow$ по θ. Търсим q_1, q_2, т.ч.

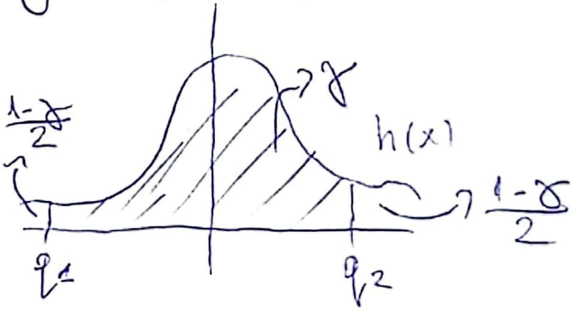
$$\gamma = \mathbb{P}(q_1 < T < q_2) = \mathbb{P}(T < q_2) - \mathbb{P}(T \le q_1) = \mathbb{P}(T^{-1}(q_1) < \theta < T^{-1}(q_2))$$

$$I_1 = T^{-1}(q_1);\ I_2 = T^{-1}(q_2) \Rightarrow \mathbb{P}(\theta \in (I_1, I_2) = H(q_2) - H(q_1)$$

Избираме q_1 и q_2, т.ч. $H(q_2) - H(q_1) = \gamma$

Имаме 2 свободни параметъра

$H'(x) = h(x)$



Най-често q_1 е $\frac{1-\gamma}{2}$ квартила

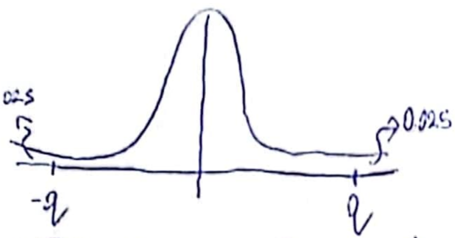
q_2 е $\frac{1-\gamma}{2}$ квартила

⊕ $X \sim N(\mu, \underset{\text{известно}}{\sigma^2})$; $\vec{X} = (X_1, \dots, X_n)$

$T = \frac{\bar{X}_n^{(1)} - \mu}{\sigma\sqrt{n}} \sim N(0,1)$; T е монот. ↓ по μ

$H(x) = P(T < x) = \Phi(x)$

$\bar{X}_n^{(1)} \sim N(\mu, \frac{\sigma^2}{n})$



$\gamma = 0.95 = P(-q < T < q) = P\left(-q < \frac{\bar{X}_n^{(1)} - \mu}{\sigma/\sqrt{n}} < q\right) = P\Big(\underbrace{\bar{X}_n^{(1)} - \frac{\sigma}{\sqrt{n}} q}_{I_1} < \mu < \underbrace{\bar{X}_n^{(1)} + \frac{\sigma}{\sqrt{n}} q}_{I_2}\Big) =$

$I_1 = \bar{X}_n^{(1)} - \frac{\sigma}{\sqrt{n}} \cdot q$ ✓

$I_2 = \bar{X}_n^{(1)} + \frac{\sigma}{\sqrt{n}} \cdot q$ ✓ $\qquad I_2 - I_1 = \frac{2q\sigma}{\sqrt{n}}$

$\mu \in \left(\bar{X}_n^{(1)} - \frac{\sigma}{\sqrt{n}} q_{0.025}, \bar{X}_n^{(1)} + \frac{\sigma}{\sqrt{n}} \cdot q_{0.975}\right)$

⊕ $X \sim N(\mu, \underset{\text{неизв.}}{\sigma^2})$ Как да конструираме Ц.С. за μ?

$\hat{\mu} = \bar{X}_n^{(1)}$; $\hat{\sigma}^2 = \sum_{j=1}^{n} \frac{(X_j - \bar{X}_n^{(1)})^2}{n\sigma^2} = \frac{n-1}{n} \frac{S^2}{\sigma^2}$

Твърдение $\hat{\mu} \perp\!\!\!\perp S^2$ $\quad \cancel{n}^{(n-1)} \frac{S^2}{\sigma^2} \sim \chi^2(n-1)$, $S^2 = \frac{1}{n-1} \sum_{j=1}^{n} (X_j - \bar{X}_n^{(1)})$

Доказателство $\sum_{j=1}^{n} Z_j^2 = \sum_{j=1}^{n} \left(\frac{X_j - \mu}{\sigma}\right)^2 =: U \sim \chi^2(n)$

$Z_j = \frac{X_j - \mu}{\sigma} \sim N(0,1)$; $U = (n-1) \sum_{j=1}^{n} \frac{(X_j - \bar{X}_n^{(1)})^2}{\sigma^2 (n-1)} + n \frac{(\bar{X}_n^{(1)} - \mu)^2}{\sigma^2} = (n-1)\frac{S^2}{\sigma^2} + \frac{n(\bar{X}_n^{(1)} - \mu)^2}{\sigma^2}$

