Точкови оценки

Постановка Сл. вел. X - описва св-во на някаква генерална съвкупност

$F_X(x) = \mathbb{P}(X<x)$

$f_X(x) = \frac{d}{dx} F_X(x)$ - ако е непр.

Цел $\vec{X} = (X_1, \dots, X_n)$, $\{X_j\}_{j=1}^{n}$ са нез. едн. разпр. сл. в. и $X_j \stackrel{d}{=} X$

На база на $\vec{X}$ искаме да определим (приближим) F_X, f_X

$\vec{x} = (x_1, x_2, \dots, x_n)$ - ст-ст на $\vec{X}$ $\xRightarrow{?!} F_X, f_X$

Допускания X принадлежи на някакъв клас от сл. вел

Пример: $X \sim Ber(p) \Rightarrow X$ има $F_X(x, p)$, $p \in (0,1)$, $\theta = p$ (клас)

$X \sim Exp(\lambda), \lambda > 0 \Rightarrow F_X(x, \lambda) = 1 - e^{-\lambda x}$, $\theta = \lambda$

$X \sim N(\mu, \sigma^2), \mu \in \mathbb{R}, \sigma > 0$, $\theta = (\mu, \sigma^2)$

Имаме: X, $F_X(x,\theta)$, $f_X(x,\theta)$

Цел Имаме X, $F_X(x,\theta)$, $\vec{X} \Rightarrow \hat{\theta}$ ще бъде оценка за θ

Деф. (Точкова оценка)

X от класа $F_X(x,\theta)$. Тогава $\hat{\theta} = \hat{\theta}(\vec{X})$ е точкова оценка за θ

Метод на максималното правдоподобие (м.п.о)

$X, f_X(x;\theta), \vec{X}$

ф-я на максимално правдоподобие наричаме съвм. плътност на $\vec{X}$, т.е.

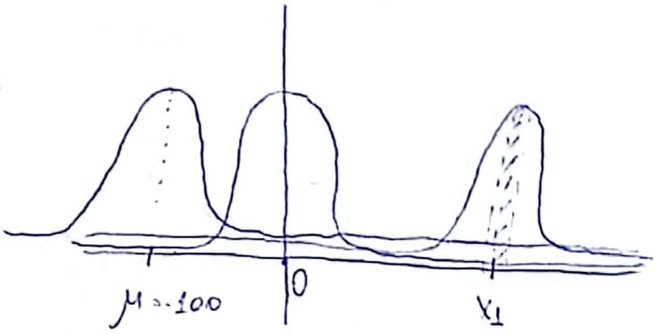
$L(\vec{X}, \theta) = \prod_{j=1}^{n} f_X(x_j;\theta)$

$\vec{x}, L(\vec{x}, \theta) = \prod_{j=1}^{n} f_X(x_j, \theta)$

Пример. $X \sim N(\mu, 1)$; $\vec{X} = X_1$; $X_1 = x_1$

$L(\vec{X}, \mu) = f_X(X_1, \mu)$

$L(x_1, \mu) = f_X(x_1, \mu) \xrightarrow{\text{избираме}} \hat{\mu} = x_1$



Деф. Нека X е сл. вел. с пл. $f_X(x, \theta)$

Нека $\vec{X}$ са n незав. набл. над X

Тогава под м.п.о. за θ разбираме $\hat{\theta}: L(\vec{X}; \hat{\theta}) = \sup_{\theta \in \Theta} L(\vec{X}; \theta)$

Ако f_X е диф. по θ, то $\frac{\partial}{\partial \theta} L = 0$, за да намерим $\hat{\theta}$

$L(\vec{X}, \theta) = \prod_{j=1}^{n} f_X(X_j, \theta)$

$\frac{\partial \ln(L)}{\partial \theta} = 0$ (защото ln е раст. ф-я и работим със сума ln, вместо произв. f)

(+) $X \sim N(\mu, \sigma^2)$ $\quad \theta = (\mu, \sigma^2)$

$$\left.\begin{cases} \frac{\partial}{\partial \mu} \ln(L) = 0 \\ \frac{\partial}{\partial \sigma^2} \ln(L) = 0 \end{cases}\right\} \Rightarrow (\hat{\mu}, \hat{\sigma}^2) = \hat{\theta}$$

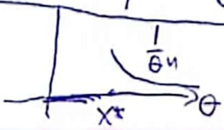
(+) $X \sim Unif(0, \theta)$, $\theta > 0$, $f_X(x) = \frac{1}{\theta} \cdot 1_{\{x \in [0,\theta]\}}$

