4.2. Wibracje akustyczne warstwy materiału Grzegorz Filipek

$\boxed{-u''(x)-u(x)=\sin x}$, $u(0)=0$, $u'(2)-u(2)=0$, $N=3$

$\Downarrow$

$u'(2)=u(2)$

$\Omega=[0,2]$

$u: [0,2]\ni x \to u(x)\in\mathbb{R}$

Ponieważ mamy warunek Dirichleta z lewej strony ~~...~~ za przestrzeń V przyjmujemy tę przestrzeń, ~~w~~ której funkcje zerują się na prawym brzegu. ~~...~~ Mnożymy nasze równanie ~~p~~ razy v i całkujemy po Ω:

$$-\int_0^2 u''v\,dx-\int_0^2 uv\,dx=\int_0^2 \sin x\, v\,dx$$

$$-\left([u'v]_0^2-\int_0^2 u'v'dx\right)-\int_0^2 uv\,dx=\int_0^2 \sin x\, v\,dx$$

$$-\underbrace{u'(2)}_{=u(2)\text{ na mocy warunku Robina}}v(2)+u'(0)\underbrace{v(0)}_{=0\text{, ponieważ } v\in V}+\int_0^2 u'v'dx-\int_0^2 uv\,dx=\int_0^2 \sin x\, v\,dx$$

$$\underbrace{-u(2)v(2)+\int_0^2 u'v'dx-\int_0^2 uv\,dx}_{B(u,v)}=\underbrace{\int_0^2 \sin x\, v\,dx}_{L(v)}$$

$u\approx u_0e_0+u_1e_1+u_2e_2+u_3e_3=w$, ale ponieważ $u(0)=0$, mamy

$u\approx u_1e_1+u_2e_2+u_3e_3=w$,

czyli $w\in \mathrm{Lin}\{e_1,e_2,e_3\}$

$B(u,v)\approx B(w,u)=L(v)$

$B\left(\sum_{i=1}^{3}u_ie_i, e_j\right)=L(e_j)$

$\sum_{i=1}^{3}\left[\cancel{B}u_i B(e_i,e_j)\right]=L(e_j)$

$u_1B(e_1,e_j)+u_2B(e_2,e_j)+u_3B(e_3,e_j)=L(e_j)$

~~dla~~ (dla $j=1,2,3$ i po zapisaniu w postaci macierzowej $\longrightarrow$

$$\begin{bmatrix}B_{11}&B_{21}&B_{31}\\B_{12}&B_{22}&B_{32}\\B_{13}&B_{23}&B_{33}\end{bmatrix}\cdot\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}L_1\\L_2\\L_3\end{bmatrix}$$

gdzie $B_{ij}=B(e_i,e_j)$ oraz $L_i=L(e_i)$

$u, v \in \{e_1, e_2, \ldots, e_i, \ldots, e_{n-1}, e_n\}$, $h = x_i - x_{i-1}$, $i \in \{1, 2, \ldots, n\}$

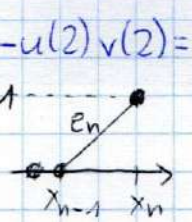
$$e_i = \begin{cases} \frac{x - x_{i-1}}{x_i - x_{i-1}} & , x \in (x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{x_{i+1} - x_i} & , x \in (x_i, x_{i+1}) \\ 0 & , \text{wpp} \end{cases} = \begin{cases} \frac{x - x_{i-1}}{h} & , x \in (x_{i-1}, x_i) \\ \frac{x_{i+1} - x}{h} & , x \in (x_i, x_{i+1}) \\ 0 & , \text{wpp} \end{cases}$$

$$e_i' = \begin{cases} \frac{1}{h} & , x \in (x_{i-1}, x_i) \\ \frac{-1}{h} & , x \in (x_i, x_{i+1}) \\ 0 & , \text{wpp} \end{cases}$$

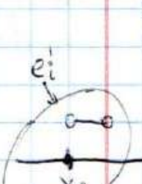
$$B(u,v) = -u(2)v(2) + \int_0^2 u'v' dx - \int_0^2 uv dx$$

Teraz analitycznie wyznaczę wartości wszystkich wyrażeń, aby móc obliczyć $B(u,v)$ dla każdego $u,v \in \{e_1, e_2, \ldots, e_n\}$

$$-u(2)v(2) = -e_i(2)e_j(2) = \begin{cases} -e_n(2) & , i=j=n \\ 0 & , \text{wpp} \end{cases} = \begin{cases} -e_n(x_n) & , i=j=n \\ 0 & , \text{wpp} \end{cases} = \begin{cases} -1 & , i=j=n \\ 0 & , \text{wpp} \end{cases}$$



$$\int_0^2 u'v' dx = \int_0^2 e_i' e_j' dx = \begin{cases} \text{dla } |i-j| \geq 2 : 0 \\ \text{dla } |i-j| = 1 : \int_{x_i}^{x_{i+1}} e_i' e_{i+1}' dx = \int_{x_i}^{x_{i+1}} \left(\frac{-1}{h}\right) \cdot \frac{1}{h} dx = -\frac{1}{h^2} \cdot [x]_{x_i}^{x_{i+1}} = -\frac{1}{h} \\ \text{dla } i=j \neq n : \int_{x_{i-1}}^{x_{i+1}} (e_i')^2 dx = \int_{x_{i-1}}^{x_{i+1}} \frac{1}{h^2} dx = \frac{1}{h^2} \cdot 2h = \frac{2}{h} \\ \text{dla } i=j=n : \frac{1}{h} \end{cases}$$



$$\int_0^2 uv dx = \begin{cases} ① & , |i-j| \geq 2 \\ ② & , |i-j| = 1 \\ ③ & , i=j \neq n \\ ④ & , i=j=n \end{cases}$$

