The Fractional Discrete Cosine Transform

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***Abstract*—The extension of the Fourier transform operator to a fractional power has received much attention in signal theory and is finding attractive applications. The paper introduces and develops the fractional discrete cosine transform (DCT) on the same lines, discussing multiplicity and computational aspects. Similarities and differences with respect to the fractional Fourier transform are pointed out.**

I. INTRODUCTION

FRACTIONAL operators, particularly the fractional Fourier transform (FRT), have been investigated in some depth in recent years. The FRT was introduced [1], [2] as an extension of the ordinary Fourier transform (FT) and applied to bulk optics [3]–[5] and to fiber optics [6], [7] as a fundamental tool for optical information processing [8]. More recently, the FRT has also been considered for discrete-time periodic signals as an extension of the discrete Fourier transform (DFT) [9]–[13]. The purpose of this paper is an investigation of the fractional discrete-cosine transform (DCT). It is well known that the DCT may be regarded as a *cosine version* of the DFT, which, due to its ability of compacting energy distribution, is used very successfully as a coding tool, particularly in the field of image compression. Nowadays, the DCT forms part of several standards [14]–[16].

Starting from the DCT operator $\mathcal{C}:\mathbb{R}^N\to\mathbb{R}^N$, which maps an N-size real sequence into another N-size real sequence, the fractional DCT (FDCT) operator $\mathcal{C}_a:\mathbb{R}^N\to\mathbb{R}^N$ is introduced in a way that is standard for fractional operators [9]. Given a real parameter $a\in\mathbb{R}$ called the "fraction," $\mathcal{C}_a$ must have the *additive property* $\mathcal{C}_{a+b}=\mathcal{C}_a\mathcal{C}_b$ and the *marginal property* $\mathcal{C}_1=\mathcal{C}$, i.e., for $a=1$, it must give the ordinary DCT operator. Since the operator $\mathcal{C}$ is represented by an $N\times N$ real orthogonal matrix $\mathbf{C}$ and $\mathcal{C}_a$ by that matrix raised to a power $\mathbf{C}^a$, matrix theory methods can be applied to investigate the FDCT. It should be expected that for $0<a<1$, the FDCT should produce a sequence that has intermediate properties between those of the original sequence and those of the DCT sequence. To the authors' knowledge, this is the first contribution to this subject.[1] A similar approach can be found in a recent paper [17] for a simplified DCT version, which testifies to the interest in the topic.

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The paper is mainly devoted to the mathematical aspects of the FDCT. Throughout, we make reference to the *standard* form of the DCT, that is, to the one generally used in image processing (see MPEG standard [16]). In Section II, we review the properties and representations of the DCT matrix $\mathbf{C}$, outlining differences with respect to the DFT matrix. In Section III, we define the FDCT and discuss its properties and multiplicity, and in Section IV, we study FDCT representations that lead to computationally simple formulas. Since there are several other forms of DCT [18] (including that of [17]), for completeness, in Section V, we briefly discuss fractionalization issues for these alternative forms. Section VI outlines some thoughts for possible applications and concludes the paper.

II. REVIEW OF THE DCT

A. Standard DCT

Let s_n be a discrete sequence of size N. Then, the standard forms of the forward and inverse DCT are given by

$$S_k=\frac{1}{\sqrt{N}}\epsilon_k\sum_{n=0}^{N-1}s_n\cos\left(2\pi\frac{(2n+1)k}{4N}\right)\qquad k=0,\ldots,N-1 \quad (1a)$$

$$s_n=\frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\epsilon_k S_k\cos\left(2\pi\frac{(2n+1)k}{4N}\right)\qquad n=0,\ldots,N-1 \quad (1b)$$

where $\epsilon_0=1$ and $\epsilon_k=\sqrt{2}$, for $k>0$. If we interpret the N-size sequences as column vectors $\mathbf{s}=[s_0,\ldots,s_{N-1}]'$ and $\mathbf{S}=[S_0,\ldots,S_{N-1}]'$ and denote the $N\times N$ DCT matrix by

$$\mathbf{C}=\left\|\frac{1}{\sqrt{N}}\epsilon_k\cos\left(2\pi\frac{(2n+1)k}{4N}\right)\right\| \quad (2)$$

then (1) takes the matrix form $\mathbf{S}=\mathbf{C}\mathbf{s},\mathbf{s}=\mathbf{C}^{-1}\mathbf{S}$, where the inverse relationship is guaranteed by imposing appropriate *orthonormality* conditions.

B. Eigenstructure of the DCT Matrix

Since the fractional DCT will be defined through the real powers of the DCT matrix $\mathbf{C}$, it is necessary to investigate in depth the properties of this matrix. The fundamental properties are 1) $\mathbf{C}$ is *real*, and 2) $\mathbf{C}$ is *orthogonal*, i.e., $\mathbf{C}\mathbf{C}'=\mathbf{I}$, where $'$ denotes transposition, and $\mathbf{I}$ is the identity matrix. Then, 3) $\mathbf{C}$ is *unitary*, i.e., $\mathbf{C}\mathbf{C}^*=\mathbf{I}$, where * denotes conjugate transposition.

Being unitary assures that $\mathbf{C}$ admits an *orthonormal* set of N eigenvectors $\mathbf{u}_n$ [20, p. 274], i.e., one with the property $\mathbf{u}_m^*\mathbf{u}_n=\delta_{mn}$, and that the corresponding eigenvalues λ_n lie on the unit circle, i.e., $\lambda_n = e^{j\varphi_n}$, with φ_n real. Moreover, $\mathbf{C}$ can be diagonalized in the form[2]

$$\mathbf{C} = \mathbf{U}\mathbf{\Lambda}\mathbf{U}^* = \sum_n \mathbf{U}_n e^{j\varphi_n} \tag{3}$$

where $\mathbf{U}$ is a unitary matrix, with columns $\mathbf{u}_n$, $\mathbf{\Lambda}$ is the diagonal matrix, with diagonal entries λ_n, and $\mathbf{U}_n \triangleq \mathbf{u}_n\mathbf{u}_n^*$ are unitary matrices having the properties

$$\mathbf{U}_m\mathbf{U}_n = \delta_{mn}\mathbf{U}_m, \quad \sum_n \mathbf{U}_n = \mathbf{I}. \tag{4}$$

The property of $\mathbf{C}$ being real assures that some $\mathbf{u}_n$ may be real (if $\lambda_n = \pm 1$) and that the rest come in conjugate pairs, with eigenvalues also in conjugate pairs. Let μ_1 and μ_{-1} be the multiplicities of the eigenvalues 1 and -1, respectively; then, for conjugate eigenvalues and eigenvectors, we can use the notation

$$\begin{aligned} &\lambda_{\pm n} = e^{\pm j\varphi_n}, \quad \mathbf{u}_{\pm n} \quad n = 1, 2, \ldots, K \\ &K = (N - \mu_1 - \mu_{-1})/2 \end{aligned} \tag{5}$$

where $0 < \varphi_n < \pi$, and $\mathbf{u}_{-n}$ and $\mathbf{u}_n$ form a conjugate pair.

From the above, we can write the following real-valued expression for the DCT matrix:

$$\begin{aligned} \mathbf{C} &= 2\Re\left[\sum_{n=1}^{K} \mathbf{U}_n e^{j\varphi_n}\right] + \mathbf{V}_1 - \mathbf{V}_{-1} \\ &= \sum_{n=1}^{K} (\mathbf{A}_n \cos\varphi_n + \mathbf{B}_n \sin\varphi_n) + \mathbf{V}_1 - \mathbf{V}_{-1} \end{aligned} \tag{6}$$

where $\mathbf{U}_n = \mathbf{u}_n\mathbf{u}_n^*$, $\mathbf{A}_n = 2\Re[\mathbf{U}_n]$, $\mathbf{B}_n = -2\Im[\mathbf{U}_n]$, and $\mathbf{V}_1$ collect the μ_1 matrices $\mathbf{U}_n$ corresponding to the eigenvalue 1 and similarly for $\mathbf{V}_{-1}$.

