

Power App Derivations

1 Exponential Distribution

1.1 Statistic: $\sum_{i=1}^n X_i$

1.1.1 Alternative: Greater than

Suppose that $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0: \theta \leq \theta_0$$

$$H_a: \theta > \theta_0$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. It can be shown that if $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, then $\sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$. Using this relationship, we derive the test.

$$Pr(\phi(\mathbf{X}) = 1 | \theta_0) = Pr\left(\sum_{i=1}^n X_i > k \middle| \theta_0\right) = \alpha$$

Let $T = \sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$. To define our test, we seek the value k such that

$$Pr(T > k | \theta_0) = 1 - Pr(T \leq k | \theta_0) = \alpha$$

The value of k that satisfies this equation is the $(1 - \alpha)^{th}$ quantile of the $\text{Gamma}(n, \theta_0)$ distribution. Letting $\Gamma_{n, \theta_0, 1-\alpha}$ denote this value, our test functions becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i > \Gamma_{n, \theta_0, 1-\alpha} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = Pr\left(\sum_{i=1}^n X_i > \Gamma_{n, \theta_0, 1-\alpha} \middle| \theta\right) = 1 - Pr\left(\sum_{i=1}^n X_i \leq \Gamma_{n, \theta_0, 1-\alpha} \middle| \theta\right)$$

1.1.2 Alternative: Less than

Suppose that $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0: \theta \geq \theta_0$$

$$H_a: \theta < \theta_0$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. Let $T = \sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$. To define our test, we seek the value k such that

$$Pr(T < k | \theta_0) = \alpha$$

The value of k that satisfies this equation is the α^{th} quantile of the $\text{Gamma}(n, \theta_0)$ distribution. Letting $\Gamma_{n, \theta_0, \alpha}$ denote this value, our test functions becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i < \Gamma_{n, \theta_0, \alpha} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = Pr\left(\sum_{i=1}^n X_i < \Gamma_{n, \theta_0, \alpha} \middle| \theta\right)$$

1.1.3 Alternative: Not equal to

Suppose that $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0: \theta = \theta_0$$

$$H_a: \theta \neq \theta_0$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i < k_1 \text{ or } \sum_{i=1}^n X_i > k_2 \\ 0 & \text{else} \end{cases},$$

where k_1 and k_2 are chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. In this case, we assume a symmetric test function in that k_1 and k_2 are chosen such that $Pr(\sum_{i=1}^n X_i < k_1) = \alpha/2$ and $Pr(\sum_{i=1}^n X_i > k_2) = \alpha/2$. Let $T = \sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$.

The values of k_1 and k_2 that satisfy these equations are the $(\alpha/2)^{th}$ and $(1 - \alpha/2)^{th}$ quantiles of the $\text{Gamma}(n, \theta_0)$ distribution. Letting $\Gamma_{n, \theta_0, \alpha/2}$ and $\Gamma_{n, \theta_0, 1-\alpha/2}$ denote these values, our test functions becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i < \Gamma_{n, \theta_0, \alpha/2} \text{ or } \sum_{i=1}^n X_i > \Gamma_{n, \theta_0, 1-\alpha/2} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = 1 - Pr\left(\sum_{i=1}^n X_i < \Gamma_{n, \theta_0, 1-\alpha/2}\right) + Pr\left(\sum_{i=1}^n X_i > \Gamma_{n, \theta_0, \alpha/2} \middle| \theta\right)$$

1.2 Statistic: $X_{(1)}$

1.2.1 Alternative: Greater than

Suppose that $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned} H_0: \theta &\leq \theta_0 \\ H_a: \theta &> \theta_0 \end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(1)} > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. It can be shown that if $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, then $X_{(1)} \sim \text{Exp}(\frac{\theta}{n})$. Using this relationship, we derive the test.