①, $\int_0^2 e_i e_j dx = 0$

②, $\int_0^2 e_i e_{i+1} dx = \int_{x_i}^{x_{i+1}} e_i e_{i+1} dx = \int_{x_i}^{x_{i+1}} \frac{x_{i+1} - x}{x_{i+1} - x_i} \cdot \frac{x - x_i}{x_{i+1} - x_i} dx =$

$= \frac{1}{h^2} \int_{x_i}^{x_{i+1}} (x_{i+1}x - x_{i+1}x_i - x^2 + x_i x) dx =$

$= \frac{1}{h^2} \int_{x_i}^{x_{i+1}} (-x^2 + x(x_{i+1} + x_i) - x_{i+1}x_i) dx = \frac{1}{h^2} \left[-\frac{1}{3}x^3 + \frac{1}{2}x^2(x_{i+1} + x_i) - x_{i+1}x_i x \right]_{x_i}^{x_{i+1}} =$

$= \frac{1}{h^2} \left(-\frac{1}{3}x_{i+1}^3 + \frac{1}{2}x_{i+1}^3 + \frac{1}{2}x_{i+1}^2 x_i - x_{i+1}^2 x_i - \left(-\frac{1}{3}x_i^3 + \frac{1}{2}x_{i+1}x_i^2 + \frac{1}{2}x_i^3 - x_{i+1}x_i^2 \right) \right) =$

$$=\frac{1}{h^2}\left(\frac{1}{6}x_{i+1}^3-\frac{1}{2}x_{i+1}^2x_i+\frac{1}{2}x_{i+1}x_i^2-\frac{1}{6}x_i^3=\frac{1}{6h^2}\left(x_{i+1}^3-3x_{i+1}^2x_i+3x_{i+1}x_i^2-x_i^3\right)=\right.$$

$$=\frac{1}{6h^2}(x_{i+1}-x_i)^3=\frac{1}{6\cancel{h^2}}\cdot h^{\cancel{3}}=\frac{h}{6}$$

(3), $$\int_0^2 e_i^2\,dx=\int_{x_{i-1}}^{x_i}\left(\frac{x-x_{i-1}}{x_i-x_{i-1}}\right)^2dx+\int_{x_i}^{x_{i+1}}\left(\frac{x_{i+1}-x}{x_{i+1}-x_i}\right)^2dx=\frac{1}{h^2}\left(\int_{x_{i-1}}^{x_i}(x^2-2x_{i-1}x+x_{i-1}^2)dx+\int_{x_i}^{x_{i+1}}(x_{i+1}^2-2x_{i+1}x+x^2)dx\right)=$$

$$=\frac{1}{h^2}\left(\left[\frac{1}{3}x^3-x_{i-1}x^2+x_{i-1}^2x\right]_{x_{i-1}}^{x_i}+\left[x_{i+1}^2x-x_{i+1}x^2+\frac{1}{3}x^3\right]_{x_i}^{x_{i+1}}\right)=$$

$$=\frac{1}{h^2}\left(\frac{1}{3}x_i^3-x_{i-1}x_i^2+x_{i-1}^2x_i-\left(\frac{1}{3}x_{i-1}^3-\cancel{x_{i-1}^3}+\cancel{x_{i-1}^3}\right)+\cancel{x_{i+1}^3}-\cancel{x_{i+1}^3}+\frac{1}{3}x_{i+1}^3-\left(x_{i+1}^2x_i-x_{i+1}x_i^2+\frac{1}{3}x_i^3\right)\right)=$$

$$=\frac{1}{3h^2}\left(\underbrace{x_i^3-x_i^2x_{i-1}+x_ix_{i-1}^2-x_{i-1}^3}+\underbrace{x_{i+1}^3-x_{i+1}^2x_i+x_{i+1}x_i^2-x_i^3}\right)=$$

$$=\frac{1}{3h^2}\cdot\left((x_i-x_{i-1})^3+(x_{i+1}-x_i)^3\right)=\frac{h^3+h^3}{3h^2}=\frac{2h}{3}$$

(4), $$\int_0^2 e_n^2\,dx=\frac{1}{3}h$$

$$L(v)=\int_0^2 \sin x\cdot v\,dx$$

$$L(e_i)=L_i=\int_0^2\sin x\cdot e_i\,dx=\int_{x_{i-1}}^{x_i}\sin x\cdot\frac{x-x_{i-1}}{h}dx+\int_{x_i}^{x_{i+1}}\sin x\cdot\frac{x_{i+1}-x}{h}dx=$$

$$=\frac{1}{h}\left(\int_{x_{i-1}}^{x_i}(\sin x\cdot x-\sin x\cdot x_{i-1})dx+\int_{x_i}^{x_{i+1}}(\sin x\cdot x_{i+1}-\sin x\cdot x)dx\right)=\ldots$$ obliczenia są długie i żmudne

$$=\frac{1}{h}\left(-\sin(x_{n-1})+2\sin(x_i)-\sin(x_{i+1})\right)$$

$$L(e_n)=L_n=\int_0^2\sin x\cdot e_n\,dx=\int_{x_{n-1}}^{x_n}\sin x\cdot\frac{x-x_{i-1}}{h}dx=\frac{1}{h}\left(\int_{x_{n-1}}^{x_n}(\sin x\cdot x-x_{i-1}\cdot\sin x)dx\right)=\ldots$$ żmudne obliczenia

$$=-\cos(x_n)-\frac{1}{h}\sin(x_{n-1})+\frac{1}{h}\sin(x_n)$$