Power App Derivations

1 Exponential Distribution

1.1 Statistic: $\sum_{i=1}^n X_i$

1.1.1 Alternative: Greater than

Suppose that $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta \leq \theta_0$$
$$H_a\colon\ \theta > \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. It can be shown that if $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, then $\sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$. Using this relationship, we derive the test.

$$Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = Pr\left(\sum_{i=1}^n X_i > k \middle| \theta_0\right) = \alpha$$

Let $T = \sum_{i=1}^n X_i \sim \text{Gamma}(n, \theta)$. To define our test, we seek the value k such that

$$Pr(T > k|\theta_0) = 1 - Pr(T \leq k|\theta_0) = \alpha$$

The value of k that satisfies this equation is the $(1-\alpha)^{th}$ quantile of the Gamma(n, θ_0) distribution. Letting $\Gamma_{n,\theta_0,1-\alpha}$ denote this value, our test functions becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i > \Gamma_{n,\theta_0,1-\alpha} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = Pr\left(\sum_{i=1}^n X_i > \Gamma_{n,\theta_0,1-\alpha} \middle| \theta\right) = 1 - Pr\left(\sum_{i=1}^n X_i \leq \Gamma_{n,\theta_0,1-\alpha} \middle| \theta\right)$$

1.1.2 Alternative: Less than

Suppose that $X_i \stackrel{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta \geq \theta_0$$
$$H_a\colon\ \theta < \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^{n} X_i < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. Let $T = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \theta)$. To define our test, we seek the value k such that

$$Pr(T < k|\theta_0) = \alpha$$

The value of k that satisfies this equation is the α^{th} quantile of the Gamman, θ_0 distribution. Letting $\Gamma_{n,\theta_0,\alpha}$ denote this value, our test functions becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^{n} X_i < \Gamma_{n,\theta_0,\alpha} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = Pr\left(\sum_{i=1}^{n} X_i < \Gamma_{n,\theta_0,\alpha} \middle| \theta\right)$$

1.1.3 Alternative: Not equal to

Suppose that $X_i \overset{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon \theta = \theta_0$$
$$H_a\colon \theta \neq \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^{n} X_i < k_1 \text{ or } \sum_{i=1}^{n} X_i > k_2 \\ 0 & \text{else} \end{cases},$$

where k_1 and k_2 are chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. In this case, we assume a symmetric test function in that k_1 and k_2 are chosen such that $Pr(\sum_{i=1}^{n} X_i < k_1) = \alpha/2$ and $Pr(\sum_{i=1}^{n} X_i > k_2) = \alpha/2$. Let $T = \sum_{i=1}^{n} X_i \sim \text{Gamma}(n, \theta)$.
The values of k_1 and k_2 that satisfy these equations are the $(\alpha/2)^{th}$ and $(1 - \alpha/2)^{th}$ quantiles of the Gamman, θ_0 distribution. Letting $\Gamma_{n,\theta_0,\alpha/2}$ and $\Gamma_{n,\theta_0,1-\alpha/2}$ denote these values, our test functions becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^{n} X_i < \Gamma_{n,\theta_0,\alpha/2} \text{ or } \sum_{i=1}^{n} X_i > \Gamma_{n,\theta_0,1-\alpha/2} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = 1 - Pr\left(\sum_{i=1}^{n} X_i < \Gamma_{n,\theta_0,1-\alpha/2}\right) + Pr\left(\sum_{i=1}^{n} X_i > \Gamma_{n,\theta_0,\alpha/2} \middle| \theta\right)$$

1.2 Statistic: $X_{(1)}$

1.2.1 Alternative: Greater than

Suppose that $X_i \overset{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta \leq \theta_0$$
$$H_a\colon\ \theta > \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(1)} > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. It can be shown that if $X_i \overset{iid}{\sim} \text{Exp}(\theta)$, then $X_{(1)} \sim \text{Exp}(\frac{\theta}{n})$. Using this relationship, we derive the test.

$$Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = Pr\left(X_{(1)} > k \middle| \theta_0\right) = \alpha$$

Let $T = X_{(1)} \sim \text{Exp}(\frac{\theta}{n})$. To define our test, we seek the value k such that

$$Pr(T > k|\theta_0) = 1 - Pr(T \leq k|\theta_0) = \alpha$$

The value of k that satisfies this equation is the $(1-\alpha)^{th}$ quantile of the $\text{Exp}(\frac{\theta_0}{n})$ distribution. Letting $\eta_{\theta_0/n,1-\alpha}$ denote this value, our test functions becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(1)} > \eta_{\theta_0/n,1-\alpha} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = Pr\left(X_{(1)} > \eta_{\theta_0/n,1-\alpha} \middle| \theta\right) = 1 - Pr\left(X_{(1)} \leq \eta_{\theta_0/n,1-\alpha} \middle| \theta\right)$$