$$Pr(\phi(\mathbf{X}) = 1 | \theta_0) = Pr(X_{(1)} > k | \theta_0) = \alpha$$

Let $T = X_{(1)} \sim \text{Exp}(\frac{\theta}{n})$. To define our test, we seek the value k such that

$$Pr(T > k | \theta_0) = 1 - Pr(T \leq k | \theta_0) = \alpha$$

The value of k that satisfies this equation is the $(1 - \alpha)^{th}$ quantile of the $\text{Exp}(\frac{\theta_0}{n})$ distribution. Letting $\eta_{\theta_0/n, 1-\alpha}$ denote this value, our test functions becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(1)} > \eta_{\theta_0/n, 1-\alpha} \\ 0 & \text{else} \end{cases},$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = Pr(X_{(1)} > \eta_{\theta_0/n, 1-\alpha} | \theta) = 1 - Pr(X_{(1)} \leq \eta_{\theta_0/n, 1-\alpha} | \theta)$$

1.2.2 Alternative: Less than

Suppose that $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned} H_0: \theta &\geq \theta_0 \\ H_a: \theta &< \theta_0 \end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(1)} < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. Let $T = X_{(1)} \sim \text{Exp}(\frac{\theta}{n})$. To define our test, we seek the value k such that

$$Pr(T < k | \theta_0) = \alpha$$

The value of k that satisfies this equation is the α^{th} quantile of the $\text{Exp}(\frac{\theta_0}{n})$ distribution. Letting $\eta_{\theta_0/n, \alpha}$ denote this value, our test functions becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(1)} < \eta_{\theta_0/n, \alpha} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = Pr\left(X_{(1)} < \eta_{\theta_0/n, \alpha} \middle| \theta\right)$$

1.2.3 Alternative: Not equal to

Suppose that $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0: \theta = \theta_0$$

$$H_a: \theta \neq \theta_0$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(1)} < k_1 \text{ or } X_{(1)} > k_2 \\ 0 & \text{else} \end{cases},$$

where k_1 and k_2 are chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. In this case, we choose k_1 and k_2 such that our test is symmetric. Let $T = X_{(1)} \sim \text{Exp}(\frac{\theta}{n})$. To define our test, we seek the values k_1 and k_2 such that $Pr(T < k_1 | \theta_0) = \alpha/2$ and $Pr(T > k_2 | \theta_0) = \alpha/2$.

The values of k_1 and k_2 that satisfy these equations are the $(\alpha/2)^{th}$ and $(1 - \alpha/2)^{th}$ quantiles of the $\text{Exp}(\frac{\theta_0}{n})$ distribution. Letting $\eta_{\theta_0/n, \alpha/2}$ and $\eta_{\theta_0/n, 1-\alpha/2}$ denote this value, our test functions becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(1)} < \eta_{\theta_0/n, \alpha/2} \text{ or } X_{(1)} > \eta_{\theta_0/n, 1-\alpha/2} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = 1 - Pr\left(X_{(1)} < \eta_{\theta_0/n, 1-\alpha/2}\right) + Pr\left(X_{(1)} < \eta_{\theta_0/n, \alpha/2} \middle| \theta\right)$$

1.3 Statistic: $X_{(n)}$

1.3.1 Alternative: Greater than

Suppose that $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0: \theta \leq \theta_0$$

$$H_a: \theta > \theta_0$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. To find this value of k , we need to know the distribution of the sample max. By definition,

$$F_{X_{(n)}}(x) = [F_X(x)]^n = \left[1 - \exp\left(-\frac{x}{\theta}\right)\right]^n$$

Let $T = X_{(n)}$. To define our test, we seek the value k such that

$$Pr(T > k | \theta_0) = 1 - Pr(T \leq k | \theta_0) = \alpha$$

Using the distribution function derived above, we have

$$\begin{aligned}
1 - \left[1 - \exp\left(-\frac{k}{\theta_0}\right)\right]^n &= \alpha \\
\left[1 - \exp\left(-\frac{k}{\theta_0}\right)\right]^n &= 1 - \alpha \\
1 - \exp\left(-\frac{k}{\theta_0}\right) &= (1 - \alpha)^{1/n} \\
\exp\left(-\frac{k}{\theta_0}\right) &= 1 - (1 - \alpha)^{1/n} \\
-\frac{k}{\theta_0} &= \log(1 - (1 - \alpha)^{1/n}) \\
k &= -\theta_0 \log(1 - (1 - \alpha)^{1/n})
\end{aligned}$$

