Homework 0

No special notes to consider for grading.

Problem 1

1. Answer: B and D belong to the $SE(3)$ group of valid transforms, but A and C do *not* belong to $SE(3)$ group of valid transforms. See the following justifications for each matrix.

   Matrix A

   A does not have the form $\begin{bmatrix} R & r \\ 0 & 1 \end{bmatrix}$ where $R \in \mathbb{R}^{3x3}$, $r \in \mathbb{R}^3$. Therefore, $A \notin SE(3)$.

   The remaining matrices B, C, and D do have the required form above.

   Matrix B

   $$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$$

   $$\det(R) = (1)\det \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = 0(0) - 1(-1) = 1$$

   $$R^T R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

   $$RR^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

   Therefore, $B \in SE(3)$.

   Matrix C

   $$R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$\det(R) = (1)\det\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = 0(0) - (-1)(-1) = -1 \neq 1$$

Therefore, $C \notin SE(3)$.

Matrix D

$$R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\det(R) = (-1)\det\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -(0(0) - 1(1)) = 1$$

$$R^T R = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

$$RR^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Therefore, $D \in SE(3)$.

2. $SE(3) \cap \{A, B, C, D\} = \{B, D\}$

Matrix B

$$[B|I] = \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] R_2 \leftrightarrow -R_3$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -3 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{r} R_1 \to R_1 - R_4 \\ R_2 \to R_2 - 5R_4 \\ R_3 \to R_3 + 3R_4 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 1 & 0 & 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = [I|B^{-1}]$$

$$\Rightarrow B^{-1} = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -5 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$B^{-1}B = \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -5 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$BB^{-1} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 1 & -5 \\ 0 & -1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Matrix D

$$[D|I] = \left[\begin{array}{cccc|cccc} -1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{r} R_1 \to -R_1 \\ R_2 \leftrightarrow R_3 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & -1 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 5 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] \begin{array}{r} R_1 \to R_1 + R_4 \\ R_2 \to R_2 - 5R_4 \\ R_3 \to R_3 - 3R_4 \end{array}$$

$$= \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \end{array}\right] = [I|D^{-1}]$$

$$\Rightarrow D^{-1} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$D^{-1}D = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$$DD^{-1} = \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & -5 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

Problem 2

1. $\begin{bmatrix} {}^1p \\ 1 \end{bmatrix} = {}^1T_2 \begin{bmatrix} {}^2p \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -3 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 6 \\ 1 \end{bmatrix} = \begin{bmatrix} -1(4) \\ -1(2) - 3(1) \\ 1(6) - 10(1) \\ 1(1) \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ -4 \\ 1 \end{bmatrix}$

   Thus ${}^1p = \begin{bmatrix} 4 & -5 & -4 \end{bmatrix}^T$.

2. ${}^0T_2 = {}^0T_1{}^1T_2 = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 10 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -3 \\ 0 & 0 & 1 & -10 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$

   Here $r = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, so there is no translation, and $R = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}$.

   Match R to the form

   $$\begin{bmatrix} \cos\alpha\cos\beta & \cos\alpha\sin\beta\sin\gamma - \sin\alpha\cos\gamma & \cos\alpha\sin\beta\sin\gamma + \sin\alpha\sin\gamma \\ \sin\alpha\cos\beta & \sin\alpha\sin\beta\sin\gamma + \cos\alpha\cos\gamma & \sin\alpha\sin\beta\sin\gamma - \cos\alpha\sin\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\sin\gamma \end{bmatrix}$$

   From entry $(3, 1)$, $-\sin\beta = -1 \Rightarrow \beta = 90°$

   Thus $R = \begin{bmatrix} 0 & -\cos\alpha\sin\gamma - \sin\alpha\cos\gamma & -\cos\alpha\sin\gamma + \sin\alpha\sin\gamma \\ 0 & -\sin\alpha\sin\gamma + \cos\alpha\cos\gamma & -\sin\alpha\sin\gamma - \cos\alpha\sin\gamma \\ 1 & 0 & 0 \end{bmatrix}$

   Trying $\alpha = 90°$ and $\gamma = 0$ matches the two matrices. Thus, $\alpha = \beta = 90°$ and $\gamma = 0$.

   Therefore, this transform consists of rotations of 90° about the y-axis and 90° about the z-axis.

Problem 3

1. **Pose** ${}^mT_w$

   The steps in intrinsic coordinates are:

   (a) Rotate $+90°$ about the x-axis of frame m.

   Thus $\alpha = +90°, \sin\alpha = 1, \cos\gamma = 0$ and $R_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix}$.

(b) Rotate $-90°$ about the y-axis of the previous frame.

Thus $\beta = -90°, \sin\beta = -1, \cos\beta = 0$ and $R_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$.

(c) In frame m, translate +4 in x and +6 in y. Thus $r = \begin{bmatrix} 4 & 6 & 0 \end{bmatrix}^T$.

Thus $R = R_1 R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & -1 \\ -1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and

$$^{m}T_w = \begin{bmatrix} 0 & 0 & -1 & 4 \\ -1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Pose $^{e}T_m$
The steps are:

(a) Rotate $+90°$ about the y-axis of frame e.

Thus $\beta = +90°, \sin\beta = 1, \cos\beta = 0$ and $R = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$.

(b) In frame e, translate -10 in x. Thus $r = \begin{bmatrix} -10 & 0 & 0 \end{bmatrix}^T$.

Thus $^{e}T_m = \begin{bmatrix} 0 & 0 & 1 & -10 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

Pose $^{e}T_w$
The steps are:

(a) Rotate $-90°$ about the z-axis of frame e.

Thus $\alpha = -90°, \sin\alpha = -1, \cos\alpha = 0$ and $R = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) In frame e, translate -10 in x, +6 in y, and -4 in z. Thus $r = \begin{bmatrix} -10 & 6 & 4 \end{bmatrix}^T$.

Thus $^{e}T_w = \begin{bmatrix} 0 & 1 & 0 & -10 \\ -1 & 0 & 0 & 6 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

2. $$ {}^eT_m {}^mT_w = \begin{bmatrix} 0 & 0 & 1 & -10 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & -1 & 4 \\ -1 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & -10 \\ -1 & 0 & 0 & 6 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^eT_w. $$

Pose ${}^wT_e$

The steps are:

(a) Rotate $+90°$ about the z-axis of frame w.

Thus $\alpha = +90°, \sin\alpha = 1, \cos\alpha = 0$ and $R = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) In frame w, translate $+6$ in x, $+10$ in y, and $+4$ in z.
Thus $r = \begin{bmatrix} 6 & 10 & 4 \end{bmatrix}^T$.

Thus $${}^wT_e = \begin{bmatrix} 0 & -1 & 0 & 6 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

$$ {}^eT_w {}^wT_e = \begin{bmatrix} 0 & 1 & 0 & -10 \\ -1 & 0 & 0 & 6 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 6 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I $$

$$ {}^wT_e {}^eT_w = \begin{bmatrix} 0 & -1 & 0 & 6 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & -10 \\ -1 & 0 & 0 & 6 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I $$

By the definition of the matrix inverse, if $AB = BA = I$ for two square matrices A and B, then A and B are inverses of each other, i.e., $B = A^{-1}$ and $A = B^{-1}$.

Given that ${}^eT_w {}^wT_e = {}^wT_e {}^eT_w = I$, it follows that ${}^eT_w = ({}^wT_e)^{-1}$.

Problem 4

No additional comments.