1.2.2 Alternative: Less than

Suppose that $X_i \overset{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta \geq \theta_0$$
$$H_a\colon\ \theta < \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(1)} < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. Let $T = X_{(1)} \sim \text{Exp}(\frac{\theta}{n})$. To define our test, we seek the value k such that

$$Pr(T < k|\theta_0) = \alpha$$

The value of k that satisfies this equation is the α^{th} quantile of the $\text{Exp}(\frac{\theta_0}{n})$ distribution. Letting $\eta_{\theta_0/n,\alpha}$ denote this value, our test functions becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(1)} < \eta_{\theta_0/n,\alpha} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = Pr\left(X_{(1)} < \eta_{\theta_0/n,\alpha}\middle|\theta\right)$$

1.2.3 Alternative: Not equal to

Suppose that $X_i \overset{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta = \theta_0$$
$$H_a\colon\ \theta \neq \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(1)} < k_1 \text{ or } X_{(1)} > k_2 \\ 0 & \text{else} \end{cases},$$

where k_1 and k_2 are chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. In this case, we choose k_1 and k_2 such that our test is symmetric. Let $T = X_{(1)} \sim \text{Exp}(\frac{\theta}{n})$. To define our test, we seek the values k_1 and k_2 such that $Pr(T < k_1|\theta_0) = \alpha/2$ and $Pr(T > k_2|\theta_0) = \alpha/2$.
The values of k_1 and k_2 that satisfy these equations are the $(\alpha/2)^{th}$ and $(1-\alpha/2)^{th}$ quantiles of the $\text{Exp}(\frac{\theta_0}{n})$ distribution. Letting $\eta_{\theta_0/n,\alpha/2}$ and $\eta_{\theta_0/n,1-\alpha/2}$ denote this value, our test functions becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(1)} < \eta_{\theta_0/n,\alpha/2} \text{ or } X_{(1)} > \eta_{\theta_0/n,1-\alpha/2} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = 1 - Pr\left(X_{(1)} < \eta_{\theta_0/n,1-\alpha/2}\right) + Pr\left(X_{(1)} < \eta_{\theta_0/n,\alpha/2}\middle|\theta\right)$$

1.3 Statistic: $X_{(n)}$

1.3.1 Alternative: Greater than

Suppose that $X_i \overset{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta \leq \theta_0$$
$$H_a\colon\ \theta > \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. To find this value of k, we need to know the distribution of the sample max. By definition,

$$F_{X_{(n)}}(x) = [F_X(x)]^n = \left[1 - \exp(-\frac{x}{\theta})\right]^n$$

Let $T = X_{(n)}$. To define our test, we seek the value k such that

$$Pr(T > k|\theta_0) = 1 - Pr(T \leq k|\theta_0) = \alpha$$

Using the distribution function derived above, we have

$$
\begin{aligned}
1-\left[1-\exp(-\frac{k}{\theta_0})\right]^n &= \alpha \\
\left[1-\exp(-\frac{k}{\theta_0})\right]^n &= 1-\alpha \\
1-\exp(-\frac{k}{\theta_0}) &= (1-\alpha)^{1/n} \\
\exp(-\frac{k}{\theta_0}) &= 1-(1-\alpha)^{1/n} \\
-\frac{k}{\theta_0} &= \log(1-(1-\alpha)^{1/n}) \\
k &= -\theta_0 \log(1-(1-\alpha)^{1/n})
\end{aligned}
$$

Therefore, our test function is

$$
\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} > -\theta_0 \log(1-(1-\alpha)^{1/n}) \\ 0 & \text{else} \end{cases}
$$

Therefore, our power function is

$$
\begin{aligned}
\beta(\theta) = Pr(\boldsymbol{X} \in RR) &= Pr\left(X_{(n)} > -\theta_0 \log(1-(1-\alpha)^{1/n}) \middle| \theta\right) \\
&= 1 - Pr\left(X_{(n)} \leq -\theta_0 \log(1-(1-\alpha)^{1/n}) \middle| \theta\right) \\
&= 1 - \left[1 - \exp\left(\frac{-\theta_0 \log(1-(1-\alpha)^{1/n})}{\theta}\right)\right]^n
\end{aligned}
$$