Therefore, our test function is

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} > -\theta_0 \log(1 - (1 - \alpha)^{1/n}) \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\begin{aligned}
\beta(\theta) &= Pr(\mathbf{X} \in RR) = Pr\left(X_{(n)} > -\theta_0 \log(1 - (1 - \alpha)^{1/n}) \middle| \theta\right) \\
&= 1 - Pr\left(X_{(n)} \leq -\theta_0 \log(1 - (1 - \alpha)^{1/n}) \middle| \theta\right) \\
&= 1 - \left[1 - \exp\left(\frac{-\theta_0 \log(1 - (1 - \alpha)^{1/n})}{\theta}\right)\right]^n
\end{aligned}$$

1.3.2 Alternative: Less than

Suppose that $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned}
H_0: \theta &\geq \theta_0 \\
H_a: \theta &< \theta_0
\end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. Let $T = X_{(n)}$. To define our test, we seek the value k such that

$$Pr(T < k | \theta_0) = \alpha$$

Using the distribution function derived above, we have

$$\begin{aligned}
\left[1 - \exp\left(-\frac{k}{\theta_0}\right)\right]^n &= \alpha \\
1 - \exp\left(-\frac{k}{\theta_0}\right) &= \alpha^{1/n} \\
\exp\left(-\frac{k}{\theta_0}\right) &= 1 - \alpha^{1/n} \\
-\frac{k}{\theta_0} &= \log(1 - \alpha^{1/n}) \\
k &= -\theta_0 \log(1 - \alpha^{1/n})
\end{aligned}$$

Therefore, our test function is

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} < -\theta_0 \log(1 - \alpha^{1/n}) \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\begin{aligned}
\beta(\theta) &= Pr(\mathbf{X} \in RR) = Pr\left(X_{(n)} < -\theta_0 \log(1 - \alpha^{1/n}) \middle| \theta\right) \\
&= Pr\left(X_{(n)} < -\theta_0 \log(1 - \alpha^{1/n}) \middle| \theta\right) \\
&= \left[1 - \exp\left(\frac{-\theta_0 \log(1 - \alpha^{1/n})}{\theta}\right)\right]^n
\end{aligned}$$

1.3.3 Alternative: Not equal to

Suppose that $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned}
H_0: \theta &= \theta_0 \\
H_a: \theta &\neq \theta_0
\end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} < k_1 \text{ or } X_{(n)} > k_2 \\ 0 & \text{else} \end{cases},$$

where k_1 and k_2 are chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. In this case, we choose a symmetric test function. Let $T = X_{(n)}$. To define our test, we seek the values k_1 and k_2 such that $Pr(T < k_1 | \theta_0) = \alpha/2$ and $Pr(T > k_2 | \theta_0) = \alpha/2$. Using the distribution function derived above, we have

$$\begin{aligned}
\left[1 - \exp\left(-\frac{k_1}{\theta_0}\right)\right]^n &= \alpha/2 \\
1 - \exp\left(-\frac{k_1}{\theta_0}\right) &= (\alpha/2)^{1/n} \\
\exp\left(-\frac{k_1}{\theta_0}\right) &= 1 - (\alpha/2)^{1/n} \\
-\frac{k_1}{\theta_0} &= \log(1 - (\alpha/2)^{1/n}) \\
k_1 &= -\theta_0 \log(1 - (\alpha/2)^{1/n})
\end{aligned}$$

Similarly,

$$\begin{aligned}
1 - \left[1 - \exp\left(-\frac{k_2}{\theta_0}\right)\right]^n &= \alpha/2 \\
\left[1 - \exp\left(-\frac{k_2}{\theta_0}\right)\right]^n &= 1 - \alpha/2 \\
1 - \exp\left(-\frac{k_2}{\theta_0}\right) &= (1 - \alpha/2)^{1/n} \\
\exp\left(-\frac{k_2}{\theta_0}\right) &= 1 - (1 - \alpha/2)^{1/n} \\
-\frac{k_2}{\theta_0} &= \log(1 - (1 - \alpha/2)^{1/n}) \\
k_2 &= -\theta_0 \log(1 - (1 - \alpha/2)^{1/n})
\end{aligned}$$