To find more subtle properties, we must further investigate the eigenvalues, i.e., the solutions of the equation $P_N(\lambda) = \det(\lambda\mathbf{I} - \mathbf{C}) = 0$, where $P_N(\lambda)$ is the characteristic polynomial of $\mathbf{C}$. From (5), we also have

$$P_N(\lambda) = (\lambda - 1)^{\mu_1}(\lambda + 1)^{\mu_{-1}} \prod_{n=1}^{K} (\lambda^2 - 2\lambda\cos(\varphi_n) + 1). \tag{7}$$

In the diagonalization (3), the distinctness or multiplicity of the eigenvalues has an effect on the choice of eigenvectors. In fact, a unitary matrix admits a unique set of eigenvectors (apart from a multiplication by a constant) if, and only if, all the eigenvalues are distinct (for example, see [22]). For the DCT matrix, we have the following.

1) Conjecture: For every order N, the eigenvalues of the DCT matrix $\mathbf{C}$ are distinct. Moreover, the multiplicity of eigenvalues ± 1 is related to N as

N	$4N_0$	$4N_0+1$	$4N_0+2$	$4N_0+3$
μ_1	0	1	1	0
μ_{-1}	0	0	1	1

(8)

[2]This *complex* representation holds for the class of *unitary* matrices. More specifically, since $\mathbf{C}$ is a *real orthogonal* matrix, a true *real* representation, i.e., one in terms of a real basis, could be considered. To complete the topic, this is outlined in the Appendix, but it is not attractive for defining the FDCT.

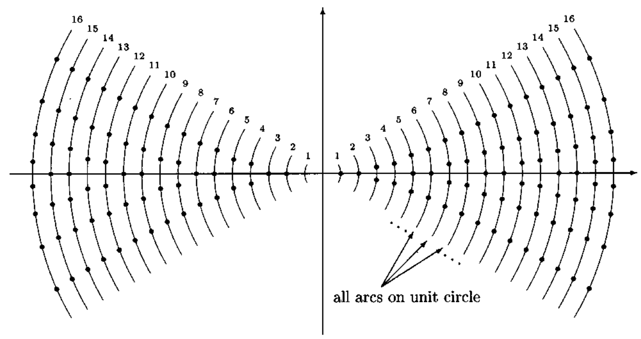

Fig. 1. Eigenvalue constellations of DCT matrix for the first 16 orders.

The conjecture is "well posed" on the following considerations. For $N \le 10$, using the standard explicit formulas for the solution of algebraic equations up to the fourth degree, with the help of symmetries in the characteristic polynomial, we can find a closed-form expression for the eigenvalues. For example, for $N = 8$, the real parts of the eigenvalues $\Re[\lambda_n] = \cos(\varphi_n)$, $n = 0, 1, 2, 3$ can be calculated as solutions of

$$x^4 + \frac{d_1}{2}x^3 + \frac{d_2 - 4}{4}x^2 + \frac{d_3 - 3d_1}{8}x + \frac{d_4 - 2d_2 + 2}{16} = 0$$

where d_i is the coefficient of order i of the characteristic polynomial $P_8(\lambda)$.

For $N > 10$, an exact evaluation of the eigenvalues would require the solution of an algebraic equation of degree ≥ 5, which is not available. Hence, for $N > 10$, we proceed numerically, setting any desired accuracy for the calculations. The numerical evidence for the distinctness of DCT eigenvalues is illustrated in Fig. 1.[3] We finally remark that the order most used in DCT applications is $N = 8$, for which value the claim of the conjecture has been proved analytically.

Having consolidated this point, we can state Property 4): The orthonormal set of DCT eigenvectors is unique. Hence, the representation of the DCT matrix given by (3) is also unique.

Considering that only the case $N = 4N_0$ is relevant in applications, hereafter, we will mainly concentrate on this case, which avoids the values $\lambda_n = \pm 1$ and real eigenvectors, and gives simpler equations. In particular, (6) simplifies to

$$\mathbf{C} = \sum_{n=1}^{N/2} (\mathbf{A}_n \cos\varphi_n + \mathbf{B}_n \sin\varphi_n). \tag{9}$$

2) Quasi-Quadrantal Symmetry: Visual inspection of Fig. 1 might suggest that for $N = 4N_0$, the eigenvalue constellation has *quadrant* symmetry, i.e., one with arguments $\pm\varphi_n$ and $\pm(\pi - \varphi_n)$, $n = 1, \ldots, N/4$. This symmetry is exact for $N = 4$, for which we have $\cos(\varphi_1) = (1/2)\sqrt{3 + \cos(\pi/8)}$, whereas for $N = 4N_0 > 4$, although very close, it is only approximate. For instance, for $N = 8$, we find that $\varphi_1 = 0.083\,883\,6$, $\varphi_2 = 0.286\,792$, $\pi - \varphi_3 = 0.079\,306\,8$, and $\pi - \varphi_4 = 0.282\,215$, where the differences are not due to numerical inaccuracy since these eigenvalues were calculated analytically.

[3]We note that the "closest eigenvalues" have a distance of about $2\pi/4N$ rad. The accuracy used in two independent numerical evaluations was at least 10^{-10} for N up to 128.

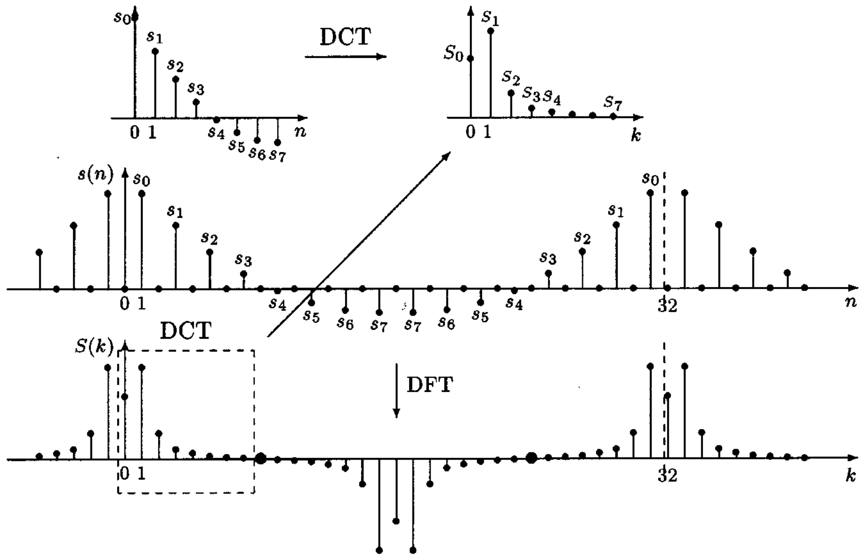

Fig. 2. Relationship between DFT and DCT: The DFT $S(k)$ of the signal $s(n)$ gives the DCT S_k.

C. Comparison of the DCT With the DFT

The DCT and the DFT are closely linked. Let a periodic discrete-time signal $s(n)$ of period $4N$ be defined as having the two symmetries: 1) $s(n)$ real and even and 2) $s(2n) = 0, \forall n$. It is easily seen that the effect of these symmetries on the signal's DFT $S(k)$ is 1) $S(k)$ real and even and 2) $S(k)+S(k-2N) = 0$, which represents an odd symmetry with respect to a quarter of the period, and gives the particular value $S(N) = 0$. Moreover

$$S(k) = \frac{1}{\sqrt{4N}} \sum_{n=0}^{4N-1} s(n)e^{-j2\pi kn/(4N)} = \frac{1}{\sqrt{N}} \sum_{n=0}^{N-1} s(2n+1) \cos\left(2\pi \frac{(2n+1)k}{4N}\right) \quad (10)$$

so that the link between the signals $s_n = s(2n+1)$ and $S_k = \varepsilon_k S(k)$ (both of length N) is given by the DCT [see (1a)]. This relation is illustrated in Fig. 2.

Apart from the above consideration, the DFT and the DCT have very different mathematical structures. The DFT matrix $\mathbf{W} = \|(1/\sqrt{N})\exp(-j2\pi kn/N)\|$ is unitary, but not orthogonal, and has only four eigenvalues [21] given by $\lambda_n = \exp(jn\pi/2)$, which are the fourth roots of unity. For $N > 4$, the DFT eigenvalues are not distinct, thus permitting many possible sets of orthonormal eigenvectors. Therefore, for the purpose of fractionalization, diagonalizable forms like (3) are also applicable to the DFT matrix, but these are not unique (and this multiplicity creates several problems in the definition of a fractional DFT [19]).