$L(\vec{x}, \theta) = \prod_{j=1}^{n} \frac{1}{\theta} 1_{\{x \in [0,\theta]\}} = \left(\frac{1}{\theta}\right)^n \cdot 1_{\{x^* \in [0,\theta]\}}$ $\quad X^* = \max_{j \in [1,n]}(X_j)$

$= \frac{1}{\theta^n} \cdot 1_{\{x^* \in [0,\theta]\}}$

$\begin{cases} \frac{1}{\theta^n}, & X^* \le \theta \\ 0, & X^* > \theta \end{cases}$

$\frac{1}{\theta^n}$ е нам. по θ



$\Rightarrow L(\vec{x}, \hat{\theta}) = \sup_{\theta > 0} L(\vec{x}, \theta) \Rightarrow \hat{\theta} = \max_{j \le n}(x_j)$, $\hat{\theta} = \hat{\theta}(\vec{X})$

Твърдение Нека $X \sim \mathcal{N}(\mu, \sigma^2)$. Тогава м.п.о. за μ е

$$\hat{\mu} = \overline{X}_n^{(1)} = \frac{1}{n}\sum_{j=1}^{n} X_j$$

м.п.о. за σ^2 зависи от това дали знаем μ, т.е.

а) $\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j-\mu)^2$, ако знаем μ

б) $\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j-\overline{X}_n^{(1)})^2$, ако не знаем μ.

Доказателство $L(\vec{X}, \theta) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \sigma^{-n} \cdot e^{\sum_{j=1}^{n}\frac{(X_j-\mu)^2}{2\sigma^2}}$, $\theta = (\mu, \sigma^2)$

$$\ln(L(\vec{X}, \theta)) = n\ln\left(\frac{1}{\sqrt{2\pi}}\right) - \frac{n}{2}\ln(\sigma^2) - \sum_{j=1}^{n}\frac{(X_j-\mu)^2}{2\sigma^2}$$

$$\frac{\partial}{\partial\mu}L = \frac{1}{\sigma^2}\sum_{j=1}^{n}(X_j-\mu) = 0 \Rightarrow \hat{\mu} = \frac{1}{n}\sum_{j=1}^{n}X_j = \overline{X}_n^{(1)}$$

$$\frac{\partial}{\partial\sigma^2}L = \frac{-n}{2\sigma^2} - \sum_{j=1}^{n}(X_j-\mu)^2 \cdot \left(-\frac{1}{2\sigma^4}\right) = 0$$

μ изв. $\Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j-\mu)^2$

μ неизв. $\Rightarrow \hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j-\overline{X}_n^{(1)})^2$

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Метод на моментите (М.М.О.)

$\theta \in \mathbb{R}$, X, $F_X(x,\theta)$. Доп. че $\mathbb{E}[X] = \mu^{(1)}(\theta)$; $(\mu^{(1)})^{-1}(\mathbb{E}[X])$

$$\overline{X}_n^{(1)} = \frac{\sum_{j=1}^{n} X_j}{n} \xrightarrow[n\to\infty]{п.с.} \mathbb{E}[X] \text{ (УЗГЧ)}$$

$$\mu^{(1)}(\hat{\theta}) = \overline{X}_n^{(1)}, \quad \hat{\theta} = (\mu^{(1)})^{(-1)}(\overline{X}_n^{(1)})$$

$$\downarrow$$

$$\theta = (\mu^{(1)})^{(-1)}(\mathbb{E}[X])$$

$X \sim Unif(-\theta, \theta)$

$\mathbb{E}[X] = 0 = \mu^{(1)}(\theta)$

$\overline{X}_n^{(j)} = \frac{1}{n}\sum_{k=1}^{n} X_k^j, \ j \geq 1$

Деф (М.М.О.)

Нека X е сл. вел., $F_X(x,\theta)$, $\theta = (\theta_1, \dots, \theta_s)$

Тогава М.М.О. оценка за θ или $\hat{\theta}$ намираме чрез реш. на с-мата:

$$\left.\begin{aligned} &\mu^{(j)}(\theta) = \overline{X}_n^{(j)}, \quad j = \overline{1,s} \\ &\text{където } \mu^{(j)}(\theta) = \mathbb{E}[X^j], \quad j = \overline{1,s} \end{aligned}\right\} \Rightarrow \begin{aligned} &\hat{\theta} = (\hat{\theta}_1, \dots, \hat{\theta}_s) \\ &\hat{\theta}_j = (\mu^{(j)})^{-1}(X_n^{(j)}) \end{aligned}$$

