Stellar structure and the Lane-Emden equation

- The full equations of stellar structure are extremely complicated. We have to deal with energy generation, nuclear physics, convection, etc. There are astrophysicists who spend their entire career thinking about these issues.
- For some cases, all we want is a simple, heuristic model describing stellar structure that allows us to paper over the details. We will trace it out below, and then discuss how it can be generalized to lend insights into particular physical cases, especially those involving compact objects, i.e., white dwarfs and neutron stars.
- If we assume stars are spherical objects, we can define the enclosed mass $m(r)$ to be the mass contained within a given radius. It satisfies a fairly straightforward identity:

$$\frac{dm}{dr} = 4\pi r^2 \rho$$

- Meanwhile, the equations of hydrostatic equilibrium link the pressure $P(r)$ to the enclosed mass $m(r)$ and density $\rho(r)$, indicating that pressure forces must balance gravity for a star – note we are neglecting things like convective effects here.

$$\nabla P = \frac{dP}{dr} = -\frac{Gm\rho}{r^2}$$

- We now have two linked ODEs in three variables: $m(r), P(r), \rho(r)$. How can we determine the density $\rho(r)$? Rather than attempt to find a third ODE, we will assume that the **equation of state** (EOS), which yields the pressure as a function of the other thermodynamic variables, may be assumed to be a function of the density *only*: $P = P(\rho) = P(\rho(r))$. Why might this be?
  - Pressure really does depend only on density. This is true in some white dwarf models when looking at Fermi pressure.
  - Pressure depends on density and temperature, but we can specify the temperature as a function of density because of some other constraint. In this case, $P = P(T, \rho) = P(T(\rho), \rho) \equiv P(\rho)$.
- Here we will make a very simplistic assumption: the pressure has power-law dependence on the density: $P = \kappa\rho^\gamma$, with κ, γ as constants. This is *consistent* with the ideal gas law if we assume that the temperature $T \propto \rho^{\gamma-1}$. We are in essence defining $\gamma \approx \frac{d\log P}{d\log\rho} = \frac{\rho}{P}\frac{dP}{d\rho}$.
- We have enough information to solve the equations. The recipe would look like this:
  1. Choose a value of γ.
  2. Choose a value for κ in the EOS.
  3. Choose either a central pressure $P(0)$ or density $\rho(0)$, and use the EOS to set the other.
  4. Set $m(0) = 0$.
  5. Evolve the ODEs until you find $\rho(R) = 0$. That will be the stellar surface. Define the total mass $M \equiv m(R)$.
  6. Divide into portions and serve hot.

Scaling

The assumption of a power law gives us a great deal of freedom to investigate scalings. Let's assume we have a fixed value of γ, but that κ is a parameter we can tune.

Assume we have a solution to the stellar structure equations, in the form $m(r), \rho(r), P(r)$ with a value $r = R$ where the density goes to zero implying the surface of a star. How can we change the mass while leaving the radius fixed? Looking at the mass equation, if we have $m \to bm = \bar{m}$, then we need $\rho \to b\rho = \bar{\rho}$ to compensate. Our pressure equation implies

$$\frac{d\bar{P}}{dr} = -\frac{G\bar{m}\bar{\rho}}{r^2} = b^2 \left[-\frac{Gm\rho}{r^2} \right]$$

We'd better have $P \to \bar{P} = b^2 P$ to compensate:

$$\underline{\mathrm{Fixed-radius:}\ \ m \to bm;\ \rho \to b\rho,\ P \to b^2 P \quad \to \quad \kappa = \frac{P}{\rho^\gamma} \propto b^{2-\gamma}}$$

What if we change the radius but leave the mass fixed? Here, we'd have $m(r) \to m(\bar{r}) = m(cr)$, and so on, with the argument changing but not the overall profile - we are horizontally stretching/compressing our functions. In this case, if $r \to \bar{r} = cr$, we find from the mass equation

$$\begin{aligned} \frac{dm}{d\bar{r}} &= c^{-1} \left[\frac{dm}{dr} \right] = c^{-1}[4\pi r^2 \rho] \\ &= 4\pi\bar{r}^2\bar{\rho} = c^2[4\pi r^2 \bar{\rho}] \quad \to \quad \bar{\rho} = c^{-3}\rho \end{aligned}$$