A further difference relates to periodicity. The minimal polynomial of the DFT matrix $\mathbf{W}$ is $\lambda^4 - 1$, which, by the Cayley-Hamilton theorem [22, p. 86], leads to the relation $\mathbf{W}^4 = \mathbf{I}$. This states that the sequence of integer powers $\{\mathbf{W}^n\}$ is periodic, with period 4. In fact, we can recover an original sequence by a forward repetition of the DFT operator. In contrast, the DCT matrix is not periodic (conjecture). DCT periodicity would require that $\mathbf{C}^M = \mathbf{I}$, for some positive integer M, and would imply that all the eigenvalue arguments φ_n are rational fractions of 2π. Periodicity may be tested using (3) or, more specifically, (6), and the failure of this condition was tested numerically up to the order $N = 1000$.

III. FDCT Definition

Let $\mathcal{C}_a$ be a linear operator that for any given "fraction" $a \in \mathbb{R}$, maps an N-size vector $\mathbf{s}$ into another N-size vector $\mathbf{S}_a = \mathcal{C}_a[\mathbf{s}]$. $\mathcal{C}_a$ will be an FDCT operator if we have the following.

1) $\mathcal{C}_a$ verifies the DCT condition $\mathcal{C}_1 = \mathcal{C}$, where $\mathcal{C}$ is defined by (1).
2) $\mathcal{C}_a$ has the *additive* property $\mathcal{C}_{a+b} = \mathcal{C}_a\mathcal{C}_b$ for every choice of real a and b.

The FDCT operator can be expressed as

$$\mathcal{C}_a : \mathbf{S}_a = \mathbf{C}_a\mathbf{s} \quad (11)$$

for a suitable $N \times N$ matrix $\mathbf{C}_a$. The DCT condition 1) is then given by $\mathbf{C}_1 = \mathbf{C}$, where $\mathbf{C}$ is the DCT matrix defined by (2). The additive property 2) becomes $\mathbf{C}_{a+b} = \mathbf{C}_a\mathbf{C}_b$, which implies the properties $\mathbf{C}_a\mathbf{C}_b = \mathbf{C}_b\mathbf{C}_a$ and $\mathbf{C}_0 = \mathbf{I}$. The class of *possible* definitions is generated by matrices $\mathbf{C}_a$ that satisfy conditions 1) and 2). When possible, that is, when permitted by the eigenstructure, we add a further constraint.

3) $\mathcal{C}_a$ is *real* in the sense that $\mathbf{s} \in \mathbb{R}^N \Rightarrow \mathbf{S}_a \in \mathbb{R}^N$ for every $a \in \mathbb{R}$, which implies that the matrix $\mathbf{C}_a$ must be real.

It becomes evident that the formulation of the FDCT operator requires a rigorous definition of a real power $\mathbf{C}_a = \mathbf{C}^a$ of the DCT matrix $\mathbf{C}$. To define $\mathbf{C}^a$, we cannot use the theory of "functions of a matrix" since the corresponding scalar function $f(z) = z^a$ is not analytic. The topic rather belongs to the theory of "matrix equations," in particular, to "roots of a square matrix."

A. Real Power of the FDCT Matrix

To get a real power of the DCT matrix, we consider the expansion (3) of $\mathbf{C}$ and replace the eigenvalues $\lambda_n = e^{j\varphi_n}$ by their ath powers λ_n^a, that is, the matrix $\mathbf{\Lambda}$ by its ath power $\mathbf{\Lambda}^a$. We then find that $\mathbf{C}_a$ can be written in the compact form

$$\mathbf{C}_a = \mathbf{U}\mathbf{\Lambda}^a\mathbf{U}^* \tag{12}$$

which satisfies both the DCT condition and the additive property.[4] This procedure of fractionalization is equivalent to that used with the DFT in [9]–[12].

Having defined $\mathbf{C}_a$, for any given sequence $\mathbf{s}$, the corresponding FDCT $\mathbf{S}_a$, with fraction a, can be calculated by (11). By the additive property, we have $\mathbf{C}_{-a}\mathbf{C}_a = \mathbf{C}_0 = \mathbf{I}$ so that the *inverse* FDCT is obtained by using the matrix $\mathbf{C}_{-a}$, namely

$$\mathbf{S}_a = \mathbf{C}_a\mathbf{s}, \quad \mathbf{s} = \mathbf{C}_{-a}\mathbf{S}_a. \tag{13}$$

The matrix $\mathbf{C}_a$, which is given by (12), can be written in alternative forms. In particular, from (6), we find

$$\boxed{\mathbf{C}_a = 2\Re\left[\sum_{n=1}^{K}\mathbf{U}_n\,\lambda_n^a\right] + \mathbf{V}_1\,(1)^a + \mathbf{V}_{-1}\,(-1)^a} \tag{14}$$

The latter shows that for $N \neq 4N_0$, the presence of the eigenvalues ± 1 leads to a complex-valued FDCT matrix since for a non-natural, $(\pm 1)^a = e^{\pm j2\pi a}$ is a complex number.

On the other hand, when $N = 4N_0$, the absence of $(\pm 1)^a$ guarantees that $\mathbf{C}_a$ becomes a real-valued matrix. Since the case $N = 4\,N_0$ is the more attractive for applications, hereafter, we confine the development of the theory to this case. However, an extension of the theory to $N \neq 4N_0$ merely complicates the formulas, but the basic idea still holds.

B. Multiplicity of the FDCT—Generating Sequences

The *nonuniqueness* of a fractional operator, which is defined according to (12), has two main aspects [19]. The first is that the choice of the eigenvector basis $\mathbf{U}$ could be nonunique, and the second is that a real power of a complex number λ_n^a is not unique. We saw that the DCT eigenvector basis $\mathbf{U}$ is unique because the eigenvalues are distinct. Therefore, we concentrate on the FDCT multiplicity due to the real power of complex numbers. Considering that $\lambda_n = e^{j\varphi_n}$, the possible values of λ_n^a are given by

$$\lambda_n^a = e^{j(\varphi_n + 2\pi q_n)a}, \quad q_n \in \mathbb{Z} \tag{15}$$

where q_n is an arbitrary sequence of integers. Different choices of q_n lead to different matrices $\mathbf{C}_a$ and, hence, to different FDCT definitions. We let (for $N = 4N_0$)

$$\omega_n = \varphi_n + 2\pi q_n, \quad n = 1, 2, \ldots, N/2 \tag{16}$$

and call the sequence $\mathbf{q} = (q_1, \ldots, q_{N/2})$ a *generating sequence* (GS) of the FDCT.

[4]The DCT condition is obvious. The key to the other is the additive property of the power to a real number, namely, $\lambda_n^{a+b} = \lambda_n^a\lambda_n^b$, combined with orthonormality of eigenvectors $\mathbf{u}_n$

Once the ambiguity on λ_n^a has been resolved, by having chosen a specific GS, the FDCT matrix $\mathbf{C}_a$ becomes unique, namely (for $N = 4N_0$)

$$\begin{aligned}\mathbf{C}_a &= 2\Re\left[\sum_{n=1}^{N/2}\mathbf{U}_n e^{j\omega_n a}\right]\\ &= \sum_{n=1}^{N/2}(\mathbf{A}_n\cos\omega_n a + \mathbf{B}_n\sin\omega_n a).\end{aligned} \tag{17}$$

For a given size N, the multiplicity of $\mathbf{C}_a$ depends on the nature of the fraction, according to the cases (for $a > 0$):

1) $a = 1/M$ with M a positive integer;
2) $a = L/M$ with L and M relative primes;
3) a irrational.