1.3.2 Alternative: Less than

Suppose that $X_i \overset{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$
\begin{aligned}
H_0&\colon\ \theta \geq \theta_0 \\
H_a&\colon\ \theta < \theta_0
\end{aligned}
$$

Consider the following test function:

$$
\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} < k \\ 0 & \text{else} \end{cases},
$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. Let $T = X_{(n)}$. To define our test, we seek the value k such that

$$
Pr(T < k|\theta_0) = \alpha
$$

Using the distribution function derived above, we have

$$\begin{aligned}
\left[1 - \exp(-\frac{k}{\theta_0})\right]^n &= \alpha \\
1 - \exp(-\frac{k}{\theta_0}) &= \alpha^{1/n} \\
\exp(-\frac{k}{\theta_0}) &= 1 - \alpha^{1/n} \\
-\frac{k}{\theta_0} &= \log(1 - \alpha^{1/n}) \\
k &= -\theta_0 \log(1 - \alpha^{1/n})
\end{aligned}$$

Therefore, our test function is

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} < -\theta_0 \log(1 - \alpha^{1/n}) \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\begin{aligned}
\beta(\theta) = Pr(\boldsymbol{X} \in RR) &= Pr\left(X_{(n)} < -\theta_0 \log(1 - \alpha^{1/n}) \middle| \theta\right) \\
&= Pr\left(X_{(n)} < -\theta_0 \log(1 - \alpha^{1/n}) \middle| \theta\right) \\
&= \left[1 - \exp\left(\frac{-\theta_0 \log(1 - \alpha^{1/n})}{\theta}\right)\right]^n
\end{aligned}$$

1.3.3 Alternative: Not equal to

Suppose that $X_i \overset{iid}{\sim} \text{Exp}(\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned}
&H_0\colon\ \theta = \theta_0 \\
&H_a\colon\ \theta \neq \theta_0
\end{aligned}$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} < k_1 \text{ or } X_{(n)} > k_2 \\ 0 & \text{else} \end{cases},$$

where k_1 and k_2 are chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. In this case, we choose a symmetric test function. Let $T = X_{(n)}$. To define our test, we seek the values k_1 and k_2 such that $Pr(T < k_1|\theta_0) = \alpha/2$ and $Pr(T > k_2|\theta_0) = \alpha/2$. Using the distribution function derived above, we have

$$\begin{aligned}
\left[1 - \exp(-\frac{k_1}{\theta_0})\right]^n &= \alpha/2 \\
1 - \exp(-\frac{k_1}{\theta_0}) &= (\alpha/2)^{1/n} \\
\exp(-\frac{k_1}{\theta_0}) &= 1 - (\alpha/2)^{1/n} \\
-\frac{k_1}{\theta_0} &= \log(1 - (\alpha/2)^{1/n}) \\
k_1 &= -\theta_0 \log(1 - (\alpha/2)^{1/n})
\end{aligned}$$

Similarly,

$$\begin{aligned}
1 - \left[1 - \exp(-\frac{k_2}{\theta_0})\right]^n &= \alpha/2 \\
\left[1 - \exp(-\frac{k_2}{\theta_0})\right]^n &= 1 - \alpha/2 \\
1 - \exp(-\frac{k_2}{\theta_0}) &= (1 - \alpha/2)^{1/n} \\
\exp(-\frac{k_2}{\theta_0}) &= 1 - (1 - \alpha/2)^{1/n} \\
-\frac{k_2}{\theta_0} &= \log(1 - (1 - \alpha/2)^{1/n}) \\
k_2 &= -\theta_0 \log(1 - (1 - \alpha/2)^{1/n})
\end{aligned}$$

Therefore, our test function is

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} < -\theta_0 \log(1 - (\alpha/2)^{1/n}) \text{ or } X_{(n)} > -\theta_0 \log(1 - (1 - \alpha/2)^{1/n}) \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\begin{aligned}
\beta(\theta) &= Pr(\boldsymbol{X} \in RR) \\
&= 1 - Pr\left(X_{(n)} < -\theta_0 \log(1 - (1 - \alpha/2)^{1/n} \middle| \theta\right) + Pr\left(X_{(n)} < -\theta_0 \log(1 - (\alpha/2)^{1/n}) \middle| \theta\right) \\
&= 1 - \left[1 - \exp\left(\frac{-\theta_0 \log(1 - (1 - \alpha/2)^{1/n})}{\theta}\right)\right]^n + \left[1 - \exp\left(\frac{-\theta_0 \log(1 - (\alpha/2)^{1/n})}{\theta}\right)\right]^n
\end{aligned}$$