⊕ $X \sim Unif(0,\theta)$, $\vec{X}$, М.М.П. $\hat{\theta} = \max_{j \leq n} X_j$

$\mathbb{E}[X] = \mu^{(1)}(\theta) = \frac{\theta}{2} \Rightarrow \theta = 2\mathbb{E}[X]$

$\overline{X}_n^{(1)} = \frac{\theta}{2} \Rightarrow \hat{\theta} = 2\overline{X}_n^{(1)}$ М.М.О

⊕ $X \sim N(\mu, \sigma^2)$ $\mathbb{E}[X] = \mu$; $\mathbb{E}[X^2] = \mathbb{D}[X] + (\mathbb{E}[X])^2 = \sigma^2 + \mu^2$

$$\begin{cases} \overline{X}_n^{(1)} = \hat{\mu}, \quad \text{М.М.О = М.М.П.} \\ \overline{X}_n^{(2)} = \mu^2 + \sigma^2 \Rightarrow \hat{\sigma}^2 = \overline{X}_n^{(2)} - (\overline{X}_n^{(1)})^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j - \overline{X}_n^{(1)})^2 \quad \text{М.М.П = М.М.О} \end{cases}$$

Свойства

а) **Неизместеност** – казваме, че $\hat{\theta}$ е неизместена оценка за θ, ако

$\mathbb{E}[\hat{\theta}] = \theta$ $\quad (\mathbb{E}[\hat{\theta}_j] = \theta_j, \ j = \overline{1,s}, \ \theta = (\theta_1, \dots, \theta_s))$

$|\mathbb{E}[\hat{\theta}] - \theta|$ системна грешка на $\hat{\theta}$.

Ⓟ $X \sim N(\mu, \sigma^2)$, $\hat{\mu} = \frac{1}{n}\sum_{j=1}^{n} X_j$, $\mathbb{E}[\hat{\mu}] = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[X_j] = \frac{n\mu}{n} = \mu$

Ако знаете μ, то $\mathbb{E}[\hat{\sigma}^2] = \frac{1}{n}\mathbb{E}\Big[\sum_{j=1}^{n}(X_j - \underbrace{\overline{X}_n^{(1)}}_{=\mu})^2\Big] = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[(X_j - \mu)^2] = \frac{n \cdot \sigma^2}{n} = \sigma^2$

Ако не знаем μ, то $\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(X_j - \overline{X}_n^{(1)})^2 = \overline{X}_n^{(2)} - (\overline{X}_n^{(1)})^2$

$$\mathbb{E}[\hat{\sigma}^2] = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[(X_j - \overline{X}_n^{(1)})^2] = \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[X_j^2 - 2X_j\overline{X}_n^{(1)} + (\overline{X}_n^{(1)})^2]$$

$$= \frac{1}{n}\sum_{j=1}^{n}\mathbb{E}[X_j^2] - \frac{2}{n}\sum_{j=1}^{n}\mathbb{E}\dots = \mathbb{E}[\overline{X}_n^{(2)}] - \mathbb{E}[(\overline{X}_n^{(1)})^2] =$$

$$= \frac{1}{n}\sum_{j=1}^{n}\underbrace{\mathbb{E}[X_j^2]}_{=\mathbb{E}[X_1^2]} - \frac{1}{n}\mathbb{E}\Big(\sum_{j=1}^{n}X_j\Big)^2 = \mathbb{E}[X_1^2] - \frac{1}{n}\mathbb{E}\Big[\sum_{j=1}^{n}X_j^2 + \sum_{i\neq j}X_iX_j\Big]$$

$$= \mathbb{E}[X_1^2] - \frac{1}{n^2}\sum_{j=1}^{n}\underbrace{\mathbb{E}[X_j^2]}_{=\mathbb{E}[X_1^2]} - \frac{1}{n^2}\mathbb{E}\Big[\sum_{i\neq j}X_iX_j\Big]$$

$$= \left(1 - \frac{1}{n}\right)\mathbb{E}[X_1^2] - \frac{1}{n^2}\sum_{i\neq j}\underbrace{\mathbb{E}[X_i]}_{=\mu}\underbrace{\mathbb{E}[X_j]}_{=\mu} = \left(1 - \frac{1}{n}\right)\mathbb{E}[X_1^2] - \frac{n(n-1)}{n^2}\mu^2 =$$

$$= \frac{n-1}{n}(\mu^2 + \sigma^2) - \left(\frac{n-1}{n}\right)\mu^2 = \frac{n-1}{n}\sigma^2 = \mathbb{E}[\hat{\sigma}^2] \neq \sigma^2 \text{ - изместена}$$