The first case corresponds to finding the Mth roots of a square matrix, i.e., the solutions of the matrix equation $\mathbf{X}^M = \mathbf{C}$. Now, from (16) and (17), we find

$$\mathbf{C}_{1/M} = 2\Re\left[\sum_{n=1}^{N/2}\mathbf{U}_n e^{j\varphi_n/M}W_M^{q_n}\right] \tag{18}$$

where $W_M = \exp(j2\pi/M)$. Since, with q_n integer, each $W_M^{q_n}$ can take M distinct values, the number of distinct matrices $\mathbf{C}_{1/M}$ is $M^{N/2}$. For instance, for $M = 2$, we find that the number of distinct real square roots of $\mathbf{C}$ is $2^{N/2}$. This result agrees with matrix theory (see [20, ch. VIII] and [23, ch. 6]). The same conclusion holds for case 2) since (18) holds for $a = L/M$, with $W_M^{q_n}$ replaced by $W_M^{Lq_n}$, and the multiplicity of each factor is again M. For instance, $W_{14}^{3q_n}$ generates 14 distinct values, as does $W_{14}^{q_n}$.

Clearly, when a is irrational, each factor $e^{j2\pi q_n a}$ has infinite multiplicity so that the matrix $\mathbf{C}_a$ takes an infinite number of distinct values. In what follows, we confine our attention to the "principal" value, which is obtained for $\omega_n = \varphi_n$, that is, with the GS $\mathbf{q} = \mathbf{0}$. However, the properties established below hold for all possible values of $\mathbf{C}_a$, and it may be of interest to consider a FDCT definition with a nontrivial GS.

We finally note an alternative form of the FDCT matrix (see the Appendix)

$$\mathbf{C}_a = \mathbf{T}e^{a\mathbf{E}}\mathbf{T}'$$

where $\mathbf{T}$ is a real and orthogonal matrix, and the matrix $\mathbf{E}$ collects the values $\omega_n = \varphi_n + 2\pi q_n$ in a 2×2 block diagonal form.

C. Properties of the FDCT

The FDCT has been defined by modifying the DCT and, more specifically, by preserving the orthonormal basis of eigenvectors $\mathbf{u}_{\pm n}$ and changing the corresponding eigenvalues $\lambda_{\pm n}^a = e^{\pm j\omega_n a}$. Since these new eigenvalues have modulus 1 and are distinct (for $a \neq 0$), for every fraction $a \neq 0$ and every GS q_n, the DFCT operator $\mathcal{C}_a$ has exactly the same properties as the original DCT operator $\mathcal{C}$, namely, it is unitary, real orthogonal, and has a unique orthonormal basis (given by $\mathbf{u}_{\pm n}$). In particular, the unitary property assures the Parseval's relationship $\mathbf{S}_a'\mathbf{S}_a = \mathbf{s}'\mathbf{s} = \mathbf{S}'\mathbf{S}$ for every a and every GS q_n.

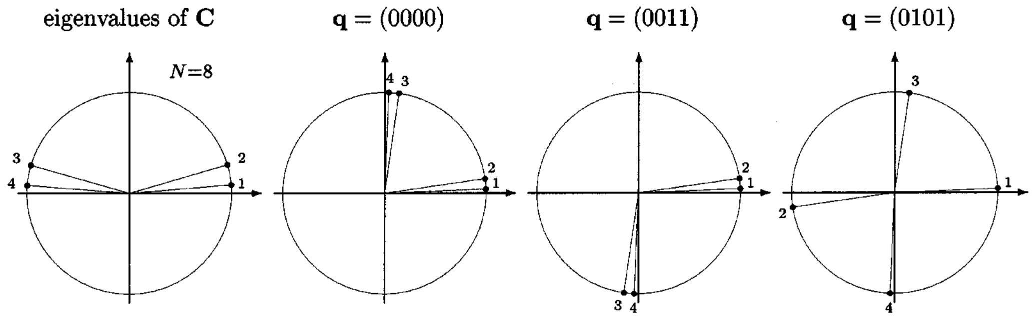

Fig. 3. Eigenvalue constellations for defining the square roots of the DCT matrix (for clarity, the conjugate eigenvectors are not indicated).

Note that the orthogonality property $\mathbf{C}'_a\mathbf{C}_a = \mathbf{I}$, compared with the additive property $\mathbf{C}_{-a}\mathbf{C}_a = \mathbf{I}$, gives

$$\mathbf{C}_a^{-1} = \mathbf{C}_{-a} = \mathbf{C}'_a \tag{19}$$

so that the matrix $\mathbf{C}_{-a}$ in the inverse FDCT (13) is simply given by the transpose of $\mathbf{C}_a$.

D. Half-DCT

The FDCT for $a = (1/2)$ is obtained by calculating the square roots of the DCT matrix, which are given by

$$\sqrt{\mathbf{C}} = 2\Re\left[\sum_{n=0}^{N/2}\mathbf{U}_n e^{j\frac{1}{2}(\varphi_n+2\pi q_n)}\right].$$

The distinct GSs $\mathbf{q}$ are given by the binary words of length $N/2$ and lead to $2^{N/2}$ distinct square roots. For example, for $N = 8$, the $2^4 = 16$ square roots are obtained for the GSs $\mathbf{q} = (0000), (0001), \ldots, (1111)$. Fig. 3 illustrates the eigenvalues constellations corresponding to three of these GSs. To complete the example for $N = 8$, we give the numerical values of the DCT matrix $\mathbf{C}$ and of the principal value of $\sqrt{\mathbf{C}}$ (which is obtained for $\mathbf{q} = (0000)$), shown in the equation at the bottom of the page.

E. FDCT Examples

Fig. 4 illustrates the FDCT of some typical signals displayed on a frame of $N = 16$ points. The diagrams are obtained for different values of the fraction a, starting with the fraction $a = 0$, which corresponds to the original sequence and including the fraction $a = 1$, which corresponds to the ordinary DCT. In this example, the FDCT was calculated with the GS $\mathbf{q} = \mathbf{0}$.

Fig. 5 further illustrates the dependence of the FDCT on the GS $\mathbf{q}$. The original sequence is the rectangular signal on the left of Fig. 4, and the fraction is fixed to the value $a = 3/4$. It shows that the GS has a strong effect on the result.

IV. Analysis in the Domain of the "Fraction"

We investigate the behavior of the FDCT matrix in the domain of the fraction a, for a given GS $\mathbf{q}$, by interpreting

$$\mathbf{C}_a = \sum_{n=1}^{N/2}(\mathbf{A}_n\cos\omega_n a + \mathbf{B}_n\sin\omega_n a) \tag{20}$$

as an aperiodic signal of continuous argument, where the "argument" is the fraction a.

A. General Interpolation Formula

Considering the trigonometric expression of $\mathbf{C}_a$ given by (20), it is easy to write an interpolation formula that allows

$$\mathbf{C} = \begin{bmatrix} 0.354 & 0.354 & 0.354 & 0.354 & 0.354 & 0.354 & 0.354 & 0.354 \\ 0.490 & 0.416 & 0.278 & 0.098 & -0.098 & -0.278 & -0.416 & -0.490 \\ 0.462 & 0.191 & -0.191 & -0.462 & -0.462 & -0.191 & 0.191 & 0.462 \\ 0.416 & -0.098 & -0.490 & -0.278 & 0.278 & 0.490 & 0.098 & -0.416 \\ 0.354 & -0.354 & -0.354 & 0.354 & 0.354 & -0.354 & -0.354 & 0.354 \\ 0.278 & -0.490 & 0.098 & 0.416 & -0.416 & -0.098 & 0.490 & -0.278 \\ 0.191 & -0.462 & 0.462 & -0.191 & -0.191 & 0.462 & -0.462 & 0.191 \\ 0.098 & -0.278 & 0.416 & -0.490 & 0.490 & -0.416 & 0.278 & -0.098 \end{bmatrix}$$

$$\sqrt{\mathbf{C}} = \begin{bmatrix} 0.703 & 0.314 & -0.110 & 0.038 & 0.096 & 0.411 & 0.080 & 0.457 \\ 0.068 & 0.743 & 0.435 & 0.121 & 0.170 & -0.121 & -0.094 & -0.433 \\ 0.492 & -0.238 & 0.466 & -0.078 & -0.477 & -0.490 & -0.064 & 0.079 \\ 0.351 & -0.132 & -0.391 & 0.393 & 0.076 & -0.236 & 0.518 & -0.471 \\ 0.212 & -0.367 & 0.121 & 0.209 & 0.717 & -0.161 & -0.471 & 0.013 \\ -0.122 & -0.237 & 0.459 & 0.664 & -0.180 & 0.484 & 0.103 & -0.009 \\ 0.153 & -0.281 & 0.344 & -0.580 & 0.234 & 0.380 & 0.336 & -0.361 \\ -0.229 & 0.072 & 0.291 & 0.041 & 0.357 & -0.341 & 0.606 & 0.495 \end{bmatrix}$$