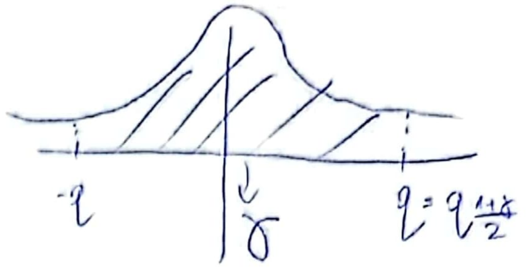
$U \in \chi^2(n) = (n-1)\frac{S^2}{\sigma^2} + \underbrace{\left(\frac{\bar{X}_n^{(1)} - \mu}{\sigma/\sqrt{n}}\right)^2}_{\chi^2(1)} \overset{S^2 \perp\!\!\!\perp \hat{\mu}}{\Longrightarrow} (n-1)\frac{S^2}{\sigma^2} \in \chi^2(n-1)$

$\Rightarrow T_n = \frac{\bar{X}_n^{(1)} - \mu}{S/\sqrt{n}} = \frac{\sqrt{n}\frac{\bar{X}_n^{(1)} - \mu}{\sigma}}{\frac{S}{\sigma}} = \frac{\frac{\bar{X}_n^{(1)} - \mu}{\sigma/\sqrt{n}}}{\sqrt{(n-1)\frac{S^2}{\sigma^2} \frac{1}{n-1}}} = \frac{Z_n}{\sqrt{\frac{U_n}{n-1}}} \in t(n-1)$

От Тв. $Z_n \perp\!\!\!\perp U_n$, $U_n \sim \chi^2(n-1)$ и $Z_n \sim N(0,1)$

$T = \frac{Z_n}{\sqrt{\frac{U_n}{n-1}}} = \frac{\bar{X}_n^{(1)} - \mu}{S/\sqrt{n}} \in t(n-1) \Rightarrow T_n$ е Ц.С. за μ

$\gamma = \mathbb{P}(I_1 < \mu < I_2) = \mathbb{P}(-q < T_n < q)$



$I_1 = \overline{X}_n^{(1)} - q_{\frac{1+\gamma}{2}} \cdot \frac{S}{\sqrt{n}}$

$I_2 = \overline{X}_n^{(1)} + q_{\frac{1+\gamma}{2}} \frac{S}{\sqrt{n}}$

$T_n = \dfrac{Z_n}{\sqrt{\frac{U_n}{n-1}}} \in t(n-1)$, $n \geq 30$ $T_n \approx Z_n \sim N(0,1)$

$U_n = \chi^2(n-1) = \sum_{j=1}^{n-1} \chi_j^2 \qquad \chi_j \sim \chi^2(1)$

$\frac{U_n}{n-1} \xrightarrow[n\to\infty]{\text{п.с.}} \mathbb{E}[\chi_1] = \mathbb{E}[Z_1^2] = 1$

Ако знаем $\mathbb{D}[X] = \sigma^2$

$T_n = \dfrac{\overline{X}_n^{(1)} - \mu}{\sigma/\sqrt{n}} \overset{ЦГТ}{\approx} N(0,1)$ $(X_1, \dots, X_n)$

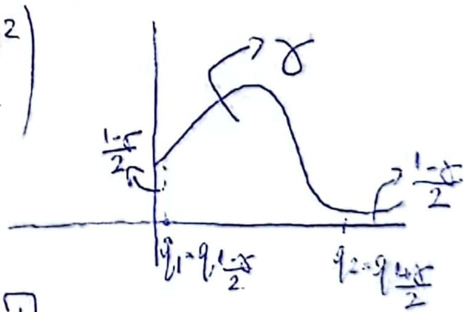
$\mathbb{E}[X] = \mu$

$\mathbb{D}[X] = \sigma^2$

$\overline{X}_n^{(1)} - q_{\frac{1+\gamma}{2}} \frac{\sigma}{\sqrt{n}} < \mu < \overline{X}_n^{(1)} + q_{\frac{1+\gamma}{2}} \frac{\sigma}{\sqrt{n}}$

$\oplus$ $X \sim N(\underbrace{\mu}_{\text{изв.}}, \sigma^2)$ $\hat{\sigma}^2 = \sum_{j=1}^{n} \frac{(X_j-\mu)^2}{n} \Rightarrow \frac{n\hat{\sigma}^2}{\sigma^2} = \sum_{j=1}^{n} \frac{(X_j-\mu)^2}{\sigma^2} \sim \chi^2(n)$

$\gamma = \mathbb{P}\left(q_1 < \frac{n\hat{\sigma}^2}{\sigma^2} < q_2\right) = \mathbb{P}\left(\underbrace{\frac{\sum_{j=1}^{n}(X_j-\mu)^2}{q_2}}_{I_1} < \sigma^2 < \underbrace{\frac{\sum_{j=1}^{n}(X_j-\mu)^2}{q_1}}_{I_2}\right)$



Ако μ е неизв., то $\frac{n\hat{\sigma}^2}{\sigma^2} = \sum_{j=1}^{n} \frac{(X_j - \overline{X}_n^{(1)})^2}{\sigma^2} \in \chi^2(n-1) = T$

$\frac{n\hat{\sigma}^2}{\sigma^2} = \frac{(n-1)S^2}{\sigma^2}$