$\alpha = a_1 e_1' + a_2 e_2' + \dots + a_m e_m' + \underbrace{b_1 e_1^* + b_2 e_2^* + \dots + b_m e_n^*}$

Let's say (A) $\in W$, as $\{e_1^*, e_2^*, \dots, e_n^*\}$ is basis of W.

$\alpha = (a_1 e_1' + a_2 e_2' + \dots + a_m e_m') + A$

$\Rightarrow W + \alpha = (W + A) + (a_1 e_1' + a_2 e_2' + \dots + a_m e_m')$

$\Rightarrow W + \alpha = W + (a_1 e_1' + a_2 e_2' + \dots + a_m e_m')$ [$\because$ since $A \in W$. Thus, $W + A = W$]

$\Rightarrow \boxed{W + \alpha = a_1(W + e_1') + a_2(W + e_2') + \dots + a_m(W + e_m')}$

Here we can see that $W + \alpha$ is expressed as an L.C. of $\{W + e_1', W + e_2', \dots, W + e_m'\} = S_1$.

So by (i) and (ii),

S_1 is basis for V/W. i.e.,

$\boxed{\dim(V/W) = m}$ — (2)

By (1) and (2) we get

$\dim(V/W) = \dim(V) - \dim(W)$

Now we need to show that $\dim(V/W) = m$

Thus let us assume that $\{W+e_1', W+e_2', \ldots, W+e_m'\} = S_1$ be a subset of V/W, where $e_1', e_2', e_3', \ldots, e_m' \in V$ defined by (i) $(W+\alpha_1) + (W+\alpha_2) = W + (\alpha_1+\alpha_2)$

(ii) $a(W+\alpha) = W + a\alpha$.

Now, we will show

a) S_1 is L.I. :

Let $a_1, a_2, \ldots, a_m \in F$ and

~~$a_1(W+\alpha_1) + a_2(W+\alpha_2) + \ldots + a_m(W+\alpha_m) = W + \vec{0}$~~

~~$(W + a\alpha_1$~~

$a_1(W+e_1') + a_2(W+e_2') + \ldots + a_m(W+e_m') = W + \bar{0}$

$\Rightarrow (W + a_1e_1') + (W + a_2e_2') + \ldots + (W + a_me_m') = W$

$\Rightarrow W + (a_1e_1' + a_2e_2' + \ldots + a_me_m') = W$

[by definition of vector addition and scalar multiplication for quotient space]

$\Rightarrow (a_1e_1' + a_2e_2' + \ldots + a_me_m') \in W$ [$\because W + a = W \Rightarrow \alpha \in W$]

But since S is basis for W. thus,

$a_1e_1' + a_2e_2' + \ldots + a_me_m' = b_1e_1 + b_2e_2 + \ldots + b_ne_n$

$\Rightarrow a_1e_1' + a_2e_2' + \ldots + a_me_m' - (b_1e_1 + b_2e_2 + \ldots + b_ne_n) = \vec{0}$

But again $\{e_1, e_2, \ldots, e_n, e_1', e_2', \ldots, e_m'\}$ is basis for $V(F)$, Thus, it will be L.I.. so

$a_1 = a_2 = \ldots = a_m = b_1 = b_2 = \ldots = b_n = 0$.

$\Rightarrow a_1 = a_2 = \ldots = a_m = 0$.

Thus $a_1(W+e_1') + a_2(W+e_2') + \ldots + a_m(W+e_m') = \vec{W}$

where, $S_1 = \{W+e_1', W+e_2', \ldots, W+e_m'\}$ is L.I.

(ii) $L(S_1) = V/m$.

Let $(W+\alpha) \in V/W$, $\alpha \in V$.

$\Rightarrow \alpha \in V$ can be expressed as a linear combination of elements of S'. i.e.,

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(x) $I(w+\alpha) = (w+\alpha)$
$\forall\ (w+\alpha)\ \exists\ 1 \in F$ s.t.
$1.(w+\alpha) = w + 1.\alpha = w+\alpha$

Since all 10 axioms/ ~~post~~ postulates for vector space are satisfied thus V/w is a vector space also called Quotient space.

Theorem :- If W is any Vector subspace of vector space $V(F)$ and $\{e_1, e_2, \dots e_n\}$ form the basis of W and $\{e_1, e_2, \dots e_m, e_1', e_2', \dots, e_m'\}$ is the basis of V, then:
(i) $w+e_1', w+e_2', w+e_3' + \dots + w+e_m'$ generates V/W
(ii) $w+e_1', w+e_2', \dots, w+e_m'$ is L.I
(iii) $Dim(V/W) = \dim V - \dim W$.

