

Quantum Mechanics

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1 Preliminaries

Quantum mechanics is characterized by the following principles:

1. The quantum state space, \mathcal{H} , is a complex, separable (has a countable, dense subset) Hilbert space. A quantum state $\phi \in \mathcal{H}$ satisfies $\langle \phi, \phi \rangle = 1$
2. The observable/measurable quantities of quantum mechanics are represented by Hermitian operators on \mathcal{H} , that is $L : \mathcal{H} \rightarrow \mathcal{H}$ s.t. $L = L^*$
3. The possible results of a measurement are the eigenvalues of the corresponding operator.
4. Given an observable L with eigenvalues $\{\lambda_i\}$ and corresponding eigenspaces $\{E_i\}$, if the state vector of the system is $\Psi \in \mathcal{H}$, then the probability of observing λ_i is

$$\mathbb{P}(\lambda_i) = \|P_{E_i}\Psi\|^2 \quad (1)$$

where $P_{E_i}\Psi$ is the projection of Ψ onto E_i .

5. The quantum state evolves across time according to a unitary operator U ($UU^* = U^*U = I$) such the

$$\Psi(t) = U(t)\Psi(0) \quad (2)$$

where $\Psi(t)$ is the quantum state at time t and $\Psi(0)$ is the quantum state at time 0. U is smooth in t .

If we define the Hamiltonian $H(t)$ as

$$H(t) = \frac{1}{i\hbar} \frac{\partial U}{\partial t}(t) \quad (3)$$

where \hbar is Planck's constant, then we have

$$i\hbar \frac{\partial \Psi(t)}{\partial t} = H(t)\Psi(t) \quad (4)$$

the generalized Schrödinger equation.

2 A single qubit

The simplest quantum system is that of a single qubit: for instance, the quantum spin of a single particle. In this case V is isomorphic to \mathbb{C}^2 and we let $\mathcal{B} = \{|0\rangle, |1\rangle\}$ denote an orthonormal basis of V .

Definition 2.1 (Spin Operators σ_x , σ_y and σ_z). The three spin operators σ_x , σ_y and σ_z are defined by

$$\begin{aligned}\sigma_x |0\rangle &= |1\rangle \text{ and } \sigma_x |1\rangle = |0\rangle \\ \sigma_y |0\rangle &= i |1\rangle \text{ and } \sigma_y |1\rangle = -i |0\rangle \\ \sigma_z |0\rangle &= |0\rangle \text{ and } \sigma_z |1\rangle = -|1\rangle\end{aligned}$$

Definition 2.2 (Pauli Matrices). The matrix representations of the spin operators σ_x , σ_y and σ_z in the basis \mathcal{B} are

$$[\sigma_x]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, [\sigma_y]_{\mathcal{B}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, [\sigma_z]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These are the **Pauli matrices**.

From the Pauli matrices it is easy to see that any Hermetian operator $L \in \text{span}\{I, \sigma_x, \sigma_y, \sigma_z\}$.

The normalized eigenvectors of σ_z are obviously $|0\rangle$ and $|1\rangle$ with corresponding eigenvalues of 1 and -1. Likewise, it is easily verified that the normalized eigenvectors of σ_x are

$$\frac{1}{\sqrt{2}} (|0\rangle \pm |1\rangle)$$

with eigenvalues ± 1 . Finally, one can also verify that the normalized eigenvectors of σ_y are

$$\frac{1}{\sqrt{2}} (|0\rangle \pm i |1\rangle)$$

with eigenvalues ± 1 .

3 Entanglement

Consider two systems, A and B , of a single qubit each. The two systems have state spaces V_A and V_B , respectively, and orthonormal bases \mathcal{B}_A and \mathcal{B}_B , respectively. We can construct a combined system with state space $V = V_A \otimes V_B$ and basis $\mathcal{B} = \mathcal{B}_A \otimes \mathcal{B}_B$. Note that V is isomorphic to $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ and in the basis \mathcal{B} :

$$[|0_A 0_B\rangle]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, [|0_A 1_B\rangle]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, [|1_A 0_B\rangle]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } [|1_A 1_B\rangle]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Definition 3.1 (Singlet State). The singlet state, $|s\rangle$, of this system is

$$|s\rangle = \frac{1}{\sqrt{2}} (|0_A 1_B\rangle - |1_A 0_B\rangle)$$

In the basis \mathcal{B} :

$$[|s\rangle]_{\mathcal{B}} = \frac{1}{\sqrt{2}} \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Now observe that

$$\begin{aligned} \langle s | \sigma_z \otimes I | s \rangle &= \frac{1}{\sqrt{2}} \langle s | (|0_A 1_B\rangle + |1_A 0_B\rangle) \\ &= \frac{1}{2} (\langle 0_A 1_B | 0_A 1_B \rangle + \langle 0_A 1_B | 1_A 0_B \rangle - \langle 1_A 0_B | 0_A 1_B \rangle - \langle 1_A 0_B | 1_A 0_B \rangle) \\ &= \frac{1}{2} (1 + 0 - 0 - 1) \\ &= 0 \end{aligned} \quad (5)$$

$$\begin{aligned} \langle s | \sigma_x \otimes I | s \rangle &= \frac{1}{\sqrt{2}} \langle s | (|1_A 1_B\rangle - |0_A 0_B\rangle) \\ &= \frac{1}{2} (\langle 0_A 1_B | 1_A 1_B \rangle - \langle 0_A 1_B | 0_A 0_B \rangle - \langle 1_A 0_B | 1_A 1_B \rangle + \langle 1_A 0_B | 0_A 0_B \rangle) \\ &= \frac{1}{2} (0 - 0 - 0 + 0) \\ &= 0 \end{aligned} \quad (6)$$

$$\begin{aligned} \langle s | \sigma_y \otimes I | s \rangle &= \frac{1}{\sqrt{2}} \langle s | (i |1_A 1_B\rangle + i |0_A 0_B\rangle) \\ &= \frac{1}{2} (i \langle 0_A 1_B | 1_A 1_B \rangle + i \langle 0_A 1_B | 0_A 0_B \rangle - i \langle 1_A 0_B | 1_A 1_B \rangle - i \langle 1_A 0_B | 0_A 0_B \rangle) \\ &= \frac{1}{2} (0 + 0 - 0 - 0) \\ &= 0 \end{aligned} \quad (7)$$

Let L be a Hermetian operator on V_A and consider the observable defined by $L \otimes I$. Since L is Hermetian, there exists $a, b, c, d \in \mathbb{C}$ such that $L = aI + b\sigma_x + c\sigma_y + d\sigma_z$