Meanwhile, the pressure equation implies

$$\frac{d\bar{P}}{d\bar{r}} = c^{-1}\frac{d\bar{P}}{dr} = -\frac{Gm\bar{\rho}}{\bar{r}^2} = c^{-5}\left[-\frac{Gm\rho}{r^2} \right] = c^{-5}\left[\frac{dP}{dr} \right] \quad \to \quad \bar{P} = c^{-4}P$$

We just found that

$$\underline{\mathrm{Fixed-mass:}\ \ r \to cr;\ \rho \to c^{-3}\rho,\ P \to c^{-4} P \quad \to \quad \kappa = \frac{P}{\rho^\gamma} \propto c^{3\gamma-4}}$$

Combining these results, we find that for fixed γ:

$$\kappa \propto M^{2-\gamma} R^{3\gamma-4}; \quad \rho \propto M/R^3 \ \to\ \rho_c \propto M/R^3$$

To find the constants, we can either a.) figure out the central density and κ value required to give us unit mass and radius; or b.) pick $\rho_c = 1,\ \kappa = 1$, figure out what mass and radius these correspond to, and compensate accordingly. For the latter, if we find $M_{1,1}$ and $R_{1,1}$ as the mass and radius, respectively, then

$$\kappa = \left(\frac{M}{M_{1,1}} \right)^{2-\gamma} \left(\frac{R}{R_{1,1}} \right)^{3\gamma-4}; \quad \rho_c = \left(\frac{M}{M_{1,1}} \right) \left(\frac{R}{R_{1,1}} \right)^{-3}$$

Thus, the problem becomes a matter of tabulating the values of $M_{1,1}$ and $R_{1,1}$ (in the appropriate units) for different values of γ.

Stellar masses and radii

In real life, the value of κ is not just a variable you can choose at will, since it reflects the underlying physics of the equation of state. For a real set of stars, we would expect κ and γ to remain fixed, and it is the central density that varies to generate a series of stars with different masses and radii. If κ is fixed, we find for a given value of γ that

$$M^{2-\gamma}R^{3\gamma-4} = \text{const.} \quad \Rightarrow \quad M^{\gamma-2} \propto R^{3\gamma-4}$$

This leads to a few interesting cases:

$\gamma > 2$: Both exponents are positive, so as we *increase* the mass of the star, the radius *increases*.

$\gamma = 2$: For this particular value, the radius depends only on κ, and is independent of the stellar mass.

$\frac{4}{3} < \gamma < 2$: The exponents have apposite signs, so as we *increase* the mass of the star, the radius **decreases**. This is the regime in which many stars and compact objects live.

$\gamma = \frac{4}{3}$: In this case, which applies to highly relativistic white dwarfs supported by degeneracy pressure, the mass is constant. This marks the **onset of instability** – such stars cannot support any change in mass, and any perturbation leads to collapse.

The Lane-Emden equations

Now that we understand the scalings involved, we still need to solve the actual equations. Historically, the standard approach is to *non-dimensionalize* all variables out of the problem to find universal solutions for each value of γ. We start by solving for the mass in the pressure evolution equation, and then plugging into the mass evolution equation:

$$m = -\frac{r^2}{\rho G}\frac{dP}{dr} \quad \rightarrow \quad \frac{1}{4\pi G r^2}\frac{d}{dr}\left[\frac{r^2}{\rho}\frac{dP}{dr}\right] = -\rho$$

This is true for any EOS model – for a polytrope, we can relate the change in pressure to the change in density:

$$\begin{aligned} P = \kappa\rho^\gamma \qquad &\Rightarrow \qquad \frac{dP}{dr} = \kappa\gamma\rho^{\gamma-1}\frac{d\rho}{dr} \\ &\Rightarrow \qquad \frac{1}{4\pi G r^2}\frac{d}{dr}\left[\frac{\kappa}{\gamma}r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right] = -\rho \end{aligned}$$