Therefore, our test function is

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} < -\theta_0 \log(1 - (\alpha/2)^{1/n}) \text{ or } X_{(n)} > -\theta_0 \log(1 - (1 - \alpha/2)^{1/n}) \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\begin{aligned}
\beta(\theta) &= Pr(\mathbf{X} \in RR) \\
&= 1 - Pr\left(X_{(n)} < -\theta_0 \log(1 - (1 - \alpha/2)^{1/n}) \middle| \theta\right) + Pr\left(X_{(n)} > -\theta_0 \log(1 - (\alpha/2)^{1/n}) \middle| \theta\right) \\
&= 1 - \left[1 - \exp\left(\frac{-\theta_0 \log(1 - (1 - \alpha/2)^{1/n})}{\theta}\right)\right]^n + \left[1 - \exp\left(\frac{-\theta_0 \log(1 - (\alpha/2)^{1/n})}{\theta}\right)\right]^n
\end{aligned}$$

2 Normal Distribution

2.1 Statistic: $\sum_{i=1}^n X_i$

2.1.1 Alternative: Greater than

Suppose that $X_i \stackrel{iid}{\sim} N(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned} H_0: \theta &\leq \theta_0 \\ H_a: \theta &> \theta_0 \end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. It can be shown that if $X_i \stackrel{iid}{\sim} N(\theta, \sigma^2)$, then $\sum_{i=1}^n X_i \sim N(n\theta, n\sigma^2)$. Using this relationship, we derive the test.

$$Pr(\phi(\mathbf{X}) = 1 | \theta_0) = Pr\left(\sum_{i=1}^n X_i > k \middle| \theta_0\right) = \alpha$$

Let $T = \sum_{i=1}^n X_i \sim N(n\theta, n\sigma^2)$. To define our test, we seek the value k such that

$$Pr(T > k | \theta_0) = 1 - Pr(T \leq k | \theta_0) = \alpha$$

The value of k that satisfies this equation is the $(1 - \alpha)^{th}$ quantile of the $N(n\theta_0, n\sigma^2)$ distribution. Letting $z_{n\theta_0, n\sigma^2, 1-\alpha}^*$ denote this value, our test functions becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i > z_{n\theta_0, n\sigma^2, 1-\alpha}^* \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = Pr\left(\sum_{i=1}^n X_i > z_{n\theta_0, n\sigma^2, 1-\alpha}^* \middle| \theta\right) = 1 - Pr\left(\sum_{i=1}^n X_i \leq z_{n\theta_0, n\sigma^2, 1-\alpha}^* \middle| \theta\right)$$

Derive in terms of typical Z test?

2.1.2 Alternative: Less than

Suppose that $X_i \stackrel{iid}{\sim} N(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned} H_0: \theta &\geq \theta_0 \\ H_a: \theta &< \theta_0 \end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. Let $T = \sum_{i=1}^n X_i \sim N(n\theta, n\sigma^2)$. To define our test, we

seek the value k such that

$$Pr(T < k | \theta_0) = \alpha$$

The value of k that satisfies this equation is the α^{th} quantile of the $N(n\theta_0, n\sigma^2)$ distribution. Letting $z_{n\theta_0, n\sigma^2, \alpha}^*$ denote this value, our test function becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i < z_{n\theta_0, n\sigma^2, \alpha}^* \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = Pr\left(\sum_{i=1}^n X_i < z_{n\theta_0, n\sigma^2, \alpha}^* \middle| \theta\right)$$

Derive in terms of typical Z test?

2.1.3 Alternative: Not equal to

Suppose that $X_i \stackrel{iid}{\sim} N(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned} H_0: \theta &= \theta_0 \\ H_a: \theta &\neq \theta_0 \end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i > |k| \\ 0 & \text{else} \end{cases},$$

where k are is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. Note that we assume a symmetric test function in that k is chosen such that $Pr(\sum_{i=1}^n X_i < -k) = \alpha/2$ and $Pr(\sum_{i=1}^n X_i > k) = \alpha/2$. Let $T = \sum_{i=1}^n X_i \sim N(n\theta, n\sigma^2)$. The value of k that satisfies this equation is the $(1-\alpha/2)^{th}$ quantile of the $Nn\theta_0, n\sigma^2$ distribution. Letting $z_{n\theta_0, n\sigma^2, 1-\alpha/2}^*$ denote this value, our test function becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i < z_{n\theta_0, n\sigma^2, \alpha/2}^* \text{ or } \sum_{i=1}^n X_i > z_{n\theta_0, n\sigma^2, 1-\alpha/2}^* \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = 1 - Pr\left(\sum_{i=1}^n X_i < z_{n\theta_0, n\sigma^2, 1-\alpha/2}^*\right) + Pr\left(\sum_{i=1}^n X_i > -z_{n\theta_0, n\sigma^2, 1-\alpha/2}^* \middle| \theta\right)$$