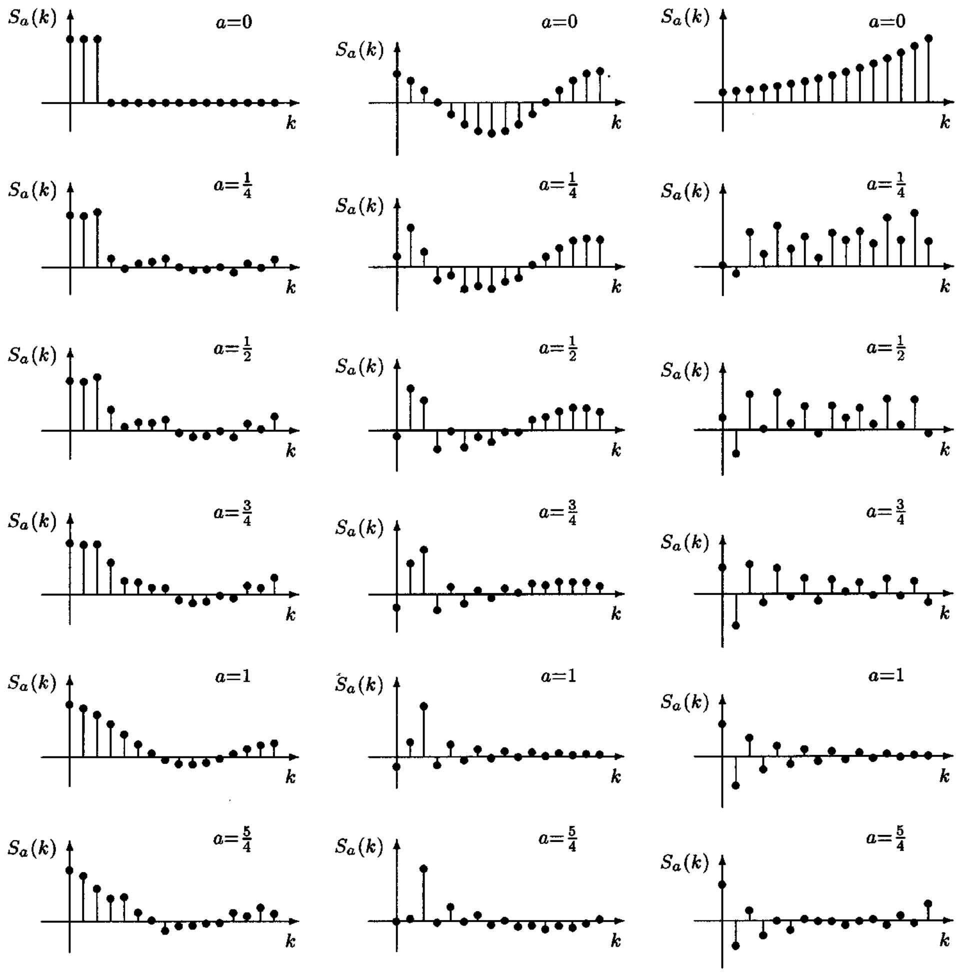

Fig. 4. FDCTs of (left) a rectangular sequence (center) a cosine-shaped sequence, and (right) an exponential sequence with $N = 16$ and GS $\mathbf{q} = \mathbf{0}$.

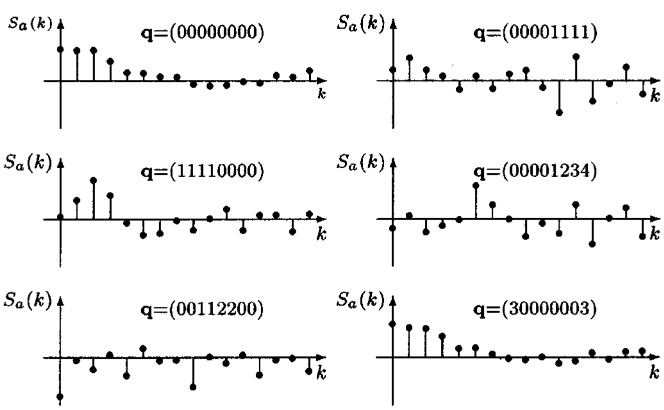

Fig. 5. FDCT of a rectangular sequence with $N = 16$ and different GSs $\mathbf{q}$ with a fixed fraction $a = (3/4)$.

constructing the whole function $\mathbf{C}_a$, which is continuous in a, from N of its samples in a. Let $a_0, a_1, \ldots, a_{N-1}$ be N distinct "fractions," and write the system of N equations

$$\mathbf{C}_{a_r} = \sum_{n=1}^{N/2} (\mathbf{A}_n \cos \omega_n a_r + \mathbf{B}_n \sin \omega_n a_r) \qquad r = 0, 1, \ldots, N-1.$$

Assuming that the values $\mathbf{C}_{a_r}$ are known, we solve the system with respect to the unknowns $\mathbf{A}_n$ and $\mathbf{B}_n$. The system's coefficient matrix is $\mathbf{H} = \|h_{rn}\|$, where

$$h_{r,2n-1} = \cos \omega_n a_r, \quad h_{r,2n} = \sin \omega_n a_r \qquad r = 0, 1, \ldots, N-1 \quad n = 1, \ldots, N/2.$$

Letting $\mathbf{K} = \|k_{nr}\| = \mathbf{H}^{-1}$, then

$$\mathbf{A}_n = \sum_{r=0}^{N-1} k_{2n-1,r} \mathbf{C}_{a_r}, \quad \mathbf{B}_n = \sum_{r=0}^{N-1} k_{2n,r} \mathbf{C}_{a_r}. \tag{21}$$

Finally, substituting (21) into (20) yields

$$\boxed{\mathbf{C}_a = \textstyle\sum_{r=0}^{N-1} \mathbf{C}_{a_r} p_r(a)} \tag{22}$$

where

$$p_r(a) = \sum_{n=1}^{N/2} (k_{2n-1,r} \cos \omega_n a + k_{2n,r} \sin \omega_n a). \tag{23}$$

This is the general interpolation formula that gives the FDCT matrix as a weighted combination of its values at the N fractions a_r. The only assumption is that the fractions a_r must be chosen such that the matrix $\mathbf{H}$ is not singular. This certainly holds when the N fractions a_r are equally spaced.

From (22), we find that the FDCT $\mathbf{S}_a$ of a given sequence $\mathbf{s}$ is obtained as

$$\mathbf{S}_a = \sum_{r=0}^{N-1} \mathbf{S}_{a_r} p_r(a).$$

Thus, to calculate $\mathbf{S}_a$, for all fractions $a \in \mathbb{R}$, it is sufficient to evaluate the FDCTs $\mathbf{S}_{a_r}$ of the given sequence $\mathbf{s}$ for N fractions a_r. To calculate the inverse FDCT, $\mathbf{s} = \mathcal{C}_a^{-1}[\mathbf{S}_a] = \mathbf{C}_a^{-1}\mathbf{S}_a$, we can use (13), that is

$$\mathbf{s} = \mathbf{C}_{-a}\mathbf{S}_a = \sum_{r=0}^{N-1} \mathbf{C}_{a_r}\mathbf{S}_a\, p_r(-a)$$

where $\mathbf{C}_{a_r}\mathbf{S}_a$ is the forward FDCT of $\mathbf{S}_a$ evaluated at the fraction a_r.

B. Polynomial Interpolation

The choice $a_r = r, r = 0, 1, \ldots, N-1$ yields

$$\boxed{\mathbf{C}_a = \textstyle\sum_{r=0}^{N-1} \mathbf{C}^r p_r(a)} \tag{24}$$

and permits evaluation of the FDCT matrix by merely calculating the first N powers of the DCT matrix, without the preliminary evaluation of $\mathbf{C}_a$ at some fractions. It also has the following advantages.

1) It does not require the evaluation of the orthonormal set of eigenvectors $\mathbf{u}_{\pm n}$ but only the arguments φ_n of the eigenvalues.
2) It allows a fast computation.