$S^2 = \underbrace{\frac{n}{n-1}}_{\xrightarrow{n\to\infty} 1}\hat{\sigma}^2$, $\mathbb{E}[S^2] = \frac{n}{n-1}\cdot\frac{n-1}{n}\sigma^2 = \sigma^2$

$S^2 = \frac{1}{n-1}\sum_{j=1}^{n}(X_j - \overline{X}_n^{(1)})^2$ – $\frac{1}{n-1}$ е, защото сме използвали 1 набл. да оценим $\overline{X}_n^{(1)}$

Ако $\lim_{n\to\infty}\mathbb{E}[\hat{\theta}] = \theta$, то $\hat{\theta}$ се нарича асимптотично неизм.

$\Rightarrow \hat{\sigma}^2$ е асимптотично неизм., докато $\hat{\mu}$ е неизм.

б) Състоятелност

Деф. Ако $\hat{\theta}(\vec{x}) \xrightarrow[n\to\infty]{\mathbb{P}} \theta$, то $\hat{\theta}$ е състоятелна оценка

$\forall \theta \in \mathbb{R}$, $\mathbb{P}(|\hat{\theta}(\vec{x}) - \theta| > \varepsilon) \xrightarrow[n\to\infty]{} 0$

Деф. (силна състоятелност)

Ако $\hat{\theta}(\vec{x}) \xrightarrow[n\to\infty]{п.с.} \theta$, то $\hat{\theta}$ е силно състоятелна

$\oplus$ $X \sim N(\mu, \sigma^2)$, $\hat{\mu} = \overline{X_n}^{(1)} \xrightarrow[n\to\infty]{п.с.} \mu$ (УЗГЧ)

$\hat{\mu}$ освен неизм. е и силно състоятелна

По-общо $\hat{\theta}_k = \dfrac{\sum_{j=1}^{n} X_j^k}{n} \xrightarrow[n\to\infty]{п.с.} \mathbb{E}[X_1^k]$ (УЗГЧ, $Y_i = X_i^k$

$\oplus$ $X \sim Unif(0, \theta)$, $\hat{\theta}_1 = \underset{j\le n}{\max}(X_j) = X^*$; $\hat{\theta}_2 = 2\overline{X_n}^{(1)}$

$\mathbb{E}[\hat{\theta}_2] = 2\mathbb{E}[\overline{X_n}^{(1)}] = \frac{2}{n}\sum_{j=1}^{n}\mathbb{E}[X_j] = \frac{2}{n}\cdot\frac{n}{2}\cdot\theta = \theta$ - неизм.

$\hat{\theta}_2 = 2\overline{X_n}^{(1)} \xrightarrow[n\to\infty]{п.с.} \frac{2\cdot\theta}{2} = \theta$ - силно със.

$F_{X^*}(x) = \mathbb{P}(X^* < x) = \mathbb{P}(\bigcap_{j=1}^{n}\{X_j < x\}) \overset{нез.}{=} \prod_{j=1}^{n}\mathbb{P}(X_j < x) \overset{еqu.}{=} \mathbb{P}(X_1 < x)^n = \left(\frac{x}{\theta}\right)^n \cdot \mathbb{1}\{x \in [0,\theta]\}$

$f_{X^*}(x) = \frac{d}{dx}F_{X^*}(x) = n\cdot\frac{x^{n-1}}{\theta^n}\cdot\mathbb{1}\{x\in[0,\theta]\}$

$\mathbb{E}[X^*] = \frac{n}{\theta^n}\int_0^\theta x\cdot x^{n-1}dx = \frac{n}{\theta(n+1)}\cdot\theta^{n+1} = \frac{n}{n+1}\theta = \mathbb{E}[\hat{\theta}_1]$ - изместена (асимп. неизм.)

$\frac{n+1}{n}\hat{\theta}_1$ е неизм. оценка.

$\hat{\theta}_1 = X^* \xrightarrow[n\to\infty]{п.с.} \theta$

$\mathbb{P}(\underset{j\in[1,n]}{\max} X_j < \theta - \varepsilon) = \mathbb{P}(X_1 < \theta - \varepsilon)^n = \left(\frac{\theta-\varepsilon}{\theta}\right)^n \xrightarrow[n\to\infty]{} 0$

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