~~proof~~:- or
~~If W is any vector spaces V(F)~~
If W is any vector subspace of the vector space $V(F)$, then
$$\dim(V/W) = \dim V - \dim W$$

Proof :- Let $V(F)$ be finite diminsional vector space and W be its any vector subspace.
So, W will also be FDVS. then, Let
$\dim(W) = n \Rightarrow \exists\ S = \{e_1, e_2, \dots, e_n\} \subseteq W$ is basis.
Now since S is L. I. it can be extended as basis for V. i.e., say
$$S' = \{e_1, e_2, \dots, e_n, e_1', e_2', \dots, e_m'\}$$
So, $\dim(V) = m+n$.
Thus $\dim(V) - \dim(W) = (m+n) - n$
$$\boxed{\dim(V) - \dim(W) = m.} \quad — \quad (1)$$

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(vi) Scalar multiplication :-

let $a \in F$ and $W+\alpha \in V/W$.

Thus $a(W+\alpha) = W + a\alpha$ (by ②)

Here by closure for scalar multiplication in V,

$\forall a \in F, \alpha \in V \Rightarrow a\alpha \in V$

So, $W + a\alpha \in V/W$ (set of all ~~vecto~~ additive vector cosets of W in V)

So, V/W is also closed for scalar multiplication.

(vii) $(ab)(W+\alpha) = a(b(W+\alpha))$ [Associative law w.r.t scalar multiplication] ~~multiplia~~

$$(ab)(W+\alpha) = W + ab\alpha \quad (\text{by ②})$$
$$= a[W + b\alpha]$$
$$= a[b(W+\alpha)]$$

(viii) $a[(W+\alpha_1)+(W+\alpha_2)] = a(W+\alpha_1) + a(W+\alpha_2)$

L.H.S $= a[(W+\alpha_1)+(W+\alpha_2)]$
$= a[W+(\alpha_1+\alpha_2)]$
$= W + a(\alpha_1+\alpha_2)$
$= W + (a\alpha_1 + a\alpha_2)$

R.H.S $= a(W+\alpha_1) + a(W+\alpha_2)$
$= (W+a\alpha_1)+(W+a\alpha_2)$
$= W + (a\alpha_1 + a\alpha_2)$

L.H.S = R.H.S

(ix) $(a+b)(W+\alpha) = a(W+\alpha) + b(W+\alpha)$

L.H.S $= (a+b)(W+\alpha)$
$= W + (a+b)\alpha$ [by ②]
$= W + (a\alpha + b\alpha)$
$= (W + a\alpha) + (W + b\alpha)$
$= a(W+\alpha) + b(W+\alpha)$

$= W + [\alpha_1 + (\alpha_2 + \alpha_3)]$ [$\because \alpha_1, \alpha_2, \alpha_3 \in V$

$= (W + \alpha_1) + [W + (\alpha_2 + \alpha_3)]$ $\because$ V is a vector space. $\therefore$ Associative

$= (W + \alpha_1) + [(W + \alpha_2) + (W + \alpha_3)]$ property of (V, +) abelian group]

$\therefore$ Associative law holds.

(iii) Existence of identity :-

~~$\because$ always $\vec{0} \in V$ (additive~~

$\because$ always $\vec{0} \in V$ [additive identity of V]

$\therefore W + \vec{0} \in V/W$. Thus $\forall (W + \alpha) \in V/W$

$(W+\alpha) + (W + \vec{0}) = W + (\alpha + \vec{0})$ [$\because \alpha, \vec{0} \in V$

$= W + \alpha$ $\because$ V is a vector space]

So, additive identity exists.

(iv) Existence of additive inverse :-

Let $(W + \alpha) \in V/W$ where $\alpha \in V$. Since V is a vector space, Thus $\forall \alpha \in V \ \exists (-\alpha) \in V$ s.t., $\alpha + (-\alpha) = \vec{0}$

$\therefore (W - \alpha) \in V/W$ [set of all additive cosets of W in V]

Thus,

$(W+\alpha) + (W + (-\alpha)) = W + (\alpha + (-\alpha))$ (by ①)

$= W + \vec{0}$ [Additive identity]

So, we can say that additive ~~identity~~ inverse exists in $V/W \ \forall (W + \alpha) \in V/W$.