2.2 Statistic: $X_{(1)}$

2.2.1 Alternative: Greater than

Suppose that $X_i \stackrel{iid}{\sim} N(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned} H_0: \theta &\leq \theta_0 \\ H_a: \theta &> \theta_0 \end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(1)} > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. To find this value, we must first find the distribution of the sample minimum. By definition,

$$F_{X_{(1)}}(x) = 1 - [1 - F_X(x)]^n = 1 - [1 - \psi(x, \theta_0, \sigma^2)]^n,$$

where $\psi(x, \theta_0, \sigma^2)$ denotes the distribution function of the $N(\theta_0, \sigma^2)$ distribution. Let $T = X_{(1)}$. We seek the value k such that

$$Pr(T > k | \theta_0) = 1 - Pr(T \leq k | \theta_0) = \alpha$$

Using the distribution function derived above, we have:

$$\begin{aligned} 1 - Pr(T \leq k | \theta_0) &= 1 - (1 - [1 - \psi(k, \theta_0, \sigma^2)]^n) = \alpha \\ [1 - \psi(k, \theta_0, \sigma^2)]^n &= \alpha \\ \psi(k, \theta_0, \sigma^2) &= 1 - \alpha^{1/n} \end{aligned}$$

The value of k that satisfies this equation is the $(1 - \alpha^{1/n})^{th}$ quantile of the $N(\theta_0, \sigma^2)$ distribution. Letting $z_{\theta_0, \sigma^2, 1 - \alpha^{1/n}}^*$ denote this value, our test functions becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(1)} > z_{\theta_0, \sigma^2, 1 - \alpha^{1/n}}^* \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = Pr\left(X_{(1)} > z_{\theta_0, \sigma^2, 1 - \alpha^{1/n}}^* \middle| \theta\right) = 1 - Pr\left(X_{(1)} \leq z_{\theta_0, \sigma^2, 1 - \alpha^{1/n}}^* \middle| \theta\right)$$

2.2.2 Alternative: Less than

Suppose that $X_i \stackrel{iid}{\sim} N(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned} H_0: \theta &\geq \theta_0 \\ H_a: \theta &< \theta_0 \end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(1)} < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. Let $T = X_{(1)}$. We seek the value k such that

$$Pr(T < k | \theta_0) = \alpha$$

Using the distribution function derived above, we have:

$$\begin{aligned} Pr(T < k | \theta_0) &= 1 - [1 - \psi(k, \theta_0, \sigma^2)]^n = \alpha \\ [1 - \psi(k, \theta_0, \sigma^2)]^n &= 1 - \alpha \\ \psi(k, \theta_0, \sigma^2) &= 1 - (1 - \alpha)^{1/n} \end{aligned}$$

The value of k that satisfies this equation is the $1 - (1 - \alpha)^{1/n}$ quantile of the $N(\theta_0, \sigma^2)$ distribution. Letting

$z_{\theta_0, \sigma^2, 1-(1-\alpha)^{1/n}}^*$ denote this value, our test functions becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(1)} < z_{\theta_0, \sigma^2, 1-(1-\alpha)^{1/n}}^* \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = Pr\left(X_{(1)} < z_{\theta_0, \sigma^2, 1-(1-\alpha)^{1/n}}^* \middle| \theta\right)$$

2.2.3 Alternative: Not equal to

Suppose that $X_i \stackrel{iid}{\sim} N(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0: \theta = \theta_0$$

$$H_a: \theta \neq \theta_0$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(1)} < k_1 \text{ or } X_{(1)} > k_2 \\ 0 & \text{else} \end{cases},$$

where k_1 and k_2 are chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. We choose a symmetric test. Let $T = X_{(1)}$. We seek the values k_1 and k_2 such that $Pr(T < k_1 | \theta_0) = \alpha/2$ and $Pr(T > k_2 | \theta_0) = \alpha/2$. Using the distribution function derived above, we have:

$$\begin{aligned} Pr(T < k_1 | \theta_0) &= 1 - [1 - \psi(k_1, \theta_0, \sigma^2)]^n = \alpha/2 \\ [1 - \psi(k_1, \theta_0, \sigma^2)]^n &= 1 - \alpha/2 \\ \psi(k_1, \theta_0, \sigma^2) &= 1 - (1 - \alpha/2)^{1/n} \end{aligned}$$