The N weights $p_r(a)$ obtained with the choice $a_r = r$ are illustrated in Fig. 6 for $N = 8$. As a check, we note that $p_r(s) = \delta_{rs}$, where $\delta_{rs} = 1$ for $r = s$ and $\delta_{rs} = 0$ otherwise. A further check can be made by using the property

$$P_N(\mathbf{C}) = \sum_{n=0}^{N} d_n \mathbf{C}^n = \mathbf{0}. \tag{25}$$

In fact, calculating $\mathbf{C}_N = \mathbf{C}^N$ from (24) and comparing with (25), we find

$$p_0(N) = -1, \quad p_r(N) = d_r, \quad 1 \le r \le N-2$$
$$p_{N-1}(N) = 1.$$

For the FDCT $\mathbf{S}_a$, the relationship (24) yields

$$\mathbf{S}_a = p_0(a)\mathbf{s} + p_1(a)\mathbf{S} + p_2(a)\mathbf{S}_2 + \cdots + p_{N-1}(a)\mathbf{S}_{N-1} \tag{26}$$

where

- $\mathbf{s}$ original sequence;
- $\mathbf{S}$ DCT of $\mathbf{s}$;
- $\mathbf{S}_2$ DCT of $\mathbf{S}$;
- $\mathbf{S}_3$ DCT of $\mathbf{S}_2$;

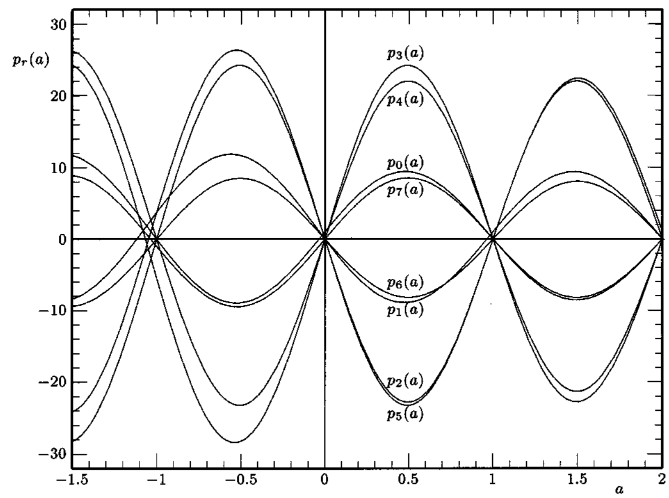

Fig. 6. Weights of FDCT for $N = 8$ with fractions 0, 1, 2, 3, 4, 5, 6, and 7.

etc., so that $\mathbf{S}_a$ can be calculated, for any fraction $a \in \mathbb{R}$, by iterating the DCT algorithm $N-1$ times. Since fast algorithms are available for the DCT, with a computational complexity of $N \log_2 N$ operations [24], the calculation of K distinct FDCTs using (26) has complexity of $(N-1)N\log_2 N + KN$ operations, instead of the KN^2 operations required for direct calculation. Therefore, with the approach of (26), an improvement can be obtained for $K > \log_2 N$.

C. Computation via the DCT and IDCT

The choice $a_r = r, r = -(N-2)/2, \ldots, -1, 0, 1, \ldots, N/2$ gives the interpolation formula

$$\mathbf{C}_a = \sum_{r=-(N-2)/2}^{N/2} \mathbf{C}^r p_r(a)$$

which is essentially equivalent to (24). For the FDCT it yields

$$\boxed{\mathbf{S}_a = \textstyle\sum_{r=-(N-2)/2}^{N/2} \mathbf{S}_r\, p_r(a)}$$

where, for $r \ge 1$, $\mathbf{S}_r$ is obtained by an r-fold evaluation of the DCT and for $r \le -1$ by a $(-r)$-fold evaluation of the IDCT. For instance, with $N = 8$, we find

$$\begin{aligned}\mathbf{S}_a = {} & p_{-3}(a)\mathbf{S}_{-3} + p_{-2}(a)\mathbf{S}_{-2} + p_{-1}(a)\mathbf{S}_{-1} + p_0(a)\mathbf{s} \\ & + p_1(a)\mathbf{S} + p_2(a)\mathbf{S}_2 + p_3(a)\mathbf{S}_3 + p_4(a)\mathbf{S}_4\end{aligned}$$

where, e.g., $\mathbf{S}_{-2}$ is the IDCT of ~~the IDCT~~ of $\mathbf{s}$. The new weights $p_r(a)$ are illustrated in Fig. 7 for $N = 8$. Note that the excursions of the $p_r(a)$ are smaller than in the previous case, at least within the range $-1.5 \le a \le 2$.

D. Other Forms of Computation

Choices of the forms

$$a_r = \frac{1}{2}r, \quad r = 0, 1, \ldots, N-1 \quad \text{or}$$
$$a_r = \frac{1}{2}r, \quad r = -N/2, \ldots, 0, \ldots, (N-2)/2$$

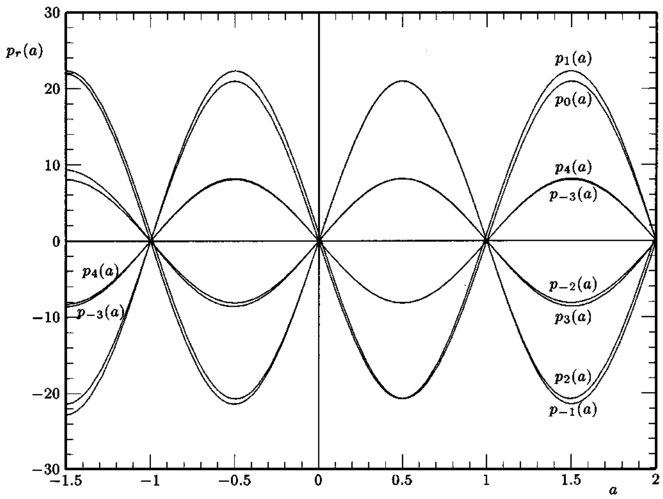

Fig. 7. Weights of FDCT for $N=8$ with fractions $-3,-2,-1,0,1,2,3,4$.

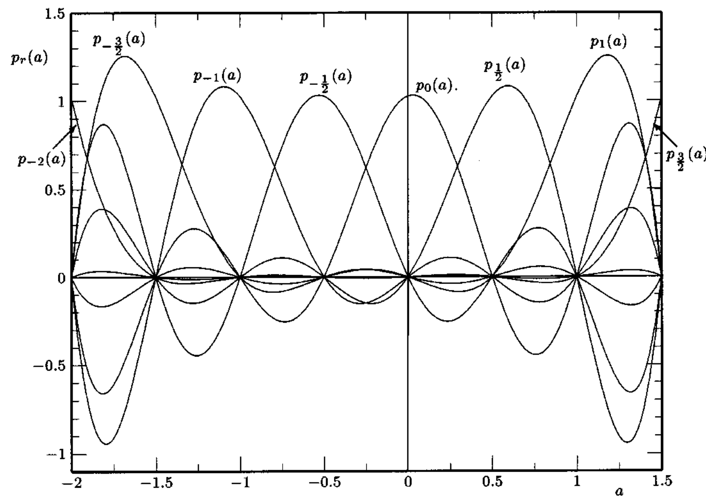

Fig. 8. Weights of FDCT for $N=8$ with fractions $-2,-(3/2),-1,-(1/2),0,(1/2),1,(3/2)$.

allow an evaluation of the FDCT by a combination of the half-DCT with the DCT. As an example, for $N=8$, the second choice gives an FDCT matrix interpolation formula

$$\begin{aligned}\mathbf{C}_a &= \sum_{r=-4}^{4}\mathbf{C}_{r/2}p_{r/2}(a)\\ &= [p_{-2}(a)+\sqrt{\mathbf{C}}p_{-3/2}(a)]\mathbf{C}^{-2}\\ &\quad+[p_{-1}(a)+\sqrt{\mathbf{C}}p_{-1/2}(a)]\mathbf{C}^{-1}\\ &\quad+[p_0(a)+\sqrt{\mathbf{C}}p_{1/2}(a)]\mathbf{I}\\ &\quad+[p_1(a)+\sqrt{\mathbf{C}}p_{1/2}(a)]\mathbf{C}.\end{aligned}$$

The FDCT can thus be evaluated by starting with the original sequence $\mathbf{s}$ and its half-DCT $\mathbf{S}_{1/2}$ and then applying two inverse DCTs and one forward DCT. It may be used, provided that a fast half-DCT algorithm is available. A possible advantage may be that the values of the weights, illustrated in Fig. 8, are small, compared with those of the previous choices.