To nondimensionalize the RHS, we can divide through by $\rho_c \equiv \rho(0)$, the central density:

$$-\left(\frac{\rho}{\rho_c}\right) = \frac{1}{4\pi G r^2 \rho_c}\frac{d}{dr}\left[\kappa\gamma r^2\rho^{\gamma-2}\frac{d\rho}{dr}\right] = \frac{\rho_c^{\gamma-2}}{4\pi G r^2}\frac{d}{dr}\left[\kappa\gamma r^2\left(\frac{\rho}{\rho_c}\right)^{\gamma-2}\frac{d(\rho/\rho_c)}{dr}\right]$$

By historical convention, we define a dimensionless *temperature* θ to be proportional to $\frac{P}{\rho}$. In this case,

$$P \propto \rho^\gamma \propto \rho\theta \quad \Rightarrow \quad \rho^{\gamma-1} \propto \theta \quad \rightarrow \quad \rho \propto \theta^{1/(\gamma-1)}$$

and we define $n = \frac{1}{\gamma-1}$ so that $\frac{1}{n} = \gamma - 1$ and thus $\gamma = 1 + \frac{1}{n} = \frac{n+1}{n}$. The quantity n is referred to as the "**polytropic index**", and γ as the "**adiabatic index**", respectively. To establish a scale, we *define*:

$$\frac{\rho}{\rho_c} \equiv \theta^{1/(\gamma-1)} = \theta^n$$

Rewriting our equation in terms of n and θ, we find

$$-\theta^n = \frac{\kappa \rho_c^{(1-n)/n}}{4\pi G r^2} \frac{d}{dr}\left[\left(\frac{n+1}{n}\right) r^2 \theta^{1-n} \left(n\theta^{n-1}\frac{d\theta}{dr}\right)\right] = \frac{(n+1)\kappa\rho_c^{(1-n)/n}}{4\pi G r^2}\frac{d}{dr}\left[r^2 \frac{d\theta}{dr}\right]$$

Note that θ is dimensionless, the factors of r starting at the derivative cancel out ($r^{-1} \cdot r^2 \cdot r^{-1}$), and thus the fraction out in front must be dimensionless. Only the length retains dimensions, so we can define a dimensionless length $\xi \equiv \frac{r}{\alpha}$ in terms of a scale length α such that

$$\alpha^2 \equiv \frac{(n+1)\kappa\rho_c^{(1-n)/n}}{4\pi G} \quad \rightarrow \quad -\theta^n = \frac{\alpha^2}{r^2}\frac{d}{dr}\left[r^2\frac{d\theta}{dr}\right] = \frac{\alpha^2}{r^2}\frac{d}{d(r/\alpha)}\left[\frac{r^2}{\alpha^2}\frac{d\theta}{d(r/\alpha)}\right] = \xi^{-2}\frac{d}{d\xi}\left[\xi^2\frac{d\theta}{d\xi}\right]$$

This is the **Lane-Emden equation**. The boundary conditions are $\theta(0) = 1,\ \theta'(0) = 0$. The star's radius Ξ is defined by the condition $\theta(\Xi) = 0$, where the density goes to zero.

Analytical solutions

There are three values of n that yield analytical solutions:

Incompressibility: $n = 0,\ \gamma \to \infty$**:** In this case, $\theta(\xi) = 1 - \frac{\xi^2}{6}$, but $\rho = \rho_c$ everywhere. This is more of a rocky planet, or maybe an undrilled bowling ball, than a star.

$n = 1,\ \gamma = 2$**:** Here $\theta(\xi) = \frac{\sin\xi}{\xi}$, $\Xi = \pi$. The density takes the same form $\rho(r) = \rho_c\frac{\sin(\pi r/R)}{\pi r/R}$. From the scaling discussion above, the stellar radius R depends only on the value of κ, and is independent of the total mass or central density at fixed κ.

The nebula?: $n = 5,\ \gamma = \frac{6}{5}$**:** Here, $\theta(\xi) = (1 + \xi^2/3)^{-1/2}$. The radius of this model is infinite, and the density only goes to zero asymptotically at infinite distance. This is a cloud, not a star.