2 Normal Distribution

2.1 Statistic: $\sum_{i=1}^{n} X_i$

2.1.1 Alternative: Greater than

Suppose that $X_i \overset{iid}{\sim} \mathrm{N}(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta \leq \theta_0$$
$$H_a\colon\ \theta > \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^{n} X_i > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. It can be shown that if $X_i \overset{iid}{\sim} \mathrm{N}(\theta, \sigma^2)$, then $\sum_{i=1}^{n} X_i \sim \mathrm{N}(n\theta, n\sigma^2)$. Using this relationship, we derive the test.

$$Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = Pr\left(\sum_{i=1}^{n} X_i > k \middle| \theta_0\right) = \alpha$$

Let $T = \sum_{i=1}^{n} X_i \sim \mathrm{N}(n\theta, n\sigma^2)$. To define our test, we seek the value k such that

$$Pr(T > k|\theta_0) = 1 - Pr(T \leq k|\theta_0) = \alpha$$

The value of k that satisfies this equation is the $(1-\alpha)^{th}$ quantile of the $\mathrm{N}(n\theta_0, n\sigma^2)$ distribution. Letting $z^*_{n\theta_0, n\sigma^2, 1-\alpha}$ denote this value, our test functions becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^{n} X_i > z^*_{n\theta_0, n\sigma^2, 1-\alpha} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = Pr\left(\sum_{i=1}^{n} X_i > z^*_{n\theta_0, n\sigma^2, 1-\alpha} \middle| \theta\right) = 1 - Pr\left(\sum_{i=1}^{n} X_i \leq z^*_{n\theta_0, n\sigma^2, 1-\alpha} \middle| \theta\right)$$

Derive in terms of typical Z test?

2.1.2 Alternative: Less than

Suppose that $X_i \overset{iid}{\sim} \mathrm{N}(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta \geq \theta_0$$
$$H_a\colon\ \theta < \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^{n} X_i < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. Let $T = \sum_{i=1}^{n} X_i \sim \mathrm{N}(n\theta, n\sigma^2)$. To define our test, we seek the value k such that

$$Pr(T < k|\theta_0) = \alpha$$

The value of k that satisfies this equation is the α^{th} quantile of the $\mathrm{N}(n\theta_0, n\sigma^2)$ distribution. Letting $z^*_{n\theta_0,n\sigma^2,\alpha}$ denote this value, our test function becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i < z^*_{n\theta_0,n\sigma^2,\alpha} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = Pr\left(\sum_{i=1}^n X_i < z^*_{n\theta_0,n\sigma^2,\alpha}\middle|\theta\right)$$

Derive in terms of typical Z test?

2.1.3 Alternative: Not equal to

Suppose that $X_i \overset{iid}{\sim} \mathrm{N}(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta = \theta_0$$
$$H_a\colon\ \theta \neq \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i > |k| \\ 0 & \text{else} \end{cases},$$

where k are is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. Note that we assume a symmetric test function in that k is chosen such that $Pr(\sum_{i=1}^n X_i < -k) = \alpha/2$ and $Pr(\sum_{i=1}^n X_i > k) = \alpha/2$. Let $T = \sum_{i=1}^n X_i \sim \mathrm{N}(n\theta, n\sigma^2)$. The value of k that satisfies this equation is the $(1-\alpha/2)^{th}$ quantile of the $\mathrm{N}n\theta_0, n\sigma^2$ distribution. Letting $z^*_{n\theta_0,n\sigma^2,1-\alpha/2}$ denote this value, our test function becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & \sum_{i=1}^n X_i < z^*_{n\theta_0,n\sigma^2,\alpha/2} \text{ or } \sum_{i=1}^n X_i > z^*_{n\theta_0,n\sigma^2,1-\alpha/2} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = 1 - Pr\left(\sum_{i=1}^n X_i < z^*_{n\theta_0,n\sigma^2,1-\alpha/2}\right) + Pr\left(\sum_{i=1}^n X_i > -z^*_{n\theta_0,n\sigma^2,1-\alpha/2}\middle|\theta\right)$$