The value of k_1 that satisfies this equation is the $1 - (1 - \alpha/2)^{1/n}$ quantile of the $N(\theta_0, \sigma^2)$ distribution. We let $z_{\theta_0, \sigma^2, 1-(1-\alpha/2)^{1/n}}^*$ denote this value. Similarly,

$$\begin{aligned} 1 - Pr(T \leq k_2 | \theta_0) &= 1 - (1 - [1 - \psi(k_2, \theta_0, \sigma^2)]^n) = \alpha/2 \\ [1 - \psi(k_2, \theta_0, \sigma^2)]^n &= \alpha/2 \\ \psi(k_2, \theta_0, \sigma^2) &= 1 - (\alpha/2)^{1/n} \end{aligned}$$

The value of k_2 that satisfies this equation is the $1 - (\alpha/2)^{1/n}$ quantile of the $N(\theta_0, \sigma^2)$ distribution. We let $z_{\theta_0, \sigma^2, 1-(\alpha/2)^{1/n}}^*$ denote this value. Therefore, our test function becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(1)} < z_{\theta_0, \sigma^2, 1-(1-\alpha/2)^{1/n}}^* \text{ or } X_{(1)} > z_{\theta_0, \sigma^2, 1-(\alpha/2)^{1/n}}^* \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = 1 - Pr\left(X_{(1)} < z_{\theta_0, \sigma^2, 1-(\alpha/2)^{1/n}}^* \middle| \theta\right) + Pr\left(X_{(1)} > z_{\theta_0, \sigma^2, 1-(\alpha/2)^{1/n}}^* \middle| \theta\right)$$

2.3 Statistic: $X_{(n)}$

2.3.1 Alternative: Greater than

Suppose that $X_i \stackrel{iid}{\sim} N(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned} H_0: \theta &\leq \theta_0 \\ H_a: \theta &> \theta_0 \end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. To find this value, we must first find the distribution of the sample maximum. By definition,

$$F_{X_{(n)}}(x) = [F_X(x)]^n = [\psi(x, \theta_0, \sigma^2)]^n,$$

where $\psi(x, \theta_0, \sigma^2)$ denotes the distribution function of the $N(\theta_0, \sigma^2)$ distribution. Let $T = X_{(n)}$. We seek the value k such that

$$Pr(T > k | \theta_0) = 1 - Pr(T \leq k | \theta_0) = \alpha$$

Using the distribution function derived above, we have:

$$\begin{aligned} 1 - Pr(T \leq k | \theta_0) &= 1 - [\psi(k, \theta_0, \sigma^2)]^n = \alpha \\ \psi(k, \theta_0, \sigma^2) &= (1 - \alpha)^{1/n} \end{aligned}$$

The value of k that satisfies this equation is the $(1 - \alpha)^{1/n}$ quantile of the $N(\theta_0, \sigma^2)$ distribution. Letting $z_{\theta_0, \sigma^2, (1-\alpha)^{1/n}}^*$ denote this value, our test functions becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} > z_{\theta_0, \sigma^2, (1-\alpha)^{1/n}}^* \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = Pr\left(X_{(n)} > z_{\theta_0, \sigma^2, (1-\alpha)^{1/n}}^* \mid \theta\right) = 1 - Pr\left(X_{(n)} \leq z_{\theta_0, \sigma^2, (1-\alpha)^{1/n}}^* \mid \theta\right)$$

2.3.2 Alternative: Less than

Suppose that $X_i \stackrel{iid}{\sim} N(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned} H_0: \theta &\geq \theta_0 \\ H_a: \theta &< \theta_0 \end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. Let $T = X_{(n)}$. We seek the value k such that

$$Pr(T < k | \theta_0) = \alpha$$

Using the distribution function derived above, we have:

$$\begin{aligned} Pr(T < k|\theta_0) &= [\psi(k, \theta_0, \sigma^2)]^n = \alpha \\ \psi(k, \theta_0, \sigma^2) &= \alpha^{1/n} \end{aligned}$$