(v) Commutative law :-

$\forall (W + \alpha_1) \& (W + \alpha_2) \in V/W$

$(W+\alpha_1) + (W + \alpha_2) = W + (\alpha_1 + \alpha_2)$ [$\because \alpha_1, \alpha_2 \in V$

$= W + (\alpha_2 + \alpha_1)$ $\because$ V is v.s

$= (W+\alpha_2) + (W + \alpha_1)$ $\because$ (V,+) is abel. grp]

Thus commutative law holds.

So (V/W, +) is abelian group.

Quotient Space

Let $V(F)$ be any vector space and $W(F)$ be any subspace of V, then the set V/W of all vector additive cosets of W in V (generated by member of V and W itself) forms a vector space w.r.t operation of vector addition and scalar multiplication called Quotient space.

proof:- Let us consider W be a subspace of V.S. $V(F)$.
$\because$ W is a subspace, Thus W is itself a ~~group~~ vector space and so it satisfies all the postulates of V.S.
Let V/W be the set of all the vector additive cosets of W in V s.t.

$$V/W = \{ W+\alpha : \alpha \in V \}$$

Now we need to show that V/W forms a V.S. w.r.t Vector addition and scalar multiplication. defined as

$$(W+\alpha_1) + (W+\alpha_2) = W + (\alpha_1+\alpha_2) \quad - (1)$$

$$a(W+\alpha) = W + a\alpha \quad - (2)$$

Now assume $(W+\alpha_1), (W+\alpha_2), (W+\alpha_3) \in V/W$, where $\alpha_1, \alpha_2, \alpha_3 \in V$, But since V is a V.S. it satisfies all its properties. now we prove V/W is V.S. :-

(i) closure:- Now,

$$\forall (W+\alpha_1) \text{ and } (W+\alpha_2) \in V/W$$

$$(W+\alpha_1)+(W+\alpha_2) = W+(\alpha_1+\alpha_2) \quad (\text{By } (1))$$

Also, $W+(\alpha_1+\alpha_2) \in V/W$ [$\because$ $(V,+)$ is abelian, $\because$ closure property for α_1 and α_2]

(ii) Associative property :-

$$\forall (W+\alpha_1), (W+\alpha_2), (W+\alpha_3) \in V/W$$

$$[(W+\alpha_1)+(W+\alpha_2)]+(W+\alpha_3) = [W+(\alpha_1+\alpha_2)]+(W+\alpha_3)$$

$$= W+(\alpha_1+\alpha_2+\alpha_3) \quad - \boxed{\text{by } (1)}$$

Again let $\alpha \in W_1 + W_2$ and S_1 and S_2 are basis of W_1 and W_2. Thus α can be expressed as

$$\alpha = \sum_{i=1}^{m} b_i \beta_i + \sum_{i=1}^{n} a_i \alpha_i + \sum_{i=1}^{k} (c_i + d_i) \gamma_i$$

[$\because \alpha$ is an element of $W_1 + W_2 = \{\alpha + \beta : \alpha \in W_1 \text{ and } \beta \in W_2\}$]

So, $\alpha \in L(S') \Rightarrow W_1 + W_2 \subseteq L(S')$ — (8)

By (7) and (8) we can say that,

$L(S') = W_1 + W_2$.

So, S' forms the basis for $W_1 + W_2$ and hence,

$\boxed{\dim(W_1 + W_2) = k + n + m.}$ — (9)

By (1) & (9),

$\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

Special case : $\dim(W_1 \oplus W_2) = \dim(W_1) + \dim(W_2)$

because in case of $W_1 \oplus W_2 = V$, $W_1 \cap W_2 = \{0\}$

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(i) <u>S' is L.I.</u> :-

Let $c_1, c_2, \ldots, c_K, a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m \in F$ be the scalars s.t.,

$$c_1\gamma_1 + c_2\gamma_2 + \ldots + c_K\gamma_K + a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n + b_1\beta_1 + \ldots + b_m\beta_m = 0 \quad - (2)$$

$$\Rightarrow b_1\beta_1 + b_2\beta_2 + \ldots + b_m\beta_m = -(c_1\gamma_1 + c_2\gamma_2 + \ldots + c_K\gamma_K + a_1\alpha_1 + \ldots + a_n\alpha_n) \quad - (3)$$

Here, $-(c_1\gamma_1 + c_2\gamma_2 + \ldots + c_K\gamma_K + a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n) \in W_1$ [$\because S_1$ is basis for W_1]

since L.H.S = R.H.S in eq (3), so,

$$b_1\beta_1 + b_2\beta_2 + \ldots + b_m\beta_m \in W_1.$$

But, $\beta_1, \beta_2, \ldots, \beta_m \in W_2 \quad - (4)$

By (3) and (4) we can say that

$$b_1\beta_1 + b_2\beta_2 + \ldots + b_m\beta_m \in W_1 \cap W_2.$$

Again since $S = \{\gamma_1, \gamma_2, \ldots, \gamma_K\}$ is basis for $W_1 \cap W_2$,