2.2 Statistic: $X_{(1)}$

2.2.1 Alternative: Greater than

Suppose that $X_i \overset{iid}{\sim} \mathrm{N}(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta \leq \theta_0$$
$$H_a\colon\ \theta > \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(1)} > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. To find this value, we must first find the distribution of the sample minimum. By definition,

$$F_{X_{(1)}}(x) = 1 - [1 - F_X(x)]^n = 1 - [1 - \psi(x, \theta_0, \sigma^2)]^n,$$

where $\psi(x, \theta_0, \sigma^2)$ denotes the distribution function of the $N(\theta_0, \sigma^2)$ distribution. Let $T = X_{(1)}$. We seek the value k such that

$$Pr(T > k|\theta_0) = 1 - Pr(T \leq k|\theta_0) = \alpha$$

Using the distribution function derived above, we have:

$$\begin{aligned} 1 - Pr(T \leq k|\theta_0) = 1 - \left(1 - [1 - \psi(k, \theta_0, \sigma^2)]^n\right) &= \alpha \\ [1 - \psi(k, \theta_0, \sigma^2)]^n &= \alpha \\ \psi(k, \theta_0, \sigma^2) &= 1 - \alpha^{1/n} \end{aligned}$$

The value of k that satisfies this equation is the $(1 - \alpha^{1/n})^{th}$ quantile of the $\mathrm{N}(\theta_0, \sigma^2)$ distribution. Letting $z^*_{\theta_0, \sigma^2, 1-\alpha^{1/n}}$ denote this value, our test functions becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(1)} > z^*_{\theta_0, \sigma^2, 1-\alpha^{1/n}} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = Pr\left(X_{(1)} > z^*_{\theta_0, \sigma^2, 1-\alpha^{1/n}} \middle| \theta\right) = 1 - Pr\left(X_{(1)} \leq z^*_{\theta_0, \sigma^2, 1-\alpha^{1/n}} \middle| \theta\right)$$

2.2.2 Alternative: Less than

Suppose that $X_i \overset{iid}{\sim} \mathrm{N}(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned} &H_0\colon\ \theta \geq \theta_0 \\ &H_a\colon\ \theta < \theta_0 \end{aligned}$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(1)} < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. Let $T = X_{(1)}$. We seek the value k such that

$$Pr(T < k|\theta_0) = \alpha$$

Using the distribution function derived above, we have:

$$\begin{aligned} Pr(T < k|\theta_0) = 1 - [1 - \psi(k, \theta_0, \sigma^2)]^n &= \alpha \\ [1 - \psi(k, \theta_0, \sigma^2)]^n &= 1 - \alpha \\ \psi(k, \theta_0, \sigma^2) &= 1 - (1 - \alpha)^{1/n} \end{aligned}$$

The value of k that satisfies this equation is the $1 - (1 - \alpha)^{1/n}$ quantile of the $\mathrm{N}(\theta_0, \sigma^2)$ distribution. Letting

$z^*_{\theta_0,\sigma^2,1-(1-\alpha)^{1/n}}$ denote this value, our test functions becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(1)} < z^*_{\theta_0,\sigma^2,1-(1-\alpha)^{1/n}} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = Pr\left(X_{(1)} < z^*_{\theta_0,\sigma^2,1-(1-\alpha)^{1/n}} \middle| \theta\right)$$

2.2.3 Alternative: Not equal to

Suppose that $X_i \overset{iid}{\sim} \mathrm{N}(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta = \theta_0$$
$$H_a\colon\ \theta \neq \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(1)} < k_1 \text{ or } X_{(1)} > k_2 \\ 0 & \text{else} \end{cases},$$

where k_1 and k_2 are chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. We choose a symmetric test. Let $T = X_{(1)}$. We seek the values k_1 and k_2 such that $Pr(T < k_1|\theta_0) = \alpha/2$ and $Pr(T > k_2|\theta_0) = \alpha/2$. Using the distribution function derived above, we have:

$$\begin{aligned} Pr(T < k_1|\theta_0) = 1 - [1 - \psi(k_1, \theta_0, \sigma^2)]^n &= \alpha/2 \\ [1 - \psi(k_1, \theta_0, \sigma^2)]^n &= 1 - \alpha/2 \\ \psi(k_1, \theta_0, \sigma^2) &= 1 - (1 - \alpha/2)^{1/n} \end{aligned}$$