The value of k that satisfies this equation is the $\alpha^{1/n}$ quantile of the $N(\theta_0, \sigma^2)$ distribution. Letting $z_{\theta_0, \sigma^2, \alpha^{1/n}}^*$ denote this value, our test functions becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} > z_{\theta_0, \sigma^2, \alpha^{1/n}}^* \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = Pr\left(X_{(n)} < z_{\theta_0, \sigma^2, \alpha^{1/n}}^* \middle| \theta\right)$$

2.3.3 Alternative: Not equal to

Suppose that $X_i \stackrel{iid}{\sim} N(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned} H_0: \theta &= \theta_0 \\ H_a: \theta &\neq \theta_0 \end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} < k_1 \text{ or } X_{(n)} > k_2 \\ 0 & \text{else} \end{cases},$$

where k_1 and k_2 are chosen such that $Pr(\phi(\mathbf{X}) = 1|\theta_0) = \alpha$. We choose a symmetric test. Let $T = X_{(n)}$. We seek the values k_1 and k_2 such that $Pr(T < k_1|\theta_0) = \alpha/2$ and $Pr(T > k_2|\theta_0) = \alpha/2$. Using the distribution function derived above, we have:

$$\begin{aligned} Pr(T < k_1|\theta_0) &= [\psi(k_1, \theta_0, \sigma^2)]^n = \alpha/2 \\ \psi(k_1, \theta_0, \sigma^2) &= (\alpha/2)^{1/n} \end{aligned}$$

The value of k_1 that satisfies this equation is the $(\alpha/2)^{1/n}$ quantile of the $N(\theta_0, \sigma^2)$ distribution. We let $z_{\theta_0, \sigma^2, (\alpha/2)^{1/n}}^*$ denote this value. Similarly,

$$\begin{aligned} 1 - Pr(T \leq k_2|\theta_0) &= 1 - [\psi(k_2, \theta_0, \sigma^2)]^n = \alpha/2 \\ [\psi(k_2, \theta_0, \sigma^2)]^n &= 1 - \alpha/2 \\ \psi(k_2, \theta_0, \sigma^2) &= (1 - \alpha/2)^{1/n} \end{aligned}$$

The value of k_2 that satisfies this equation is the $(1 - \alpha/2)^{1/n}$ quantile of the $N(\theta_0, \sigma^2)$ distribution. We let $z_{\theta_0, \sigma^2, (1-\alpha/2)^{1/n}}^*$ denote this value. Therefore, our test function becomes

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} < z_{\theta_0, \sigma^2, (\alpha/2)^{1/n}}^* \text{ or } X_{(n)} > z_{\theta_0, \sigma^2, (1-\alpha/2)^{1/n}}^* \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\mathbf{X} \in RR) = 1 - Pr\left(X_{(n)} < z_{\theta_0, \sigma^2, (1-\alpha/2)^{1/n}}^* \middle| \theta\right) + Pr\left(X_{(n)} < z_{\theta_0, \sigma^2, (\alpha/2)^{1/n}}^* \middle| \theta\right)$$

3 Uniform Distribution

3.1 Statistic: $\sum_{i=1}^n X_i$

3.1.1 Alternative: Greater than

3.1.2 Alternative: Less than

3.1.3 Alternative: Not equal to

3.2 Statistic: $X_{(1)}$

3.2.1 Alternative: Greater than

3.2.2 Alternative: Less than

3.2.3 Alternative: Not equal to

3.3 Statistic: $X_{(n)}$

3.3.1 Alternative: Greater than

Suppose that $X_i \stackrel{iid}{\sim} \text{Unif}(0, \theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0: \theta \leq \theta_0$$

$$H_a: \theta > \theta_0$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. To find this value of k , we need to know the distribution of the sample max. By definition,

$$F_{X_{(n)}}(x) = [F_X(x)]^n = \left[\frac{x}{\theta}\right]^n = \frac{x^n}{\theta^n}$$

Let $T = X_{(n)}$. To define our test, we seek the value k such that

$$Pr(T > k | \theta_0) = 1 - Pr(T \leq k | \theta_0) = \alpha$$

Using the distribution function derived above, we have

$$\begin{aligned} 1 - \frac{k^n}{\theta_0^n} &= \alpha \\ \frac{k^n}{\theta_0^n} &= 1 - \alpha \\ k^n &= \theta_0^n(1 - \alpha) \\ k &= \theta_0(1 - \alpha)^{1/n} \end{aligned}$$

