Quantum Mechanics

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1 Preliminaries

Quantum mechanics is characterized by the following principles:

1. The quantum state space, $\mathcal{H}$, is a complex, separable Hilbert space. A quantum state $\phi \in \mathcal{H}$ satisfies $\langle \phi, \phi \rangle = 1$
2. The observable/measurable quantities of quantum mechanics are represented by Hermitian operators on $\mathcal{H}$, that is $L : \mathcal{H} \to \mathcal{H}$ $s.t.$ $L = L^*$
3. The possible results of a measurement are the eigenvalues of the corresponding operator.
4. Given an observable L with eigenvalues $\{\lambda_i\}$ and corresponding eigenspaces $\{E_i\}$, if the state vector of the system is $|\Psi\rangle \in \mathcal{H}$, then the probability of observing λ_i is
$$\mathbb{P}\left(\lambda_i\right) = \left|\left|P_{E_i}\Psi\right\rangle\right|^2 \tag{1}$$
where $|P_{E_i}\Psi\rangle$ is the projection of $|\Psi\rangle$ onto E_i.
5. The quantum state evolves across time according to a unitary operator U ($UU^* = U^*U = I$) such the
$$|\Psi\left(t\right)\rangle = U\left(t\right)|\Psi\left(0\right)\rangle \tag{2}$$
where $|\Psi\left(t\right)\rangle$ is the quantum state at time t and $|\Psi\left(0\right)\rangle$ is the quantum state at time 0. U is smooth in t.

If we define the Hamiltonian $H\left(t\right)$ as

$$H\left(t\right) = \frac{1}{i\hbar}\frac{\partial U}{\partial t}\left(t\right) \tag{3}$$

where $\hbar$ is Planck's constant, then we have

$$i\hbar\frac{\partial|\Psi\left(t\right)\rangle}{\partial t} = H\left(t\right)|\Psi\left(t\right)\rangle \tag{4}$$

the generalized Schrödinger equation.

2 A single qubit

The simplest quantum system is that of a single qubit: for instance, the quantum spin of a single particle. In this case V is isomorphic to $\mathbb{C}^2$ and we let $\mathcal{B} = \{|0\rangle, |1\rangle\}$ denote an orthonormal basis of V.

Definition 2.1 (Spin Operators σ_x, σ_y and σ_z)**.** The three spin operators σ_x, σ_y and σ_z are defined by

$$\sigma_x |0\rangle = |1\rangle \text{ and } \sigma_x |1\rangle = |0\rangle$$
$$\sigma_y |0\rangle = i|1\rangle \text{ and } \sigma_y |1\rangle = -i|0\rangle$$
$$\sigma_z |0\rangle = |0\rangle \text{ and } \sigma_z |1\rangle = -|1\rangle$$

Definition 2.2 (Pauli Matrices)**.** The matrix representations of the spin operators σ_x, σ_y and σ_z in the basis $\mathcal{B}$ are

$$[\sigma_x]_{\mathcal{B}} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \ [\sigma_y]_{\mathcal{B}} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \ [\sigma_z]_{\mathcal{B}} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

These are the **Pauli matrices**.

From the Pauli matrices it is easy to see that any Hermetian operator $L \in span\{I, \sigma_x, \sigma_y, \sigma_z\}$.

The normalized eigenvectors of σ_z are obviously $|0\rangle$ and $|1\rangle$ with corresponding eigenvalues of 1 and -1. Likewise, it is easily verified that the normalized eigenvectors of σ_x are

$$\frac{1}{\sqrt{2}}(|0\rangle \pm |1\rangle)$$

with eigenvalues ± 1. Finally, one can also verify that the normalized eigenvectors of σ_y are

$$\frac{1}{\sqrt{2}}(|0\rangle \pm i|1\rangle)$$

with eigenvalues ± 1.

3 Entanglement

Consider two systems, A and B, of a single qubit each. The two systems have state spaces V_A and V_B, respectively, and orthonormal bases $\mathcal{B}_A$ and $\mathcal{B}_B$, respectively. We can construct a combined system with state space $V = V_A \otimes V_B$ and basis $\mathcal{B} = \mathcal{B}_A \otimes \mathcal{B}_B$. Note that V is isomorphic to $\mathbb{C}^2 \otimes \mathbb{C}^2 = \mathbb{C}^4$ and in the basis $\mathcal{B}$:

$$[|0_A 0_B\rangle]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \ [|0_A 1_B\rangle]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ [|1_A 0_B\rangle]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } [|1_A 1_B\rangle]_{\mathcal{B}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Definition 3.1 (Singlet State)**.** The singlet state, $|s\rangle$, of this system is

$$|s\rangle = \frac{1}{\sqrt{2}}\left(|0_A 1_B\rangle - |1_A 0_B\rangle\right)$$

In the basis $\mathcal{B}$:

$$[|s\rangle]_{\mathcal{B}} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}$$

Now observe that

$$\begin{aligned}
\langle s|\, \sigma_z \otimes I \,|s\rangle &= \frac{1}{\sqrt{2}} \langle s| \left(|0_A 1_B\rangle + |1_A 0_B\rangle\right) \\
&= \frac{1}{2}\left(\langle 0_A 1_B | 0_A 1_B\rangle + \langle 0_A 1_B | 1_A 0_B\rangle - \langle 1_A 0_B | 0_A 1_B\rangle - \langle 1_A 0_B | 1_A 0_B\rangle\right) \\
&= \frac{1}{2}\left(1 + 0 - 0 - 1\right) \\
&= 0
\end{aligned} \tag{5}$$

$$\begin{aligned}
\langle s|\, \sigma_x \otimes I \,|s\rangle &= \frac{1}{\sqrt{2}} \langle s| \left(|1_A 1_B\rangle - |0_A 0_B\rangle\right) \\
&= \frac{1}{2}\left(\langle 0_A 1_B | 1_A 1_B\rangle - \langle 0_A 1_B | 0_A 0_B\rangle - \langle 1_A 0_B | 1_A 1_B\rangle + \langle 1_A 0_B | 0_A 0_B\rangle\right) \\
&= \frac{1}{2}\left(0 - 0 - 0 + 0\right) \\
&= 0
\end{aligned} \tag{6}$$

$$\begin{aligned}
\langle s|\, \sigma_y \otimes I \,|s\rangle &= \frac{1}{\sqrt{2}} \langle s| \left(i\,|1_A 1_B\rangle + i\,|0_A 0_B\rangle\right) \\
&= \frac{1}{2}\left(i\,\langle 0_A 1_B | 1_A 1_B\rangle + i\,\langle 0_A 1_B | 0_A 0_B\rangle - i\,\langle 1_A 0_B | 1_A 1_B\rangle - i\,\langle 1_A 0_B | 0_A 0_B\rangle\right) \\
&= \frac{1}{2}\left(0 + 0 - 0 - 0\right) \\
&= 0
\end{aligned} \tag{7}$$

Let L be a Hermetian operator on V_A and consider the observable defined by $L \otimes I$. Since L is Hermetian, there exists $a, b, c, d \in \mathbb{C}$ such that $L = aI + b\sigma_x + c\sigma_y + d\sigma_z$