SPATIAL EXPLORATION

Gaussian Process

We begin with the simple Gaussian process, with exponential covariance structure.

Sampling Model:

$$\boldsymbol{y} \sim \text{mvnormal}(\boldsymbol{X\beta}, \boldsymbol{\Sigma}), \quad \boldsymbol{\Sigma} = \sigma^2 \boldsymbol{H}(\phi), \quad \boldsymbol{H}(\phi) = exp\left(-\frac{d_{ij}}{\phi}\right)$$

Priors:

$$\begin{aligned} \boldsymbol{\beta} &\sim \text{mvnormal}(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \\ \sigma^2 &\sim \text{inverse-gamma}(\frac{\nu_0}{2}, \frac{\sigma_0^2}{2}) \\ \phi &\sim \text{gamma}(a, b) \end{aligned}$$

Full conditional distribution of $\boldsymbol{\beta}$

$$
\begin{aligned}
p(\boldsymbol{\beta}|\boldsymbol{y},\sigma^2,\phi) &\propto p(\boldsymbol{y}|\boldsymbol{\beta},\sigma^2,\phi)p(\boldsymbol{\beta}) \\
&\propto exp\left[-\frac{1}{2}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})\right] exp\left[-\frac{1}{2}(\boldsymbol{\beta}-\boldsymbol{\mu}_0)^T\boldsymbol{\Sigma}_0^{-1}(\boldsymbol{\beta}-\boldsymbol{\mu}_0)\right] \\
&\propto exp\left[-\frac{1}{2}\left(\boldsymbol{y}^T\boldsymbol{\Sigma}^{-1}-\boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\right)(\boldsymbol{y}-\boldsymbol{X}\boldsymbol{\beta})-\frac{1}{2}\left(\boldsymbol{\beta}^T\boldsymbol{\Sigma}_0^{-1}-\boldsymbol{\mu}_0^T\boldsymbol{\Sigma}_0^{-1}\right)(\boldsymbol{\beta}-\boldsymbol{\mu}_0)\right] \\
&\propto exp\left[-\frac{1}{2}\left(\boldsymbol{y}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{y}-\boldsymbol{y}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\boldsymbol{\beta}-\boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{y}+\boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\boldsymbol{\beta}+\boldsymbol{\beta}^T\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\beta}-\boldsymbol{\beta}^T\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0-\boldsymbol{\mu}_0^T\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\beta}+\boldsymbol{\mu}_0^T\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0\right)\right] \\
&\propto exp\left[-\frac{1}{2}\left(\boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\boldsymbol{\beta}+\boldsymbol{\beta}^T\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\beta}-\boldsymbol{\beta}^T\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{y}-\boldsymbol{\beta}^T\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0-\boldsymbol{y}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X}\boldsymbol{\beta}-\boldsymbol{\mu}_0^T\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\beta}\right)\right] \\
&\propto exp\left[-\frac{1}{2}\left(\boldsymbol{\beta}^T\left(\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X}+\boldsymbol{\Sigma}_0^{-1}\right)\boldsymbol{\beta}-\boldsymbol{\beta}^T\left(\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{y}-\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0\right)-\left(\boldsymbol{y}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X}-\boldsymbol{\mu}_0^T\boldsymbol{\Sigma}_0^{-1}\right)\boldsymbol{\beta}\right)\right]
\end{aligned}
$$

This is the kernel of a multivariate normal distribution. That is,

$$
\boldsymbol{\beta}|\boldsymbol{y},\sigma^2,\phi \sim \text{mvnormal}\left(\left(\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X}+\boldsymbol{\Sigma}_0^{-1}\right)^{-1}\left(\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{y}-\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0\right),\left(\boldsymbol{X}^T\boldsymbol{\Sigma}^{-1}\boldsymbol{X}+\boldsymbol{\Sigma}_0^{-1}\right)^{-1}\right)
$$

Full conditional distribution of σ^2

$$\begin{aligned}
p(\sigma^2|\boldsymbol{y},\boldsymbol{\beta},\phi) &\propto p(\boldsymbol{y}|\boldsymbol{\beta},\sigma^2,\phi)p(\sigma^2) \\
&\propto \left(\sigma^2\right)^{-\frac{n}{2}} exp\left[-\frac{1}{2\sigma^2}(\boldsymbol{y}-\boldsymbol{X\beta})^T(\boldsymbol{H}(\phi))^{-1}(\boldsymbol{y}-\boldsymbol{X\beta})\right](\sigma^2)^{-\frac{\nu_0}{2}-1} exp\left[-\frac{\sigma_0^2}{2\sigma^2}\right] \\
&\propto (\sigma^2)^{-\frac{\nu_0+n}{2}-1} exp\left[-\frac{1}{2\sigma^2}\left((\boldsymbol{y}-\boldsymbol{X\beta})^T(\boldsymbol{H}(\phi))^{-1}(\boldsymbol{y}-\boldsymbol{X\beta})+\sigma_0^2\right)\right]
\end{aligned}$$

This is the kernel of an inverse-gamma distribution. That is,

$$\sigma^2|\boldsymbol{y},\boldsymbol{\beta},\phi \sim \text{inverse-gamma}\left(\frac{\nu_0+n}{2},\frac{(\boldsymbol{y}-\boldsymbol{X\beta})^T(\boldsymbol{H}(\phi))^{-1}(\boldsymbol{y}-\boldsymbol{X\beta})+\sigma_0^2}{2}\right)$$