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The Mathematics of LLMs

Given an embedding $E \in \mathbb{R}^N$. Define a question $q \in Q \subset \mathbb{R}^{N \times M}$:

$$q = \sum_{i=j}^{M} q_j \mathbf{r}_{i_j}$$

where $\mathbf{r}_1, \ldots, \mathbf{r}_N$ is a pseudo-basis of E. Then, there exists a function:

$$F_{\text{LLM}} : Q \to A$$

where A is the answer space.

Research Questions (Mathematical Formulation):

1. **Conjecture**: The answer space A of an LLM forms a **pseudo-Riemannian manifold** (A, g), where g is a metric tensor:

$$g : TA \times TA \to \mathbb{R}$$

and in local coordinates, it takes the form:

$$g = g_{uv} \, dx^u \otimes dx^v$$

where g_{uv} are the metric components, which may exhibit singularities.

Hypothesis: The answer space evolves according to the Ricci flow equation*:

$$\frac{\partial g_{ij}}{\partial t} = -2 \operatorname{Ric}_{ij}$$

where Ric_{ij} is the Ricci curvature tensor of the manifold.

Key Idea: Applying Perelman's surgery can remove singularities, effectively smoothing the manifold of generated answers.

Research Goal:Can Ricci flow + surgery provide insights into how LLMs structure and generate responses?

2. Mapping Between LLMs and the Space of All LLMs: Given two LLMs, each with an associated function $F_{\text{LLM}}^{(1)}$ and $F_{\text{LLM}}^{(2)}$, there exists a transformation:

$$\varphi : E_1 \rightarrow E_2$$

where E_1, E_2 are embedding spaces of different LLMs.

Key Concept: The space of all LLMs is defined as:

$$\widetilde{F_{\text{LLM}}} = \{F_{\text{LLM}}^{(\alpha)}\}_{\alpha \in \mathcal{I}}$$

where $\mathcal{I}$ indexes different LLM architectures.

Research Goal: Can we construct a universal embedding transformation φ that enables seamless knowledge transfer across different LLMs?

3. **Knowledge Graphs and Answer Spaces**: For every answer $a \in A$, there exists a knowledge graph KG in a structured knowledge space $\mathcal{K}$.

Definition:We define a function mapping answers to structured knowledge graphs:

$$\Psi : A \rightarrow \mathcal{K}$$

Research Goal: Does a well-defined mapping Ψ improve retrievability and factual consistency in LLM-generated responses?

The work on these research questions will improve our understanding of how LLMs store knowledge and generate responses.

Reference:

Chami, I., Wolf, A., Juan, D.-C., Sala, F., Ravi, S., & Ré, C. (2020).
"Low-Dimensional Hyperbolic Knowledge Graph Embeddings." arXiv preprint [arXiv:2005.00545](`https://arxiv.org/abs/2005.00545`).