Therefore, our test function is

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} > \theta_0(1 - \alpha)^{1/n} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is piecewise and looks like the following:

For $0 < \theta < k$:

$$\beta(\theta) = 0$$

For $\theta > k$:

$$\begin{aligned}\beta(\theta) &= Pr(\mathbf{X} \in RR) = Pr\left(X_{(n)} > \theta_0(1 - \alpha)^{1/n} \middle| \theta\right) \\ &= 1 - Pr\left(X_{(n)} \leq \theta_0(1 - \alpha)^{1/n} \middle| \theta\right) \\ &= 1 - \frac{\theta_0^n(1 - \alpha)}{\theta^n}\end{aligned}$$

3.3.2 Alternative: Less than

Suppose that $X_i \stackrel{iid}{\sim} \text{Unif}(0, \theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned}H_0: \theta &\geq \theta_0 \\ H_a: \theta &< \theta_0\end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. Let $T = X_{(n)}$. To define our test, we seek the value k such that

$$Pr(T < k | \theta_0) = \alpha$$

Using the distribution function derived above, we have

$$\begin{aligned}\frac{k^n}{\theta_0^n} &= \alpha \\ k^n &= \theta_0^n \alpha \\ k &= \theta_0 \alpha^{1/n}\end{aligned}$$

Therefore, our test function is

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} < \theta_0 \alpha^{1/n} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is piecewise and looks like:

For $0 < \theta < k$

$$\beta(\theta) = 1$$

For $\theta \geq k$

$$\begin{aligned}\beta(\theta) &= Pr(\mathbf{X} \in RR) = Pr\left(X_{(n)} < \theta_0 \alpha^{1/n} \middle| \theta\right) \\ &= \frac{\theta_0^n \alpha}{\theta^n}\end{aligned}$$

3.3.3 Alternative: Not equal to

Suppose that $X_i \stackrel{iid}{\sim} \text{Unif}(0, \theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned}H_0: \theta &= \theta_0 \\ H_a: \theta &\neq \theta_0\end{aligned}$$

Consider the following test function:

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} < k_1 \text{ or } X_{(n)} > k_2 \\ 0 & \text{else} \end{cases},$$

where k_1 and k_2 are chosen such that $Pr(\phi(\mathbf{X}) = 1 | \theta_0) = \alpha$. We choose a symmetric test here. Let $T = X_{(n)}$. To define our test, we seek the values k_1 and k_2 such that $Pr(T < k_1) = \alpha/2$ and $Pr(T > k_2) = \alpha/2$. Using the distribution function derived above, we have

$$\begin{aligned} \frac{k_1^n}{\theta_0^n} &= \alpha/2 \\ k_1^n &= \theta_0^n \alpha/2 \\ k_1 &= \theta_0 (\alpha/2)^{1/n} \end{aligned}$$

Similarly,

$$\begin{aligned} 1 - \frac{k_2^n}{\theta_0^n} &= \alpha/2 \\ \frac{k_2^n}{\theta_0^n} &= 1 - \alpha/2 \\ k_2^n &= \theta_0^n (1 - \alpha/2) \\ k_2 &= \theta_0 (1 - \alpha/2)^{1/n} \end{aligned}$$

Therefore, our test function is

$$\phi(\mathbf{X}) = \begin{cases} 1 & X_{(n)} < \theta_0 (\alpha/2)^{1/n} \text{ or } X_{(n)} > \theta_0 (1 - \alpha/2)^{1/n} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is piecewise and looks like:

For $0 < \theta \leq k_1$

$$\beta(\theta) = 1$$

For $k_1 < \theta \leq k_2$

$$\begin{aligned} \beta(\theta) &= Pr(\mathbf{X} \in RR) = Pr\left(X_{(n)} < \theta_0 (1 - \alpha/2)^{1/n} \middle| \theta\right) - Pr\left(X_{(n)} < \theta_0 (1 - \alpha/2)^{1/n} \middle| \theta\right) \\ &= \frac{\theta_0^n \alpha}{\theta^n} \end{aligned}$$