Classical Mechanics

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1 Preliminaries

Definition 1.1 (Mechanical System)**.** A **Mechanical System** is a tuple (M, L) where

1. M is the configuration space - a manifold in a vector space of generalized coordinates
2. L is the Lagrangian - a function $L : TM \times \mathbb{R}_t \to \mathbb{R}$ where TM is the tangent bundle to M and $\mathbb{R}_t \subseteq \mathbb{R}$

In many applications, the Lagrangian is equal to the kinetic energy minus the potential energy of the system.

Definition 1.2 (Principle of Stationary Action)**.** Given time instances t_1 and t_2 ($t_1 < t_2$), the time evolution of the mechanical system (M, L) is given by a path $q_0 : [t_1, t_2] \to M$ that minimizes

$$A(q) = \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt \tag{1}$$

The path q_0 is a **stationary point**.

2 Deriving Lagrange's Equations

Consider adding a variation η to a stationary point q_0. Then

$$\begin{aligned}
0 = \frac{d}{d\epsilon} A(q_0 + \epsilon\eta)|_{\epsilon=0} &= \frac{d}{d\epsilon} \int_{t_1}^{t_2} L(q_0 + \epsilon\eta, \dot{q}_0 + \epsilon\dot{\eta}, t)\, dt|_{\epsilon=0} \\
&= \int_{t_1}^{t_2} \left[L_q(q, \dot{q}, t)\,\eta + L_{\dot{q}}(q, \dot{q}, t)\,\dot{\eta}\right] dt \\
&= \int_{t_1}^{t_2} L_q(q, \dot{q}, t)\,\eta dt + L_{\dot{q}}(q, \dot{q}, t)\,\eta|_{t_1}^{t_2} - \int_{t_1}^{t_2} \frac{\partial L_{\dot{q}}}{\partial t}(q, \dot{q}, t)\,\eta dt \\
&= \int_{t_1}^{t_2} \left[L_q(q, \dot{q}, t) - \frac{\partial L_{\dot{q}}}{\partial t}(q, \dot{q}, t)\right] \eta dt
\end{aligned} \tag{2}$$

By the fundamental lemma of the calculus of variations, it follows that

$$L_q\left(q,\dot{q},t\right)=\frac{\partial L_{\dot{q}}}{\partial t}\left(q,\dot{q},t\right) \tag{3}$$

3 Deriving Hamilton's Equations

Consider a physical system comprised of trajectories q in configuration space, velocities $\dot{q}$ and Lagrangian $L\left(q,\dot{q}\right)$. The conjugate momenta to the trajectories is

$$p=\frac{\partial L}{\partial \dot{q}}\left(q,\dot{q}\right) \tag{4}$$

Under certain regularity conditions (the conditions of the Inverse Function Theorem), we may invert this map to recover $\dot{q}$ from p and q:

$$\dot{q}=f\left(q,p\right) \tag{5}$$

The Hamiltonian for this system is

$$H\left(q,p\right)=\sum_i p_i f_i\left(q,p\right)-L\left(q,f\left(q,p\right)\right) \tag{6}$$

By adding a variation to individual components of the phase space, we find

$$\begin{aligned}\frac{\partial H}{\partial p_j}\eta &= \frac{\partial}{\partial \epsilon}H\left(q,p_{-j},p_j+\epsilon\eta\right)|_{\epsilon=0}\\ &= \sum_i p_i\frac{\partial f_i}{\partial p_j}\eta+f_j\eta-\sum_i\frac{\partial L}{\partial \dot{q}_i}\frac{\partial f_i}{\partial p_j}\eta\\ &= f_j\eta\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{\partial H}{\partial q_j}\eta &= \frac{\partial}{\partial \epsilon}H\left(q_{-j},q_j+\epsilon\eta,p\right)|_{\epsilon=0}\\ &= \sum_i p_i\frac{\partial f_i}{\partial q_j}\eta-\sum_i\frac{\partial L}{\partial \dot{q}_i}\frac{\partial f_i}{\partial q_j}\eta-\frac{\partial L}{\partial q_j}\eta\\ &= -\frac{\partial L}{\partial q_j}\eta\end{aligned} \tag{8}$$