The value of k_1 that satisfies this equation is the $1 - (1 - \alpha/2)^{1/n}$ quantile of the $\mathrm{N}(\theta_0, \sigma^2)$ distribution. We let $z^*_{\theta_0,\sigma^2,1-(1-\alpha/2)^{1/n}}$ denote this value. Similarly,

$$\begin{aligned} 1 - Pr(T \leq k_2|\theta_0) = 1 - \left(1 - [1 - \psi(k_2, \theta_0, \sigma^2)]^n\right) &= \alpha/2 \\ [1 - \psi(k_2, \theta_0, \sigma^2)]^n &= \alpha/2 \\ \psi(k_2, \theta_0, \sigma^2) &= 1 - (\alpha/2)^{1/n} \end{aligned}$$

The value of k_2 that satisfies this equation is the $1 - (\alpha/2)^{1/n}$ quantile of the $\mathrm{N}(\theta_0, \sigma^2)$ distribution. We let $z^*_{\theta_0,\sigma^2,1-(\alpha/2)^{1/n}}$ denote this value. Therefore, our test function becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(1)} < z^*_{\theta_0,\sigma^2,1-(1-\alpha/2)^{1/n}} \text{ or } X_{(1)} > z^*_{\theta_0,\sigma^2,1-(\alpha/2)^{1/n}} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = 1 - Pr\left(X_{(1)} < z^*_{\theta_0,\sigma^2,1-(\alpha/2)^{1/n}} \middle| \theta\right) + Pr\left(X_{(1)} < z^*_{\theta_0,\sigma^2,1-(1-\alpha/2)^{1/n}} \middle| \theta\right)$$

2.3 Statistic: $X_{(n)}$

2.3.1 Alternative: Greater than

Suppose that $X_i \overset{iid}{\sim} \mathrm{N}(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta \leq \theta_0$$
$$H_a\colon\ \theta > \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. To find this value, we must first find the distribution of the sample maximum. By definition,

$$F_{X_{(n)}}(x) = [F_X(x)]^n = [\psi(x, \theta_0, \sigma^2)]^n,$$

where $\psi(x, \theta_0, \sigma^2)$ denotes the distribution function of the $N(\theta_0, \sigma^2)$ distribution. Let $T = X_{(n)}$. We seek the value k such that

$$Pr(T > k|\theta_0) = 1 - Pr(T \leq k|\theta_0) = \alpha$$

Using the distribution function derived above, we have:

$$1 - Pr(T \leq k|\theta_0) = 1 - [\psi(k, \theta_0, \sigma^2)]^n = \alpha$$
$$\psi(k, \theta_0, \sigma^2) = (1-\alpha)^{1/n}$$

The value of k that satisfies this equation is the $(1-\alpha)^{1/n}$ quantile of the $\mathrm{N}(\theta_0, \sigma^2)$ distribution. Letting $z^*_{\theta_0, \sigma^2, (1-\alpha)^{1/n}}$ denote this value, our test functions becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} > z^*_{\theta_0, \sigma^2, (1-\alpha)^{1/n}} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = Pr\left(X_{(n)} > z^*_{\theta_0, \sigma^2, (1-\alpha)^{1/n}} \middle| \theta\right) = 1 - Pr\left(X_{(n)} \leq z^*_{\theta_0, \sigma^2, (1-\alpha)^{1/n}} \middle| \theta\right)$$

2.3.2 Alternative: Less than

Suppose that $X_i \overset{iid}{\sim} \mathrm{N}(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta \geq \theta_0$$
$$H_a\colon\ \theta < \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. Let $T = X_{(n)}$. We seek the value k such that

$$Pr(T < k|\theta_0) = \alpha$$

Using the distribution function derived above, we have:

$$\begin{aligned} Pr(T < k|\theta_0) = [\psi(k, \theta_0, \sigma^2)]^n &= \alpha \\ \psi(k, \theta_0, \sigma^2) &= \alpha^{1/n} \end{aligned}$$

The value of k that satisfies this equation is the $\alpha^{1/n}$ quantile of the $\mathrm{N}(\theta_0, \sigma^2)$ distribution. Letting $z^*_{\theta_0,\sigma^2,\alpha^{1/n}}$ denote this value, our test functions becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} > z^*_{\theta_0,\sigma^2,\alpha^{1/n}} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = Pr\left(X_{(n)} < z^*_{\theta_0,\sigma^2,\alpha^{1/n}} \middle| \theta\right)$$

