Midterm 1- Standard 11

Due Date ....TODO
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Contents

1 Instructions

- The solutions **should be typed**, using proper mathematical notation. We cannot accept hand-written solutions. Here's a short intro to LaTeX.

- You should submit your work through the **class Canvas page** only. Please submit one PDF file, compiled using this LaTeX template.

- You may not need a full page for your solutions; pagebreaks are there to help Gradescope automatically find where each problem is. Even if you do not attempt every problem, please submit this document with no fewer pages than the blank template (or Gradescope has issues with it).

- You **may not collaborate with other students**. **Copying from any source is an Honor Code violation. Furthermore, all submissions must be in your own words and reflect your understanding of the material.** If there is any confusion about this policy, it is your responsibility to clarify before the due date.

- Posting to **any** service including, but not limited to Chegg, Discord, Reddit, StackExchange, etc., for help on an assignment is a violation of the Honor Code.

- You **must** virtually sign the Honor Code (see Section 2). Failure to do so will result in your assignment not being graded.

2 Honor Code (Make Sure to Virtually Sign)

Problem 1.

- My submission is in my own words and reflects my understanding of the material.

- I have not collaborated with any other person.

- I have not posted to external services including, but not limited to Chegg, Discord, Reddit, StackExchange, etc.

- I have neither copied nor provided others solutions they can copy.

Agreed (John Blackburn). □

3 Standard 11- Network Flows: Reductions

Problem 2. In this problem we will consider a new reduction from finding a maximum matching in a unweighted, undirected bipartite graph to finding a maximum (s,t)-flow in a flow network. Let $G = (L \dot\cup R, E)$ be a bipartite graph with the vertices partitioned into L and R. We construct an (s,t)-flow network $\mathcal{N} = (H, c, s, t)$ from G as follows:

- Let $V(H) = V(G) \cup \{s, t\}$.

- For each edge $\{u, v\}$ in $E(G)$ with $u \in L$ and $v \in R$, add a directed edge (u, v) to $E(H)$. For each of these edges, set $c(u, v) = 1$.

- For each $v \in L$, add a directed edge (s, v) to $E(H)$. For each of these edges, set $c(s, v) = 2$.

- For each $v \in R$, add a directed edge (v, t) to $E(H)$. For each of these edges, set $c(v, t) = 1$.

To help you understand the reduction we have drawn in Figure 1 an example of a bipartite graph and the resulting flow network. If you need to draw a graph as part of your answer you may copy and modify the tikzpictures or submit hand-drawn examples.
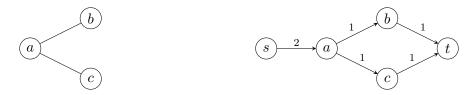

Figure 1: A bipartite graph and the flow network we get by applying the above reduction. The number next to each edge represents the capacity of that edge.

Do the following. Let $G(L \dot\cup R, E)$ be a bipartite graph, and let $\mathcal{N}$ the corresponding flow network obtained by applying the reduction.

(a) Let $\mathcal{M}$ be a matching of G, where $|\mathcal{M}| = k$. Is it the case that $\mathcal{N}$ has a feasible flow f where val$(f) = k$? If so, explain how to construct such a flow f from $\mathcal{M}$. If not, carefully explain your reasoning.

Answer. From $\mathcal{M}$ we have a subset of edges such that no two edges share an endpoint. We can construct a flow f as follows. for each edge of $\mathcal{M}$ we can push a single unit of flow through $s \to u \to v \to t$. Since no two edges within $\mathcal{M}$ share an endpoint, f is a feasible flow. For each edge in $\mathcal{M}$, there can only be 1 unit of flow pushed across it, and since they don't share an endpoint they can each be used summing up the total flow to be equivalent to $|\mathcal{M}|$. Therefore, we can build a feasible flow such that val$(f) = k$. $\square$

(b) Let f' be a feasible flow of $\mathcal{N}$ where for each edge $(u, v) \in E(H)$, $f'(u, v)$ is an integer. Suppose that val$(f') = k$. Does the existence of f' imply that there is a matching $\mathcal{M}$ of size k in G? That is, can we necessarily recover a matching of size k of G from f'? If so, explain how to construct such a matching $\mathcal{M}$ from f'. If not, give a counterexample.

Answer. Yes, it does imply a matching $\mathcal{M}$ of size k. A flow can be built across each edge that isn't connected to s or t. Since only 1 can be pushed across each flow path, we know that k is just a sum of all the edges that would make up $|\mathcal{M}|$. Therefore we can build $\mathcal{M}$ from a feasible flow because we know how many edges will there will be in $\mathcal{M}$ from the amount of flow being pushed through. $\square$