Remark: In the above illustrations of weights, we tacitly assumed the minimal GS $\mathbf{q}=\mathbf{0}$. In general, an interpolation formula depends on the GS $\mathbf{q}$, as is clear from (23), where $\omega_n=\varphi_n+2\pi q_n$, and from (22), where the samples $\mathbf{C}_{a_r}$ depend on $\mathbf{q}$. When the "fractions" are integer, the latter are independent of $\mathbf{q}$.

V. Other Forms of DCT

Although the DCT and IDCT expressions (1) are by far the most common, many other formulations do exist. Strang [18] lists eight different forms, which we summarize in Table I. In particular, DCT-2 and DCT-3 correspond to the forward and inverse DCTs defined by (1a) and (1b), respectively. All the other transforms of Table I also have *real*, *unitary* ($\mathbf{CC}^*=\mathbf{I}$), and *orthogonal* ($\mathbf{CC}'=\mathbf{I}$) matrices. However, they differ from (1) in their eigenvalue structure.

We recognize two separate groups. The first (DCT-1,4,5,8) has eigenvalues 1 and -1 only, as can be easily derived from the symmetric structure of the matrices, which gives

$$\mathbf{C}'=\mathbf{C}\rightarrow\mathbf{C}^{-1}=\mathbf{C}\rightarrow\mathbf{C}^2=\mathbf{I}. \tag{27}$$

TABLE I
Summary of Strang's DCTs

Name	DCT Matrix Entry	Eigenvalues	Inverse DCT
DCT-1	$\sqrt{\frac{2}{N-1}}\,\nu_k\,\nu_n\,\cos(2\pi\frac{kn}{2(N-1)})$	$1,-1$	DCT-1
DCT-2	$\sqrt{\frac{2}{N}}\,\epsilon_k\,\cos(2\pi\frac{(2n+1)k}{4N})$	Quasi Quad. Symm.	DCT-3
DCT-3	$\sqrt{\frac{2}{N}}\,\epsilon_n\,\cos(2\pi\frac{n(2k+1)}{4N})$	Quasi Quad. Symm.	DCT-2
DCT-4	$\sqrt{\frac{2}{N}}\,\cos(2\pi\frac{(2n+1)(2k+1)}{8N})$	$1,-1$	DCT-4
DCT-5	$\sqrt{\frac{4}{2N-1}}\,\epsilon_n\,\epsilon_k\,\cos(2\pi\,\frac{nk}{2N-1})$	$1,-1$	DCT-5
DCT-6	$\sqrt{\frac{4}{2N-1}}\,\mu_n\,\epsilon_k\,\cos(2\pi\,\frac{(2n+1)\,2k}{8N-4})$	Quadrant Symm.	DCT-7
DCT-7	$\sqrt{\frac{4}{2N-1}}\,\epsilon_n\,\mu_k\,\cos(2\pi\,\frac{2n\,(2k+1)}{8N-4})$	Quadrant Symm.	DCT-6
DCT-8	$\sqrt{\frac{4}{2N+1}}\,\cos(2\pi\frac{(2n+1)\,(2k+1)}{8N+4})$	$1,-1$	DCT-8

$$\epsilon_i=\begin{cases}\frac{1}{\sqrt{2}} & i=0\\ 1 & i>0\end{cases}\qquad \mu_i=\begin{cases}1 & i<N-1\\ \frac{1}{\sqrt{2}} & i=N-1\end{cases}\qquad \nu_i=\epsilon_i\,\mu_i$$

Therefore, for a given order N, two separate eigenspaces exist: one for 1 and one for -1, and, for $N>2$, the dimension of at least one of these eigenspaces is greater than one [25]. This implies that there is no unique eigenvector basis and that the DCT fractionalization rule (17) has two-fold multiplicity given by

a) the choice of an orthonormal set of N eigenvectors $\mathbf{u}_n$;
b) the choice of a generating sequence $\mathbf{q}$.

An in-depth dissertation on the relation between fractional transforms derived from different choices of the eigenvector basis can be found in [19]. It deals with a similar case of FT fractionalization, having four eigenspaces with eigenvectors $\{1,-1,j,-j\}$. An explicit reference to the fractionalization of DCT-1 can be found in [17].

The second group is given by DCT-6,7, which are the inverse of each other. In particular, the DCT-6 matrix exhibits an eigenstructure that is similar to the standard DCT, with distinct eigenvalues. This implies that the orthonormal set of eigenvectors is unique so that multiplicity is confined to the choice of GS. Unlike the DCT-2, for $N = 4N_0$, the eigenvalues of the DCT-6 have an exact *quadrant* symmetry for every N_0. This can be tested by calculating the polynomial $\det(\mathbf{C} - \lambda\mathbf{I})$, which turns out to be a polynomial $P_N(\lambda^2)$ that is symmetric in λ^2. Unfortunately, although it is an elegant property, the simplifications due to quadrant symmetry do not appear to be very useful for fractionalization purposes. By setting the eigenvector basis $\mathbf{u}_n$ to that unique DCT-6 basis, the FDCT-6 can be evaluated by following the steps of the DCT-2, that is, of the standard DCT, whose fractionalization is the main topic of this paper.

VI. Possible Applications and Conclusions

Several areas of signal design and analysis can be envisaged for the FDCT. We briefly outline two possible applications.

One could be in transform coding, for data compression purposes, in which the FDCT may generalize and optimize the performance of the ordinary DCT. We can search for optimum values of the fraction a, with regard to compression capability (in two-dimensional applications, two fractions could be optimized). In some preliminary work on the well-known test image "Lena," using FDCT's on 8×8 pixel blocks, the fraction $a = 1.01$ gave the best compression gain, but the improvement on the ordinary DCT was disappointing (0.012 dB).

A second application may be in cryptography. The exact recovery of a sequence from its FDCT requires that both the fraction a and the GS $\mathbf{q}$ be known. If an FDCT $\mathbf{S}_a$, produced with the fraction a, is recovered with a fraction b, the result is the sequence $\mathbf{S}_{a-b}$, which is different from the original sequence $\mathbf{s}$, whenever $b \neq a$. If an FDCT $\mathbf{S}_a = \mathbf{S}_a(\mathbf{q})$, produced with the GS $\mathbf{q}$, is recovered with another GS $\mathbf{p} \neq \mathbf{q}$, the result is again incorrect. In fact, the overall matrix is given by

$$\begin{aligned}\mathbf{C}_{-a}(\mathbf{p})\mathbf{C}_a(\mathbf{q}) &= \sum_m \sum_n \mathbf{U}_m \mathbf{U}_n e^{-j(\varphi_m + 2\pi p_m - \varphi_n - 2\pi q_m)a} \\ &= \sum_m \mathbf{U}_m e^{j2\pi(q_m - p_m)}\end{aligned}$$

where we used the orthonormality of the matrices $\mathbf{U}_m$ [see (4)]. Since $\sum_m \mathbf{U}_m = \mathbf{I}$, a correct recovery requires that $\mathbf{p} = \mathbf{q}$. In conclusion, for encryption, we can use both the fraction a, which is a real parameter, and the GS $\mathbf{q}$, which may have the cardinality of $\mathbb{Z}^{N/2}$ (we recall that for a rational fraction L/M, the number of effective GSs is $M^{N/2}$).

As a closing note, we believe that if the aforementioned applications tourned out to be really attractive, an effort should be made to find fast algorithms for practical implementation.