$\therefore b_1\beta_1 + b_2\beta_2 + \ldots + b_m\beta_m = d_1\gamma_1 + d_2\gamma_2 + \ldots + d_K\gamma_K$ [$\because d_1, d_2, \ldots d_K \in F$]

$\Rightarrow b_1\beta_1 + b_2\beta_2 + \ldots + b_m\beta_m - (d_1\gamma_1 + d_2\gamma_2 + \ldots + d_K\gamma_K) = \vec{0}$

$\Rightarrow$ is a L.C in between $\beta_1, \beta_2, \ldots, \beta_m, \gamma_1, \gamma_2, \ldots \gamma_K$ which forms basis for W_2 so, L.I. i.e., $\boxed{b_1\beta_1 + b_2\beta_2 + \ldots + b_m\beta_m = \bar{0}}$

$$b_1 = b_2 = \ldots = b_m = d_1 = d_2 = \ldots = d_K = 0 \quad - (5)$$

Now by (2) and (5)

$$c_1\gamma_1 + c_2\gamma_2 + \ldots + c_K\gamma_K + a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n = \vec{0}$$

$\Rightarrow \boxed{c_1 = c_2 = \ldots = c_K = a_1 = a_2 = \ldots = a_n = 0}$ [$\because S_2$ is L.I.] $- (6)$

Now by (5) and (6) we can say that S' is Linearly Independent.

(ii) <u>$L(S') = W_1 + W_2$</u> :- In order to prove this we need to prove that $L(S') \subseteq W_1 + W_2$ & $L(S') \supseteq W_1 + W_2$

So, let

$L(S') = c_1\gamma_1 + c_2\gamma_2 + \ldots + c_K\gamma_K + a_1\alpha_1 + a_2\alpha_2 + \ldots + a_n\alpha_n + b_1\beta_1 + \ldots + b_m\beta_m$

since $S_1 = \{\gamma_1, \gamma_2, \ldots \gamma_K, \alpha_1, \alpha_2, \ldots \alpha_n\}$ and $S_2 = \{\gamma_1, \ldots \gamma_K, \beta_1, \ldots \beta_m\}$ are basis their L.C. $\in W_1$ and W_2 respectively. Then we can say that $L(S') \in W_1 + W_2$.

$$\Rightarrow L(S') \subseteq W_1 + W_2 \quad - (7)$$

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Dimension of sum of subspaces:

Theorem :- If W_1 and W_2 are the two subspace of the vector space $V(F)$, then,
$$\dim(W_1+W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$$

proof :
Let us consider $V(F)$ be a vector space over the field F and W_1 and W_2 are its two subspace such that,
$$\dim(W_1 \cap W_2) = k.$$
Then, $\exists$ a basis say $S = \{\gamma_1, \gamma_2, \dots \gamma_k\}$ for $W_1 \cap W_2$
i.e., S is L.I. and $L(S) = W_1 \cap W_2$.
Also, $S \subseteq W_1$ and $S \subseteq W_2$ $[\because S \subseteq W_1 \cap W_2 \subseteq W_1 \,\&\, W_2$
We know it is not necessary that S also form the basis for W_1 and W_2 But by the extension theorem S can be extended to form the basis for W_1 and W_2. Thus,
Let $S_1 = \{\gamma_1, \gamma_2, \dots, \gamma_k, \alpha_1, \alpha_2, \dots, \alpha_n\}$
$S_2 = \{\gamma_1, \gamma_2, \dots, \gamma_k, \beta_1, \beta_2, \dots, \beta_m\}$.
be the extended basis for W_1 and W_2 ~~respective~~ respectively. i.e., $L(S_1) = W_1$ and $L(S_2) = W_2$ and S_1, S_2 are L.I.

$$\boxed{\therefore \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2) = (k+n) + (k+m) - k = k+n+m} \quad -(1)$$

(In Order to prove the theorem we need to show that $\dim(W_1+W_2) = k+m+n$).
for the, Let's assume
$$S' = \{\gamma_1, \gamma_2, \dots, \gamma_k, \alpha_1, \alpha_2, \dots, \alpha_n, \beta_1, \beta_2, \dots, \beta_m\}$$
To show : (i) S' is L.I.
(ii) $L(S') = W_1 + W_2$. Thus forms basis for $W_1 + W_2$