2.3.3 Alternative: Not equal to

Suppose that $X_i \overset{iid}{\sim} \mathrm{N}(\theta, \sigma^2)$, where σ^2 is known, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned} &H_0\colon\ \theta = \theta_0 \\ &H_a\colon\ \theta \neq \theta_0 \end{aligned}$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} < k_1 \text{ or } X_{(n)} > k_2 \\ 0 & \text{else} \end{cases},$$

where k_1 and k_2 are chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. We choose a symmetric test. Let $T = X_{(n)}$. We seek the values k_1 and k_2 such that $Pr(T < k_1|\theta_0) = \alpha/2$ and $Pr(T > k_2|\theta_0) = \alpha/2$. Using the distribution function derived above, we have:

$$\begin{aligned} Pr(T < k_1|\theta_0) = [\psi(k_1, \theta_0, \sigma^2)]^n &= \alpha/2 \\ \psi(k_1, \theta_0, \sigma^2) &= (\alpha/2)^{1/n} \end{aligned}$$

The value of k_1 that satisfies this equation is the $(\alpha/2)^{1/n}$ quantile of the $\mathrm{N}(\theta_0, \sigma^2)$ distribution. We let $z^*_{\theta_0,\sigma^2,(\alpha/2)^{1/n}}$ denote this value. Similarly,

$$\begin{aligned} 1 - Pr(T \leq k_2|\theta_0) = 1 - [\psi(k_2, \theta_0, \sigma^2)]^n &= \alpha/2 \\ [\psi(k_2, \theta_0, \sigma^2)]^n &= 1 - \alpha/2 \\ \psi(k_2, \theta_0, \sigma^2) &= (1 - \alpha/2)^{1/n} \end{aligned}$$

The value of k_2 that satisfies this equation is the $(1-\alpha/2)^{1/n}$ quantile of the $\mathrm{N}(\theta_0, \sigma^2)$ distribution. We let $z^*_{\theta_0,\sigma^2,(1-\alpha/2)^{1/n}}$ denote this value. Therefore, our test function becomes

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} < z^*_{\theta_0,\sigma^2,(\alpha/2)^{1/n}} \text{ or } X_{(n)} > z^*_{\theta_0,\sigma^2,(1-\alpha/2)^{1/n}} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is

$$\beta(\theta) = Pr(\boldsymbol{X} \in RR) = 1 - Pr\left(X_{(n)} < z^*_{\theta_0,\sigma^2,(1-\alpha/2)^{1/n}} \middle| \theta\right) + Pr\left(X_{(n)} < z^*_{\theta_0,\sigma^2,(\alpha/2)^{1/n}} \middle| \theta\right)$$

3 Uniform Distribution

3.1 Statistic: $\sum_{i=1}^{n} X_i$

3.1.1 Alternative: Greater than

3.1.2 Alternative: Less than

3.1.3 Alternative: Not equal to

3.2 Statistic: $X_{(1)}$

3.2.1 Alternative: Greater than

3.2.2 Alternative: Less than

3.2.3 Alternative: Not equal to

3.3 Statistic: $X_{(n)}$

3.3.1 Alternative: Greater than

Suppose that $X_i \overset{iid}{\sim} \text{Unif}(0, \theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$H_0\colon\ \theta \leq \theta_0$$
$$H_a\colon\ \theta > \theta_0$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} > k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. To find this value of k, we need to know the distribution of the sample max. By definition,

$$F_{X_{(n)}}(x) = [F_X(x)]^n = \left[\frac{x}{\theta}\right]^n = \frac{x^n}{\theta^n}$$

Let $T = X_{(n)}$. To define our test, we seek the value k such that

$$Pr(T > k|\theta_0) = 1 - Pr(T \leq k|\theta_0) = \alpha$$

Using the distribution function derived above, we have

$$\begin{aligned} 1 - \frac{k^n}{\theta_0^n} &= \alpha \\ \frac{k^n}{\theta_0^n} &= 1 - \alpha \\ k^n &= \theta_0^n(1 - \alpha) \\ k &= \theta_0(1 - \alpha)^{1/n} \end{aligned}$$

Therefore, our test function is

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} > \theta_0(1 - \alpha)^{1/n} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is piecewise and looks like the following:
For $0 < \theta < k$:

$$\beta(\theta) = 0$$

For $\theta > k$:

$$\begin{aligned}\beta(\theta) &= Pr(\boldsymbol{X} \in RR) = Pr\left(X_{(n)} > \theta_0(1-\alpha)^{1/n} \middle| \theta\right)\\ &= 1 - Pr\left(X_{(n)} \leq \theta_0(1-\alpha)^{1/n} \middle| \theta\right)\\ &= 1 - \frac{\theta_0^n(1-\alpha)}{\theta^n}\end{aligned}$$