Appendix
Other Representations of DCT and FDCT Matrices

The complex representation (3) holds for the class of *unitary* matrices. More specifically, since $\mathbf{C}$ is *real orthogonal*, a real representation may be considered. Let $\mathbf{a}_n = \sqrt{2}\Re[\mathbf{u}_n]$ and $\mathbf{b}_n = \sqrt{2}\Im[\mathbf{u}_n]$, and let $\mathbf{T}$ be the real $N \times N$ matrix $\mathbf{T} = [\mathbf{a}_1, \mathbf{b}_1, \ldots, \mathbf{a}_{N/2}, \mathbf{b}_{N/2}]$. Then, $\mathbf{T}$ is real orthogonal ($\mathbf{T}\mathbf{T}' = \mathbf{I}$), and its columns form an orthogonal basis in $\mathbb{R}^N$ [20, p. 278]. The matrix $\mathbf{C}$ can be written as

$$\mathbf{C} = \mathbf{T}\mathbf{D}\mathbf{T}' \tag{28}$$

where $\mathbf{D}$ is a quasidiagonal matrix, consisting of 2×2 blocks along the main diagonal

$$\mathbf{D} = \text{diag}[\mathbf{D}_1, \ldots, \mathbf{D}_{N/2}], \quad \mathbf{D}_n = \begin{bmatrix} \cos\varphi_n & \sin\varphi_n \\ -\sin\varphi_n & \cos\varphi_n \end{bmatrix}.$$

By expanding the blocks in (28), we obtain the trigonometric form given by (9), with $\mathbf{A}_n = \mathbf{a}_n\mathbf{a}_n' + \mathbf{b}_n\mathbf{b}_n'$ and $\mathbf{B}_n = \mathbf{a}_n\mathbf{b}_n' - \mathbf{b}_n\mathbf{a}_n'$.

Considering that each $\mathbf{D}_n$ can be written in the exponential form

$$\mathbf{D}_n = e^{\mathbf{E}_n} \quad \text{with} \quad \mathbf{E}_n = \begin{bmatrix} 0 & \varphi_n \\ -\varphi_n & 0 \end{bmatrix}$$

we also have $\mathbf{D} = e^{\mathbf{E}}$ with $\mathbf{E} = \text{diag}[\mathbf{E}_1, \ldots, \mathbf{E}_{N/2}]$, and hence

$$\mathbf{C} = \mathbf{T}e^{\mathbf{E}}\mathbf{T}'.$$

Note that the matrices $\mathbf{E}_n$ are not unique since we can add an arbitrary multiple of 2π to each φ_n, without changing $e^{\mathbf{E}}$, that is, we can replace φ_n by $\omega_n = \varphi_n + 2\pi q_n$, as was done in Section III. Note also that $\mathbf{D}$ is real *orthogonal* and $\mathbf{E}$ is *skew-symmetric*, i.e., with real elements, and $e_{ji} = -e_{ij}$.

We finally note that the DCT matrix $\mathbf{C}$ can be directly expressed in terms of a skew-symmetric matrix $\mathbf{Q}$ according to

$$\mathbf{C} = (\mathbf{I} - \mathbf{Q})(\mathbf{I} + \mathbf{Q})^{-1} \quad \text{with} \quad \mathbf{Q} = (\mathbf{I} - \mathbf{C})(\mathbf{I} + \mathbf{C})^{-1}. \tag{29}$$

These are Cayley's formulas, which establish a connection between orthogonal and skew-symmetric matrices.

Since the FDCT matrix $\mathbf{C}_a$ has exactly the same properties as the real orthogonal DCT matrix, the above representations can be used for $\mathbf{C}_a$. In particular, we have

$$\mathbf{C}_a = \mathbf{T}e^{a\mathbf{E}}\mathbf{T}'$$

where we need to introduce a generic GS $\mathbf{q}$ in the definition of $\mathbf{E}$, that is

$$\mathbf{E}_n = \begin{bmatrix} 0 & \omega_n \\ -\omega_n & 0 \end{bmatrix} \quad \text{with} \quad \omega_n = \varphi_n + 2\pi q_n$$

to take multiplicity into account.

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References

[1] V. Namias, "The fractional order Fourier transform and its applications to quantum mechanics," *Inst. Math. Appl.*, vol. 25, pp. 241–265, 1980.

[2] A. C. McBride and F. H. Kerr, "On Namias's fractional Fourier transform," *IMA J. Appl. Math.*, vol. 39, pp. 159–175, 1987.

[3] A. W. Lohmann, "Image rotation, wigner rotation, and the fractional Fourier transform," *J. Opt. Soc. Amer. A*, vol. 10, no. 10, pp. 2181–2186, Oct. 1993.

[4] L. M. Bernardo and O. D. D. Soares, "Fractional Fourier transforms and imaging," *J. Opt. Soc. Amer. A*, vol. 11, no. 10, pp. 2622–2626, Oct. 1994.
[5] H. M. Ozaktas and D. Mendlovic, "Fractional Fourier optics," *J. Opt. Soc. Amer. A*, vol. 12, no. 4, pp. 743–751, Apr. 1995.
[6] D. Mendlovic and H. M. Ozaktas, "Fractional Fourier transforms and their optical implementation: I," *J. Opt. Soc. Amer. A*, vol. 10, no. 9, pp. 1875–1881, Sept. 1993.
[7] H. M. Ozaktas and D. Mendlovic, "Fractional Fourier transforms and their optical implementation: II," *J. Opt. Soc. Amer. A*, vol. 10, no. 12, pp. 2522–2531, Dec. 1993.
[8] A. W. Lohmann and B. H. Soffer, "Relationship between the Radon-Wigner and fractional Fourier transforms," *J. Opt. Soc. Amer. A*, vol. 11, no. 6, pp. 1798–1801, June 1994.
[9] G. Cariolaro, T. Erseghe, P. Kraniauskas, and N. Laurenti, "A unified framework for the fractional Fourier transform," *IEEE Trans. Signal Processing*, vol. 46, pp. 3206–3219, Dec. 1998.
[10] S. C. Pei, M. H. Yeh, and C. C. Tseng, "Discrete fractional Fourier transform based on orthogonal projections," *IEEE Trans. Signal Processing*, vol. 47, pp. 1335–1348, May 1999.
[11] L. Barker, C. Candan, and T. Hakioglu *et al.*, "The discrete harmonic oscillator, Harper's equation and the discrete fractional Fourier transform," *J. Phys. A-Meth. Gen.*, vol. 33, pp. 2209–2222, Mar. 2000.
[12] C. Candan, M. A. Kutay, and H. M. Ozaktas, "The discrete fractional Fourier transform," *IEEE Trans. Signal Processing*, vol. 48, pp. 1329–1337, May 2000.
[13] S. C. Pei and J. J. Ding, "Closed-form discrete fractional and affine Fourier transforms," *IEEE Trans. Signal Processing*, vol. 48, pp. 1338–1353, May 2000.
[14] W. B. Pennebaker and J. L. Mitchell, *JPEG Still Image Data Compression Standard*. New York: Van Nostrand Reinhold, 1993.
[15] B. Haskell, A. Puri, and A. N. Netravali, *Digital Video: An Introduction to MPEG-2*. London, U.K.: Chapman & Hall, 1997.
[16] J. L. Mitchell, *MPEG Video Compression Standard*. London, U.K.: Chapman & Hall, 1997.
[17] S. C. Pei and M. H. Yeh, "The discrete fractional cosine and sine transforms," *IEEE Trans. Signal Processing*, vol. 49, pp. 1198–1207, June 2001.
[18] G. Strang, "The discrete cosine transform," *SIAM Rev.*, vol. 41, no. 1, pp. 135–147, 1999.
[19] G. Cariolaro, T. Erseghe, P. Kraniauskas, and N. Laurenti, "Multiplicity of fractional Fourier transforms and their relationships," *IEEE Trans. Signal Processing*, vol. 48, pp. 227–241, Jan. 2000.
[20] F. R. Gantmacher, *The Theory of Matrices*. New York: Chelsea, 1960, vol. I.
[21] B. W. Dickinson and K. Steiglitz, "Eigenvectors and functions of the discrete Fourier transform," *IEEE Trans. Acoust., Speech, Signal Processing*, vol. ASSP-30, pp. 25–31, Jan. 1982.
[22] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, MA: Cambridge Univ. Press, 1985.
[23] ——, *Topics in Matrix Analysis*. Cambridge, MA: Cambridge Univ. Press, 1991.
[24] R. J. Clark, *Transform Coding of Images*, London, U.K.: Academic, 1985.
[25] R. Bellmann, *Introduction to Matrix Analysis*. New York: McGraw-Hill, 1970.

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