It follows that

$$\frac{\partial H}{\partial p_j}=f_j=\dot{q}_j \tag{9}$$

In addition, if the trajectories through phase space minimize the action of the system, then

$$\frac{\partial L}{\partial q_i}(q,\dot{q}) = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i}(q,\dot{q}) \tag{10}$$

so that

$$\frac{\partial H}{\partial q_i} = -\dot{p}_i \tag{11}$$

We have derived Hamilton's equations:

$$\frac{\partial H}{\partial p_j} = \dot{q}_j \qquad \text{and} \qquad \frac{\partial H}{\partial q_i} = -\dot{p}_i \tag{12}$$

4 Poisson Brackets

Two equivalent definitions of the Poisson Bracket:

$$\{F,G\} = \sum_i \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i} \tag{13}$$

Axiomatic definition of Poisson Bracket:

1. Anticommutativity: $\forall f,g : \{f,g\} = -\{g,f\}$
2. Linearity: $\forall a,b \in \mathbb{R}, \forall f,g,h : \{af+bg,h\} = a\{f,h\} + b\{g,h\}$
3. Leibniz's Rule: $\forall f,g,h : \{fg,h\} = f\{g,h\} + g\{f,h\}$
4. Canonical Coordinates: $\forall i,j : \{q_i,q_j\} = 0, \{p_i,p_j\} = 0, \{q_i,p_j\} = \delta_{ij}$

5 Curl and Divergence

Suppose $\overrightarrow{v} = (f(x,y,z), g(x,y,z), h(x,y,z))$ is a vector field such that $\overrightarrow{\nabla} \cdot \overrightarrow{v} = f_x + g_y + h_z = 0$. Then there is another vector field $\overrightarrow{w}$, such that $\overrightarrow{v} = \overrightarrow{\nabla} \times \overrightarrow{w}$. Let

$$\overrightarrow{w}_1 = \left(0, \int_0^x h(t,y,z)\,dt, -\int_0^x g(t,y,z)\,dt\right)$$

Now

$$\begin{aligned}
\overrightarrow{\nabla} \times \overrightarrow{w}_1 &= \left(\frac{-\partial \int_0^x g(t,y,z)\,dt}{\partial y} - \frac{\partial \int_0^x h(t,y,z)\,dt}{\partial z}, \frac{\partial \int_0^x g(t,y,z)\,dt}{\partial x}, \frac{\partial \int_0^x h(t,y,z)\,dt}{\partial x}\right) \\
&= \left(-\int_0^x \left[\frac{\partial g}{\partial y}(t,y,z) + \frac{\partial h}{\partial x}(t,y,z)\right] dt, g(x,y,z), h(x,y,z)\right) \\
&= (A(x,y,z), g(x,y,z), h(x,y,z))
\end{aligned} \tag{14}$$

where

$$A\left(x,y,z\right)=-\int_{0}^{x}\left[\frac{\partial g}{\partial y}\left(t,y,z\right)+\frac{\partial h}{\partial x}\left(t,y,z\right)\right]dt \tag{15}$$

Let

$$\overrightarrow{v}_1=\overrightarrow{v}-\overrightarrow{\bigtriangledown}\times\overrightarrow{w}_1=\left(f-A,0,0\right) \tag{16}$$

Then

$$\frac{\partial}{\partial x}\left(f-A\right)=\overrightarrow{\bigtriangledown}\cdot\overrightarrow{v}_1=\overrightarrow{\bigtriangledown}\cdot\overrightarrow{v}-\overrightarrow{\bigtriangledown}\cdot\overrightarrow{\bigtriangledown}\times\overrightarrow{w}_1=0 \tag{17}$$

so that $f-A$ does NOT depend on x. Let $k\left(y,z\right)=f-A$ and

$$\overrightarrow{w}_2=\left(0,0,\int_{0}^{y}k\left(t,z\right)dt\right) \tag{18}$$

Then

$$\overrightarrow{\bigtriangledown}\times\overrightarrow{w}_2=\left(k\left(y,z\right),0,0\right)=\left(f-A,0,0\right) \tag{19}$$

Finally,

$$\begin{aligned}\overrightarrow{\bigtriangledown}\times\left(\overrightarrow{w}_1+\overrightarrow{w}_2\right)&=\overrightarrow{\bigtriangledown}\times\overrightarrow{w}_1+\overrightarrow{\bigtriangledown}\times\overrightarrow{w}_2\\&=\left(A,g,h\right)+\left(f-A,0,0\right)\\&=\left(f,g,h\right)\end{aligned} \tag{20}$$

so that $\overrightarrow{v}=\overrightarrow{\bigtriangledown}\times\left(\overrightarrow{w}_1+\overrightarrow{w}_2\right)$