3.3.2 Alternative: Less than

Suppose that $X_i \overset{iid}{\sim} \text{Unif}(0,\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned}&H_0\colon\ \theta \geq \theta_0\\ &H_a\colon\ \theta < \theta_0\end{aligned}$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} < k \\ 0 & \text{else} \end{cases},$$

where k is chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. Let $T = X_{(n)}$. To define our test, we seek the value k such that

$$Pr(T < k|\theta_0) = \alpha$$

Using the distribution function derived above, we have

$$\begin{aligned}\frac{k^n}{\theta_0^n} &= \alpha\\ k^n &= \theta_0^n\alpha\\ k &= \theta_0\alpha^{1/n}\end{aligned}$$

Therefore, our test function is

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} < \theta_0\alpha^{1/n} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is piecewise and looks like:
For $0 < \theta < k$

$$\beta(\theta) = 1$$

For $\theta \geq k$

$$\begin{aligned}\beta(\theta) &= Pr(\boldsymbol{X} \in RR) = Pr\left(X_{(n)} < \theta_0\alpha^{1/n} \middle| \theta\right)\\ &= \frac{\theta_0^n\alpha}{\theta^n}\end{aligned}$$

3.3.3 Alternative: Not equal to

Suppose that $X_i \overset{iid}{\sim} \text{Unif}(0,\theta)$, and we take a random sample of size n from this population. Further suppose that we wish to test the following hypotheses:

$$\begin{aligned}&H_0\colon\ \theta = \theta_0\\ &H_a\colon\ \theta \neq \theta_0\end{aligned}$$

Consider the following test function:

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} < k_1 \text{ or } X_{(n)} > k_2 \\ 0 & \text{else} \end{cases},$$

where k_1 and k_2 are chosen such that $Pr(\phi(\boldsymbol{X}) = 1|\theta_0) = \alpha$. We choose a symmetric test here. Let $T = X_{(n)}$. To define our test, we seek the values k_1 and k_2 such that $Pr(T < k_1) = \alpha/2$ and $Pr(T > k_2) = \alpha/2$. Using the distribution function derived above, we have

$$\begin{aligned} \frac{k_1^n}{\theta_0^n} &= \alpha/2 \\ k_1^n &= \theta_0^n \alpha/2 \\ k_1 &= \theta_0(\alpha/2)^{1/n} \end{aligned}$$

Similarly,

$$\begin{aligned} 1 - \frac{k_2^n}{\theta_0^n} &= \alpha/2 \\ \frac{k^n}{\theta_0^n} &= 1 - \alpha/2 \\ k_2^n &= \theta_0^n(1 - \alpha/2) \\ k_2 &= \theta_0(1 - \alpha/2)^{1/n} \end{aligned}$$

Therefore, our test function is

$$\phi(\boldsymbol{X}) = \begin{cases} 1 & X_{(n)} < \theta_0(\alpha/2)^{1/n} \text{ or } X_{(n)} > \theta_0(1 - \alpha/2)^{1/n} \\ 0 & \text{else} \end{cases}$$

Therefore, our power function is piecewise and looks like:
For $0 < \theta \leq k_1$

$$\beta(\theta) = 1$$

For $k_1 < \theta \leq k_2$

$$\begin{aligned} \beta(\theta) = Pr(\boldsymbol{X} \in RR) &= Pr\left(X_{(n)} < \theta_0(\alpha/2)^{1/n} \middle| \theta\right) \\ &= \frac{\theta_0^n \alpha}{2\theta^n} \end{aligned}$$

For $\theta > k_2$

$$\begin{aligned} \beta(\theta) = Pr(\boldsymbol{X} \in RR) &= Pr\left(X_{(n)} > \theta_0(1 - \alpha/2)^{1/n} \middle| \theta\right) + Pr\left(X_{(n)} < \theta_0(\alpha/2)^{1/n} \middle| \theta\right) \\ &= 1 - Pr\left(X_{(n)} \leq \theta_0(1 - \alpha/2)^{1/n} \middle| \theta\right) + Pr\left(X_{(n)} < \theta_0(\alpha/2)^{1/n} \middle| \theta\right) \\ &= 1 - \frac{\theta_0^n(1 - \alpha/2)}{\theta^n} + \frac{\theta_0^n \alpha}{2\theta^n} \end{aligned}$$